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Green's function coupled with perturbation approach to dynamic analysis of inhomogeneous beams with eigenfrequency and rotational effect's investigations

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Abstract. The elastic theory of beams is fundamental in engineering of design and structure. In this study, we construct Green's function for inhomogeneous fourth−order differential operators subjected to associated constraints that arises in dealing with dynamic problems in the Rayleigh beam. We obtain solutions for homogeneous and completely inhomogeneous beam problems using Green's function. This enables us to consider rotational influences in determining the eigenfrequency of beam vibrations. Additionally, we investigate the dynamic vibration model of inhomogeneous beams incorporating rotational effects. The eigenvalues of Rayleigh beams, including first−order correction terms, are also computed and displayed in tabular forms.

Keywords: Euler−Bernoulli Beam; Green's function; inhomogeneous; perturbation; Rayleigh beam; vibrations

1. Introduction

The application of beams is widespread in the design of structures, applied mechanics, civil and mechanical engineering. The theory of elastic beams was first studied in the 18^{th} century. The simplest model of that era was the Euler−Bernoulli beam, which only took into account bending moments. The fourth−order Euler−Bernoulli beam model depicts the connection between the deflection of the beam and the applied load. The Rayleigh beam theory incorporates axial, shear and bending deformations. It considers both bending as well as shear effects in deformations. Rayleigh beam theory is more applicable in cases where shear is significant than the Euler-Bernoulli theory as involving shear and axial deformations, leading to a more accurate representation of beam behavior. Eigenvalues, or eigenfrequencies in the context of vibrations, play a crucial role in structural engineering and design, especially in the analysis of systems such as Rayleigh beam's vibration analysis, eigenvalues represent the natural frequencies at which a structure vibrates when subjected to external forces or disturbances. Understanding these

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frequencies is vital to prevent resonance. Engineers design structures to ensure that external forces do not match the eigenfrequencies, as this could lead to excessive vibrations and potential structural failure.

By analyzing the eigenvalues, we can assess the structural integrity of beams. If the eigenvalues fall within a specific range, it indicates that the structure is stable and not liable to excessive vibrations. On the other hand, eigenvalues outside the acceptable range could signal a flaw in the design or the need for structural modifications. We optimize the design of a structure by manipulating its geometry and material properties to obtain desired eigenvalues. For instance, in the case of a Rayleigh beam, parameters such as length, width, or material composition can be arranged to achieve specific eigenfrequencies, ensuring that the structure performs optimally under given conditions. Eigenvalues are associated with mode shapes, which describe the patterns of vibration in a structure. Model analysis helps to identify how different parts of a structure move during vibration. This information is important for designing components such as damping systems or tuning the structure to avoid unwanted vibrations. Changes in structural properties due to wear, fatigue, or other factors can alter eigenvalues. Monitoring eigenvalues over time allows to predict structural failures before they occur. Regular eigenfrequency analysis can thus be part of a structural health monitoring system. Eigenvalues are fundamental in predicting how a structure responds to dynamic loads, such as earthquakes or wind. Understanding these responses enables engineers to design buildings and bridges that can withstand various environmental forces.

In the realm of literature, we have a glance on the previous work. (Alshorbagy *et al.* 2011) examined the dynamic behavior and free vibrations of functionally graded materials numerically using the finite element method. (Bakalah *et al.* 2018) constructed Green's function and employed the perturbation technique simultaneously to investigate the dynamic and static bending transversely for fourth−order boundary value problems arising in the inhomogeneous model of the Euler−Bernoulli beam subjected to various boundary conditions, such as clamped-free, fixed-clamped, and hinged boundary conditions. However (Bakalah *et al.* 2018) did not consider the torsion effects that are present in Rayleigh model. (Rizov 2018, 2020, 2021) studied analytical solutions of nonlinear inhomogeneous functionally graded beams. He also developed J−integral approach to prove that nonlinearity is responsible for the increase of the strain energy release rate in materials. Additionally, a delamination analysis of inhomogeneous cantilever beam was also analyzed under a torsion moment. (Hadzalic *et al.* 2020) analyzed the three-dimensional thermo-hydro-mechanical coupled discrete beam model using finite element method. (Ibrahimbegovic *et al.* 2021, 2022) analyzed the Euler-Bernoulli and Rayleigh beam models via finite element method. He also considered a perturbed stochastic equation with damping term. (Cheng and Batra 2000) investigated the steady-state vibrations and buckling in the case of a simply supported functionally graded polygonal isotropic plate lying on the Pasternak-Winkler elastic basis subjected to uniform hydrostatic loads in-plane, using Reddy's third-order plate theory. The Poisson's ratio and Young's modulus for the plate material were assumed to vary only along the thickness direction, and the rotary effects of inertia were also considered. (Han *et al.* 1999) mathematically derived models based solely on bending, without considering rotary inertial effects.

Another intriguing case involves incorporating rotational effects alongside simple bending in a beam, known as the Rayleigh beam. The Euler−Bernoulli beam model differs from the Rayleigh beam model due to rotational effects. (Shariati *et al.* 2020) investigated and compared the vibration of viscoelastic axially functional graded moving Euler–Bernoulli and Rayleigh beams. (Ebrahimi Mamaghani *et al.* 2020) focused on enhancing the performance of transportation systems, considering both forced and free vibrations of axially functional graded Euler−Bernoulli and Rayleigh beams. (Russillo and Failla 2022) studied a small planar beam lattice and proposed two innovative computational approaches for examining elastic wave propagation. (Yang *et al.* 2021) presented a novel element model based on the Rayleigh theory of beams. (Jočković *et al.* 2019) dealt with a linear free vibration investigation of Euler−Bernoulli and Rayleigh curved beams employing an isogeometric approach. (Panchore 2022) solved the free vibration problem of a rotating Rayleigh beam using the meshless Petrov-Galerkin method. (Olotu *et al.* 2023) analyzed the effects of variable radical parameters on the natural frequencies of a prestressed tapered Rayleigh beam with general elastically 2 restrained terminals. (Nieves and Movchan 2023) developed a pointwise description of the system's response and demonstrated that when the separation of the resonators is small, the structure can be approximated by the generalized Rayleigh beam. (Molina-Villegas *et al.* 2023) presented the construction of Green's function stiffness technique for the static analysis of nonuniform Euler−Bernoulli frames subjected to arbitrary external loads and bending moments. (Hoskoti *et al.* 2021) discussed the free vibration analysis of a flipped, double-tapered blade mounted on a rotating disk undergoing overall motion. (Hong *et al.* 2022) provided a thorough analysis of the dynamics of axially moving beams.

One may refer to (Kato 2013) and (Rellich 1969) for a brief idea of the perturbation technique. A good resource for constructing the Green's function of the fourth−order linear model is available in (Logan 2013), (Orucoglu 2005), (Stakgold and Holst 2011) and (Taterina 2013) work. A treatise of (Love 1892) and (Truessdell 2013) provides the basics for an account of the mathematical theory of elasticity. (Sankar 2001) calculated the elasticity solution for a simply supported functionally graded beam constrained to transverse sinusoidal loading. (Xu and Ma 2017) computed the eigenfunctions corresponding to the respective eigenvalues for the fourth order linear boundary-value problem. (Yiğit *et al.* 2016) examined functionally graded composite materials using the Adomian decomposition method. The deflections of composite beams for various inhomogeneity parameters were obtained, and the resultant values showed that smaller parametric values for inhomogeneous materials were closely related to the resultant values for homogeneous materials. The rotational effects on stoneley waves have been examined numerically and graphically by (Lata and Himansi 2022) in orthotropic magneto thermo-elastic media.

In this paper, we have considered a more general model of beam incorporating a nonhomogeneous term. This has been dealt with introducing Green's function for a fourth-order differential equation and thereby finding solutions for some typical nonlinear loading under a variety of boundary conditions. The dynamic problem of vibration of beam of Rayleigh type with torsion term has been studied using perturbation technique coupled with Green's function. The eigenfrequencies have been found using different boundary conditions. Our approach is analytic rather than numerical used by other authors cited above. The methods employed in most of these studies are numerical whereas we use analytical methods. This enables us to study torsion effect on eigenfrequency of the beam. The remainder of this paper is organized as follows: Section 2 includes the formulation of the problem. In section 3 and section 4, a fourth−order boundary value problem is defined along with boundary conditions. Green's function is defined in section 5 and constructed in section 6. Solutions to inhomogeneous problems are computed and displayed graphically in section 7. Section 8 addresses solutions to completely inhomogeneous problems. Section 9 discusses the perturbation technique. In section 10 eigenvalues are computed, assuming small rotational effects in the Rayleigh beam model.

2. Problem formulation

A fourth−order differential equation is utilized to describe the behavior of a beam in various scenarios. This equation can manifest as either an ODE or a PDE. In the context of a static beam, the ODE offers a suitable model, whereas the PDE serves to model a dynamic beam. Numerous beam models exist; examples include the Euler−Bernoulli and Rayleigh beam models. The Euler−Bernoulli beam relies on linear elasticity theory and assumes transverse deflections exclusively.

For simplicity, we focus on the depiction of an elastic rectangular Euler−Bernoulli beam in cartesian coordinates, examining both static and dynamic instances across diverse boundary conditions. The static representation results in a fourth-order ordinary differential equation ODE, while the dynamic counterpart gives rise to an eigenvalue problem governed by a fourth-order operator. In the Euler−Bernoulli beam model, our consideration revolves around slight transverse deflections. If we introduce rotational effects into the beam's analysis, the resultant configuration is recognized as the Rayleigh beam.

2.1 Derivation of the static case of the Euler−Bernoulli beam equation

In this subsection, we present a concise construction of the governing equation for an Euler−Bernoulli beam. We focus on a homogeneous elastic beam with length $l = 1$ unit and a cross−sectional area A. The beam's symmetry axis aligns with the x-axis. We assume that the acting force is only its weight, say $f_1(x)$, which acts solely along the y-axis, as shown in Fig. 1.

When the weight acts along the y-axis as a load, deflection occurs connecting the same terminals as depicted in Fig. 2.

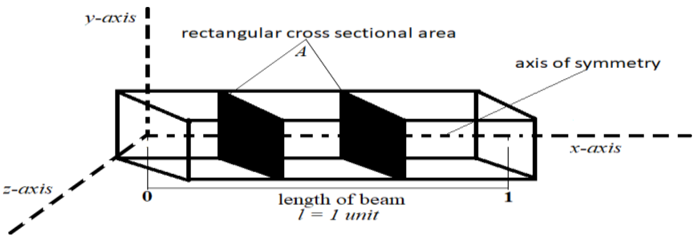

Fig. 1 A rectangular shaped elastic homogeneous beam

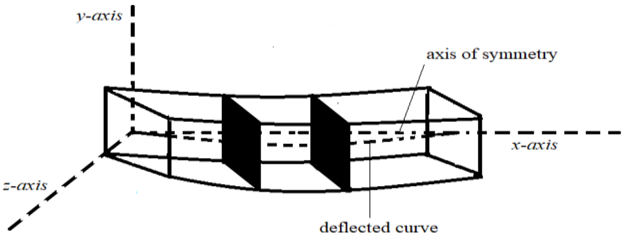

Fig. 2 Effect of weight as deflection

In the field of elastic theory, it is established that the bending moment $M(x)$ at a specific location (referred to as y) along a beam can be correlated with applied load as follows

$$\frac{d^2}{dx^2}\big(M(x)\big) = \quad f_1(x) \tag{1}$$

where, an external transverse distributed load applied to a beam is a force $f_1(x)$ that acts perpendicular to the axis of the beam and is distributed along its length. This type of load can cause a beam to bend and deflect, and it can also induce shear forces and bending moments within the beam. Additionally, the bending rule which describes the relation versus the bending moment and stress, i.e.

$$\frac{M}{I} = \frac{\tau}{z} = \frac{E}{R} \tag{2}$$

where, M is a moment of bending, I is an inertial moment, τ stands for stress, z signifies a distance from the axis of symmetry to the deflected curve of the beam, E corresponds to Young's elasticity modulus and R represents the radius of curvature. Therefore, Eq. (2) becomes

$$M(x) = EI\kappa \tag{3}$$

where, κ represents the curvature defined as the reciprocal of the radius of curvature and the EI corresponds flexural rigidity for a beam which signifies the resistance towards bending. The curvature is defined mathematically as

$$\kappa = \frac{d^2v/dx^2}{\sqrt[3]{((dv/dx)^2+1)^2}} \tag{4}$$

In this scenario, the deflection change or tangential slope, represented as dv/dx, is exceedingly small, it can be approximated that $(dv/dx)^2 + 1 \approx 1$. This approximation simplifies the expression and allows for a more manageable analysis then

$$\kappa = \frac{d^2v}{dx^2} \tag{5}$$

We use the Eq. (1) along with Eq. (3) and Eq. (5) so

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v(x)}{dx^2}\right) = \quad f_1(x) \tag{6}$$

If we assume the constant nature of EI, we can proceed to rearrange Eq. (6) in the following manner

$$\frac{d^4v(x)}{dx^4} = \quad f(x) \tag{7}$$

where $f(x) = \frac{f_1(x)}{\mathrm{EI}}$. For a homogeneous elastic beam with inhomogeneity $f(x)$, Eq. (7) so called Euler–Bernoulli beam model.

3. Homogeneous boundary conditions

Various categories of boundary conditions arise based on the beam support.

Table 1 Conditions in the case of a homogeneous beam with terminals at $x = 1$ and $x = 0$.

Beam type	**Boundary Conditions**
Cantilever	$v'''(1) = 0, v''(1) = 0, v'(0) = 0. v(0) = 0.$
Fixed Supported	$v'(1) = 0, v(1) = 0, v'(0) = 0. v(0) = 0.$
Simply Supported	$v''(1) = 0, v(1) = 0, v''(0) = 0. v(0) = 0.$

4. Fourth−order boundary value problem (B.V.P.)

The general expression for a fourth−order differential model can be formulated as

$$a_4(x)v''''(x) + a_3(x)v'''(x) + a_2(x)v''(x) + a_1(x)v'(x) + a_0(x)v(x) = f(x),\ x\epsilon[0,1] \quad (8)$$

Here $a_4 \neq 0, \forall\ x\ \epsilon\ [0,1]. a_l(x)\ \epsilon\ \mathrm{C}[0,1]$, for all $l = 1,2,3,4$. Consider a function $f(x)$, which exhibits piecewise continuity over the closed interval $[0,1]$. Denoting the general solution for the aforementioned differential equation as $v(x)$, it becomes possible to establish the subsequent boundary conditions

$$\begin{aligned} b_3(x)v'''(0) + b_2(x)v''(0) + b_1(x)v'(0) + b_0(x)v(0) &= g_0, \\ c_3(x)v'''(0) + c_2(x)v''(0) + c_1(x)v'(0) + c_0(x)v(0) &= g_1, \\ d_3(x)v'''(1) + d_2(x)v''(1) + d_1(x)v'(1) + d_0(x)v(1) &= g_2, \\ e_3(x)v'''(1) + e_2(x)v''(1) + e_1(x)v'(1) + e_0(x)v(1) &= g_4 \end{aligned} \quad (9)$$

where, b_i, c_i, d_i, e_i and g_i are linearly independent vectors. Eq. (8) and the boundary conditions in Eq. (9) are known as a fourth−order B.V.P. with data $\{f, g_0, g_1, g_2, g_3\}$.

- A B.V.P. with data $\{0,0,0,0,0\}$. is called a homogeneous B.V.P.
- A B.V.P. with data $\{f, 0,0,0,0\}$. is called an inhomogeneous B.V.P.
- A B.V.P. with data $\{f, g_0, g_1, g_2, g_3\}$. is called a completely inhomogeneous B.V.P.

5. Green's function

The utilization of Green's function holds significant importance, particularly when seeking solutions for inhomogeneous differential equations accompanied by specific categories of boundary conditions. This approach offers a notable advantage (upon deriving a solution utilizing Green's function of a designated differential equation with specific boundary conditions) that the solution can subsequently be applied to solve the same differential equation featuring identical boundary conditions but incorporating any form of inhomogeneity, denoted as $f(x)$. Let us consider the operator form of an inhomogeneous boundary value problem, characterized by the data set $\{f, 0,0,0,0\}$.

$$S\{v\} = f,\ x\epsilon[0,1] \quad (10)$$

$$B_1\{v\} = B_3\{v\} = 0$$

$$B_2\{v\} = B_4\{v\} = 0 \quad (11)$$

Consider the corresponding homogeneous B.V.P. with specified data: $\{\delta(x - \varsigma),0,0,0,0\}$

$$S\{G\} = \delta(x-\varsigma),\ x\epsilon[0,1] \tag{12}$$

$$G(0,\varsigma) = G(1,\varsigma) = 0$$
$$G'(0,\varsigma) = G'(1,\varsigma) = 0 \tag{13}$$

Green's function $G(x,\varsigma)$ is stated as a solution of Eq. (12) fulfilling the constraints at $x = 1$ and $x = 0$, given in Eq. (13), and $\delta(x-\varsigma)$ is the Dirac delta function. Thus, the solution of Eq. (10) utilizing Green's function can be written as

$$v(x) = \int_0^1 f(\varsigma)G(x,\varsigma)d\varsigma \tag{14}$$

The function $v(x)$, as provided in Eq. (14), not only fulfills the differential operator described in Eq. (10), but also satisfies the constraints stated in Eq. (11). The Eq. (14) highlights the fact that if $G(x,\varsigma)$ of the associated problem is found then the solution $v(x)$ can be determined for a variety of inhomogeneous term f.

6. Formation of Green's function with appropriate constraints

We construct the Green's function for various boundary conditions.

6.1 Case of fixed supported beam

A Green's function associated with Eq. (7), fulfills the subsequent relationship in the context of a fixed-supported case

$$G''''(x,\varsigma) = \delta(x-\varsigma),\ x\epsilon[0,1] \tag{15}$$

$$G(0,\varsigma) = G(1,\varsigma) = 0, \quad G'(0,\varsigma) = G'(1,\varsigma) = 0 \tag{16}$$

Ensure that Green's function fulfills the continuity condition when evaluated at $x = \varsigma$ i.e.

$$G''(x,\varsigma)|_{x=\varsigma^-} - G''(x,\varsigma)|_{x=\varsigma^+} = 0,$$
$$G'(x,\varsigma)|_{x=\varsigma^-} - G'(x,\varsigma)|_{x=\varsigma^+} = 0,$$
$$G(x,\varsigma)|_{x=\varsigma^-} - G(x,\varsigma)|_{x=\varsigma^+} = 0 \tag{17}$$

It fulfills jump discontinuity at a point $x = \varsigma$ i.e.

$$G'''(x,\varsigma)|_{x=\varsigma^-} - G'''(x,\varsigma)|_{x=\varsigma^+} = 1 \tag{18}$$

$x^3, x^2, x, 1$ at point $x = 0$, and $(1-x)^3, (1-x)^2, (1-x), 1$ at point $x = 1$ are four linearly independent solutions to Eq. (15). Initially, we assume the general form of Green's function

$$G(x,\varsigma) = \begin{cases} B_3(1-x)^3 + B_2\ (1-x)^2 + B_1(1-x) + B_0, & x > \varsigma, \\ A_3x^3 + A_2\ x^2 + A_1\ x + A_0, & x < \varsigma. \end{cases} \tag{19}$$

Here, capital letters are the functions of variable ς. Now by the second property it satisfies the boundary conditions stated in Eq. (16) so, A_1, A_0, B_1 and B_0 vanish. Now, we update the Green's function from Eq. (19).

$$G(x,\varsigma)=\begin{cases}B_3(1-x)^3+B_2\ (1-x)^2, & x>\varsigma,\\ A_3x^3+A_2\ x^2, & x<\varsigma.\end{cases} \tag{20}$$

By using continuity property at $x=\varsigma$ results

$$\begin{cases}-6A_3\varsigma-2A_2+6B_3(1-\varsigma)+2B_2=0,\\ -3A_3\varsigma^2-2A_2\varsigma-3B_3(1-\varsigma)^2-2B_2(1-\varsigma)=0,\\ -A_3\varsigma^3-A_2\varsigma^2+B_3(1-\varsigma)^3+B_2(1-\varsigma)^2=0.\end{cases} \tag{21}$$

It also fulfills the jump discontinuity at $x=\varsigma$ so

$$-6A_3-6B_3=1 \tag{22}$$

The augmented matrix for the system of equations comprising Eqs. (21) and (22) can be constructed as follow

$$M_1=\begin{bmatrix}-6\varsigma & -2 & 6(1-\varsigma) & 2 & 0\\ -3\varsigma^2 & -2\varsigma & -3(1-\varsigma)^2 & -2(1-\varsigma) & 0\\ -\varsigma^3 & -\varsigma^2 & (1-\varsigma)^3 & (1-\varsigma)^2 & 0\\ 0 & -6 & 0 & -6 & 1\end{bmatrix} \tag{23}$$

We compute unknown values of A_3, A_2, B_3 and B_2 by reduced row Echelon form of M_1 as

$$A_3=\left(-\frac{\varsigma^3}{3}-\frac{1}{6}+\frac{\varsigma^2}{2}\right),$$
$$A_2=\left(\frac{\varsigma^3}{2}+\frac{\varsigma}{2}-\varsigma^2\right),$$
$$B_3=\left(\frac{\varsigma^3}{3}-\frac{\varsigma^2}{2}\right),$$
$$B_2=\left(-\frac{\varsigma^3}{2}+\frac{\varsigma^2}{2}\right)$$

Once more, refine it by inserting corresponding values of A_3, A_2, B_3 and B_2 into Eq. (20). so that

$$G(x,\varsigma)=\begin{cases}\left(\frac{\varsigma^2}{2}-\frac{\varsigma^3}{2}\right)(1-x)^2+\left(\frac{\varsigma^3}{3}-\frac{\varsigma^2}{2}\right)(1-x)^3, & x>\varsigma,\\ \left(\frac{\varsigma^3}{2}-\varsigma^2+\frac{\varsigma}{2}\right)x^2+\left(\frac{\varsigma^2}{2}+\frac{\varsigma^3}{3}-\frac{1}{6}\right)x^3, & x<\varsigma\end{cases} \tag{24}$$

This is the expression for Green's function $G(x,\varsigma)$ in the case of a fixed supported beam. It exhibits nonnegativity and symmetry, represented by $G(x,\varsigma)=G(\varsigma,x)$.

6.2 Case of cantilever beam

In the same fashion mentioned in the above section, Green's function in the case of a cantilever beam results as

$$G(x,\varsigma)=\begin{cases}\left(\frac{\varsigma^2}{2}-\frac{\varsigma^3}{6}\right)-\left(\frac{\varsigma^2}{2}\right)(1-x), & x>\varsigma,\\ \left(\frac{\varsigma}{2}\right)x^2-\left(\frac{1}{6}\right)x^3, & x<\varsigma\end{cases} \tag{25}$$

Green's function associated with the cantilever beam exhibits both symmetry and nonnegativity.

6.3 Case of simply supported beam

Similarly, Green's function in the case of a simply supported beam results as

$$G(x,\varsigma)=\begin{cases}\left(\frac{\varsigma}{6}-\frac{\varsigma^3}{6}\right)(1-x)-\left(\frac{\varsigma}{6}\right)(1-x)^3, & x>\varsigma,\\ \left(\frac{\varsigma^3}{6}-\frac{\varsigma^2}{2}+\frac{\varsigma}{3}\right)x+\left(\frac{\varsigma}{6}-\frac{1}{6}\right)x^3, & x<\varsigma\end{cases} \tag{26}$$

Green's function associated with the simply supported beam exhibits both symmetry and nonnegativity.

7. Solutions to inhomogeneous problems

We will determine general solutions for inhomogeneous problems, considering suitable boundary conditions. As stated above, Eq. (14) will be used to find the solutions using Green's functions found in above section 6.

7.1 Solutions to fixed supported beam

We formulate Green's function in accordance with the details provided in Eq. (24), specifically addressing the scenario of a fixed supported beam in subsection (6.1). Upon deriving Eq. (24), we proceed by introducing this expression into Eq. (14), resulting in the following expression

$$\begin{aligned}v(x)= &\int_0^x f(\varsigma)\left(\left(\frac{\varsigma^2}{2}-\frac{\varsigma^3}{2}\right)(1-x)^2+\left(\frac{\varsigma^3}{3}-\frac{\varsigma^2}{2}\right)(1-x)^3\right)d\varsigma\\ &+\int_x^1 f(\varsigma)\left(\left(\frac{\varsigma^3}{2}-\varsigma^2+\frac{\varsigma}{2}\right)x^2+\left(\frac{\varsigma^2}{2}+\frac{\varsigma^3}{3}-\frac{1}{6}\right)x^3\right)d\varsigma\end{aligned} \tag{27}$$

Assuming the function is defined as $f(\varsigma)$ as ς^2, the solution to the inhomogeneous problem can be described as follows

$$v(x)=\frac{x^6}{360}-\frac{x^3}{90}+\frac{x^2}{120} \tag{28}$$

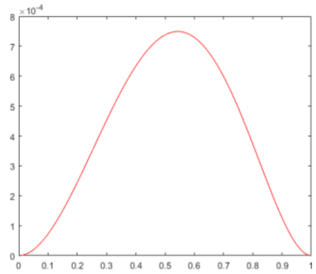

Fig. 3 The solution curve in the case of a fixed supported beam along inhomogeneity $f(\varsigma)=\varsigma^2$

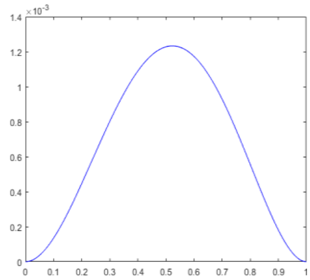

Fig. 4 The solution curve in the case of a fixed supported beam along inhomogeneity $f(\varsigma) = \sin(\varsigma)$

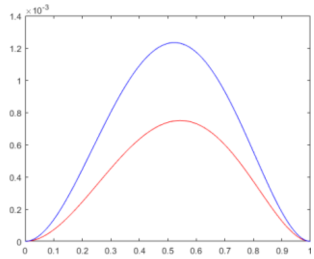

Fig. 5 The contrast of solution curves to the fixed supported beam in a graph

Now consider the function $f(\varsigma)$ as $\sin(\varsigma)$. With this choice, the solution can be described as follows

$$v(x) = (2\sin(1) - \cos(1) - 1)x^3 + (\cos(1) - 3\sin(1) + 2)x^2 + \sin(x) - x \tag{29}$$

The solution for the fixed-supported beam can be visually represented through the composite graph shown in Fig. 5.

7.2 Solutions to the cantilever beam

We develop the Green's function using the approach outlined in Eq. (25), specifically for the

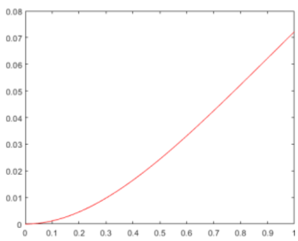

Fig. 6 The solution curve in the case of cantilever beam along inhomogeneity $f(\varsigma) = \varsigma^2$

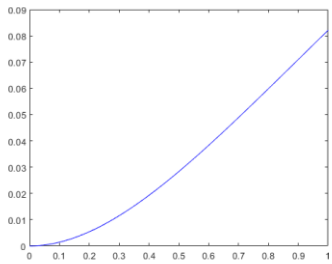

Fig. 7 The solution curve in the case of cantilever beam along inhomogeneity $f(\varsigma) = \sin(\varsigma)$

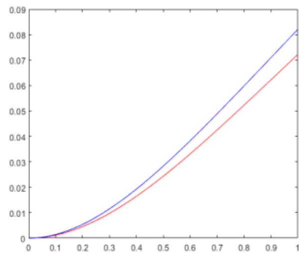

Fig. 8 The contrast of solution curves to the cantilever beam in a graph

cantilever beam discussed in subsect. (6.2). Next, by inserting Eq. (25) into Eq. (14) we obtain

$$v(x) = \int_0^x f(\varsigma)\left(\left(\frac{\varsigma^2}{2}-\frac{\varsigma^3}{6}\right)-\left(\frac{\varsigma^2}{2}\right)(1-x)\right)d\varsigma + \int_x^1 f(\varsigma)\left(\left(\frac{\varsigma}{2}\right)x^2-\left(\frac{1}{6}\right)x^3\right)d\varsigma \tag{30}$$

If we consider the function $f(\varsigma)$ as ς^2, the solution to the inhomogeneous problem can be described as follows

$$v(x) = x^2\left(\frac{x^4}{360}-\frac{x}{18}+\frac{1}{8}\right) \tag{31}$$

If we consider the function $f(\varsigma) = \sin(\varsigma)$, then the solution is presented as

$$v(x) = -x-\frac{x^2}{2}\cos(1)+\frac{x^3}{6}\cos(1)+\frac{x^2}{2}\sin(1)+\sin(x) \tag{32}$$

To represent the solution for the cantilever beam graphically, we can utilize the composite diagram shown in Fig. 8.

7.3 Solutions to simply supported beam

We formulate Green's function according to the expression presented in Eq. (26) for a simply supported beam, which is detailed in subsect. (6.3). By inserting Eq. (26) into Eq. (14), we obtain the following result

$$v(x) = \int_0^x f(\varsigma)\left(\left(\frac{\varsigma}{6}-\frac{\varsigma^3}{6}\right)(1-x)-\left(\frac{\varsigma}{6}\right)(1-x)^3\right)d\varsigma$$

$$+\int_x^1 f(\varsigma)\left(\left(\frac{\varsigma^3}{6}-\frac{\varsigma^2}{2}+\frac{\varsigma}{3}\right)x+\left(\frac{\varsigma}{6}-\frac{1}{6}\right)x^3\right)d\varsigma \tag{33}$$

If we consider the function $f(\varsigma)$ as ς^2, the solution to the inhomogeneous problem can be written as

$$v(x)=\frac{x^7}{12}-\frac{13x^6}{120}+\frac{x^3}{72}+\frac{x}{90} \tag{34}$$

Suppose we consider the function $f(\varsigma)$ as $sin(\varsigma)$. In this case, solution to problem can be presented as

$$v(x)=\sin(x)+x\left(\frac{x^2}{3}\cos(x)-\frac{x^3}{3}\cos(x)+\frac{x^2}{3}\sin(x)-\frac{7}{6}\sin(1)-\frac{x^2}{6}\sin(1)\right) \tag{35}$$

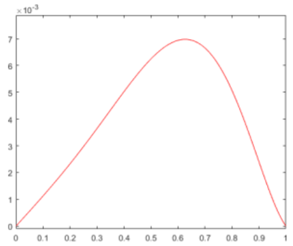

Fig. 9 The solution curve in the case of a simply supported beam along inhomogeneity $f(\varsigma)=\varsigma^2$

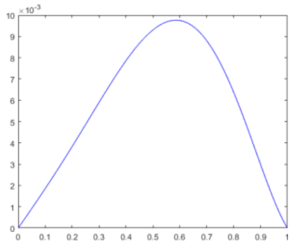

Fig. 10 The solution curve in the case of a simply supported beam along inhomogeneity $f(\varsigma)=\sin(\varsigma)$

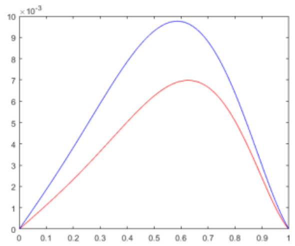

Fig. 11 The contrast of solution curves to the simply supported beam in a graph

The solution for the simply supported beam can be visually represented through the composite graph depicted in Fig. 11.

8. Solutions to completely inhomogeneous problem

Take into account the equation given in Eq. (7), accompanied by inhomogeneous boundary conditions depicted as follows

$$\frac{d^4v(x)}{dx^4} = f(x),\ x\epsilon[0,1] \tag{36}$$

$$B_1v = \zeta_1, B_3v = \zeta_3, \quad B_2v = \zeta_2, B_4v = \zeta_4 \tag{37}$$

The completely inhomogeneous solution in Eq. (36) with constraints in Eq. (37) can be expressed as

$$v(x) = v^1(x) + v^2(x) \tag{38}$$

where, $v^1(x)$ denotes the solution of the following problem

$$\frac{d^4v^1(x)}{dx^4} = f(x),\ x\epsilon[0,1] \tag{39}$$

with homogeneous constraints

$$B_1v = 0, B_3v = 0, \quad B_2v = 0, B_4v = 0 \tag{40}$$

and $v^2(x)$ depicts the solution of the following problem

$$\frac{d^4v^2(x)}{dx^4} = 0,\ x\epsilon[0,1] \tag{41}$$

with inhomogeneous constraints

$$B_1v = \zeta_1, B_3v = \zeta_3, \quad B_2v = \zeta_2, B_4v = \zeta_4 \tag{42}$$

We computed the solutions to the inhomogeneous problem in Eq. (39) with homogeneous boundary conditions in Eq. (40) in the previous section 5. In the subsequent discussion, we focus on the problem defined by Eq. (41) with inhomogeneous constraints in Eq. (42). The solution $v^2(x)$ is represented as

$$v^2(x) = b_3v_3(x) + b_2v_2(x) + b_1v_1(x) + b_0v_0(x) \tag{43}$$

Let $v_3(x), v_2(x), v_1(x)$, and $v_0(x)$ represent the distinct nontrivial solutions of the homogeneous Eq. (41), satisfying the relevant constraints. Next, we insert the values of $v^1(x)$ and $v^2(x)$ into Eq. (38) to derive the solution for completely inhomogeneous problem.

$$v(x) = \int_0^x f(\varsigma)G(x,\varsigma)d\varsigma + b_3v_3(x) + b_2v_2(x) + b_1v_1(x) + b_0v_0(x) \tag{44}$$

8.1 Solutions to fixed supported beam

We examine Eq. (36) under the presence of completely inhomogeneous boundary conditions described as follows

$$\frac{d^4v(x)}{dx^4} = f(x),\ x\epsilon[0,1] \tag{45}$$

$$v(0) = \zeta_1, v'(0) = \zeta_2, \qquad v(1) = \zeta_3, v'(1) = \zeta_4 \tag{46}$$

The solution of the completely inhomogeneous problem is given as

$$v(x) = v^1(x) + v^2(x)$$

If we consider the function $f(\varsigma) = \varsigma^2$ and calculate $v^2(x)$ using the expression provided in Eq. (28), the solution to the inhomogeneous problem can be expressed as

$$v(x) = \frac{x^6}{360} - \frac{x^3}{90} + \frac{x^2}{120} + \zeta_1 + \zeta_2 x + (2\zeta_1 + \zeta_2 - 2\zeta_3 + \zeta_4)x^3 - (3\zeta_1 + 2\zeta_2 - 3\zeta_3 + \zeta_4)x^2 \tag{47}$$

However, if we consider the case where $f(\varsigma)$ as $\sin(\varsigma)$ and determine the solution using the method outlined in Eq. (28), the solution to the inhomogeneous problem is as follows

$$v(x) = (2\sin(1) - \cos(1) - 1)x^3 + (\cos(1) - 3\sin(1) + 2)x^2 + \sin(x) - x$$
$$+\ \zeta_1 + \zeta_2 x + (2\zeta_1 + \zeta_2 - 2\zeta_3 + \zeta_4)x^3 - (3\zeta_1 + 2\zeta_2 - 3\zeta_3 + \zeta_4)x^2 \tag{48}$$

8.2 Solutions to the cantilever beam

We examine Eq. (36) under the presence of completely inhomogeneous boundary conditions as

$$\frac{d^4v(x)}{dx^4} = f(x),\ x\epsilon[0,1] \tag{49}$$

$$v(0) = \zeta_1, v'(0) = \zeta_2, \qquad v''(1) = \zeta_3, v'''(1) = \zeta_4 \tag{50}$$

The solution of aforementioned completely inhomogeneous problem is presented as follows

$$v(x) = v^1(x) + v^2(x)$$

If we consider the function $f(\varsigma) = \varsigma^2$ and find $v^2(x)$ using the expression provided in Eq. (31), the solution of inhomogeneous problem can be described as

$$v(x) = \frac{x^6}{360} - \frac{x^3}{18} + \frac{x^2}{8} + \zeta_1 + \zeta_2 x - \frac{(\zeta_2 - \zeta_3)x^2}{2} + \frac{\zeta_4 x^3}{6} \tag{51}$$

However, considering the scenario where $f(\varsigma)$ as $\sin(\varsigma)$, and utilizing the solution method outlined in Eq. (32), we can determine the solution to the inhomogeneous problem in a subsequent manner

$$v(x) = -x - \frac{x^2}{2}\cos(1) + \frac{x^3}{6}\cos(1) + \frac{x^2}{2}\sin(1) + \sin(x)\,\zeta_1 + \zeta_2 x - \frac{(\zeta_2 - \zeta_3)x^2}{2} + \frac{\zeta_4 x^3}{6} \tag{52}$$

8.3 Solutions to simply supported beam

We examine Eq. (36) in the presence of completely inhomogeneous boundary conditions as

$$\frac{d^4v(x)}{dx^4} = f(x),\ x\epsilon[0,1] \tag{53}$$

$$v(0) = \zeta_1, v''(0) = \zeta_2, \qquad v(1) = \zeta_3, v''(1) = \zeta_4 \tag{54}$$

The solution of the above problem is given as

$$v(x) = v^1(x) + v^2(x)$$

If we take $f(\varsigma)$ as ς^2 and find $v^2(x)$ via Eq. (34), then solution is as follow

$$v(x) = \frac{x^7}{12} - \frac{13x^6}{120} + \frac{x^3}{72} + \frac{x}{90} + \zeta_1 - \left(\zeta_1 + \frac{\zeta_2}{3} - \zeta_3 + \frac{\zeta_4}{6}\right)x - \frac{(\zeta_2-\zeta_4)x^3}{6} + \frac{\zeta_2 x^2}{2} \quad (55)$$

Alternatively, considering $f(\varsigma)$ as $\sin(\varsigma)$ and utilizing the solution method outlined in Eq. (35), the solution to a completely inhomogeneous problem is obtained as

$$v(x) = \sin(x) + x\left(\frac{x^2}{3}\cos(x) - \frac{x^3}{3}\cos(x) + \frac{x^2}{3}\sin(x) - \frac{7}{6}\sin(1) - \frac{x^2}{6}\sin(1)\right)$$
$$+\zeta_1 - \left(\zeta_1 + \frac{\zeta_2}{3} - \zeta_3 + \frac{\zeta_4}{6}\right)x - \frac{(\zeta_2-\zeta_4)x^3}{6} + \frac{\zeta_2 x^2}{2} \quad (56)$$

9. Perturbation technique

We consider a problem regarding the eigenvalue as

$$Tv = \lambda v, \ v \in \mathfrak{D}(T) \quad (57)$$

Here T is a linear operator, and its domain is denoted as $\mathfrak{D}(T)$. The closure of $\mathfrak{D}(T)$ is the entire Hilbert space $\mathcal{H}$, which possesses an inner product denoted as $\langle\cdot,\cdot\rangle$. In numerous real-world scenarios, achieving an exact solution to this problem is often impossible. Assuming that the problem defined by Eq. (57) cannot be solved with complete accuracy, an alternative approach is to consider an approximation of the operator T as follows

$$T = \varepsilon A + S \quad (58)$$

Assuming that ε is a small parameter, S is a symmetric operator and A is unbounded, it is essential to impose specific regularity conditions to ensure the validity of the following method. (For further insights, we refer to the work of (Kato 2013) and (Rellich 1969) The operator S is characterized as linear and symmetric i.e.

$$\langle Sv_1 \ , \ v_2\rangle = \langle v_1 \ , \ Sv_2\rangle, \qquad v_1, v_2 \ \epsilon \ \mathfrak{D}(S)$$

In addition to addressing the mathematical expression $\overline{\mathfrak{D}(S)} = \mathcal{H}$, we proceed with the assumption that the eigenvalue equation $Sv = \lambda v$ is solvable. Due to the symmetry (or self-adjoint nature) of matrix S, the eigenvalues of this problem are guaranteed to be real and can be organized in a subsequent manner as

$$\mu_1, \ \mu_2, \ \mu_3$$

These eigenvalues correspond to the following eigenfunctions

$$\varphi_1, \ \varphi_2, \ \varphi_3$$

The eigenvectors corresponding to different eigenvalues are mutually orthogonal i.e.

$$\left\langle \ \varphi_i \ , \ \varphi_j \ \right\rangle = 0, \quad i \neq j$$

Additionally, it is possible to standardize these eigenfunctions i.e., we can bring them to a normalized state

$$\left\langle \varphi_i \ , \ \varphi_j \right\rangle = \delta_{ij}$$

The eigenfunctions under consideration are commonly referred to as orthonormalized eigenfunctions. Referring to Eq. (57), when we substitute T with $\varepsilon A + S$, a perturbed problem arises, which can be expressed as follows

$$\varepsilon A v + S v = \lambda v \tag{59}$$

Suppose we have a set of eigenfunctions denoted as v_m, where m takes on values 1, 2, 3, and so forth. These eigenfunctions are associated with the operator T and correspond to specific eigenvalues λ_m, where again m follows the sequence 1, 2, 3, and beyond. We will utilize an expansion that is expressed as a series involving the variable ε

$$v_m = {v_m}^{(0)} + {v_m}^{(1)}\varepsilon + {v_m}^{(2)}\varepsilon^2 + \quad \cdots \tag{60}$$

$$\lambda_m = {\lambda_m}^{(0)} + v{\lambda_m}^{(1)}\varepsilon + {\lambda_m}^{(2)}\varepsilon^2 + \quad \cdots \tag{61}$$

So, Eq. (59) becomes

$$S\left({v_m}^{(0)} + {v_m}^{(1)}\varepsilon + {v_m}^{(2)}\varepsilon^2\ \right) + A\left({v_m}^{(0)} + {v_m}^{(1)}\varepsilon + {v_m}^{(2)}\varepsilon^2\right)\varepsilon = \left({\lambda_m}^{(0)} + v{\lambda_m}^{(1)}\varepsilon + {\lambda_m}^{(2)}\varepsilon^2\right)\left({v_m}^{(0)} + {v_m}^{(1)}\varepsilon + {v_m}^{(2)}\varepsilon^2\ \right) \tag{62}$$

We evaluate the similar magnitudes of ε by examining the coefficients associated with ε^0. This leads us to an unperturbed problem, which can be stated as follows

$$S\left({v_m}^{(0)}\right) = {\lambda_m}^{(0)}{v_m}^{(0)} \tag{63}$$

by analyzing the coefficients of ε in a comparative manner, we encounter a perturbed problem in the process

$$S{v_m}^{(1)} + A{v_m}^{(0)} = {\lambda_m}^{(0)}{v_m}^{(1)} + {\lambda_m}^{(1)}{v_m}^{(0)} \tag{64}$$

by examining the coefficients associated with ε^2, we can derive an altered problem through perturbation

$$S{v_m}^{(2)} + A{v_m}^{(1)} = {\lambda_m}^{(2)}{v_m}^{(0)} + {\lambda_m}^{(1)}{v_m}^{(1)} + {\lambda_m}^{(0)}{v_m}^{(2)} \tag{65}$$

It has come to our attention that Eq. (63) represents an unperturbed equation. We can readily solve for its solution and express it in the subsequent format

$${v_m}^{(0)} \ \in \ \{\varphi_1, \ \varphi_2, \ \varphi_3, \quad \cdot \quad \cdot \quad \cdot\}$$

$${\lambda_m}^{(0)} \ \in \ \{\mu_1, \ \mu_2, \ \mu_3, \quad \cdot \quad \cdot \quad \cdot\}$$

We insert values of ${\lambda_m}^{(0)}$ and ${v_m}^{(0)}$ into Eq. (64) i.e.

$${v_m}^{(0)} = \ \varphi_m, \qquad m = 1, 2, 3, \quad \cdot \quad \cdot \quad \cdot$$

$${\lambda_m}^{(0)} = \ \mu_m, \qquad m = 1, 2, 3, \quad \cdot \quad \cdot \quad \cdot$$

$$Sv_m{}^{(1)} + A\varphi_m = \lambda_m{}^{(1)}\varphi_m + \mu_m v_m{}^{(1)} \tag{66}$$

As we can expand functions in the form of infinite series in terms of eigenfunctions of a symmetric operator i.e., $v_m{}^{(1)} = \sum_{k=1}^{\infty} \beta_{km}\varphi_k$ so

$$S\sum_{k=1}^{\infty} \beta_{km}\varphi_k + A\varphi_m = \lambda_m{}^{(1)}\varphi_m + \mu_m \sum_{k=1}^{\infty} \beta_{km}\varphi_k \tag{67}$$

As S is linear operator so

$$\sum_{k=1}^{\infty} \beta_{km}S\varphi_k + A\varphi_m = \lambda_m{}^{(1)}\varphi_m + \mu_m \sum_{k=1}^{\infty} \beta_{km}\varphi_k \tag{68}$$

Scalarly multiplying Eq. (68) by φ_m

$$\langle \sum_{k=1}^{\infty} \beta_{km}S\varphi_k \ , \ \varphi_m \rangle + \langle A\varphi_m \ , \ \varphi_m \rangle = \langle \ \lambda_m{}^{(1)} \ \varphi_m \ , \ \varphi_m \ \rangle + \langle \mu_m \sum_{k=1}^{\infty} \beta_{km}\varphi_k \ , \ \varphi_m \rangle \tag{69}$$

Now we use $S\varphi_k = \mu_m\varphi_k$ and inner product properties so

$$\langle \sum_{k=1}^{\infty} \beta_{km}\mu_m\varphi_k \ , \ \varphi_m \rangle + \langle A\varphi_m \ , \ \varphi_m \rangle = \lambda_m{}^{(1)}\langle \ \varphi_m \ , \ \varphi_m \ \rangle + \mu_m\langle \sum_{k=1}^{\infty} \beta_{km}\varphi_k \ , \ \varphi_m \rangle \tag{70}$$

Since $\langle \ \varphi_k \ , \ \varphi_m \ \rangle = 0$, unless k = m, we get

$$\beta_{mm}\mu_m\langle \ \varphi_m \ , \ \varphi_m \ \rangle + \langle A\varphi_m \ , \ \varphi_m \rangle = \lambda_m{}^{(1)}\langle \ \varphi_m \ , \ \varphi_m \ \rangle + \beta_{mm}\mu_m\langle \ \varphi_m \ , \ \varphi_m \ \rangle \tag{71}$$

Due to the normalizing property of eigenfunctions $\langle \ \varphi_m \ , \ \varphi_m \ \rangle = 1$, the above expression becomes

$$\lambda_m{}^{(1)} = \langle A\varphi_m \ , \ \varphi_m \ \rangle \tag{72}$$

The initial term in the power series for λ_m can be derived. Given that $\lambda_m{}^{(0)}$ is already established, we can express λ_m as $\lambda_m = \lambda_m{}^{(0)} + \varepsilon \ \lambda_m{}^{(1)}$, where ε is accurate to the first–order. To determine the initial correction term in the expansion of v_m, we can refer to Eq. (68). By performing a scalar multiplication with φ_n (where $n \neq m$), we can follow a similar process iteratively.

$$\langle \ v_m{}^{(1)} \ , S\varphi_n\rangle + \langle A\varphi_m \ , \ \varphi_n \ \rangle = \lambda_m{}^{(1)}\langle \ \varphi_m \ , \ \varphi_n \ \rangle + \mu_m\langle \ v_m{}^{(1)} \ , \varphi_n\rangle \tag{73}$$

We use $S\varphi_n = \mu_n\varphi_n$ and the inner product properties so that

$$\langle \ v_m{}^{(1)} \ , \mu_n\varphi_n\rangle + \langle A\varphi_m \ , \ \varphi_n \ \rangle = \mu_m\langle \ v_m{}^{(1)} \ , \varphi_n\rangle \tag{74}$$

or

$$\mu_m\langle \ v_m{}^{(1)} \ , \varphi_n\rangle = \frac{1}{(\mu_m - \mu_n)}\langle A\varphi_m \ , \ \varphi_n \ \rangle, \qquad n \neq m \tag{75}$$

Upon examination, it becomes evident that Eq. (75) provides us with the Fourier coefficients denoted as β_{mm}. The representation of $v_m{}^{(1)}$ by means of the basis functions φ_m can be expressed as follows

$$v_m{}^{(1)} = \sum_{j=1}^{\infty} \frac{\beta_{jm}}{(\mu_m - \mu_j)}, \qquad m \neq j \tag{76}$$

10. Rayleigh beam's small rotational effects

Consider an eigenvalue problem emerging in the Rayleigh beam model, where a very small rotational effect is considered alongside the deflection of the beam. The fundamental equation governing the Rayleigh beam model (Han *et al.* 1999) is presented below

$$\frac{d^4 v(x)}{dx^4} + \lambda \frac{d^2 v(x)}{dx^2}\varepsilon = \lambda v(x) \tag{77}$$

$$\lambda_n = \lambda_n{}^{(0)} + v\lambda_n{}^{(1)}\varepsilon + \lambda_n{}^{(2)}\varepsilon^2 + \quad \cdots \tag{78}$$

$$v_n = v_n{}^{(0)} + v_n{}^{(1)}\varepsilon + v_n{}^{(2)}\varepsilon^2 + \quad \cdots \tag{79}$$

Next, we insert the Eq. (78) and Eq. (79) into the Eq. (77), resulting in

$$\frac{d^4}{dx^4}\left(v_n{}^{(0)} + v_n{}^{(1)}\varepsilon + \cdot \cdot \cdot\right) + \varepsilon\left(\lambda_n{}^{(0)} + v\lambda_n{}^{(1)}\varepsilon + \cdot \cdot \cdot\right)\frac{d^2}{dx^2}\left(v_n{}^{(0)} + v_n{}^{(1)}\varepsilon + \cdot \cdot \cdot .\right)$$
$$= \left(\lambda_n{}^{(0)} + v\lambda_n{}^{(1)}\varepsilon + \cdot \cdot \cdot\right)\left(v_n{}^{(0)} + v_n{}^{(1)}\varepsilon + \cdot \cdot \cdot\right) \tag{80}$$

By comparing the coefficients ε we find

$$\frac{d^4 v_n{}^{(0)}}{dx^4} = \lambda_n{}^{(0)} v_n{}^{(0)} \tag{81}$$

$$\frac{d^4 v_n{}^{(1)}}{dx^4} + \lambda_n{}^{(0)}\frac{d^2 v_n{}^{(0)}}{dx^2} = \lambda_n{}^{(0)} v_n{}^{(1)} + \lambda_n{}^{(1)} v_n{}^{(0)} \tag{82}$$

We start with the given information about the known eigenpair $(\lambda_n{}^{(0)}, v_n{}^{(0)})$. Given that the operator $\frac{d^4}{dx^4}$ is symmetric, we can conclude that its eigenvalues are distinct, and consequently, the associated eigenfunctions are orthogonal.

$$\lambda_1{}^{(0)}, \lambda_2{}^{(0)}, \lambda_3{}^{(0)} \quad \cdots \tag{83}$$

$$v_1{}^{(0)}, v_2{}^{(0)}, v_3{}^{(0)} \quad \cdots \tag{84}$$

If a function is selected from the set $\mathfrak{D}(S)$, it can be represented as an infinite series composed of orthonormal eigenfunctions. Specifically, let us take the eigenfunction $v_n{}^{(1)}$ and express it as

$$v_n{}^{(1)} = \sum_{k=1}^{\infty} \beta_{nk} v_n{}^{(0)} \tag{85}$$

consider the relationship $\beta_{nk} = \langle\ v_n{}^{(1)}\ ,\ v_k{}^{(0)}\rangle$. Next, if we insert the expansion from Eq. (85) into Eq. (82), the result will become

$$\frac{d^4}{dx^4}\left(\sum_{k=1}^{\infty} \beta_{nk} v_n{}^{(0)}\right) + \lambda_n{}^{(0)}\frac{d^2 v_n{}^{(0)}}{dx^2} = \lambda_n{}^{(1)} v_n{}^{(0)} + \lambda_n{}^{(0)}\left(\sum_{k=1}^{\infty} \beta_{nk} v_n{}^{(0)}\right) \tag{86}$$

Table 2 The eigenvalues $\lambda_n{}^{(0)}$ for three cases of conditions

Eigenvalue $\lambda_n{}^{(0)}$	**Beam type**
$+12.3622560703$	Cantilever
$+500.54665441$	Fixed Supported
$+4\pi^4$	Simply Supported

To determine the value of $\lambda_n{}^{(1)}$, we perform a scalar multiplication on Eq. (86) using $v_n{}^{(0)}$. This yields

$$\left\langle \frac{d^4}{dx^4}\left(\sum_{k=1}^{\infty}\beta_{nk}v_n{}^{(0)}\right), v_n{}^{(0)}\right\rangle + \left\langle \lambda_n{}^{(0)}\frac{d^2 v_n{}^{(0)}}{dx^2}, v_n{}^{(0)}\right\rangle = \left\langle \lambda_n{}^{(1)} v_n{}^{(0)}, v_n{}^{(0)}\right\rangle \quad + \left\langle \lambda_n{}^{(0)}\left(\sum_{k=1}^{\infty}\beta_{nk}v_n{}^{(0)}\right), v_n{}^{(0)}\right\rangle \quad \lambda_n{}^{(0)} \tag{87}$$

Utilizing the linearity of the operator and incorporating the unperturbed solution from Eq. (81), we can simplify Eq. (87) as follows

$$\left\langle \lambda_n{}^{(0)}\left(\sum_{k=1}^{\infty}\beta_{nk}v_n{}^{(0)}\right), v_n{}^{(0)}\right\rangle + \left\langle \lambda_n{}^{(0)}\frac{d^2 v_n{}^{(0)}}{dx^2}, v_n{}^{(0)}\right\rangle = \left\langle \lambda_n{}^{(1)} v_n{}^{(0)}, v_n{}^{(0)}\right\rangle \tag{88}$$

$$+\left\langle \lambda_n{}^{(0)}\left(\sum_{k=1}^{\infty}\beta_{nk}v_n{}^{(0)}\right), v_n{}^{(0)}\right\rangle \quad \lambda_n{}^{(0)}.$$

By applying the principles of orthogonality and orthonormality, we can streamline the expression in Eq. (88). This process yields a simplified representation for the value of $\lambda_n{}^{(1)}$ as presented below

$$\lambda_n{}^{(1)} = \lambda_n{}^{(0)}\left\langle \frac{d^2 v_n{}^{(0)}}{dx^2}, v_n{}^{(0)}\right\rangle \tag{89}$$

or

$$\lambda_n{}^{(1)} = \lambda_n{}^{(0)}\int_0^1 v_n{}^{(0)}\frac{d^2 v_n{}^{(0)}}{dx^2}dx \tag{90}$$

Eq. (90) provides the initial correction term within the power series concerning λ_n. In prior works, specifically in reference (Bakalah *et al*. 2018) as well as in reference (Xu and Ma 2017), an assessment of the eigenvalues $\lambda_n{}^{(0)}$ has been conducted, alongside the corresponding eigenfunctions $v_n{}^{(0)}$, while considering relevant boundary conditions. The specific eigenvalues are documented in Table 2.

The eigenfunctions denoted as $v_n{}^{(0)}$ for the eigenvalues provided in the preceding table can be described as follows:

i. The eigenfunction associated with the fixed-supported beam.

$$v_n{}^{(0)} = \frac{\left(-\sinh\left(\lambda_n{}^{(0)}\right)+\sin\left(\lambda_n{}^{(0)}\right)\right)}{\left(\cos\left(\lambda_n{}^{(0)}\right)+\cosh\left(\lambda_n{}^{(0)}\right)\right)}\left(\cos\left(\lambda_n{}^{(0)}x\right)-\cosh\left(\lambda_n{}^{(0)}x\right)\right)-\sinh\left(\lambda_n{}^{(0)}x\right)+\sin\left(\lambda_n{}^{(0)}x\right) \tag{91}$$

ii. The eigenvalue associated with the cantilever beam.

$$v_n{}^{(0)} = \frac{\left(\sinh\left(\lambda_n{}^{(0)}\right)+\sin\left(\lambda_n{}^{(0)}\right)\right)}{\left(\cos\left(\lambda_n{}^{(0)}\right)+\cosh\left(\lambda_n{}^{(0)}\right)\right)}\left(\cos\left(\lambda_n{}^{(0)}x\right)-\cosh\left(\lambda_n{}^{(0)}x\right)\right)-\sinh\left(\lambda_n{}^{(0)}x\right)+\sin\left(\lambda_n{}^{(0)}x\right) \tag{92}$$

iii. The eigenvalue associated with the simply supported beam.

$$v_n{}^{(0)} = \sqrt{2}\sin(\pi n x) \tag{93}$$

After evaluating the initial correction term $\lambda_n{}^{(1)}$ for three distinct boundary conditions using the data provided in the aforementioned Table 2, the outcomes are presented in the subsequent Table 3.

Table 3 The first correction terms ${\lambda_n}^{(1)}$ for three cases of beams

1st correction term ${\lambda_n}^{(1)}$	**Beam type**
${\lambda_n}^{(1)} = -1036.138995$	Cantilever
${\lambda_n}^{(1)} = -1.252193068 \times 10^8$	Fixed Supported
${\lambda_n}^{(1)} = -4\pi^6$	Simply Supported

Table 4 The eigenvalues in the case of a fixed supported beam corrected to the first power of ε

ε **variation**	**Eigenvalue** $\boldsymbol{\lambda_n} = \varepsilon{\boldsymbol{\lambda_n}}^{(\mathbf{1})} + {\boldsymbol{\lambda_n}}^{(\mathbf{0})}$
$\varepsilon = 0.3$	$\lambda_n = -3.756529149 \times 10^7$
$\varepsilon = 0.2$	$\lambda_n = -2.504336081 \times 10^7$
$\varepsilon = 0.1$	$\lambda_n = -1.252143013 \times 10^7$
$\varepsilon = 0$	$\lambda_n = +500.54665441$

Table 5 The eigenvalues in the case of cantilever beam corrected to the first power of ε

ε **variation**	**Eigenvalue** $\boldsymbol{\lambda_n} = \varepsilon{\boldsymbol{\lambda_n}}^{(\mathbf{1})} + {\boldsymbol{\lambda_n}}^{(\mathbf{0})}$
$\varepsilon = 0.3$	$\lambda_n = -298.4794424$
$\varepsilon = 0.2$	$\lambda_n = -194.8655429$
$\varepsilon = 0.1$	$\lambda_n = -91.25164343$
$\varepsilon = 0$	$\lambda_n = +12.3622560703$

Table 6 The eigenvalues in the case of simply supported beam corrected to the first power of ε

ε **variation**	**Eigenvalue** $\boldsymbol{\lambda_n} = \varepsilon{\boldsymbol{\lambda_n}}^{(\mathbf{1})} + {\boldsymbol{\lambda_n}}^{(\mathbf{0})}$
$\varepsilon = 0.3$	$\lambda_n = -1.2\pi^6 + 4\pi^4$
$\varepsilon = 0.2$	$\lambda_n = -0.8\pi^6 + 4\pi^4$
$\varepsilon = 0.1$	$\lambda_n = -0.4\pi^6 + 4\pi^4$
$\varepsilon = 0$	$\lambda_n = 0 + 4\pi^4$

In the following tables we have computed the perturbed eigenvalues of the Rayleigh beam model correct to the first power of ε for three types of boundary conditions. When we substitute $\varepsilon = 0$ in the above problem its values match with (Bakalah *et al*. 2018).

11. Conclusions

In this study, we use the analytical methods based upon Green's function and perturbation theory to study fourth-order differential equations arising from Euler-Bernoulli and Rayleigh beam models. The advantage of Green's function is that we can use it for any type of boundary conditions and inhomogeneous terms. We computed solutions for both inhomogeneous and completely inhomogeneous problems in a static case. The basic models used are the Euler Bernoulli beam and the Rayleigh beam. The Rayleigh beam model corresponds to an eigenvalue problem. The eigenvalues of the Rayleigh beam model were also obtained up to the first correction term using the perturbation method. These eigenvalues, particularly in the context of vibrations such as those in Rayleigh beams, serve as critical indicators of a structure's stability, integrity, and performance. Engineers hold this information to design safe, efficient, and durable structures for various applications in civil, mechanical, and aerospace engineering.

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