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Global stabilization of three-dimensional flexible marine risers by boundary control

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Abstract. A method to design a boundary controller for global stabilization of three-dimensional nonlinear dynamics of flexible marine risers is presented in this paper. Equations of motion of the risers are first developed in a vector form. The boundary controller at the top end of the risers is then designed based on Lyapunov's direct method. Proof of existence and uniqueness of the solutions of the closed loop control system is carried out by using the Galerkin approximation method. It is shown that when there are no environmental disturbances, the proposed boundary controller is able to force the riser to be globally exponentially stable at its equilibrium position. When there are environmental disturbances, the riser is stabilized in the neighborhood of its equilibrium position by the proposed boundary controller.

Keywords: marine risers, boundary control, nonlinear dynamics, equations of motion.

1. Introduction

The need for production of oil and/or gas from the sea bed has made control of the dynamics of a marine riser, which is a structure connecting a oil and/or gas oshore platform with a well at the sea bed, a necessity for both ocean and control engineers. A typical configuration of an oshore platform is depicted in Fig. 1. The riser is considered in this paper as a slender thin walled circular beam because of its large length to diameter ratio. In general, the riser is subject to nonlinear deformation dependent hydrodynamic loads induced by waves, ocean currents, tension exerted at the top, distributed/concentrated buoyancy from attached modules, its own weight, inertia forces and distributed/concentrated torsional couples. Before reviewing control techniques for the flexible marine risers, we here mention some early work on static analysis of the risers. In Huang and Chucheepsakul (1985), Bernitsas *et al.* (1985) and Huang and Kang (1991), the static models of both two-and three-dimensional risers are first presented based on the work in Love (1920). Then numerical simulations are carried out to analyze the effect of the system parameters on the riser equilibria. It should be also mentioned the recent work in Ramos and Pesce (2003), where the authors carry out static stability of a riser based on the variational method. Since the riser dynamics is essentially a distributed system and its motion is governed by a set of partial differential equations (PDE) in both time and space variables, modal control and boundary control approaches are often used to control the riser in the literature.

In the modal control approach, see Meirovitch (1997) and Gawronski (1998), distributed systems

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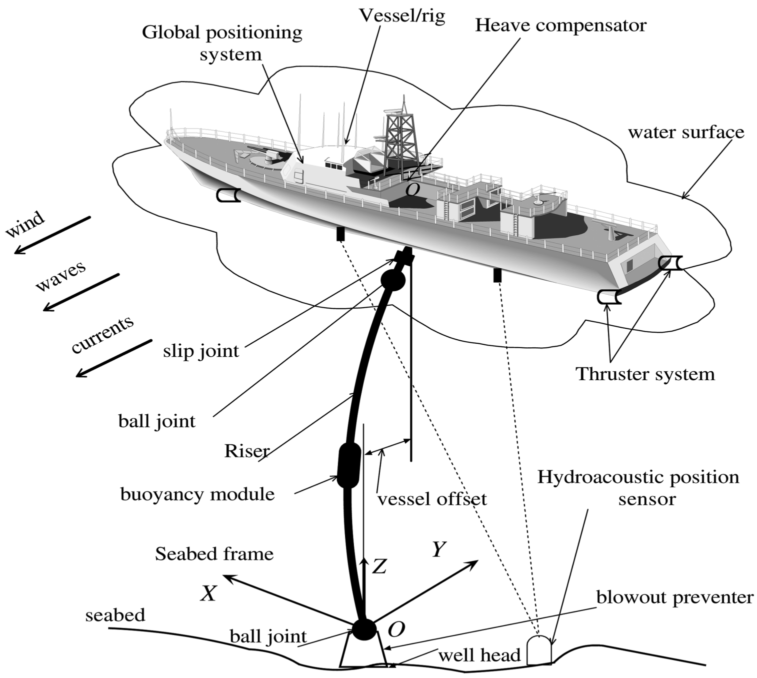


Fig. 1 A typical riser system

are controlled by controlling their modes. As a result, many concepts developed for lumped-parameter systems in Khalil (2002) and Krstic *et al*. (1995) can be used for controlling the distributed ones, since both types can be described in terms of modal coordinates. The main difficulty is computation of infinite dimensional gain matrices. This difficulty can be avoided by using the independent modal-space control method, but this method requires a distributed control force, which can be problematic to implement. One way to overcome this problem is to construct a truncated model consisting of a limited number of modes. In order to describe the behavior of a flexible system in a satisfactory fashion, it is necessary to include a large number of modes into the model. Thus, a characteristic of a truncated model is its large dimension, i.e., it is impractical to control all modes. Therefore the control of such truncated systems are restricted to a few critical modes. This also means that other modes are not controlled, and could be unstable. In fact, truncation of the infinite dimensional model divides the system into three modes: modeled and controlled, modeled and uncontrolled (residual), and un-modeled. Only the modeled modes are considered in the control design. In addition, observers are needed to provide the system output for these modeled modes from the actual distributed system. The use of these observers in combination with truncated models of distributed system leads to a spill-over phenomenon meaning that the control from actuators not only affects the controlled modes but also influences the residual and un-modeled modes, which can be unstable, Balas (1977).

The boundary control approach is more practical and efficient than the modal control approach since it excludes the effect of both observation and control spill-over phenomenon. In the boundary control approach, distributed actuators and sensors are not required. In addition control design based on the original PDE model instead of a truncated model, improves the performance of the control system. In recent years, boundary control has received a lot of attention from the control community. Design of boundary controllers for distributed systems has been usually based on functional analysis and semi-group theory, see Chen *et al.* (2001) and Curtain and Zwart (1995), and the Lyapunov's direct method, see Queiroz *et al.* (2000) and Junkins and Kim (1993). The Lyapunov's direct method is widely used since the control Lyapunov functions/functionals directly relate to the kinetic and potential energies of the distributed systems. Using the Lyapunov's direct method, various boundary controllers have been proposed for flexible beam-like systems. In Yang *et al.* (2004), a robust and adaptive boundary controller is proposed for reducing transverse vibration of an axially moving string under a varying tension and an unknown boundary disturbance force based on the Lyapunov function, which is the sum of kinetic and potential energies of the string system, plus a coupling term. In Fung *et al.* (1999) and Fung and Tseng (1999), asymptotic and exponential stability of an axially moving string is proven by using a linear and nonlinear state feedback boundary control, respectively. They proved that the mechanical energy of the system decreases exponentially in the nonlinear feedback case. In Fard and Sagatun (2001), the boundary stabilization of a beam in free transverse vibration is considered. The control law is a nonlinear function of the slopes and velocity at the boundary of the beam to provide exponential stabilization a free transversely vibrating beam via boundary control without restoring to truncation of the model. The coupling between longitudinal and transversal displacements is also taken into account. Recently, in Tanaka and Iwamoto (2007) an active boundary control is proposed for an Euler-Bernoulli beam, which enables one to generate a desired boundary condition at designated positions of a target beam based on structure transfer matrix and the optimal control methods. It should be noted that the active boundary control in Tanaka and Iwamoto (2007) is implemented at various locations of the beam. Therefore, this method closely relates to the modal control approach although it is called boundary control. In Do and Pan (2008), Ge *et al.* (2010), How *et al.* (2009), boundary controllers were proposed for controlling vibration of marine risers in two dimensional space based on the Lyapunov direct method. In Nguyen *et al.* (2010), Nguyen *et al.* (2011), different control strategies were proposed to control the angle of the marine risers' top end based on the authors' algorithms for chasing an optimal set-point. In Yang *et al.* (2004), Fung *et al.* (1999), Fung and Tseng (1999), Fard and Sagatun (2001), Tanaka and Iwamoto (2007) and Queiroz *et al.* (2000), two-dimensional strings and beams are considered, and distributed forces including the structures' own weight are ignored. Mathematical work in Tsay and Kingsbury (1998) shows that even slight space curvature introduces significant changes in the beam natural frequencies and especially on mode shapes, i.e., the coupling of the out-of-plane wave types, and extensional and flexural waves exhibits in the flexible beams. The coupling between these wave types due to the curved shape of the riser, boundary constraints and external forces made the energy exchange from one wave type to other possible. Moreover, in Yang *et al.* (2004), Fung *et al.* (1999), Fung and Tseng (1999), Fard and Sagatun (2001), Tanaka and Iwamoto (2007), Queiroz *et al.* (2000), Ge *et al.* (2010), How *et al.* (2009), no proof of existence and uniqueness of the solutions of closed loop systems was given. It is well-known that there are systems governed by initial-boundary PDEs, whose solutions do not exist or are not unique. For any control systems to be useful in practice, existence and uniqueness of the solutions of the closed loop control systems are as vital as stability.

In this paper, we consider a problem of global stabilization of three-dimensional nonlinear flexible marine risers. A set of partial differential equations and boundary conditions describing motion of the risers is presented based on balancing internal and external forces/moments. Using the Lyapunov's direct method, a boundary controller at the top end of the risers is designed. Proof of existence and uniqueness of the solutions of the closed loop system is given. The proposed controller guarantees that when there are no environmental disturbances, the riser is globally exponentially stabilized at its equilibrium position, and that when there are environmental disturbances, the riser is stabilized in the neighborhood of its equilibrium position.

2. Mathematical model and control objectives

2.1. Mathematical model

In this section, we develop equations of motion of the riser. These equations will be used for the boundary control design in the next section. In developing the equations of motion of the riser, we make the following assumption:

Assumption 2.1.
1. The riser can be modeled as a beam rather than a shell since the diameter-to-length of the riser is small, i.e., we consider the riser as a slender structure.
2. Plane sections remain plane after deformation, i.e., warping is neglected.
3. The riser is locally stiff, i.e., cross sections do not deform and Poisson effect is neglected.
4. The riser material is homogeneous, isotropic and linearly elastic, i.e., it obeys Hookes's law.
5. The riser is initially straight and vertical.
6. Torsional and distributed moments induced by environmental disturbances are neglected.

Remark 2.1. Items 1)-4) mean that the riser will be modeled as a Bernoulli-type of beam and not a Timoshenkotype, and that the extension of the riser axis small. Bernoulli-Euler models are satisfactory for modeling low frequency vibrations of beams. Item 5) generally holds in practice, and is made to simplify the development of the mathematical model and boundary controller. This item can be readily removed. Item 6) implies that we consider fluid/gas transportation risers rather than drilling risers, and that moment induced by the asymmetry of the relative flow due to vortex shedding is ignored.

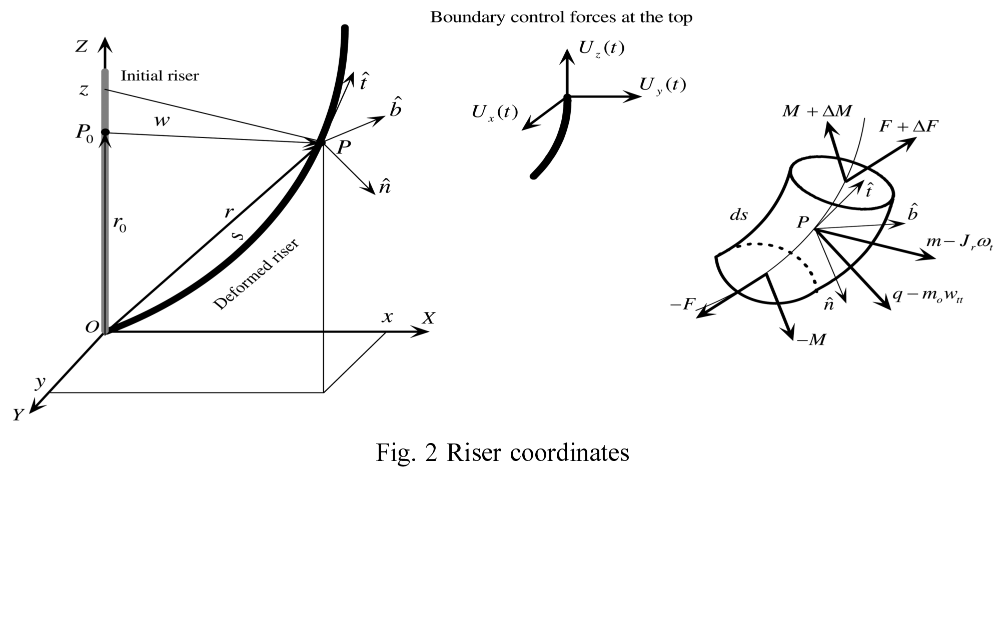


Fig. 2 Riser coordinates

2.1.1. Preliminaries

The riser coordinates are presented in Fig. 2. In this figure, we have two coordinate systems. The earth-fixed system is ($OXYZ$), where O is the bottom ball-joint of the riser, and the OZ axis is along the initial riser. Let $r^0(s_0, t_0) = [x_0, y_0, z_0]$ be the position vector of the point P_0 of the initial riser centerline at the time t_0 and the arc length s_0 from the point O. Hence at the time $t > t_0$, the point P_0 moves to the point P of the deformed riser centerline, whose position is denoted by $r(s, t)= [x(s, t), y(s, t), z(s, t)]$ at the arc length s from the point O. Moreover, let $w(s, t) = [w_x(s, t), w_y(s, t), w_z(s, t))]^T$ be the vector from the point P_0 to the point P. Then we have

$$r = r^0 + w \tag{1}$$

where from now onward whenever it is not confusing, we drop the arguments (t, s) and (t_0, s_0) of r, w and r^0, respectively for clarity. The body-fixed system is $(\hat{t}, \hat{n}, \hat{b})$, whose axes are the tangent, principal normal and binormal and unit vectors. These vectors can be expressed in terms of the fixed system as

$$\hat{t} = r_s, \qquad \hat{n} = \hat{t}_s/\kappa, \qquad \hat{b} = \hat{t}\times\hat{n} \tag{2}$$

where the subscript s denotes the partial derivative with respect to the arc-length s, and κ is curvature of the riser center line at s depicting the rate of change of the orientation of the normal plane $(\hat{n}, \hat{b})$ defined by $\kappa = \|r_{ss}\|$. The above definition of the body-fixed coordinate system means that $(\hat{t}, \hat{n}, \hat{b})$form a right handed orthonormal triad. The derivatives of the unit body-fixed vectors are given by the well-known Frenet-Serret relations Widder (1989):

$$\hat{n}_s = \tau\hat{b} - \kappa\hat{t}, \qquad \hat{b}_s = -\tau\hat{n}, \qquad \hat{t}_s = \kappa\hat{n} \tag{3}$$

where τ is the geometric torsion of the riser centerline depicting the rate of change of the orientation of the osculating plane $(\hat{n}, \hat{t})$ defined by $\tau = r_s \cdot (r_{ss} \times r_{sss})/\kappa^2$. Now from the right hand side sub-figure of Fig. 2, balancing the forces and moments on a component ds of the deformed riser results in

$$\begin{aligned} m_o w_{tt} &= F_s + q \\ J\omega_t &= M_s + \hat{t}\times F + m \end{aligned} \tag{4}$$

where from now onward, we use the subscript t to denote the partial derivative with respect to the time t, $m_o = \rho A$ is the oscillating mass of the riser per unit length with A being the riser cross section area, and ρ being the density of the riser, $J = \rho I$ with I being the second moment of the riser cross section area about the $\hat{b}$ axis, F and M are internal force and moment vectors, q and m are the external distributed force and moment vectors, and $\omega_t = \hat{n}\times\hat{n}_{tt} + \hat{b}\times\hat{b}_{tt}$ is the angular acceleration of a point on the centerline. The distributed moment vector m is induced by the asymmetry of the relative flow due to vortex shedding. Let $(F_{\hat{t}}, F_{\hat{n}}, F_{\hat{b}})$ and $(M_{\hat{t}}, M_{\hat{n}}, M_{\hat{b}})$ be the components of F and M along the $\hat{t}, \hat{n}, \hat{b}$ axes of the body-fixed system, respectively. We then can write F and M as

$$\begin{aligned} F &= F_{\hat{t}}\hat{t} + F_{\hat{n}}\hat{n} + F_{\hat{b}}\hat{b} \\ M &= M_{\hat{t}}\hat{t} + M_{\hat{n}}\hat{n} + M_{\hat{b}}\hat{b} \end{aligned} \tag{5}$$

Since the riser is assumed to be straight at the initial time t_0, we have the following constitutive relations, see Love (1920) and Bernitsas (1982)

$$F_{\hat{t}} = EA_{\in} + T_0 + \rho_w g\frac{\pi D_o^2}{4}(H_w - z) - \rho_m g\frac{\pi D_i^2}{4}(H_m - z)$$
$$M_{\hat{b}} = B\kappa, \qquad M_{\hat{n}} = 0, \qquad M_{\hat{t}} = G\tau + H \tag{6}$$

where E is Young's modulus, T_0 is the initial tension in the riser; H_w and H_m are the vertical coordinates of the free surface of the water and mud, respectively; ρ_w and ρ_m are the density of the water and mud, respectively; D_o and D_i are the external and internal diameters of the riser; z is the vertical coordinate of the point P; $B = EI$ is the bending rigidity of the riser; H is the initial torsional moment around the $\hat{t}$ axis; $G = 2\mu I$ is the torsional rigidity of the riser with μ being the shear modulus, $\in$ is the extension of the riser centerline given by Dill (1992)

$$\in = \frac{ds}{ds_0} - 1 = \sqrt{\frac{dr}{ds_0}\cdot\frac{dr}{ds_0}} - \tag{7}$$

It is noted that since we assumed that extension of the riser centerline is small and the riser centerline is stretched, hence $0 \leq \in \leq 1$. The case where $\in = 0$ corresponds to an inextensible riser. Moreover, $F_{\hat{t}}$ in (6) is referred to as the effective tension, while the actual tension is $EA_{\in}$.

Remark 2.2. In Dill (1992), a local coordinate system (a_1, a_2, a_3) where a_3 coincides with $\hat{t}$, different from the local coordinate $(\hat{t}, \hat{n}, \hat{b})$ in this paper is used. Using the local coordinate (a_1, a_2, a_3) results in complexities in calculating the curvatures of the riser in the (a_1, a_3) and (a_2, a_3) planes. Indeed, one can rotate the coordinate system (a_1, a_2, a_3) round the $\hat{t}$ axis a angle to have the coordinate system $(\hat{t}, \hat{n}, \hat{b})$, which has nice properties in (3). In Bernitsas (1982), the constitutive equation for the moment in the normal direction, $M_{\hat{n}}$, is misgiven, since $M_{\hat{n}}$ is always zero for the riser under consideration.

2.1.2. Equations of motion

From (5) and the second equation of (6), we have

$$\begin{aligned} M_s &= (B\kappa\hat{b})_s + (\overline{H}\hat{t})_s = (B\hat{t}\times\hat{t}_s)_s + (\overline{H}\hat{t})_s = \hat{t}\times(B\hat{t}_s)_s + \overline{H}_s\hat{t} + \overline{H}\hat{t}_s \\ &= \hat{t}\times(B\kappa\hat{n})_s + \overline{H}\kappa\hat{n} + \overline{H}_s\hat{t} = \hat{t}\times(B\kappa\hat{n})_s - \overline{H}\kappa\hat{t}\times\hat{b} + \overline{H}_s\hat{t} \\ &= \hat{t}\times((B\kappa\hat{n})_s - \overline{H}\kappa\hat{b}) + \overline{H}_s\hat{t} \end{aligned} \tag{8}$$

where $\overline{H} = H + G\tau$ and we have used the vector algebra properties given in (A.1). Now substituting (8) into the second equation of (4) results in

$$J\omega_t = \hat{t}\times((B\kappa\hat{n})_s - \overline{H}\kappa\hat{b} + F) + \overline{H}_s\hat{t} + m \tag{9}$$

Now producting vector both sides of (9) with $\hat{t}$ gives

$$\hat{t}.(J\omega_t) = \hat{t}.(\hat{t}\times((B\kappa\hat{n})_s - \overline{H}\kappa\hat{b} + F)) + H_s\hat{t}.\hat{t} + m.\hat{t} \;\Rightarrow\; r_s.(J\omega_t) = \overline{H}_s + m.r_s \tag{10}$$

where we have used the vector algebra properties given in (A.1) and the definition of $\hat{t}$ in (2). On the other hand, vectoring both sides of (9) with $\hat{t}$ gives

$$\hat{t}\times(J\omega_t) = \hat{t}\times(\hat{t}\times(B\kappa\hat{n})_s) - \hat{i}\times(\hat{t}\times H\kappa\hat{b}) + \hat{t}\times(\hat{t}\times F) + \hat{t}\times(\overline{H}_s\hat{t}) + \hat{t}\times m \tag{11}$$

Let us calculate the first three terms of the right hand side of (11) using the vector algebra properties given in (A.1) and the definitions of $\hat{t}, \hat{n}$ and $\hat{b}$ in (2) as follows

$$\begin{aligned}
&\hat{t}\times(\hat{t}\times(B\kappa\hat{n})_s) = -\hat{t}.\hat{t}(B\kappa\hat{n})_s + (\hat{t}.(B\kappa\hat{n})_s)\hat{t} = (B\hat{t}_s)_s + (\hat{t}.(B_s\hat{t}_s + B\hat{t}_{ss}))\hat{t} \\
&\qquad = -(B\hat{t}_s)_s + (B\hat{t}.\hat{t}_{ss})\hat{t} = -(B\hat{t}_s)_s + B\hat{t}.(\kappa_s\hat{n} - k^2\hat{t})\hat{t} = -(Br_{ss})_s - B\kappa^2 r_s, \\
&\hat{t}\times(\hat{t}\times\overline{H}\kappa\hat{b}) = \overline{H}\kappa(-\hat{t}.\hat{t}\hat{b} + (\hat{t}.\hat{b})\hat{t}) = -\overline{H}\kappa\hat{b} = -\overline{H}r_s\times r_{ss} \\
&\hat{t}\times(\hat{t}\times F) = -\hat{t}.\hat{t}F + (\hat{t}.F)\hat{t} = -F + (F.r_s)r_s
\end{aligned} \tag{12}$$

Substituting (12) into (11) gives

$$r_s\times(J\omega_t) = -(Br_{ss})_s - B\kappa^2 r_s + \overline{H}r_s\times r_{ss} - F + (F.r_s)r_s + r_s\times m \tag{13}$$

Now substituting F from (13) into the first equation of (6) and combining the second equation of (10) result in the equations of motion of the riser as follows

$$\begin{aligned}
&m_o w_{tt} = -(Br_{ss})_{ss} + (F_{\tilde{t}} - B\kappa^2)r_s)_s + (\overline{H}r_s\times r_{ss})_s + (r_s\times m)_s - (r_s\times(J\omega_t))_s + q \\
&r_s.(J\omega_t) = \overline{H}_s + m.r_s
\end{aligned} \tag{14}$$

It is noted that we have assumed the torsional moment $\overline{H}$ and the distributed moment m are negligible, and that the riser has a constant cross section. Furthermore, since the riser is initially straight, we have $r_{ss} = w_{ss}$, $r_{ssss} = w_{ssss}$ and $w_s = r_s - r_s^0$ where we take $s \simeq s_0$ due to the small extension assumption, see Dill (1992). With these in mind, we now have the equations of motion of the riser from (14) for the boundary control design in the next section

$$\begin{aligned}
&m_o w_{tt} = -Bw_{ssss} + (F_{\tilde{t}} - B\kappa^2)_s(w_s + r_s^0) + (F_{\tilde{t}} - B\kappa^2)w_{ss} + q \\
&\kappa = \|w_{ss}\|
\end{aligned} \tag{15}$$

Remark 3. The riser dynamics (14) or (15) is one dimensional (with respect to the spatial variable s). This means that a point on the riser cross section, other than the point on the centerline, cannot be traced after deformation takes place. In this paper, we consider the deformation of the riser centerline, which is, in general, a three-dimensional space curve.

2.1.3. Initial and boundary conditions

The initial conditions of the riser consist of the initial position and velocity functions. They are

$$w(s,t_0) = g_1(s), \qquad w_t(s,t_0) = g_2(s), \qquad \forall s\in(0,L) \tag{16}$$

where $g_1(s)$ and $g_2(s)$ are sufficiently smooth and bounded function vectors of s, and compatible with the boundary conditions. Next, we will apply Hamilton's principle to derive the boundary conditions for the riser under consideration.We first provide the kinetic and potential energies, then use the first variation of the Lagrangian of the system to derive the boundary conditions. As such, the kinetic energy K_E and the potential energy P_E of the riser with a length of L are

$$\begin{aligned}
&K_E = \frac{1}{2}\int_0^L m_o r_t.r_t ds \\
&P_E = \frac{1}{2}\int_0^L Br_{ss}.r_{ss}ds - \int_0^L qrds + F(0)r(0) - F(L)r(L) + M(0)r_s(0) - M(L)r_s(L)
\end{aligned} \tag{17}$$

where we have used $r_t = w_t$ and $r_{ss} = w_{ss}$. The Lagrangian L_A of the riser is

$$L_A = \int_{t_1}^{t_2} (K_E - P_E)dt \tag{18}$$

where t_1 and t_2 denote time. Moreover, the riser response must satisfy the kinetic constraint of the unit tangent vector $\hat{t}$. In terms of deformation, this constraint is expressed by

$$r_s.r_s = 1 \tag{19}$$

The above constraint is applied along the riser by modifying the Lagrangian of the riser and by embedding a continuous multiplies $(F_{\hat{t}} - B\kappa^2)/2$. As such, the modified Lagrangian L_{MA} is

$$L_{MA} = \int_{t_1}^{t_2}\left[K_E - P_E + \frac{(F_{\hat{t}} - B\kappa^2)}{2}\int_0^L (r_s.r_s - 1)ds\right]dt \tag{20}$$

It is noted that including the term $\int_{t_1}^{t_2}\left[\frac{(F_{\hat{t}} - B\kappa^2)}{2}\int_0^L (r_s.r_s - 1)ds\right]dt$ in the modified Lagrangian physically means that the modified Lagrangian takes the contribution of the axial deformation into account in the potential energy. From (20), the first variation of L_{MA} is given by

$$\begin{aligned} \delta L_{MA} = & \int_{t_1}^{t_2}\int_0^L [-(Br_{ss})_{ss} + ((F_{\hat{t}} - B\kappa^2)r_s)_s + q - m_o r_{tt}]\delta r ds dt + \\ & \int_{t_1}^{t_2}(r_s \times M - Br_{ss})\delta r_s\Big|_0^L dt + \int_{t_1}^{t_2}(-(Br_{ss})_s + (F_{\hat{t}} - B\kappa^2)r_s - F)\delta r\Big|_0^L dt \end{aligned} \tag{21}$$

Since δr is arbitrary over the domain $0 < s < L$, letting $\delta L_{MA} = 0$ results in

$$-(Br_{ss})_{ss} + ((F_{\hat{t}} - B\kappa^2)r_s)_s + q - m_o r_{tt} = 0, \qquad \forall s \in [0, L], t \in \mathrm{R}^+ \tag{22}$$

and

$$r_x \times M - Br_{ss} = 0 \quad \text{or} \quad r_s = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad s = L \quad \forall t \in \mathrm{R}^+ \tag{23}$$

and

$$-(Br_{ss})_s + (F_{\hat{t}} - B\kappa^2)r_s - F = 0 \quad \text{or} \quad r = 0 \quad \text{at} \quad s = 0 \quad \text{and} \quad s = L \quad \forall t \in \mathrm{R}^+$$

The Eq. (22) is exactly the same as (15). The Eqs. (23) and (24) specify the boundary conditions of the riser at top and bottom ends. Choosing proper conditions from (23) and (24) depends on the riser configuration. For the riser considered in this paper, ball joints at both ends imply that the moments acting at both ends are zero, i.e., $M(L, t) = M(0, t) = 0$, and the force vector $U(t)$ as the boundary control inputs at the top end. With this observation in mind, the boundary conditions (23) and (24) for the riser considered in this paper become

$$\begin{aligned} & w_{ss}(0, t) = 0, \quad w_{ss}(L, t) = 0, \qquad w(0, t) = 0 \\ & -Bw_{sss}(L, t) + F_{\hat{t}}(L, t)[w_s(L, t) + r_s^0(L)] = F(L, t) := U(t) \end{aligned} \tag{25}$$

where $F_{\hat{t}}(L, t)$ is calculated from (6) as follows

$$F_{\hat{t}}(L, t) = EA \in (L, t) + T_0 + \rho_w g\frac{\pi D_o^2}{4}(H_w - z(L, t)) - \rho_m g\frac{\pi D_i^2}{4}(H_m - z(L, t)) \tag{26}$$

2.1.4. Environmental disturbance vector q

The external disturbance vector q per unit length consists of fluid drag force, anyconcentrated forces exerted on the riser by attached cables and/or buoys modeled by dirac distributions, and effective riser weight defined as the weight of the riser plus contents in water. It is noted that the effective rather than the actual riser weight is used because the effective tension is used instead of the actual tension. In this paper, we do not consider cables or buoys attached to the riser. The fluid drag force is found by the use of a generalization of Morison's formula to account for cylinders, which are not oriented normal to the relative flow Borgman (1958). Taking the effective riser weight into account, we have

$$q(s,t) = \hat{t}\times(W_{re}\times\hat{t})+\frac{1}{2}\rho_w C_{LD}D_H V_n+\frac{1}{2}\rho_w C_{ND}D_H\|V_n\|V_n \tag{27}$$

where C_{LD} and C_{ND} are the linear and nonlinear drag coefficients, respectively; D_H is the local riser hydrodynamic diameter; $W_{re} = -[0\ 0\ w_{re}]^T$ with w_{re} is the effective riser weight per unit length; V_n is the component of the relative flow velocity normal to the riser centerline. Letting V be the (bounded) liquid flow velocity due to waves and currents. Then taking the riser motion into account, the relative flow velocity normal to the riser centerline, V_n, is given by

$$V_n = \hat{t}\times((V-w_t)\times\hat{t}) = (I_{3\times3}-r_s r_s^T)(V-w_t) \tag{28}$$

where $I_{3\times3}$ is the three dimensional identity matrix. Substituting (28) into (27) results in the equation for external disturbance vector q as follows

$$\begin{aligned} q(s,t) = {} & (I_{3\times3}-r_s r_s^T)W_{re}+\frac{1}{2}\rho_w C_{LD}D_H(I_{3\times3}-r_s r_s^T)(V-w_t) \\ & +\frac{1}{2}\rho_w C_{ND}D_H\|(I_{3\times3}-r_s r_s^T)(V-w_t)\|(I_{3\times3}-r_s r_s^T)(V-w_t) \end{aligned} \tag{29}$$

2.2. Control objectives

Under Assumption 2.1, design the boundary control $U(t)$ for the riser dynamics given by (15) subject to the boundary conditions given by (25) to globally stabilize the riser at its vertical position, i.e., finding the boundary control $U(t)$ of the form $U(t) = \Omega(w_s(L,t), w_t(L,t))$ such that:

1. when the external disturbance vector q is ignored, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to zero for all $s\in[0,L]$ and $t\geq t_0$,
2. when the external disturbance vector q is present, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t)ds$ exponentially converge to some small positive constants for all $s\in[0,L]$and $t\geq t_0$

It is seen that the control objective imposes on both the displacement and integration of square of the slop, velocity, and curvature of the riser along the riser length.

3. Boundary control design

To design the boundary control $U(t)$, we use Lyapunov's direct method. Consider the following Lyapunov function candidate

$$w = \frac{m_o}{2}\int_0^L w_t.w_t ds+\frac{B}{2}\int_0^L w_{ss}.w_{ss}ds+\frac{\lambda}{2}\int_0^L w_s.w_s ds+\alpha\int_0^L s w_t.w_s ds \tag{30}$$

where λ and α are positive constants to be specified later. In comparison with the conventional Lyapunov, the, Lyapunov function candidate (C.7) contains the constant λ and the term $\alpha\int_0^L sw_t.w_s ds$, which together with Lemmas B1 and B2 in Section Appendix B play important role in designing boundary controllers later. Since for all $t \geq t_0$

$$-L\rho_0\int_0^L w_t.w_t ds - \frac{1}{4\rho_0}\int_0^L w_s.w_s ds \leq \int_0^L sw_t.w_s ds \leq L\rho_0\int_0^L w_t.w_t ds + \frac{1}{4\rho_0}\int_0^L w_s.w_s ds \tag{31}$$

where ρ_0 is a positive constant, the function W satisfies

$$\begin{aligned} W &\geq \left(\frac{m_o}{2} - \alpha L\rho_0\right)\int_0^L w_t.w_t ds + \frac{B}{2}\int_0^L w_{ss}.w_{ss} ds + \left(\frac{\lambda}{2} - \frac{\alpha L}{4\rho_0}\right)\int_0^L w_s.w_s ds \\ W &\leq \left(\frac{m_o}{2} + \alpha L\rho_0\right)\int_0^L w_t.w_t ds + \frac{B}{2}\int_0^L w_{ss}.w_{ss} ds + \left(\frac{\lambda}{2} + \frac{\alpha L}{4\rho_0}\right)\int_0^L w_s.w_s ds \end{aligned} \tag{32}$$

Hence if we choose λ, α and ρ_0 such that

$$\frac{m_o}{2} - \alpha L\rho_0 = c_1, \qquad \frac{\lambda}{2} - \frac{\alpha L}{4\rho_0} = c_2 \tag{33}$$

where c_1 and c_2 are strictly positive constants, then the function W defined in (C.7) is a proper function of $\int_0^L w_t.w_t ds$, $\int_0^L w_{ss}.w_{ss} ds$, and $\int_0^L w_s.w_s ds$. We do not detail the conditions (33) at the moment, but deal with them after the boundary control $U(t)$ is designed since the constants λ, α and ρ_0 need to satisfy some more conditions later. It is noted that we do not include the riser displacement w, like $\int_0^L w.w ds$, in the function W because this term causes difficulties in designing the boundary control $U(t)$ later. As such, after proof of convergence of $\int_0^L w_t.w_t ds$, $\int_0^L w_{ss}.w_{ss} ds$ and $\int_0^L w_s.w_s ds$, we will use Lemma B.1 in Appendix B to prove convergence of $\int_0^L w.w ds$ and the riser displacement w. Differentiating both sides of (C.7) with respect to the time t, along the solutions of the riser dynamics (15) results in

$$\dot{W} = \Delta_1 + \Delta_2 \tag{34}$$

where

$$\begin{aligned} \Delta_1 &= \int_0^L w_t.(-Bw_{ssss} + (F_{\tilde{t}} - B\kappa^2)_s(w_s + r_s^0) + (F_{\tilde{t}} - B\kappa^2)w_{ss} + q)ds + \\ &\quad B\int_0^L w_{ss}.w_{sst} ds + \lambda\int_0^L w_s.w_{st} ds + \alpha\int_0^L sw_t.w_{ts} ds, \\ \Delta_2 &= \frac{\alpha}{m_o}\int_0^L sw_s.(-Bw_{ssss} + (F_{\tilde{t}} - B\kappa^2)_s(w_s + r_s^0) + (F_{\tilde{t}} - B\kappa^2)w_{ss} + q)ds. \end{aligned} \tag{35}$$

Using integration by part rules, we have

$$\begin{aligned} \Delta_1 &= -B(w_{ssss}.w_t|_0^L - w_{ss}.w_{st}|_0^L) + (F_{\tilde{t}} - B\kappa^2)(w_s + r_s^0).w_t|_0^L - \\ &\quad \int_0^L (F_{\tilde{t}} - B\kappa^2)(w_{ss}.w_t + (w_s + r_s^0).w_{st}) + \int_0^L (F_{\tilde{t}} - B\kappa^2)w_{ss}.w_t ds + \\ &\quad \int_0^L w_t.q ds + \lambda w_s.w_t|_0^L - \int_0^L w_{ss}.w_t ds + \frac{\alpha}{2}sw_t.w_t|_0^L - \frac{\alpha}{2}\int_0^L w_t.w_t ds \end{aligned} \tag{36}$$

Since $r_s.r_s = 1$, we have $(w_s + r_s^0)\,.\,w_{st} = 0$, which is substituted into (36) to yield

$$\begin{aligned}\Delta_1 =& -B(w_{sss}.w_t|_0^L - w_{ss}.w_{st}|_0^L) + (F_{\hat{t}} - B\kappa^2)(w_s + r_s^0).w_t|_0^L + \lambda w_s.w_t|_0^L + \frac{\alpha}{2} s w_t.w_t|_0^L + \\ & \int_0^L w_t.q ds - \lambda \int_0^L w_{ss}.w_t ds - \frac{\alpha}{2}\int_0^L w_t.w_t ds.\end{aligned} \tag{37}$$

We now focus on the term Δ_2. Expanding this term gives

$$\Delta_2 = \Delta_{21} + \Delta_{22} + \frac{\alpha}{m_o}\int_0^L s w_s.q ds \tag{38}$$

where

$$\Delta_{21} = -\frac{\alpha B}{m_o}\int_0^L s w_s.w_{ssss} ds,\ \Delta_{22} = \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2)_s (w_s + r_s^0) ds + \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2) w_{ss} ds \tag{39}$$

Using integration by part rules, we can calculate the term Δ_{21} as

$$\Delta_{21} = -\frac{\alpha B}{m_o} s w_s.w_{sss}|_0^L + \frac{\alpha B}{2m_o} s w_{ss}.w_{ss}|_0^L + \frac{\alpha B}{m_o} w_s.w_{ss}|_0^L - \frac{3\alpha B}{2m_o}\int_0^L w_{ss}.w_{ss} ds. \tag{40}$$

Similarly, the term Δ_{22} is calculated as

$$\begin{aligned}\Delta_{22} =& \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2) s w_s.(w_s + r_s^0)|_0^L + \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2) s w_s.w_{ss} ds - \\ & \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2)(w_s.(w_s + r_s^0) + s w_{ss}.(w_s + r_s^0) + s w_s.w_{ss}) ds \\ =& \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2) s w_s.(w_s + r_s^0)|_0^L - \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2) w_s.(w_s + r_s^0) ds \\ =& \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2) s w_s.(w_s + r_s^0)|_0^L - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}} w_s.w_s ds - \\ & \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}(1 - r_s^0.r_s^0) ds + \frac{\alpha B}{m_o}\int_0^L w_{ss}.w_{ss}.w_s.(w_s + r_s^0) ds\end{aligned} \tag{41}$$

where we have used $(w_s + r_s^0)\,.w_{ss} = 0$ and $w_s.w_s + 2r_s^0.w_s + r_s^0.r_s^0 = 1$ since $r_s.r_s = 1$ and $r_{ss}^0 = 0$ due to the riser is initially straight. Now substituting (41) and (40) into (38), then substituting (38) and (37) into (34) results in

$$\begin{aligned}\dot{W} =& -B(w_{sss}.w_t|_0^L - w_{ss}.w_{st}|_0^L) + (F_{\hat{t}} - B\kappa^2)(w_s + r_s^0).w_r|_0^L + \frac{\alpha}{2} s w_t.w_t|_0^L \\ & -\frac{\alpha B}{m_o} s w_s.w_{sss}|_0^L + \frac{\alpha B}{2m_o} s w_{ss}.w_{ss}|_0^L + \frac{\alpha B}{m_o} w_s.w_{ss}|_0^L + \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2) s w_s.(w_s + r_s^0)|_0^L \\ & -\lambda\int_0^L w_{ss}.w_t ds - \frac{\alpha}{2}\int_0^L w_t.w_t ds - \frac{3\alpha B}{2m_o}\int_0^L w_{ss}.w_{ss} ds - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}} w_s.w_s ds \\ & -\frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}(1 - r_s^0.r_s^0) ds + \frac{\alpha B}{m_o}\int_0^L w_{ss}.w_{ss} w_s.(w_s + r_s^0) ds + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds\end{aligned} \tag{42}$$

Before going further, we find maximum and minimum values of $F_{\hat{t}}$, and maximum value of

$w_s.(w_s+r_s^0)$ and $r_s^0 r_s^0$. From (6), we have

$$
\begin{aligned}
&F_{\hat{t}} \leq EA + T_0 + \rho_w g\frac{\pi D_o^2}{4}H_w - \rho_m g\frac{\pi D_i^2}{4}(H_m - L) := F_{\hat{t}}^{max} \\
&F_{\hat{t}} \geq T_0 + \rho_w g\frac{\pi D_o^2}{4}(H_w - L) - \rho_m g\frac{\pi D_i^2}{4}H_m := F_{\hat{t}}^{max}
\end{aligned}
\tag{43}
$$

where we have used $0 \leq \epsilon(s,t) \leq 1$ and $0 \leq z(s,t) \leq L$ for all $s \in [0, L]$ and $t \geq t_0 \geq 0$. On the other hand, from (7) we have

$$
\begin{aligned}
&w_s.(r_s^0 + w_s) = r_s.r_s - r_s.r_s^0 = 1 - \frac{\cos(\theta)\|dr\|\|dr^0\|}{(ds)^2} \leq 1 \\
&r_s^0.r_s^0 = r_{s_0}^0.r_{s_0}^0\left(\frac{ds_0}{ds}\right)^2 = \left(\frac{ds_0}{ds}\right)^2 \leq 1
\end{aligned}
\tag{44}
$$

where we have used the fact that the angle θ between the vectors r and r^0 is in the range $[-\pi/2, +\pi/2]$ due to the initial straight and vertical position of the riser. Using (43) and (44), and $F_{\hat{t}}^{min} > 0$, which holds when T_0 is sufficiently large, i.e.,

$$
T_0 \geq -\rho_w g\frac{\pi D_o^2}{4}(H_w - L) + \rho_m g\frac{\pi D_i^2}{4}H_m + \overline{T}_0 \tag{45}
$$

where $\overline{T}_0$ is a strictly positive constant, we can write (42) as follows

$$
\begin{aligned}
\dot{W} \leq & -B(w_{sss}.w_t|_0^L - w_{ss}.w_{st}|_0^L) + (F_{\hat{t}} - B\kappa^2)(w_w + r_s^0).w_t|_0^L + \lambda w_s.w_t|_0^L + \frac{\alpha}{2}sw_t.w_t|_0^L \\
& -\frac{\alpha B}{m_o}sw_s.w_{ss}|_0^L + \frac{\alpha B}{2m_o}sw_{ss}.w_{ss}|_0^L + \frac{\alpha B}{m_o}w_s.w_{ss}|_0^L + \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2)sw_s.(w_s + r_s^0)|_0^L \\
& -\left(\frac{\alpha}{2} - \lambda\rho_1\right)\int_0^L w_t.w_t ds - \left(\frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1}\right)\int_0^L w_{ss}.w_{ss}ds - \frac{\alpha F_{\hat{t}}^{min}}{2m_o}\int_0^L w_s.w_s ds \\
& + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L sw_s.q ds
\end{aligned}
\tag{46}
$$

where ρ_1 is a positive constant to be specify later. Now substituting the boundary conditions (25) into (42) results in

$$
\begin{aligned}
\dot{W} \leq & \ U(t).w_t(L,t) + \lambda w_s(L,t).w_t(L,t) + \frac{\alpha L}{2}w_t(L,t).w_t(L,t) - \frac{\alpha LB}{m_o}w_s(L,t).w_{sss}(L,t) \\
& + \frac{\alpha L}{m_o}F_{\hat{t}}(L,t)w_s(L,t).[w_s(L,t) + r_s^0(L)] + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L sw_s.q ds \\
& -\left(\frac{\alpha}{2} - \lambda\rho_1\right)\int_0^L w_t.w_t ds - \left(\frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1}\right)\int_0^L w_{ss}.w_{ss}ds - \frac{\alpha F_{\hat{t}}^{min}}{2m_o}\int_0^L w_s.w_s ds
\end{aligned}
\tag{47}
$$

From (47), we choose the boundary control $U(t)$ as follows

$$
U(t) = -k_1 w_t(L,t) - k_2 w_s(L,t) \tag{48}
$$

where k_1 and k_2 are positive constants to be specified later. It is recalled from (25) that $U(t) = -Bw_{sss}(L, t) + F_t(L,t)[w_s(L,t)+r_s^0(L)]$. Hence from (48), we have

$$-Bw_{sss}(L,t) = -k_1 w_t(L,t) - k_2 w_s(L,t) - F_t(L,t)[w_s(L,t)+r_s^0(L)] \tag{49}$$

Now substituting (48) and (49) into (47) gives

$$\begin{aligned}\dot{W} \le & -\left(k_1 - \frac{\alpha L}{2}\right) w_t(L,t).w_t(L,t) - \frac{\alpha L k_2}{m_o} w_s(L,t).w_s(L,t) + \left(\lambda - k_2 - \frac{\alpha L k_1}{m_o}\right) w_s(L,t).w_t(L,t) \\ & -\left(\frac{\alpha}{2} - \lambda\rho_1\right)\int_0^L w_t.w_t ds - \left(\frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1}\right)\int_0^L w_{ss}.w_{ss} ds - \frac{\alpha F_t^{min}}{2m_o}\int_0^L w_s.w_s ds \\ & + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds\end{aligned} \tag{50}$$

From (50), we specify the positive constants ρ_1, λ, α, k_1 and k_2 such that

$$k_1 - \frac{\alpha L}{2} = c_3, \lambda - k_2 - \frac{\alpha L k_1}{m_o} = 0, \frac{\alpha}{2} - \lambda\rho_1 = c_4, \frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1} = c_5 \tag{51}$$

where c_3, c_4 and c_5 are strictly positive constants. Using the conditions given in (51) and the upper bound of W given in (32), we can write (50) as follows

$$\dot{W} \le -c_3 w_t(L,t).w_t(L,t) - \frac{\alpha L k_2}{m_o} w_s(L,t).w_s(L,t) - cW + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds \tag{52}$$

where

$$c = \frac{\min\left(c_4, c_5, \frac{\alpha \overline{T}_0}{2m_o}\right)}{\max\left(\left(\frac{m_o}{2} + \alpha L\rho_0\right), \frac{B}{2}, \left(\frac{\lambda}{2} + \frac{\alpha L}{4\rho_0}\right)\right)} \tag{53}$$

where $\overline{T}_0$ is the strictly positive constant in (45). Before going further, we show that there always exist constants ρ_0, ρ_1, λ, α, k_1 and k_2 such that the conditions specified in (33) and (51) hold with ci, $i = 1, ..., 5$ strictly positive constants. For simplicity, we choose $\rho_0 = L\sqrt{m_o/B}$ and $\rho_1 = \sqrt{m_o/4B}$. Acalculation shows that as long as the positive constants λ, α, k_1 and k_2 are chosen such that the following inequalities strictly hold

$$\alpha < \frac{1}{2L^2}\sqrt{\frac{B}{m_o}}, \ \frac{\alpha}{2}\sqrt{\frac{B}{m_o}} < \frac{\alpha L k_1}{m_o} + k_2 < \alpha\sqrt{\frac{B}{m_o}}, \ k_1 > \frac{\alpha L}{2}, \ \lambda = \frac{\alpha L k_1}{m_o} + k_2 \tag{54}$$

then there exist strictly positive constants ci, $i = 1, ..., 5$ satisfying the conditions specified in (33) and (51). For given practical values of B, L and m_o, it is not hard to find the constants λ, α, k_1 and k_2 such that all equalities in (54) strictly hold. For example, we can take

$$\begin{aligned} & k_1 = \frac{1}{3L}\sqrt{Bm_o}, \ k_2 = \frac{\alpha}{3}\sqrt{\frac{B}{m_o}}, \ \alpha = \frac{1}{2}\min\left(\frac{1}{2L^2}\sqrt{\frac{B}{m_o}}, \frac{2}{3L^2}\sqrt{Bm_o}\right), \ \lambda = \frac{2\alpha}{3}\sqrt{\frac{B}{m_o}} \\ & \Rightarrow c_1 = \frac{m_o}{2} - \min\left(\frac{1}{4}, \frac{m_o}{3}\right), \ c_2 = \frac{\alpha}{12}\sqrt{\frac{B}{m_o}}, \ c_3 = \frac{\sqrt{Bm_o}}{L}\left(\frac{1}{3} - \min\left(\frac{1}{8m_o}, \frac{1}{6}\right)\right), \ c_4 = \frac{\alpha}{6}, \ c_5 = \frac{B\alpha}{6m_o}\end{aligned} \tag{55}$$

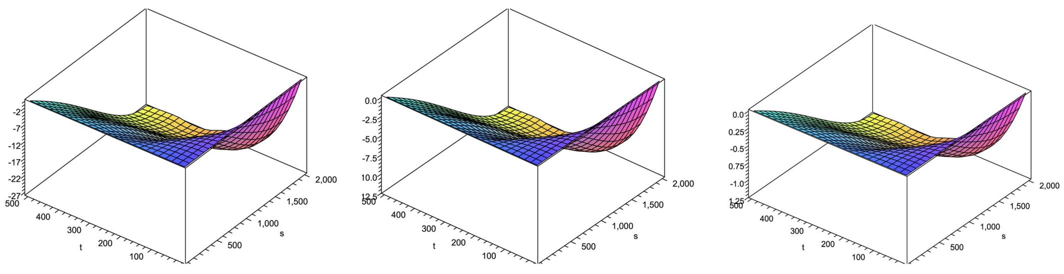
We are ready to state the main result of our paper in the following theorem.

Theorem 3.1. Under Assumption 1, the boundary control U(t) given in (48) solves the control objective provided that the initial tension T_0 is suciently large, i.e., the condition (45) holds, and the design constants k_1 and k_2 are chosen such that the conditions given in (54) hold. In particular, the solutions of the closed loop system consisting of (15), (25) and (48) exist and are unique. Moreover, when the external disturbance vector q is zero, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to zero for all $s \in [0, L]$ and $t \geq t_0$, and when the external disturbance vector q is dierent from zero but bounded, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to some small positive constants for all $s \in [0, L]$ and $t \geq t_0$. In addition, the boundary control U(t) is bounded.

Proof. See Appendix C.

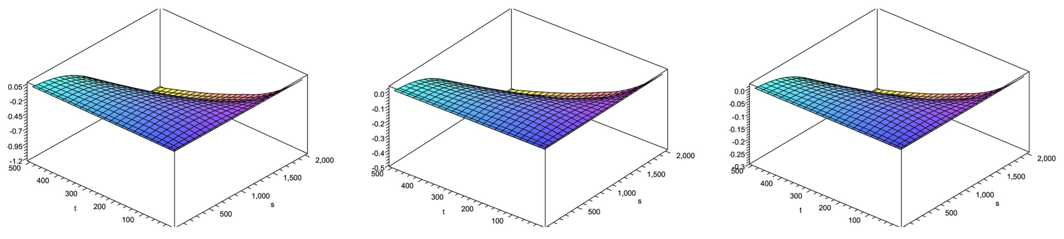
4. Simulations

In this section, we carry out some numerical simulations to illustrate the effectiveness of the proposed boundary controller. The riser parameters are taken from Bernitsas *et al.* (1985) as follows: L = 2000 m, D_o = 0.61 m, D_i = 0.575 m, D_H = 0.87 m, w_e = 1.132 KN/m, ρ_w = 1025 kg/m^3, ρ_m = 1250 kg/m^3, C_D = 0.7, ρ = 8200 kg/m^3, $E = 2 \times 10^{10}$ kg/m, H_w = L, H_m = L/3. The initial conditions are taken as $w(s, t_0)= [0, 0, 0]^T$, $w_t(s, t_0)= [0, 0, 0]^T$. The design constants k_1 and k_2 are taken according to (55), i.e., k_1 = 17.37 and k_2 = 0.0032. The ocean current velocity vector is assumed to



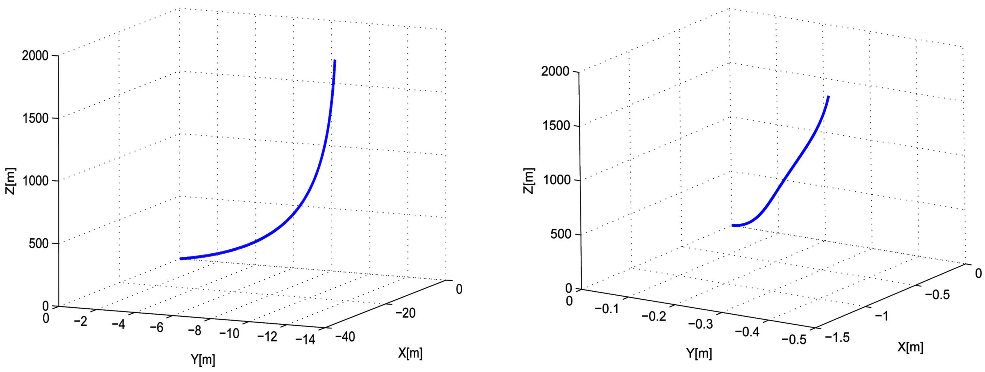
(a) w_x (b) w_y (c) w_z

Fig. 3 Simulation results without control: Displacements w_x, w_y, w_z



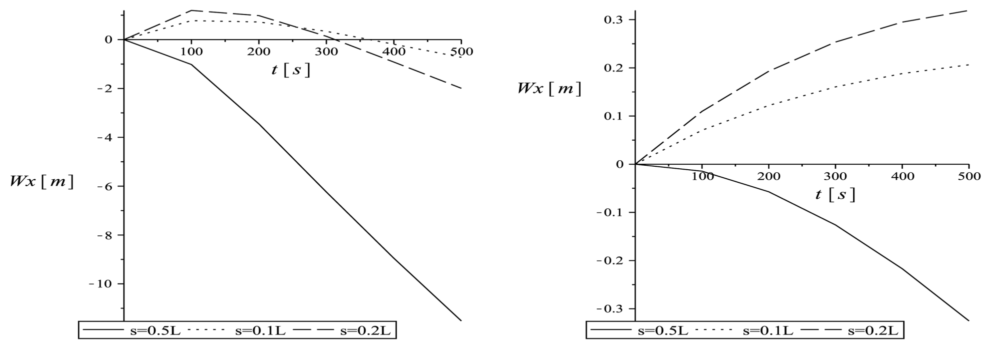
(a) w_x (b) w_y (c) w_z

Fig. 4 Simulation results with control: Displacements w_x, w_y, w_z



(a) Uncontrolled results: Displacement in 3D (b) Controlled results: Displacement in 3D

Fig. 5 Simulation results without and with control: Displacements in 3D at t = 500 second



(a) Uncontrolled results: Displacement in x-direction (b) Controlled results: Displacement in x-direction

Fig. 6 Simulation results without and with control: Displacements in x-direction at $s = 0.1L$, $s = 0.2L$ and $s = 0.5L$

be generated from wind at the ocean surface and dropped to zero at the sea bed Gaythwaite (1981): $V = [(1/L)s, (0.5/L)s, 0]^T$. The initial tension is assumed to be $\overline{T}_0 = 2.15 \times 10^6\,N$. Hence from (45), we have $\overline{T}_0 = 29 \times 10^3\,N$. We run simulations without the proposed boundary controller, i.e., $k_1 = 0$ and $k_2 = 0$, and with the proposed boundary controller, i.e., $k_1 = 17.37$ and $k_2 = 0.0032$. The length of simulation time for both cases is 500 seconds. Displacements $w = [w_x, w_y, w_z]^T$ for the uncontrolled and controlled cases are displayed in Figs 3and 4, respectively. The displacements $r = [x, y, z]^T$ for the uncontrolled and controlled cases are plotted in the left and right of Fig. 5, respectively. Moreover, the displacements wx for the uncontrolled and controlled cases are plotted in the left and right of Fig. 6, respectively. It is seen from these figures that the proposed boundary controller can reduce deflections of the riser in all directions (x, y, z) significantly, i.e., the displacement magnitudes are significantly reduced. For example, in the x direction, the displacement magnitude reduces from 27 m to 1.2 m at the top end of the riser. This illustrates the effectiveness of the proposed boundary controller in the sense that it is able to drive the riser to the small neighborhood of its equilibrium position.

5. Conclusions

Based on the Kirchho's rod theory, the equations of motion of a flexible marine riser were presented. The equations of motion were then used for the design of the boundary controller at the top end of the riser based on Lyapunov's direct method. The proposed boundary controller guaranteed that when there are no environmental disturbances, the riser is be globally exponentially stable at its equilibrium position, and that environmental disturbances are present, the riser is stabilized in the neighborhood of its equilibrium position. Proof of existence and uniqueness of the solutions of the closed loop system was given. Future work focuses on relaxing items made in Assumption 1, and carrying out experiments to test the effectiveness of the proposed boundary controller. Particularly, an immediate task is to consider an arbitrarily initial position of the riser and to take the effect of the torsional moments into account in the boundary control design.

Appendix A. Vector algebra

For convenience of the reader, we here present some basic vector algebra, which will be used in the development of the equations of motion of the risers. Let r, s, t, u be three dimensional vectors, i.e., $r = [r_1\ r_2\ r_3]^T$, $s = [s_1\ s_2\ s_3]^T$, $t = [t_1\ t_2\ t_3]^T$ and $u = [u_1\ u_2\ u_3]^T$ with $(\bullet)^T$ being the transpose of $(\bullet)$. We will use $r.s$ and $r \times s$ to denote the inner (or dot) and vector (or cross) products, respectively, of vectors r and s. We have the following properties of vector inner and cross products

$$\begin{array}{ll} 1)\ r.(s\times t) = s.(t\times r) = t.(r\times s) & 2)\ r\times s = -s\times r \\ 3)\ r\times(s\times t) = -(s.r)t + (t.r)s & 4)\ (r\times s)\times t = (r.t)s-(s.t)r \\ 5)\ (r\times s).(t\times u) = (r.t)(s.u)-(r.u)(s.t) & \end{array} \tag{A.1}$$

Appendix B. Useful lemmas

Lemma B.1 For any $y = [y_1, ..y_i, ..., y_n]^T$ with $y_i \in C^1[0, L]$, $i = 1, ..., n$, the following inequalities hold

$$\int_0^L y(s).y(s)ds \le 2Ly(0).y(0) + 4L^2\int_0^L y_s(s).y_s(s)ds \tag{B.1}$$

$$\int_0^L y(s).y(s)ds \le 2Ly(L).y(L) + 4L^2\int_0^L y_s(s).y_s(s)ds \tag{B.2}$$

Proof. We prove (B.2). The proof of (B.1) is similar by using a change of coordinate $\xi = L - s$. Using integration by parts, we have

$$\begin{aligned} \int_0^L y(s).y(s)ds &= y(s).y(s)s|_0^L - 2\int_0^L sy(s).y_s(s)ds \\ &\le Ly(L).y(L) + \frac{1}{2}\int_0^L y(s).y(s)ds + 2\int_0^L s^2 y_s(s).y_s(s)ds \\ &\le Ly(L).y(L) + \frac{1}{2}\int_0^L y(s).y(s)ds + 2L^2\int_0^L y_s(s).y_s(s)ds \end{aligned} \tag{B.3}$$

which gives (B.2). □

Lemma B.2 For any $y = [y_1, ..y_i, ..., y_n]^T$ with $y_i \in C^1[0, L]$, $i = 1, ..., n$, the following inequalities hold

$$\max_{s\in[0,L]} (y(s).y(s)) \le y(0).y(0) + 2\sqrt{\int_0^L y(s).y(s)ds}\sqrt{\int_0^L y_s(s).y_s(s)ds} \quad \text{(B.4)}$$

$$\max_{s\in[0,L]} (y(s).y(s)) \le y(L).y(L) + 2\sqrt{\int_0^L y(s).y(s)ds}\sqrt{\int_0^L y_s(s).y_s(s)ds} \quad \text{(B.5)}$$

Proof. We prove (B.4). The proof of (B.5) is similar by using a change of coordinate $\xi = L - s$. From fundamental of calculus, we have

$$\begin{aligned} y(s).y(s) &= y(0).y(0) + 2\int_0^s y(\zeta).y_\zeta(\zeta)d\zeta \\ &\le y(0).y(0) + 2\sqrt{\int_0^s y(\zeta).y(\zeta)d\zeta}\sqrt{\int_0^s y_\zeta(\zeta).y_\zeta(\zeta)d\zeta} \end{aligned} \quad \text{(B.6)}$$

where we have used the Cauchy-Schwartz inequality.

Appendix C. Proof of Theorem 3.1

C.1 Existence and uniqueness Let $H^2(0, L)$ be the usual Hilbert space Adams and Fournie. Our analysis is based on the Sobolev spaces

$$VS = w \in H^2(0, L) \mid w(0, t) = 0 \quad \text{(C.1)}$$

equipped with the norm $\|u\|_{V_s} = \|u_{ss}\|_2$, and

$$W_S = w \in V_S \cap H^4(0,L) | w_{ss}(0,t) = 0, w_{ss}(L,t) = 0 \quad \text{(C.2)}$$

equipped with the norm $\|u\|w_s = \|w_{ss}\|_2 + \|w_{ssss}\|_2$ where $\|\cdot\|_p$ denotes the L^p norms. From the Poincare' inequality, it follows that $\|\cdot\|_{V_S}$ and $\|\cdot\|_{W_S}$ are equivalent to the standard norms of H^2 (0, L) and H^4 (0, L), respectively. Next, we consider $\phi \in V_S$. Now inner producting both sides of the first equation of (15) by ϕ then integrating from 0 to L by parts result in

$$\begin{aligned} m_o\int_0^L w_{tt}.\phi ds + B\int_0^L w_{ss}.\phi_{ss}ds + \int_0^L (F_t - B\kappa^2)(w_s + r_s^0).\phi_s ds - \int_0^L q.\phi ds \\ +(k_1 w_t(L,t) + k_2 w_s(L,t)).\phi(L,t) = 0 \end{aligned} \quad \text{(C.3)}$$

where we have used (48).We will use the Galerkin approximation to show that for all $\phi \in V_S$ there exists $w \in W_S$ such that (C.3) holds. Let ϕ^j be a vector whose each component is a complete orthogonal system of W_S for which $\{w(s, t_0), w_t(s, t_0)\}$. Span$\{\phi^1, \phi^2\}$. For each $n \in N$, let W_{Sn} = Span $\{\phi^1, \phi^2, ..., \phi^n\}$. We search for a function $w^n(s, t) = \Sigma_{j=1}^n k^j(t)\phi^j$ such that for any $\phi \in W_{Sn}$, it satisfies the approximate equation

$$\begin{aligned} m_o\int_0^L w_{tt}^n.\phi ds + B\int_0^L w_{ss}^n.\phi_{ss}ds + \int_0^L (F_t^n - B\kappa^{n2})(w_s^n + r_s^{n0}).\phi_s ds - \int_0^L q.\phi ds \\ +(k_1 w_t^n(L,t)) + k_2 w_w^n(L,t).\phi(L,t) = 0 \end{aligned} \quad \text{(C.4)}$$

where $F_{\hat{t}}^{n}$ and κ^{n2} denote $F_{\hat{t}}$ and κ with w and w_{ss} replaced by w^n and w_{ss}^n, respectively, and with the initial conditions

$$w^n(s,t_0) = w(s,t_0), \qquad w_t^n(s,t_0) = w_t(s,t_0) \tag{C.5}$$

which are possible since each element of ($w(s,\ t_0)$, $w_t(s,\ t_0)$) belongs to W_{Sn} for $n \geq 2$. Noticing that (C.4) and (C.5) are in fact a $3n \times 3n$ system of ordinary dierential equations in the variable t, which has a local solution in [0, t_n). After the estimates below, the approximate solution will be extended to the interval [0, T] for any given $T > 0$.

Estimate I: Upper bound of $\int_0^L w_t^n.w_t^n ds + \int_0^L w_{ss}^n.w_{ss}^n ds$. In (C.4), taking $\phi = w_t^n$ results in

$$\begin{gathered} m_o\int_0^L w_{tt}^n.w_t^n ds + B\int_0^L w_{ss}^n.w_{sst}^n ds + \int_0^L (F_{\hat{t}}^{n} - B\kappa^{n2})(w_s^n + r_s^{n0}).w_{st}^n ds - \int_0^L q.w_t^n ds \\ +(k_1 w_t^n(L,t) + k_2 w_s^n(L,t)).w_t(L,t) = 0 \end{gathered} \tag{C.6}$$

We consider the following Lyapunov function candidate

$$W_n = \frac{m_o}{2}\int_0^L w_t^n.w_t^n ds + \frac{B}{2}\int_0^L w_{ss}^n.w_{ss}^n ds + \frac{\lambda}{2}\int_0^L w_s^n.w_s^n ds + \alpha\int_0^L s w_t^n.w_s^n ds \tag{C.7}$$

where λ and α are positive constants specified as in Section 3. Indeed, as in Section 3, the function W_n satisfies

$$\begin{aligned} W_n &\geq c_1\int_0^L w_t^n.w_t^n ds + \frac{B}{2}\int_0^L w_{ss}^n.w_{ss}^n ds + c_2\int_0^L w_s^n w_s^n ds \\ W_n &\leq (c_1 + 2\alpha L\rho_0)\int_0^L w_t^n.w_t^n ds + \frac{B}{2}\int_0^L w_{ss}^n.w_{ss}^n ds + \left(c_2 + \frac{\alpha L}{2\rho_0}\right)\int_0^L w_s^n.w_s^n ds \end{aligned} \tag{C.8}$$

where the positive constants ρ_0, c_1 and c_2 are specified in Section 3. We use the same technique in Section 3 to calculate the time derivative of the function W_n along the solutions of (C.6) as follows

$$\dot{W}_n \leq -c_3 w_t^n(L,t).w_t^n(L,t) - \frac{\alpha L k_2}{m_o} w_s^n(L,t).w_s^n(L,t) - cW_n + \Delta_n \tag{C.9}$$

where the positive constants c and c_3 are specified in Section 3, see (51), (53) and (54), and

$$\Delta_n = \int_0^L w_t^n.q ds + \frac{\alpha}{m_o}\int_0^L s w_s^n.q ds \tag{C.10}$$

Substituting the expression of q given in (29) into (C.10) gives

$$\begin{aligned} \Delta_n = \int_0^L & w_t^n.((I_{3\times 3} - r_s^n r_s^{nT})W_{re} + \frac{1}{2}\rho_w C_{LD} D_H (I_{3\times 3} - r_s^n r_s^{nT})(V - w_t^n) \\ &+ \frac{1}{2}\rho_w C_{ND} D_H \|(I_{3\times 3} - r_s^n r_s^T)(V - w_t)\|(I_{3\times 3} - r_s^n r_s^{nT})(V - w_t^n) ds \\ &+ \frac{\alpha}{m_o}\int_0^L s w_s^n.\Big((I_{3\times 3} - r_s^n r_s^{nT})W_{re} + \frac{1}{2}\rho_w C_{LD} D_H (I_{3\times 3} - r_s^n r_s^{nT})(V - w_t^n) \\ &+ \frac{1}{2}\rho_w C_{ND} D_H \|(I_{3\times 3} - r_s^n r_s^{nT})(V - w_t^n)\|(I_{3\times 3} - r_s^n r_s^{nT})(V - w_t^n))ds \end{aligned} \tag{C.11}$$

Since $r_s^n.r_s^n = 1$, i.e., $\|r_s^n\|$ is bounded by 1, there exists an arbitrarily positive constant ϱ such that

$$\Delta_n \leq \varrho\left(\int_0^L w_t^n.w_t^n ds + \int_0^L w_s^n.w_s^n ds\right) + \frac{1}{4\varrho}Q \tag{C.12}$$

where the nonnegative constant Q depends on the maximum value of $\|V\|$. Substituting (C.12) into (C.9) results in

$$\dot{W}_n \leq -c_3 w_t^n(L,t).w_t^n(L,t) - \frac{\alpha L k_2}{m_o} w_s^n(L,t).w_s^n(L,t) - \left(c - \frac{\varrho}{\min(c_1, 0.5B, c_2)}\right)W_n + \frac{1}{4\varrho}Q \tag{C.13}$$

Now picking ϱ such that $\bar{c} = c - \dfrac{\varrho}{\min(c_1, 0.5B, c_2)}$ is strictly positive, we can write (C.13) as

$$\dot{W}_n \leq -\bar{c}W_n + \frac{1}{4\varrho}Q \Rightarrow W_n(t) \leq \left(W_n(t_0) + \frac{Q}{4\varrho\bar{c}}\right)e^{-\bar{c}(t+t_0)} + \frac{1}{4\varrho\bar{c}}Q, \quad \forall t \geq t_0 \geq 0 \tag{C.14}$$

Hence from (C.8) and (C.14), we deduce that there exists a nonnegative constant M_1 such that

$$\int_0^L w_t^n.w_t^n ds + \int_0^L w_s^n.w_s^n ds + \int_0^L w_{ss}^n.w_{ss}^n ds \leq M_1 \qquad \forall t \in [0,T], n \in N \tag{C.15}$$

Estimate II: Upper bound of $w_{tt}(s, t_0)$ in L^2-norm. In (C.4), taking $\phi = w_{tt}^n(s,t_0)$ and $t = t_0$ gives

$$m_o\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)ds + B\int_0^L w_{ss}^n(s,t_0).w_{sstt}^n(s,t_0)ds - \int_0^L q.w_{tt}^n(s,t_0)ds +$$

$$\int_0^L (F_{\hat{t}}^n(s,t_0) - B\kappa^{n2}(s,t_0))(w_s^n(s,t_0) + r_s^{n0}).w_{stt}^n(s,t_0)ds + (k_1 w_t^n(L,t_0) + k_2 w_s^n(L,t_0)).w_{tt}^n(L,t_0) = 0 \tag{C.16}$$

By integrating (C.16) by parts and by the compatibility condition $Bw_{sss}^n = (L,t_0) - F_{\hat{t}}^n(L,t_0)r_s(L,t_0) = k_1 w_t^n(L,t_0) + k_2 w_s(L,t_0)$, and the boundary conditions $w_{ss}^n(0,t_0) = 0, w_{ss}^n(L,t_0)$, and $w_{sss}^n(0,t_0) = 0$ since $r_s^n.r_s^n = 1$ and $r_{ss}^{n0} = 0$ (due to the straight initial position of the riser), we have

$$m_o\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)ds + B\int_0^L w_{ssss}^n(s,t_0).w_{tt}^n(s,t_0)ds - \int_0^L q.w_{tt}^n(s,t_0)ds +$$

$$-F_{\hat{t}}^n(0,t_0)r_s^n(0,t_0).w_{tt}(0,t_0) - \int_0^L [(F_{\hat{t}}^n(s,t_0) - B\kappa^{n2}(s,t_0))r_s^n(s,t_0)]_s w_{tt}^n(s,t_0)ds = 0$$

$$\Rightarrow \frac{m_o}{4}\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)ds \leq \frac{B^2}{m_o}\int_0^L w_{ssss}^n(s,t_0).w_{ssss}^n(s,t_0)ds \tag{C.17}$$

$$+\frac{1}{m_o}\int_0^L q.qds + F_{\hat{t}}^n(0,t_0)r_s^n(0,t_0).w_{tt}(0,t_0)$$

$$+\frac{1}{m_o}\int_0^L [(F_{\hat{t}}^n(s,t_0) - B\kappa^{n2}(s,t_0))r_s^n(s,t_0)]_s.[(F_{\hat{t}}^n(s,t_0) - B\kappa^{n2}(s,t_0))r_s^n(s,t_0)]_s ds$$

Since the initial values $w(s, t_0)$ and $w_t(s, t_0)$ are suciently smooth, and we have already proved that $\int_0^L w_t^n(s,t).w_t^n(s,t)ds$, $\int_0^L w_s^n(s,t).w_s^n(s,t)ds$, $\int_0^L w_{ss}^n(s,t)ds$ are bounded, see Estimate I section, from (C.18) there exists a non-negative constant M_2 such that

$$\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)ds \leq M_2, \quad \forall t \in [0,T], n \in N \tag{C.18}$$

Estimate III: Upper bound of $w_{tt}(s, t)$ *and* $w_{sst}(s, t)$ *in* L^2*-norm*. To estimate the upper bound of these terms, we use difference approach. Let us fix t and ξ such that $\xi < T - t$. Now taking the difference of (C.4) with $t = t + \xi$ and $t = t$, and then letting $\phi = w_t^n(t+\xi) - w_t^n(t)$ result in

$$\begin{aligned}&\frac{m_o}{2}\int_0^L \frac{d}{dt}[(w_t^n(s,t+\xi)-w_t^n(s,t)).(w_t^n(s,t+\xi)-w_t^n(s,t))]ds\\&\quad+\frac{B}{2}\int_0^L \frac{d}{dt}[(w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t)).(w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t))]ds+\Omega = 0\end{aligned} \tag{C.19}$$

where

$$\begin{aligned}\Omega &= -\int_0^L (q(s,t+\xi)-q(s,t)).(w_s^n(s,t+\xi)-w_t^n(s,t))ds\\&\quad+\Delta^F.(w_t^n(s,t+\xi)-w_t^n(s,t))\Big|_0^L-\int_0^L \Delta_s^F.(w_t^n(s,t+\xi)-w_t^n(s,t))ds\\&\quad+(k_1(w_t^n(L,t+\xi)-w_t^n(L,t))+k_2(w_s^n(L,t+\xi)-w_s^n(L,t))).(w_t^n(L,t+\xi)-w_t^n(L,t))\\\Delta^F &= (F_t^n(s,t+\xi)-B\kappa^{n2}(t+\xi))(w_s^n(s,t+\xi)+r_s^{n0})-(F_t^n(s,t)-B\kappa^{n2}(t))(w_s^n(s,t)+r_s^{n0})\end{aligned} \tag{C.20}$$

Since the initial values $w(s, t_0)$ and $w_t(s, t_0)$ are sufficiently smooth, $w(0, t) + r^0 = 0$, $w_{ss}(0, t) = 0$, $w_{ss}(L, t) = 0$ for $w \in W_S$ and all the terms $\int_0^L w_t^n(s,t).w_t^n(s,t)ds$, $\int_0^L w_s^n(s,t).w_s^n(s,t)ds$, $\int_0^L w_{ss}^n(s,t).w_{ss}^n(s,t)ds$ are bounded, see Estimate I section, using the Mean Value Theorem and Lemmas B.1 and B.2 shows that there exist nonnegative constants M_{31} and M_{32} such that

$$\begin{aligned}|\Omega| &\le M_{31}\int_0^L (w_t^n(s,t+\xi)-w_t^n(s,t)).(w_t^n(s,t+\xi)-w_t^n(s,t))ds\\&\quad+M_{32}\int_0^L (w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t)).(w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t))ds\end{aligned} \tag{C.21}$$

Using (C.21), we can write (C.19) as

$$\frac{d\Phi^n}{dt}(t,\xi) \le M_{33}\Phi^n(t,\xi) \Rightarrow \Phi(t,\xi) \le \Phi(t_0,\xi)e^{M_{33}(t-t_0)} \tag{C.22}$$

where M_{33} is a nonnegative constant, and

$$\begin{aligned}\Phi^n(t,\xi) &= \frac{m_o}{2}\int_0^L (w_t^n(s,t+\xi)-w_t^n(s,t)).(w_t^n(s,t+\xi)-w_t^n(s,t))ds\\&\quad+\frac{B}{2}\int_0^L (w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t)).(w_{ss}^n(s,t+\xi)-w_{ss}^n(s,t))ds\end{aligned} \tag{C.23}$$

Dividing both sides of the last inequality in (C.22) by ξ^2 then taking the limit $\xi \to 0$ gives

$$\begin{aligned}&m_o\int_0^L w_{tt}^n(s,t).w_{tt}^n(s,t)ds+B\int_0^L w_{ss}^n(s,t).w_{sst}^n(s,t)ds \le\\&\left[m_o\int_0^L w_{tt}^n(s,t_0).w_{tt}^n(s,t_0)ds+B\int_0^L w_{sst}^n(s,t_0).w_{sst}^n(s,t_0)ds\right]e^{M_{33}(t-t_0)}\end{aligned} \tag{C.24}$$

for all $t_0 \le t \le T$. Now from the estimates given in (C.15) and (C.18), we can deduce from (C.24) that there exists $M_3 > 0$ depending on T such that

$$m_o\int_0^L w_{tt}^n(s,t).w_{tt}^n(s,t)ds + B\int_0^L w_{sst}^n(s,t).w_{sst}^n(s,t)ds \le M_3 \tag{C.25}$$

From the estimates given in (C.15), (C.18) and (C.25), we can use the Lions-Aubin theorem to get the necessary compactness to pass the nonlinear system (C.4) to the limit. Then it is a matter of routine to conclude the existence of global solutions in [0, T].

Uniqueness. Let u and v be two solutions of the closed loop system consisting of (15), (25) and (48). Letting $z = u - v$, we have $z(s, t_0) = 0$ and $z_t(s, t_0) = 0$ and from (C.3) we have

$$\begin{aligned} m_o\int_0^L z_{tt}.\phi ds + B\int_0^L z_{ss}.\phi_{ss}ds + \int_0^L ((F_t^u - B\kappa^{u2})(u_s + r_s^0) - (F_t^v - B\kappa^{v2})(v_s + r_s^0)).\phi_s ds \\ -\int_0^L (q^u - q^v).\phi ds + (k_1 z_t(L,t) + k_2 z_s(L,t)).\phi(L,t) = 0 \end{aligned} \tag{C.26}$$

By taking $\phi = z_t(s, t)$ in (C.26) and using the Mean Value Theorem and passing of the limit of all the estimates given in (C.15), (C.18) and (C.25) previously, we readily have

$$\frac{d}{dt}\left(\int_0^L z_t.z_t ds + \int_0^L z_{ss}.z_{ss}ds\right) \le M_4\left(\int_0^L z_t.z_t ds + \int_0^L z_{ss}.z_{ss}ds\right) \tag{C.27}$$

where M_4 is a positive constant. Since $z(s, t_0) = 0$ and $z_t(s, t_0) = 0$, using Gronwall's Lemma shows that $z = 0$, i.e. $u = v$ for all $t \ge t_0 \ge 0$ and $s \in [0, L]$. □

C.2 Proof of convergence. First, we consider the case $q = 0$ then move to the case $q \ne 0$.

C.2.1 Case $q = 0$. With $q = 0$, (52) becomes

$$\dot{W} \le -c_3 w_t(L,t).w_t(L,t) - \frac{\alpha L k_2}{m_o} w_s(L,t).w_s(L,t) - cW \le -cW \tag{C.28}$$

Therefore

$$W(t) \le W(t_0)e^{(-c(t-t_0))}, \forall t \ge t_0 \ge 0 \tag{C.29}$$

which combines with the low and upper bounds of $W(t)$ given in (32), we have

$$\begin{aligned} & c_1\int_0^L w_t(s,t).w_t(s,t)ds + \frac{B}{2}\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds + c_2\int_0^L w_s(s,t).w_s(s,t)ds \\ & \le \left[\left(\frac{m_o}{2} + \alpha L\rho_0\right)\int_0^L w_t(s,t_0).w_t(s,t_0)ds + \frac{B}{2}\int_0^L w_{ss}(s,t_0), w_{ss}(s,t_0)ds \right. \\ & \left. +\left(\frac{\lambda}{2} + \frac{\alpha L}{4\rho_0}\right)\int_0^L w_s(s,t_0).w_s(s,t_0)ds\right]e^{(-c(t-t_0))}, \forall t \ge t_0 \ge 0 \end{aligned} \tag{C.30}$$

Since the initial values of $w_t(s, t_0)$, $w_s(s, t_0)$, $w_{ss}(s, t_0)$ for all $s \in [0, L]$ are bounded and sufficiently smooth, all the integral terms in side the square bracket in the right hand side of (C.30) are bounded. Hence, the right hand side of (C.30) is bounded and exponentially converge to zero. Boundedness and exponential convergence of the right hand side of (C.30) to zero imply that the left hand side of (C.30) must be bounded and exponentially converge to zero. This in turn implies that all the terms $\int_0^L w_t(s,t).w_t(s,t)ds$, $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$, and $\int_0^L w_s(s,t).w_s(s,t)ds$ are bounded and exponentially converge to zero. Next, we use Lemmas B.1 and B.2 to show that $\int_0^L w(s,t).w(s,t)ds$ and $\|w(s,t)\|$ are bounded and exponentially converge to zero. An application of Lemma B.1 gives

$$\int_0^L w(s,t).w(s,t)ds \le 2w(0,t).w(0,t)+4L^2\int_0^L w_s(s,t).w_s(s,t)ds \tag{C.31}$$

Since $w(0,t)=0$ and we have already proved that $\int_0^L w(s,t).w_s(s,t)ds$ is bounded and exponentially converges to zero, (C.31) implies that $\int_0^L w(s,t).w(s,t)ds$ must be bounded and exponentially converges to zero. On the other hand, an application of Lemma B.2 shows that

$$\max_{s\in[0,L]}(w(s,t).w(s,t)) \le w(0,t).w(0,t)+2\sqrt{\int_0^L w(s,t).w(s,t)ds}\sqrt{\int_0^L w_s(s,t).w_s(s,t)ds} \tag{C.32}$$

Since $w(0,t)=0$ and we have already proved that $\int_0^L w_s(s,t).w_s(s,t)ds$ and $\int_0^L w(s,t).w(s,t)ds$ are bounded and exponentially converge to zero, (C.32) implies that $\|w(s,t)\|$ must be bounded and exponentially converges to zero. Next, to prove boundedness of the boundary control $U(t)$, we need to show that $w_s(L,t)$ and $w_t(L,t)$ must be bounded.To prove boundedness of $w_s(L,t)$, we use the constraint $r_s(s,t).r_s(s,t)=1$ for all $s\in[0,L]$ and $t\ge t_0\ge 0$. This constraint implies that

$$\begin{aligned}&(w_s(s,t)+r_s^0(s)).(w_s(s,t)+r_s^0(s)) = 1 \Rightarrow w_s(s,t).w_s(s,t)+2r_s^0(s).w_s(s,t)+r_s^0(s).r_s^0(s) = 1\\ &\Rightarrow w_s(s,t).w_s(s,t)\le 1+\frac{1}{2}w_s(s,t).w_s(s,t)+r_s^0(s).r_s^0(s)\Rightarrow w_s(s,t).w_s(s,t)\le 4\end{aligned} \tag{C.33}$$

for all $s\in[0,L]$ and $t\ge t_0\ge 0$. Hence $\|w_s(s,t)\|$ is bounded for all $s\in[0,L]$ and $t\ge t_0\ge 0$. To show that $w_t(L,t)$ is bounded, we consider the following Lyapunov function candidate

$$W_{wt}=\frac{1}{2m_o}w_t(L,t).w_t(L,t) \tag{C.34}$$

whose time derivative along the solutions of (15) with $s=L$ satisfies

$$\begin{aligned}\dot W_{wt} &= -k_1w_t(L,t).w_t(L,t)-k_2w_t(L,t).w_s(L,t)\\ &\le -\frac{k_1}{2}w_t(L,t).w_t(L,t)+\frac{k_2^2}{2k_1}w_s(L,t).w_s(L,t)\le -m_ok_1W_{wt}+\frac{2k_2^2}{k_1}\\ &\Rightarrow W_{wt}(t)\le\left(W_{st}(t_0)+\frac{2k_2^2}{m_ok_1^2}\right)e^{-2m_ok_1(t-t_0)}+\frac{2k_2^2}{m_ok_1^2}\\ &\Rightarrow \|w_t(L,t)\|\le\sqrt{\left(\|w_t(L,t_0)\|^2+\frac{4k_2^2}{k_1^2}\right)e^{-2m_ok_1(t-t_0)}+\frac{4k_2^2}{k_1^2}},\qquad \forall t\ge t_0\ge 0\end{aligned} \tag{C.35}$$

where we have used (48), (25), (C.33), and (C.34). Since the initial value $w_t(s,t_0)$ is bounded for all $s\in[0,L]$, we have from (C.35) that $\|w_t(L,t)\|$ is bounded for all $t\ge t_0\ge 0$. □

C.2.2 Case $q \neq 0$. Substituting (29) into (52) gives

$$\dot W\le -c_3w_t(L,t).w_t(L,t)-\frac{\alpha Lk_2}{m_o}w_s(L,t).w_s(L,t)-cW+\Delta_c \tag{C.36}$$

where

$$
\begin{aligned}
\Delta_c = & \int_0^L w_t.((I_{3\times 3}-r_s r_s^T)W_{re}+\frac{1}{2}\rho_w C_{LD}D_H(I_{3\times 3}-r_s r_s^T)(V-w_t) \\
& +\frac{1}{2}\rho_w C_{ND}D_H\|I_{3\times 3}-r_s r_s^T(V-w_t)\|(I_{3\times 3}-r_s r_s^T)(V-w_t))ds \\
& +\frac{\alpha}{m_o}\int_0^L s w_s.((I_{3\times 3}-r_s r_s^T)W_{re}+\frac{1}{2}\rho_w C_{LD}D_H(I_{3\times 3}-r_s r_s^T)(V-w_t) \\
& +\frac{1}{2}\rho_w C_{ND}D_H\|(I_{3\times 3}-r_s r_s^T)(V-w_t)\|(I_{3\times 3}-r_s r_s^T)V-w_t)ds
\end{aligned}
\tag{C.37}
$$

Since $r_s.r_s = 1$, i.e., $\|r_s\|$ is bounded by 1, there exists an arbitrarily positive constant ϱ such that

$$
\Delta_c \le \varrho\Big(\int_0^L w_t.w_t ds+\int_0^L w_s.w_s ds\Big)+\frac{1}{4\varrho}Q \tag{C.38}
$$

where the nonnegative constant Q depends on the maximum value of $\|V\|$. Hence, substituting (C.38) into (C.36) results in

$$
\dot{W}\le -c_3 w_t(L,t).w_t(L,t)-\frac{\alpha L k_2}{m_o}w_s(L,t).w_s(L,t)-\Big(c-\frac{\varrho}{\min(c_1, 0.5B, c_2)}\Big)W+\frac{1}{4\varrho}Q \tag{C.39}
$$

where we have used (32) and (33) to yield $\varrho\Big(\int_0^L w_t.w_t ds+\int_0^L w_s.w_s ds\Big)\le \min(c_1, 0.5B, c_2)W$. Now picking ϱ such that $\bar{c} = c-\frac{\varrho}{\min(c_1, 0.5B, c_2)}$ is strictly positive, we can write (C.40) as

$$
\dot{W}\le -\bar{c}W+\frac{1}{4\varrho}Q \Rightarrow W(t)\le\Big(W(t_0)+\frac{Q}{4\varrho\bar{c}}\Big)e^{-\bar{c}(t-t_0)}+\frac{1}{4\varrho\bar{c}}Q, \qquad \forall t\ge t_0\ge 0 \tag{C.40}
$$

Hence $W(t)$ exponentially converges to the nonnegative constant $\frac{1}{4\varrho\bar{c}}Q$. This in turn implies that all the terms $\int_0^L w_t(s,t).w_t(s,t)ds$, $\int_0^L w_s(s,t).w_s(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to some nonnega tive constant less than $\frac{1}{4\varrho\bar{c}\min(c_1, 0.5B, c_2)}Q$ due to (32) and (33). Proof of boundedness (not exponential convergence to zero of) $\|w(s,t)\|$, $\int_0^L w(s,t).w(s,t)ds$, $\|w_s(s,t)\|$, $\|w_t(L,t)\|$, and the boundary control $U(t)$ can be carried out in the same lines as in the case where q= 0. □

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