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Bi-stability in a vertically excited rectangular tank with finite liquid depth

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Abstract. We discuss the bi - stability that is possibly exhibited by a liquid free surface in a parametrically - driven two-dimensional (2D) rectangular tank with finite liquid depth. Following the method of adaptive mode ordering, assuming two dominant modes and retaining polynomial nonlinearities up to third-order, a nonlinear finite-dimensional nonlinear modal system approximation is obtained. A "continuation method" of nonlinear dynamics is then used in order to elicit efficiently the instability boundary in parameters' space and to predict how steady surface elevation changes as the frequency and/ or the amplitude of excitation are varied. Results are compared against those of the linear version of the system (that is a Mathieu-type model) and furthermore, against an intermediate model also derived with formal mode ordering, that is based on a second - order ordinary differential equation having nonlinearities due to products of elevation with elevation velocity or acceleration. The investigation verifies that, in parameters space, there must be a region, inside the quiescent region, where liquid surface instability is exhibited. There, behaviour depends on initial conditions and a wave form would be realised only if the free surface was substantially disturbed initially.

Keywords: parametric sloshing; modal method; adaptive ordering; continuation analysis

1. Introduction

Parametric sloshing is the motion of a liquid's free surface due to an excitation that acts perpendicularly to the plane of the undisturbed free surface (Ibrahim 2005). Although parametric sloshing is less often analyzed in comparison with cases of directly excited sloshing, it has a long history of investigation in the sciences, initiated by an observation by Faraday ("Faraday resonance"). In engineering this phenomenon can have catastrophic effects for vibrated structures containing liquids, if not properly prevented by design (Dodge 1966). For ships, such vertical excitation arises physically in combination with excitations in other modes of ship motion; for example in pitch and in roll. Nonetheless, it is important to understand whether heave dominated dynamics could excite sloshing motion. Furthermore such motion will be an asset on the way to studying the coupled system at a later stage.

Standing waves generated in vertically oscillating tanks were firstly studied experimentally by Faraday (1831), Mathiessen (1868, 1870) and Lord Rayleigh (1883a,b, 1887). The same problem was investigated theoretically later by Lewis (1950), Taylor (1950), Benjamin and Ursell (1954),

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Konstantinov *et al*. (1978), Perlin and Schultz (1996). In most of these cases, parametric sloshing was approached by means of linear theoretical models. Whilst such an approach can capture the region of linear instability, it does not suffice for predicting the ensuing free surface elevations, thus producing unrealistic infinite amplitudes inside the instability region.

Faraday resonance was captured by the weakly nonlinear model proposed by Miles (1994), deriving from an averaged Lagrangian approach. This model was extended by Decent (1995) and Decent and Craik (1995) to include fifth-order frequency detuning effects, in order to capture (for shallow liquid depth) the essential features of a hysterisis phenomenon observed in experiments. Simonelli and Gollub 1989, Craik and Armitage 1995). In addition, in the last two decades, Takizawa and Kondo (1995), Chen *et al*. (1996), Chern *et al*. (1999), Turnbull *et al*. (2003), Fradnsen (2004), Kim *et al.* (2007), Wu *et al.* (2007) and others, have employed various numerical approaches for sloshing prediction, treating the moving free surface either by using Lagrangian tracking of free surface nodes with regrinding; or by mappings. The last whilst being more accurate in predicting surface velocity, they are less flexible for application to irregular geometries or to cases where submerged bodies are present in the flow domain (Frandsen 2003). Also, a drawback of these numerical approaches is that they are not appropriate for long time simulations. A concise overview of research efforts concerning the nonlinear behaviour of liquids contained in tanks of various shapes and subjected to parametric excitation can be found in Ibrahim (2005).

We have recently looked into the problem of 2-D liquid sloshing in a rectangular and vertically excited tank, based on modal modelling motivated by the adaptive analysis introduced by Faltinsen and Timokha (2001). The investigation was confined to finite liquid depths and, in the first step, a priority had been set on producing, through a rational reduction process, the minimal nonlinear ODE system that could be suitable for the problem (Spandonidis and Spyrou 2011). A theoretical finding was, further corroboration that an area of bi-stability is likely to need to be added to the linear stability chart of free surface elevation associated with a parametrically forced liquid. This area lies entirely inside the domain considered from a linear perspective as globally quiescent. The interesting feature of this region is that it is initial-conditions-dependent; i.e. one may obtain a stable wavy surface or a flat surface depending on how much disturbed the free surface was when the harmonic excitation was firstly applied. Whilst similar observations have appeared also earlier (e.g., Decent 1995), in our case this has come from a fundamentally different modelling approach and through systematic dimension reduction.

An extension of this work to the immediately higher order non-linear model is reported here. This model allows approximate prediction of the free surface dynamics (and of the associated stability properties of the obtained surface profiles), without severe limitations on the excitation amplitude or frequency. The specific aim was to further support (or reject) the above finding, namely the existence of an area of bi-stability, lying to the left in terms of frequency of the area of primary resonance). The novelty of the current work accrues mainly from the proposed advanced investigation of free surface oscillations by means of a 3rd order nonlinear ODE system, that confirmed the existence of the new (in parameters' space) region of liquid surface instability for the case of finite liquid depth. Since there was no immediate plan to consider secondary or higher resonances, "sufficiently" small amplitude could be assumed. It was found that, the extended model is also characterised by an area of bi-stability.

To predict the amplitudes of steady liquid surface oscillations as the frequency and/or the amplitude of excitation (represented the characteristics of the imposed motion on the container) are varied without performing multiple simulations, the discussed models were coupled with a "continuation analysis" algorithm. Use of such numerical algorithms is popular in nonlinear dynamics investigations.

2. Formulation of the problem

For the purpose of our work we consider the mobile, rectangular, smooth and rigid tank shown in Fig. 1. Tank's velocity is $\vec{u}_0(t) = \dot{n}_3 \cdot \vec{e}_3$, where n_3 is the magnitude of the applied vertical tank displacement and $\vec{e}_3$ is the unit vector in the z- axis ($\dot{n}_3 = \frac{dn_3}{dt}$). The tank is partly filled by an inviscid and incompressible fluid. Liquid depth is finite corresponding to a liquid-height-to-tank-length-ratio of 0.4, but the tank top is high enough so that it is never reached by the moving liquid. The flow is two-dimensional and irrotational. Such a problem is formulated in terms of the Laplace equation for the velocity potential $\Phi(y, z, t)$ in the fluid volume $Q(t)$, with suitable boundary conditions on the free surface $\Sigma(t)$ and on the tank surface below it $S(t)$

$$\Delta\Phi = 0 \text{ in } Q(t),\ \frac{\partial \Phi}{\partial n} = \vec{u}_0 \cdot \vec{n} \text{ on } S(t)$$

$$\frac{\partial \Phi}{\partial n} = \vec{u}_0 \cdot \vec{n} - \frac{\partial Z/\partial t}{|\nabla Z|} \text{ and } \frac{\partial \Phi}{\partial t} - \nabla\Phi \cdot \vec{u}_0 + \frac{1}{2}(\nabla\Phi)^2 + U_g = 0 \text{ on } \Sigma(t) \tag{1}$$

where U_g stands for the gravity potential.

The so called multimodal method uses a Fourier series representation of the solution with time-dependent unknown coefficients (Narimanov 1957). Faltinsen *et al.* (2000) postulated Fourier series representations describing the free-surface elevation and the velocity potential, as follows

$$\zeta(y,t) = \sum_{i=1}^{\infty} \beta_i(t) f_i(y),\ \Phi(y,z,t) = \sum_{i=1}^{\infty} R_i(t)\varphi_i(y,z) \tag{2}$$

The modal representation shown in Eq. (2) is based upon the functions $f_i(y)$ and $\varphi_i(y, z)$ which must be a complete basis for the representation of the mean free surface and the whole liquid domain, respectively. The most common choice for the modal basis $f_i(y)$ and the set of functions

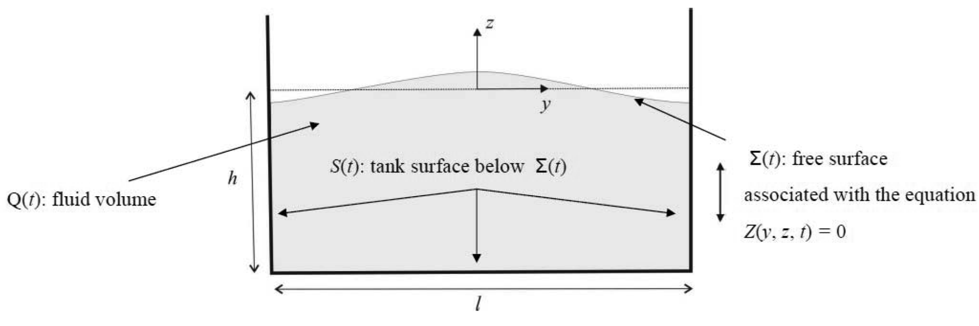

Fig. 1 System of coordinates and other definitions

$\varphi_i(y, z)$ is the set of linear natural modes shown below

$$\varphi_i(y,z) = \cos(\pi i(y+l/2)/l)\cdot\cosh(\pi i(z)/l)/\cosh(\pi ih/l)$$
$$f_i(y) = \cos(\pi i(y+l/2)/l),\ i\ge 1 \tag{3}$$

Since natural modes are theoretically defined only in the unperturbed hydrostatic domain, we follow Faltinsen and Timokha (2002) in interpreting the natural modes as an asymptotic basis, assuming that the free surface is ("to some extent") asymptotically close to its unperturbed state. This could be claimed as legitimate because the procedure needs only the completeness of $\varphi_n(y,z)$ in the unperturbed liquid domain. Faltinsen and Timokha (2001) have converted the governing equations into an infinite-dimensional system of ODEs by following what they call an "adaptive approach". Keeping, in the first instance, nonlinear terms up to third-order, this system of ODEs is reduced to the following form in the case of purely vertical excitation

$$\sum_{i=1}^{p}\ddot{\beta}_i\left(\delta_{p\mu}+d^{1,\mu}_{p,q}\sum_{i=1}^{q}\beta_i+d^{2,\mu}_{p,q,r}\sum_{i=1}^{q}\beta_i\sum_{i=1}^{r}\beta_i\right)+\sum_{i=1}^{p}\dot{\beta}_i\sum_{i=1}^{q}\dot{\beta}_i\,(t^{0,\mu}_{p,q})+\sum_{i=1}^{p}\dot{\beta}_i\sum_{i=1}^{q}\dot{\beta}_i\sum_{i=1}^{r}\beta_i\,(t^{1,\mu}_{p,q,r})+$$
$$\left[\sigma^2_\mu+\frac{\ddot{n}_3}{l}\pi\mu\tanh\left(\frac{\pi\mu h}{l}\right)\right]\beta_\mu = 0$$
$$\sigma_\mu = \sqrt{g\left(\frac{\pi\mu}{l}\right)\tanh\left(\frac{\pi\mu}{lh}\right)},\quad \mu = 1,2... \tag{4}$$

δ is Kronecker's delta, p, q, r indicate upper summation limits and σ_i represents the μ^{th} natural frequency. The d and t coefficients can be expressed as functions of liquid's height-to-tank-length ratio (such analytical expressions can be found in Faltinsen and Timokha 2001). The system described by Eq. (4) will be referred-to as "the modal system".

In contrast to a Moiseev type of ordering (Faltinsen 1974), an ordering of the form $\beta_n= O(\varepsilon^{1/3})$, $\beta_i = O(\varepsilon)$, $i \ge n + 1$ leads to a finite-dimensional nonlinear modal system ("model - k" where the integer k denotes the number of dominant modes) that is able to describe sloshing associated with the critical depth, large-amplitude response and secondary resonance. The coefficient $\varepsilon = n_{3\alpha}/l$ is an indicator of the smallness of excitation. Here it is assumed that $\varepsilon << 1$.

3. Simplified models (linear model and Model-1)

Let us now restrict our investigation to the case of vertical excitation with relatively small amplitude, with excitation frequency in the vicinity of the principal parametric resonance of the first mode; i.e., $(\sigma_1/\sigma)^2 \approx 1/4$. Liquid's height-to-tank-length-ratio is chosen to be larger than the critical depth (h/l = 0.3368) thus restricting the scope of the current investigation to finite liquid depths. We do this because, according to Fulzt (1962), in the vicinity of critical depth strong changes and amplifications in the liquid behaviour occur.

Faltinsen and Timokha (2009) introduced to the modal system an empirical linear damping term, represented by the damping ratio ζ_1. From a physical perspective, this damping term could empirically account for boundary–layer damping. Linearization of the modal system for the dominant mode β_1, after incorporating damping and perpendicular harmonic excitation $n_3 = n_{3\alpha}\cos(\sigma t)$, results in the following Mathieu-type equation

$$\ddot{\beta}_1 + 2\sigma_1\zeta_1\dot{\beta}_1 + \sigma_1^2\left(1 - n_{3a}\frac{\sigma^2}{g}\cdot\cos(\sigma\cdot t)\right)\beta_1 = 0 \tag{5}$$

where, β_1 stands for the time-dependant response of the free surface at $y = -l/2$; i.e., at its interface with the left tank wall. Benjamin and Ursell (1954) had investigated the free surface elevation under similar forcing, using Mathieu functions for expressing analytically the solution.

On the other hand, for mode ordering $\beta_1 = O(\varepsilon^{1/3})$, $\beta_\mu = O(\varepsilon)$, $\mu > 1$, by retaining only β_1 and following thinking as before (as regards external excitation and damping), the modal system can generate the following equation

$$\ddot{\beta}_1 + 2\zeta_1\sigma_1\dot{\beta}_1 + \sigma_1^2\left(1 - \frac{n_{3a}\sigma^2}{g}\cdot\cos(\sigma\cdot t)\right)\beta_1 + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) = 0$$

$$d_2 = \frac{\pi^2}{4}\left[1 - 2\cdot\tanh\left(\pi\frac{h}{l}\right)\cdot\tanh\left(2\pi\frac{h}{l}\right)\right] \tag{6}$$

By $\beta \approx \beta_1$ is meant the time-dependant response of the free surface, tracking the elevation at $y = -l/2$; i.e., at its interface with the left tank wall. To obtain the elevation for all other points $(y, 0)$ of the liquid surface, β should be multiplied by $\cos(\pi(y + l/2)/l)$.

4. Non-linear Model-2

For mode ordering $\beta_i = O(\varepsilon^{1/3})$, $i = 1, 2$ and $\beta_\mu = O(\varepsilon)$, $\mu > 2$, a set of modal equations can be obtained from the modal system that can cater, in principle, for intermodal interactions. The first two nonlinear equations couple β_1 with β_2. The equations are linear in β_u when $\mu = 3, 4, 5, 6$, although, nonlinear terms are found in β_1 and β_2. The higher modes ($\mu > 6$) are handled by the linear modal equations. Assuming harmonic external excitation $n_3 = n_{3\alpha}cos(\sigma t)$ and restricting ourselves only to the frequency region of primary resonance (only the two first equations are kept) the nonlinear equations for β_1 and β_2 receive the following form

$$((\ddot{\beta}_1 + 2\zeta_1\sigma_1\dot{\beta}_1 + \sigma_1^2\beta_1 - \pi\cdot(\tanh(\pi h/l))\cdot\sigma^2 n_{3\alpha}\cos(\sigma\cdot t))\beta_1) + d_1(\ddot{\beta}_1\beta_2 + \dot{\beta}_1\dot{\beta}_2) +$$

$$+ d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) + d_3\ddot{\beta}_2\beta_1 + \overline{d}_1\ddot{\beta}_1\beta_2^2 + \overline{d}_2\ddot{\beta}_2\beta_2\beta_1 + \overline{d}_3\dot{\beta}_2^2\beta_1 + \overline{d}_4\dot{\beta}_1\dot{\beta}_2\beta_2 = 0$$

$$(\ddot{\beta}_2 + 2\zeta_2\sigma_2\dot{\beta}_2 + \sigma_2^2\beta_2) + \overline{d}_4\ddot{\beta}_1\beta_1 + d_5\dot{\beta}_1^2 + \overline{d}_5\ddot{\beta}_1\beta_1\beta_2 + \overline{d}_6\ddot{\beta}_2\beta_1^2$$

$$+ \overline{d}_7(\ddot{\beta}_2\beta_2^2 + \dot{\beta}_2^2\beta_2) + \overline{d}_8\dot{\beta}_1^2\beta_2 + \overline{d}_9\dot{\beta}_1\dot{\beta}_2\beta_1 = 0 \tag{7}$$

In Table 1 were collected the numerical values for the coefficients of the model, corresponding to a liquid-height-to-tank-length-ratio of 0.4. As pointed out by Faltinsen (2001), this system is better

Table 1 Numerical values for Model-2 coefficients.

parameter	value	parameter	value	parameter	value	parameter	value
d_1	3.1830916	d_5	–4.29018	$\overline{d}_4$	–3.0415866	$\overline{d}_8$	–23.082502
d_2	3.4140089	$\overline{d}_1$	7.1955657	$\overline{d}_5$	10.6047	$\overline{d}_9$	2.3162401
d_3	–0.25615791	$\overline{d}_2$	4.5672544	$\overline{d}_6$	13.512814	σ_1	5.119
d_4	–0.5947726	$\overline{d}_3$	4.0684728	$\overline{d}_7$	–7.9163157	σ_2	7.8

than a model like that of Eq. (6) since it allows calculations for depths below the critical depth. Also, it could be used for the region of secondary resonance. The elevation at $y = -l/2$ should be $\beta \approx \beta_1 + \beta_2$ while at other locations it can easily deduced by combining with Eq. (3).

5. Investigation of dynamics

To elicit efficiently the stability chart of a nonlinear dynamical system in the presence of state coexistence one can use a "continuation" method. Such numerical methods usually accept the mathematical model in the so called "autonomous canonical form" $d\mathbf{x}/dt = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{b})$. Here $\boldsymbol{x}$ and $\boldsymbol{b}$ are, respectively, the state and control parameters' vectors of the o.d.e. problem at hand. Variation of one or more components of the control vector $\boldsymbol{b}$ creates, through solution of the above vector differential equation, branches of steady-state solutions. Mathematical details of the applied algorithm can be found in Dhooge *et al.* (2003).

Since Model-1 as well as Model-2 are characterised by explicit time - dependence in β_1's equation, a dummy pair of differential equations is introduced with respect to the new variables $x = \sin(\sigma t)$, $y = \cos(\sigma t)$. As a result, the model is converted into a system of six 1st - order *o.d.e.s* that, whilst retaining dynamical equivalence, at steady-state, with the original system of *o.d.e.s*, it is relieved from the explicit time-dependence (Spyrou and Tigkas 2011)

$$\dot{\beta}_1 = \phi_1 \qquad \dot{\phi}_1 = \frac{\gamma\varepsilon - \beta\xi}{\alpha\varepsilon - \beta\delta} \qquad \dot{x} = x + \sigma\cdot y - x\cdot(x^2+y^2)$$

$$\dot{\beta}_2 = \phi_2 \qquad \dot{\phi}_2 = \frac{\gamma\delta - \xi\alpha}{\beta\delta - \alpha\varepsilon} \qquad \dot{y} = y - \sigma\cdot x - y\cdot(x^2+y^2)$$

where

$$\alpha = 1 + d_1\beta_2 + d_2\beta_1^2 + \overline{d}_1\beta_2^2, \quad \beta = d_3\beta_1 + \overline{d}_2\beta_2\beta_1, \quad \delta = d_4\beta_1 + \overline{d}_5\beta_1\beta_2, \quad \varepsilon = 1 + \overline{d}_6\beta_1 + \overline{d}_7\beta_2^2$$

$$\gamma = 2\zeta_1\sigma_1\phi_1 + \sigma_1^2\beta_1 + \pi\cdot(\tanh(\pi h/l))\cdot\sigma^2 n_{3\alpha}\cos(\sigma\cdot t)\beta_1 + d_1\phi_1\phi_2 + d_2\phi_1^2\beta_1 + \overline{d}_3\phi_2^2\beta_1 + \overline{d}_4\phi_1\phi_2\beta_2$$

$$\xi = 2\zeta_2\sigma_2\phi_2 + \sigma_2^2\beta_2 + d_5\phi_1^2 + \overline{d}_7\phi_2^2\beta_2 + \overline{d}_8\phi_1^2\beta_2 + \overline{d}_9\phi_1\phi_2\beta_1 \qquad (8)$$

Due to the parametric nature of the above nonlinear system, one should expect to find an instability region in parameters' space in which, stable limit-cycle behaviour would be exhibited. Comparison of the obtained stability charts for Model-1 and Model-2 is shown in Fig. 2. They both

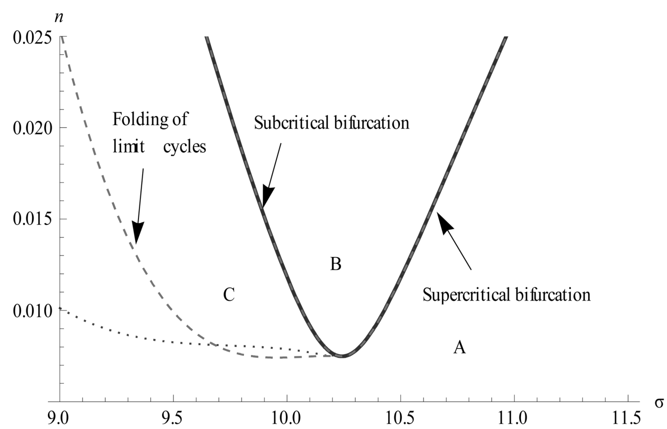

Fig. 2 Nonlinear stability chart of Model-1 (dotted line) and Model-2 (dashed line), for an assumed damping ratio $\zeta_1 = 0.02$. The two models have the same linear stability chart (solid line) since their linearised version coincide. In contrast to the linear system's stability region (solid line), in the chart of the non-linear models (dotted and dashed line) a new area of bi-stability appears (indicated by *C*). In there, an initial surface disturbance may (or may not) persist, depending on the initial condition. The nonlinear boundaries predicted by Model-1 and Model-2 differ significantly from each other as the frequency becomes lower

differ from the diagram presented by Benjamin and Ursell (1954) in the region of bi-stability (area *C*). Area *A* is the one where every external excitation (with the fitting specification) leads invariably to a flat liquid surface (region of global stability). In area *B*, an external excitation generates persistent periodic oscillations in liquid's free surface. To realise this in a computational environment one should assume a small initial surface disturbance when the excitation is applied (this is not necessary in a physical experiment). Area *C* defines the region of system bi-stability where, the same external excitation leads either to a quiescent surface or to a wavy one, depending on the initial condition (in terms of β, φ).

The phase space structure of the considered dynamical system is nevertheless quite an ordinary one. The domains of attraction of the two competing stable patterns occupy certain complementary regions of phase space. They are separated by a surface defined by the incoming (stable) manifold of the unstable periodic solution. The results obtained from Models 1 and 2 are in qualitative agreement although there is quite substantial quantitative difference. Model 2, as well as any other higher models that, in principle, one can systematically derive by the discussed adaptive mode ordering method, is expected to be more accurate than Model 1, for larger free surface motions, under the condition that no additional assumptions are introduced. However, no experimental work in this area that could be used as reference for the presented results exists. Next step of our work is to experimentally reproduce the identified types of motion using the NTUA shaking table facility.

A glimpse of system dynamics can be obtained also by simply selecting several points that lie in the regions *A*, *B* and *C* of Fig. 2, and then integrating the non-linear system, trying a variety of initial conditions. In Fig. 3 are presented the time histories of elevation β for a point inside the area *C*, for two, slightly different, initial elevations, when $\zeta_1 = \zeta_2 = 0.02$ and $n_{3a} = 0.02$ m. Initial elevation $\beta(0) = 0.19$ m leads to a stable point (zero elevation) while $\beta(0)= 0.194$ m leads to a stable periodic pattern. This indicates what is meant as "bi-stability".

Different types of behaviour of Model-2 are shown in Fig. 4, for three frequency values in the

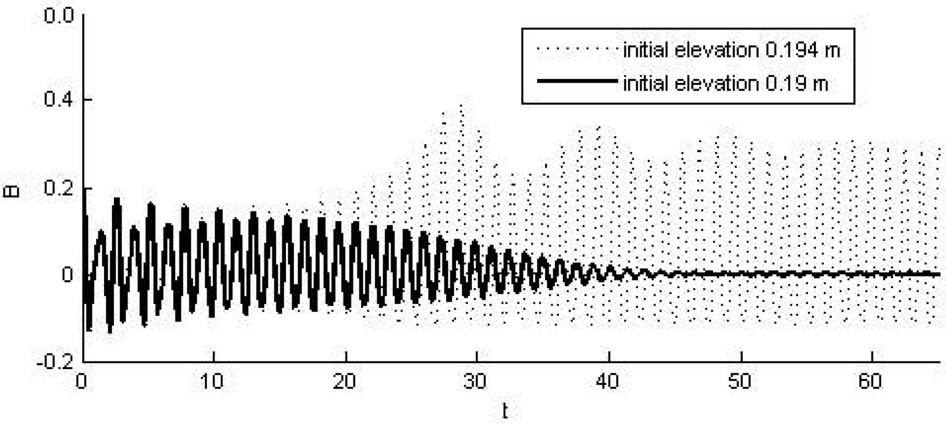

Fig. 3 Time history of β for σ = 9.7 rad/s. Slightly different initial elevations lead to different stable responses

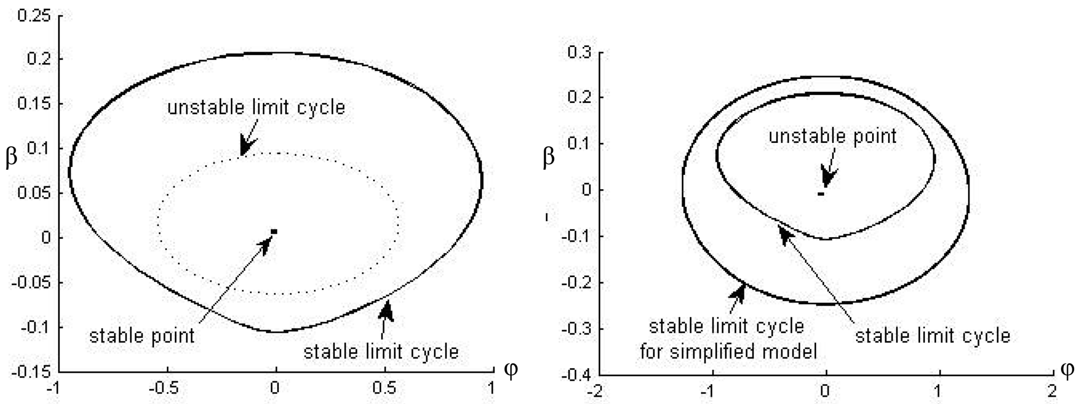

Fig. 4 Phase plots for $\zeta_1 = \zeta_2$ = 0.02 m and n_{3a} = 0.02 m: (a) Limit sets in the initial-condition-dependent area (σ = 9.7 rad/s) and (b) *A* stable limit cycle dominating in the region of linear instability (σ = 10.2 rad/s). For comparison, the outer closed orbit shown is the limit-cycle derived from the simplified nonlinear model (Model-1)

vicinity of $\sigma = 2\sigma_1$. At low frequency, every low-to-moderate excitation produces a response that eventually settles to the trivial point (that, as a matter of fact, is stable). Increase of the excitation frequency leads, through fold of limit-cycles bifurcation, to the initial-conditions-dependent area where the trivial state coexists with a stable limit cycle. In-between those (in phase plane) is found an unstable limit-cycle (Fig. 4(a)). Further increase of frequency leads, through a subcritical bifurcation, to the "classical" area of instability associated with principal parametric resonance (Fig. 4(b)), where the unstable limit-cycle disappears and the stable trivial solution becomes unstable. A supercritical ("smooth") bifurcation locus represents in fact the boundary with the stability area to the right of the "classical" linear instability area.

6. Conclusions

2-D liquid sloshing in a rectangular and vertically excited tank has been investigated with focus on the existence of a bi-stability area. The work was limited to finite liquid depth, corresponding to a tank-height-to-depth ratio of 0.4. A global picture of liquid surface dynamics was theoretically obtained, pertinent to model nonlinearity up to third-order. A relatively sophisticated nonlinear model has been derived, based on modal modelling and according to the adaptive analysis of Faltinsen and Timokha (2001). Continuation analysis results support the contention that a new area of bi-stability should be added to the stability chart of free surface oscillations. Investigation of the influence of the damping term on the size of that area is a matter to be considered.

A next step towards confirming the validity of results will be the investigation of the dynamic behaviour associated with the immediately higher order non-linear model. Areas of secondary resonance (higher excitation frequency) and also the chaotic areas (higher excitation amplitude) discussed by Ibrahim (2005) are also very interesting topics of further research.

Of course, experimental reproduction of the identified types of numerical solutions will be required before these are considered as established patterns of the behaviour of the physical system under consideration.

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References

Abramson, H.N. (1966), *The dynamics of liquids in moving containers*, NASA Report SP 106.

Benjamin, T.B. and Ursell, F. (1954), "The stability of the plane free surface of a liquid in a vertical periodic motion", *Proceedings of the Royal Society A225*.

Bredmose, H., Brocchini, M., Peregrine, D.H. and Thais, L. (2003), "Experimental investigation and numerical modelling of steep forced water waves", *J. Fluid. Mech*., 490, 217-49.

Chen, W., Haroun, M.A. and Liu, F. (1996) "Large amplitude liquid sloshing in seismically excited tanks", *Earthq. Eng. Struct. D*., **25**(7), 653-669.

Chern, M.J., Borthwick, A.G.L. and Taylor, R. (1999), "A pseudospectral s-transformation model of 2-D nonlinear waves", *J. Fluid. Struct*., **13**, 607-630.

Craik, A.D.D. and Armitage, J. (1995), "Faraday excitation, hysteresis and wave instability in a narrow rectangular wave tank", *Fluid Dyn. Res*., **15**(3), 129-143.

Decent, S.P. (1995), *Hysteresis and mode competition in Faraday waves*, Ph.D. thesis, University of St Andrews, Scotland.

Decent, S.P. and Craik, A.D.D. (1995), "Hysteresis in Faraday resonance", *J. Fluid Mech*., **293**, 237-268.

Dhooge, A., Govaerts,W., Kuznetsov, Y.A., Mestrom,W., Riet, A.M. and Sautois, B. (2003) *MATCONT and CL_MATCONT: Continuation Toolboxes for MATLAB*, Report of Gent and Utrecht Universities.

Dodge, F.T. (1966), "*Verticall excitation of propellant tanks. In the book: the dynamics of liquids in moving containers*, NASA Report, SP 106.

Faltinsen, O.M. (1974), "A nonlinear theory of sloshing in rectangular tanks", *J. Ship Res*., **18**(4), 224-241.

Faltinsen, O.M. and Rognebakke, O.F. (2000), "Sloshing. Keynote lecture". *Proceedings of the International Conference on Ship and Shipping Research (NAV2000)*, Venice, September.

Faltinsen, O.M., Rognebakke, O.F., Lukovsky, I.A. and Timokha, A.N. (2000), "Multidimensional modal

analysis of nonlinear sloshing in a rectangular tank with finite water depth", *J. Fluid Mech*., **407**, 201–234.
Faltinsen, O.M. and Timokha, A.N. (2001), "Adaptive multimodal approach to nonlinear sloshing in a rectangular tank", *J. Fluid Mech*., **432**, 167-200.
Faltinsen, O.M and Timokha, A.N. (2002), "Asymptotic modal approximation of nonlinear resonant sloshing in a rectangular tank with small fluid depth", *J. Fluid Mech*., **470**, 319–357.
Faltinsen, O.M. and Timokha, A.N. (2009), *Sloshing*, ISBN: 978-0-521-88111-1, Cambridge University Press, New York.
Faraday, M. (1831), "On a peculiar class of acoustical figures, and on certain forms assumed by groups of particles upon vibrating elastic surfaces", *Philos. T. R. Soc*., **121**, 299-340.
Frandsen, J.B. and Borthwick, L. (2003), "Simulation of sloshing motions in fixed and vertically excited containers using a 2-D inviscid transformed finite difference solver", *J. Fluid. Struct*., **18**(2), 197–214.
Frandsen, J.B. (2004), "Sloshing motions in the excited tanks", *J. Comput. Phys*., **196**(1), 53-87.
Ibrahim, R.A. (2005), *Liquid Sloshing Dynamics*, ISBN: 978-0-521-83885-6, Cambridge University Press, New York.
Fultz, D. (1962), "An experimental note on finite-amplitude standing gravity waves", *J. Fluid Mech*., **13**(2), 193-212.
Kim, Y., Nam, B.W., Kim, D.W. and Kim, Y.S. (2007), "Study on coupling effects of ship motion and sloshing", *Ocean Eng*., **34**(16), 2176–2187.
Konstantinov, A., Mikityuk, Y. and Pal'ko, L.S. (1978), "Study of the dynamic stability of a liquid in a cylindrical cavity, in dynamics of elastic systems with continuous-discrete parameters", *Naukova Dumka*, 69-72.
Lewis, D.J. (1950), "The instability of liquid surfaces when accelerated in a direction perpendicular to their planes", *Proceedings of the Royal Society A*, A 202.
Mathiessen, L. (1870),"Uber die transversal-schwingungen tonender tropharer und elastisher flussigkeiten", *Annalen der Physik*, **141**, 375–393.
Miles, J.W. (1994), "Faraday waves: rolls versus squares", *J. Fluid Mech*., **269**, 353-371.
Narimanov, G.S. (1957), "Movement of a tank partly filled by a fluid: the taking into account of non smallness of amplitude", J. *Appl.Math. Mech. (PMM)*, **21**, 513-524 (in Russian).
Perlin, M. and Schultz, W.W. (1996), "On the boundary conditions at an oscillating contact-line: a physical/ numerical experimental program", *Proceedings of the NASA 3rd Microgravity Fluid Physics Conference*, Cleveland, OH, 615–620.
Rayleigh, J.W.S. (1883a.), "On the crispations of fluid resting upon a vibrating support", *Philos. Mag*., **15**, 229-235.
Rayleigh, J.W.S. (1883b), "On maintained vibrations", *Philos. Mag*.,**15**, 229-235.
Rayleigh, J.W.S. (1887), "On the maintenance of vibrations by forces of double frequency and on the propagation of waves through a medium endowed with a periodic structure", *Philos. Mag*., **24**, 145-159.
Simoneli, F. and Gollup, J.P. (1989), "Surface wave mode interactions: effects of symetry and degeneracy", *J. Fluid Mech*., **199**, 471-494.
Spandonidis, C. (2010), *Parametric sloshing in a 2D rectangular tank*, Postgraduate Thesis, National Technical University of Athens (in Greek).
Spandonidis, C. and Spyrou, K.J. (2011), "Parametric Sloshing in A 2D Rectangular Tank with finite liquid depth", *Proceedings of the 14th "International Congress of the International Maritime Association of the Mediterranean (IMAM)*, Genova, September.
Spyrou, K.J., Tigkas, I., Scanferla, G., Pallikaropoulos, N. and Themelis, N. (2008) "Prediction potential of the parametric rolling behaviour of a post-panamax containership", *Ocean Eng*., **35**(11-12), 1235-1244.
Spyrou, K.J., and Tigkas, I. (2011), "Nonlinear surge dynamics of a ship in astern seas: continuation analysis of periodic states with hydrodynamic memory", *J. Ship Res*., **55**, 19-28.
Takizawa, A. and Kondo, S. (1995), "Computer discovery of the mechanism of flow-induced sloshing. fluid-sloshing and fluid–structure interaction", *Proceedings of the ASME Pressure Vessel Piping Conference*, PVP-314.
Taylor, G.I. (1950), "The instability of liquid surfaces when accelerated in a direction perpendicular to their planes", *Proceedings of the Royal Society*, A 201.
Turnbull, M.S., Borthwick, A.G.L. and Eatock Taylor, R. (2003), "Numerical wave tank based on a stransformed finite element inviscid flow solver", *Int. J. Numer. Meth. Fl*., **42**, 641-663.
Wu, G.X. (2007), "Second -order resonance of sloshing in a tank", *Ocean Eng*., **34**, 2345-2349.