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Surface elasticity and residual stress effect on the elastic field of a nanoscale elastic layer

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Abstract. The influence of surface elasticity and surface residual stress on the elastic field of an isotropic nanoscale elastic layer of finite thickness bonded to a rigid material base is considered by employing the Gurtin-Murdoch continuum theory of elastic material surfaces. The fundamental solutions corresponding to buried vertical and horizontal line loads are obtained by using Fourier integral transform techniques. Selected numerical results are presented for the cases of a finite elastic layer and a semi-infinite elastic medium to portray the influence of surface elasticity and residual surface stress on the bulk stress field. It is found that the bulk stress field depends significantly on both surface elastic constants and residual surface stress. The consideration of out-of-plane terms of the surface stress yields significantly different solutions compared to previous studies. The solutions presented in this study can be used to examine a variety of practical problems involving nanoscale/soft material systems and to develop boundary integral equations methods for such systems.

Keywords: continuum mechanics; elasticity; nanomechanics; residual stress; surface energy; surface stress; thin films.

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1. Introduction

Nanomaterials and nanostructures are increasingly used in advanced engineering applications due to their superior mechanical, electronic and optical properties (Wong *et al*. 1997). In nanoscale systems, the surface-to-volume ratio is relatively high compared to macroscale systems and the influence of surface/interface free energy becomes an important factor in their mechanical properties and behavior (Yakobson 2003). Surface energy effects are also important in soft materials such as polymer gels and biomaterials (Peter *et al*. 2000, Srinivasan *et al*. 2001). Although atomistic methods (e.g. Chen *et al*. 2008, Chen and Lee 2010) are considered very accurate for nanoscale systems, the associated computational resources are significantly large. Modified continuum methods are therefore considered very efficient in obtaining a first-approximation to nanoscale systems. Gurtin and Murdoch (1975, 1978) developed a rigorous theory based on continuum mechanics concepts to incorporate the surface and interfacial energy effects. The surface is modeled

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as a layer with zero thickness perfectly bonded to the underlying bulk material. The surface elastic constants can be obtained from atomistic simulations (Miller and Shenoy 2000, Shenoy 2005, Dingreville and Qu 2007).

Over the past decade, Gurtin-Murdoch theory of deformable material surfaces has been extensively applied to study problems in nanotechnology and soft materials. He and Lim (2006) derived the surface Green's function for a soft incompressible isotropic elastic half-space by assuming that the surface elastic properties are the same as bulk properties. The elastic field of a half-plane subjected to surface loading in the presence of surface stresses was considered by Huang and Yu (2007). Zhao and Rajapakse (2009) studied the plane-strain and axisymmetric response of an isotropic elastic layer bonded to a rigid base under vertical and horizontal surface loads. Recently, Intarit *et al*. (2010) derived the fundamental solutions of an elastic half-plane with surface effects under internal loading and dislocations.

In the above studies, the surface stress tensor is considered a 2D quantity and its out-of-plane components are excluded. A recent study by Wang *et al.* (2010), who formulated the surface elasticity theory in the Lagrangian and Eulerian frameworks, indicated that the deformed and undeformed configurations should be discriminated even in the case of small deformations. The out-of-plane terms of the surface displacement gradient could be significant particularly for curved and rotated surfaces. Povstenko (1993) studied the influence of residual surface stress gradient on the elastic field of a half-space that has a jump in residual surface stress over a circular area.

This paper examines the elastic field of an isotropic nanoscale or soft elastic material layer of finite thickness bonded to a rigid material base and subjected to internal and surface loading. The surface elasticity and residual surface stresses are considered in the formulation. This class of problems has extensive applications in the study of nanocoatings and nanoscale surface layers that are used in electronic devices, tribological and biomaterial applications, advanced industrial materials, communication devices, etc. The boundary-value problems involve non-classical boundary conditions due to surface stresses are solved by using Fourier integral transforms. Selected numerical results are presented to demonstrate the influence of surface elasticity and residual surface stress on the elastic field.

2. Governing equations and general solutions

Consider a finite elastic layer of thickness t bonded to a rigid material base, and subjected to vertical and horizontal loading at a depth h below the free surface as shown in Fig. 1. In the absence of body forces, the equilibrium equations, constitutive laws and strain-displacement relations of an isotropic bulk material are given by

$$\sigma_{ij,j} = 0 \tag{1}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

where u_i, σ_{ij} and ε_{ij} denote respectively the components of displacement, stress and strain tensors. In addition, μ and λ are Lamé constants of the bulk material.

For the surface, the equilibrium equation, constitutive laws and strain-displacement relations can be expressed as (Gurtin and Murdoch 1975, 1978).

$$\sigma^s_{i\alpha,\alpha} + \sigma_{ij}n_j = 0 \tag{4}$$

$$\sigma^s_{\beta\alpha} = \tau^s\delta_{\beta\alpha} + 2(\mu^s - \tau^s)\varepsilon_{\beta\alpha} + (\lambda^s + \tau^s)\varepsilon_{\gamma\gamma}\delta_{\beta\alpha} + \tau^s u^s_{\beta,\alpha}, \quad \sigma^s_{3\alpha} = \tau^s u_{3,\alpha} \tag{5}$$

$$\varepsilon^s_{\alpha\beta} = \frac{1}{2}(u^s_{\alpha,\beta} + u^s_{\beta,\alpha}) \tag{6}$$

where the superscript '*s*' is used to denote the quantities corresponding to the surface; μ^s and λ^s are surface Lamé constants; τ^s is the surface residual stress (or surface tension) under unstrained conditions; and n_i denotes the components of the unit normal vector of the surface. It is noted that the value of τ^s is constant for a given surface orientation of a pure metal/semiconductor at a specific temperature (Zhao and Rajapakse 2009).

In the above equations, Greek subscripts denote the field quantities associated with the surface and take the value of 1 or 2, while the Latin subscripts adopt values from 1 to 3. A majority of existing studies based on the Gurtin-Murdoch theory has formulated the problems in undeformed configuration due to the assumption of infinitesimal deformations thus the out-of-plane component of surface stresses given by the second equation in Eq. (5) is normally ignored. The term $\tau^s u_{3,\alpha}$ can simply be viewed as the out-of-plane component of the pre-existing surface tension τ^s in the deformed configuration whereas the surface gradient of the displacement $u_{3,\alpha}$ act as the out-of-plane component of the unit vector tangent to the surface in the deformed state. While the component $\tau^s u_{3,\alpha}$ has physical meaning only in the deformed state and identically vanishes in the undeformed configuration, its contribution to the constitutive Eq. (5) is of the same order as other terms. As recently pointed out by Wang *et al.* (2010), these out-of-plane terms could become significant even in the case of small deformations.

It is assumed that the deformations under consideration are plane-strain in the *xz*-plane, i.e. $\varepsilon_{xy} = \varepsilon_{yy} = \varepsilon_{yz} = 0$. The general solutions for the bulk stresses and displacements can be expressed with respect to a Cartesian coordinate system (Fig. 1) by using Fourier integral transforms as (Sneddon 1951)

$$\sigma_{zz} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\xi^2\Phi e^{-i\xi x}d\xi \tag{7}$$

$$\sigma_{xx} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d^2\Phi}{dz^2}e^{-i\xi x}d\xi \tag{8}$$

$$\sigma_{xz} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}i\xi\frac{d\Phi}{dz}e^{-i\xi x}d\xi \tag{9}$$

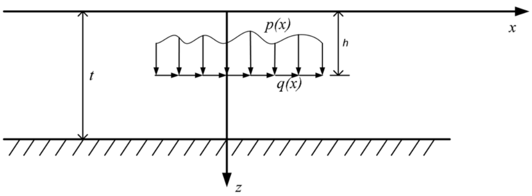


Fig. 1 An isotropic elastic layer subjected to internal vertical and horizontal loading

$$u_z = \frac{1}{8\pi\mu(\lambda+\mu)}\int_{-\infty}^{+\infty}\left[(\lambda+2\mu)\frac{d^3\Phi}{dz^3} - (3\lambda+4\mu)\xi^2\frac{d\Phi}{dz}\right]e^{-i\xi x}\frac{d\xi}{\xi^2} \tag{10}$$

$$u_x = \frac{1}{8\pi\mu(\lambda+\mu)}\int_{-\infty}^{+\infty}\left[(\lambda+2\mu)\frac{d^2\Phi}{dz^2} + \lambda\xi^2\Phi\right]ie^{-i\xi x}\frac{d\xi}{\xi} \tag{11}$$

where

$$\Phi(\xi, z) = (A+Bz)e^{-|\xi|z} + (C+Dz)e^{|\xi|z} \tag{12}$$

Note that Φ is the Fourier transform of Airy stress function that satisfies the two-dimensional biharmonic equation. In addition, the arbitrary functions A, B, C and D are determined from the appropriate boundary conditions.

3. Solutions of boundary-value problems

The solution to the elastic layer problem shown in Fig. 1 can be derived by dividing the elastic layer into two sub-domains. The sub-domain ‘1’ corresponds to the region where $0 \le z \le h$ and the sub-domain ‘2’ corresponds to the region where $h \le z \le t$. The general solution of the sub-domain ‘1’ is given by Eqs. (7) - (12) whereas the general solution of the sub-domain ‘2’ is also given by Eqs. (7) - (12) with the arbitrary functions A to D being replaced by E to H respectively. A superscript ‘i’ ($i = 1, 2$) is used hereafter to denote quantities associated with each sub-domain. The arbitrary functions A to H corresponding to each sub-domain can be obtained by solving the following boundary value problem.

$$\sigma_{zz}^{(1)}\Big|_{z=0} + \left(\frac{d\tau^s}{dx}\frac{du_z}{dx} + \tau^s\frac{d^2u_z}{dx^2}\right)_{z=0} = 0 \tag{13}$$

$$\sigma_{zx}^{(1)}\Big|_{z=0} + \left(\frac{d\tau^s}{dx} + \kappa^s\frac{d^2u_x}{dx^2}\right)_{z=0} = 0 \tag{14}$$

$$\sigma_{zz}^{(1)}\Big|_{z=h^-} - \sigma_{zz}^{(2)}\Big|_{z=h^+} = p(x) \tag{15}$$

$$\sigma_{zx}^{(1)}\Big|_{z=h^-} - \sigma_{zx}^{(2)}\Big|_{z=h^+} = q(x) \tag{16}$$

$$u_z^{(1)}\Big|_{z=h^-} = u_z^{(2)}\Big|_{z=h^+} \tag{17}$$

$$u_x^{(1)}\Big|_{z=h^-} = u_x^{(2)}\Big|_{z=h^+} \tag{18}$$

$$u_z^{(2)}\Big|_{z=t} = 0 \tag{19}$$

$$u_x^{(2)}\Big|_{z=t} = 0 \tag{20}$$

where $\kappa^s = 2\mu^s + \lambda^s$ is a surface material constant. In addition, $p(x)$ and $q(x)$ denote the jump of the normal traction and shear traction across the line $z = h$ due to the applied internal vertical and horizontal loads respectively (see Fig. 1). The Fourier transforms of $p(x)$ and $q(x)$ are given respectively by

$$\bar{p}(\xi) = \int_{-\infty}^{+\infty} p(x) e^{i\xi x} dx \tag{21}$$

$$\bar{q}(\xi) = \int_{-\infty}^{+\infty} q(x) e^{i\xi x} dx \tag{22}$$

It should be noted that both Eqs. (13) and (14) are non-classical boundary conditions obtained from Eqs. (4) and (5). In addition, Eq. (13) contains the out-of-plane component of surface stresses associated with residual surface stress, which has generally been ignored in most previous studies. For a flat surface, it can be seen from Eqs. (13) and (14) that the influence of residual surface stress τ^s will be neglected if the out-of-plane component of surface stresses is disregarded (the second term on the left-hand side of Eq. (13) vanishes) and the residual surface stress is assumed to be constant. In view of Eqs. (7) - (11), a set of linear simultaneous equations for determining the arbitrary functions can be constituted by applying Fourier integral transforms to Eqs. (13) - (20) together with the assumption that the surface residual stress is constant. The following solutions are obtained for the arbitrary functions A to H.

$$A = \frac{(A_p + iA_q)}{I}; \qquad B = \frac{(B_p + iB_q)}{I} \tag{23}$$

$$C = \frac{(C_p + iC_q)}{I}; \qquad D = \frac{(D_p + iD_q)}{I} \tag{24}$$

$$E = \frac{(E_p + iE_q)}{I}; \qquad F = \frac{(F_p + iF_q)}{I} \tag{25}$$

$$G = \frac{(G_p + iG_q)}{I}; \qquad H = \frac{(H_p + iH_q)}{I} \tag{26}$$

where the explicit expressions of A_p, A_q, B_p, B_q, C_p, C_q, D_p, D_q, E_p, E_q, F_p, F_q, G_p, G_q, H_p, H_q and I are given in Appendix.

In the following subsections, the explicit expressions of the arbitrary functions for the special cases of surface loading $h \to 0$ and a semi-infinite medium $t \to \infty$ are presented.

3.1 Surface loading on a finite layer

The surface loading of a nanoscale layer has many practical applications. The elastic field corresponding to this case can be obtained by taking the limit of $h \to 0$ in Eqs. (23)-(26). The corresponding arbitrary functions are given by Eqs. (23) and (24) with A_i to D_i ($i = p, q$) defined as follows

$$A_p = \frac{\bar{p}(\xi)}{2\xi^2}\left\{(\lambda + 3\mu)[(1 + \Lambda|\xi|)e^{2|\xi|t} - \Lambda|\xi|] + 2t\xi^2(\lambda + \mu)(\Lambda + t) - \frac{2(\lambda + \mu)^2}{(\lambda + 2\mu)}\Lambda t^2|\xi|^3 - 2(\lambda + \mu)t|\xi| + \frac{\lambda^2 + 4\lambda\mu + 5\mu^2}{\lambda + \mu}\right\} \tag{27}$$

$$B_P = \frac{\bar{p}(\xi)}{2|\xi|}\left\{(\lambda+3\mu)\left[\left(1+\frac{\lambda+\mu}{\lambda+2\mu}\Lambda|\xi|\right)e^{2|\xi|t} - \frac{\lambda+\mu}{\lambda+2\mu}\Lambda|\xi|\right] + \frac{2(\lambda+\mu)^2}{(\lambda+2\mu)}\Lambda t\xi^2 + (\lambda+\mu)(1-2t|\xi|)\right\} \quad (28)$$

$$C_P = \frac{\bar{p}(\xi)}{2\xi^2}\left\{(\lambda+3\mu)[(1-\Lambda|\xi|)e^{-2|\xi|t} + \Lambda|\xi|] + 2t\xi^2(\lambda+\mu)(\Lambda+t) + \frac{2(\lambda+\mu)^2}{(\lambda+2\mu)}\Lambda t^2|\xi|^3 + 2(\lambda+\mu)t|\xi| + \frac{\lambda^2+4\lambda\mu+5\mu^2}{\lambda+\mu}\right\} \quad (29)$$

$$D_P = -\frac{\bar{p}(\xi)}{2|\xi|}\left\{(\lambda+3\mu)\left[\left(1-\frac{\lambda+\mu}{\lambda+2\mu}\Lambda|\xi|\right)e^{-2|\xi|t} + \frac{\lambda+\mu}{\lambda+2\mu}\Lambda|\xi|\right] + \frac{2(\lambda+\mu)^2}{(\lambda+2\mu)}\Lambda t\xi^2 + (\lambda+\mu)(1+2t|\xi|)\right\} \quad (30)$$

$$A_q = -\frac{\bar{q}(\xi)}{|\xi|\xi}\left\{(\lambda+3\mu)\left[\frac{\tau^s|\xi|}{4(\lambda+\mu)}e^{2|\xi|t} - \frac{\tau^s|\xi|}{4(\lambda+\mu)}\right] + \frac{\mu(\lambda+2\mu)}{\lambda+\mu} + (\lambda+\mu)t^2\xi^2 - \frac{\tau^s t\xi^2}{2} - \frac{(\lambda+\mu)}{2\mu}\tau^s t^2|\xi|^3\right\} \quad (31)$$

$$B_q = \frac{\bar{q}(\xi)}{2\xi}\left\{(\lambda+3\mu)\left[\left(1+\frac{\tau^s|\xi|}{2\mu}\right)e^{2|\xi|t} - \frac{\tau^s|\xi|}{2\mu}\right] + (\lambda+\mu)(1+2t|\xi|) - \frac{(\lambda+\mu)}{\mu}\tau^s t\xi^2\right\} \quad (32)$$

$$C_q = \frac{\bar{q}(\xi)}{|\xi|\xi}\left\{(\lambda+3\mu)\left[-\frac{\tau^s|\xi|}{4(\lambda+\mu)}e^{-2|\xi|t} + \frac{\tau^s|\xi|}{4(\lambda+\mu)}\right] + \frac{\mu(\lambda+2\mu)}{\lambda+\mu} + (\lambda+\mu)t^2\xi^2 - \frac{\tau^s t\xi^2}{2} + \frac{(\lambda+\mu)}{2\mu}\tau^s t^2|\xi|^3\right\} \quad (33)$$

$$D_q = \frac{\bar{q}(\xi)}{2\xi}\left\{(\lambda+3\mu)\left[\left(1-\frac{\tau^s|\xi|}{2\mu}\right)e^{-2|\xi|t} + \frac{\tau^s|\xi|}{2\mu}\right] + (\lambda+\mu)(1-2t|\xi|) - \frac{(\lambda+\mu)}{\mu}\tau^s t\xi^2\right\} \quad (34)$$

The fundamental solutions corresponding to an elastic layer subjected to a vertical line load P_0 and a horizontal line load Q_0 can be obtained by substituting $\bar{p}(\xi) = P_0$ and $\bar{q}(\xi) = Q_0$ in the above solutions.

For the cases of vertical strip load of constant magnitude p_0 and horizontal strip load of constant magnitude q_0 over the region $-a \le x \le a$

$$\bar{p}(\xi) = \frac{2\sin(\xi a)}{\xi}p_0 \quad (35)$$

$$\bar{q}(\xi) = \frac{2\sin(\xi a)}{\xi}q_0 \quad (36)$$

Note that $\Lambda = \kappa^s(\lambda+2\mu)/2\mu(\lambda+\mu)$ is a parameter with a dimension of length. This parameter can be viewed as a material characteristic length that represents the influence of surface stress. It is clear from the above solutions that the influence from surface stresses does not only come from the surface material constant κ^s (or Λ) but also from the residual surface stress τ^s. In the absence of surface stress effects, Λ and τ^s vanish and the above solutions reduce to the classical elasticity solutions.

The elastic field of a semi-infinite medium under surface loading can readily be obtained from the solutions in Eqs. (23) and (24), with A_i to D_i $(i = p, q)$ given by Eqs. (27) to (34), by taking the limit of $t \to \infty$. Note that the arbitrary functions C and $D \equiv 0$ to ensure the regularity of the solutions at infinity. In the case of the vertical load, the arbitrary functions A and B take the form,

$$A = \frac{\bar{p}(\xi)}{\eta \xi^2}(1 + \Lambda|\xi|) \tag{37}$$

$$B = \frac{\bar{p}(\xi)}{\eta|\xi|}\left[1 + \frac{(\lambda+\mu)}{(\lambda+2\mu)}\Lambda|\xi|\right] \tag{38}$$

where

$$\eta = (1 + \Lambda|\xi|) + \tau^s\left[\frac{(\lambda+2\mu)}{2\mu(\lambda+\mu)}|\xi| + \frac{(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\Lambda\xi^2\right] \tag{39}$$

In the case of the horizontal loading

$$A = -i\frac{\bar{q}(\xi)}{\eta\xi}\left[\frac{\tau^s}{2(\lambda+\mu)}\right] \tag{40}$$

$$B = i\frac{\bar{q}(\xi)}{\eta\xi}\left(1 + \frac{\tau^s|\xi|}{2\mu}\right) \tag{41}$$

3.2 Internal loading in a semi-infinite medium

The stress and displacement fields of a semi-infinite medium under vertical and horizontal loads applied at a depth h below free surface can also be obtained from the solutions in Eqs. (23)-(26) by taking the limit of $t \to \infty$. Note that the arbitrary functions G and $H \equiv 0$ to ensure the regularity of the solutions at infinity and the arbitrary functions A to F can be specialized to the case of a half-plane as follows.

3.2.1 Arbitrary functions for internal vertical loading

$$A = \frac{\bar{p}(\xi)}{2\eta\xi^2}e^{-|\xi|h}\left\{(1+\Lambda|\xi|)\left(1+\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right) - \frac{\tau^s}{2\mu}|\xi|\left[\frac{\lambda+2\mu}{\lambda+\mu} + h|\xi| + \frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi| + \frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2}\Lambda h\xi^2\right]\right\} \tag{42}$$

$$B = \frac{\bar{p}(\xi)}{2(\lambda+2\mu)\eta|\xi|}e^{-|\xi|h}\left\{\lambda+3\mu+(\lambda+\mu)(\Lambda+2h)|\xi| - \frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi| + \frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right]\right\} \tag{43}$$

$$C = -\frac{\bar{p}(\xi)e^{-|\xi|h}}{2\xi^2}\left[1 + \frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right] \tag{44}$$

$$D = \frac{\bar{p}(\xi)e^{-|\xi|h}(\lambda+\mu)}{2|\xi|(\lambda+2\mu)} \tag{45}$$

$$E = \frac{\bar{p}(\xi)}{2\eta\xi^2}\left\{\begin{aligned} & e^{|\xi|h}\left\{(1+\Lambda|\xi|)\left(1-\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right) + \frac{\tau^s}{2\mu}|\xi|\left[\frac{\lambda+2\mu}{\lambda+\mu} - h|\xi| + \frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi| - \frac{(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2}\Lambda h\xi^2\right]\right\} \\ & + e^{-|\xi|h}\left\{(1+\Lambda|\xi|)\left(1+\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right) - \frac{\tau^s}{2\mu}|\xi|\left[\frac{\lambda+2\mu}{\lambda+\mu} + h|\xi| + \frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi| + \frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2}\Lambda h\xi^2\right]\right\}\end{aligned}\right\} \tag{46}$$

$$F = \frac{\bar{p}(\xi)}{2(\lambda+2\mu)\eta|\xi|}\left\{\begin{array}{l} e^{|\xi|h}\left\{(\lambda+\mu)(1+\Lambda|\xi|)+\frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right]\right\} \\ +e^{-|\xi|h}\left\{\lambda+3\mu+(\lambda+\mu)(\Lambda+2h)|\xi|-\frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|+\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right]\right\} \end{array}\right\} \tag{47}$$

3.2.2 Arbitrary functions for internal horizontal loading

$$A = \frac{i\bar{q}(\xi)}{2(\lambda+2\mu)\eta|\xi|\xi}e^{-|\xi|h}\left\{[\mu-(\lambda+\mu)h|\xi|](1+\Lambda|\xi|)-\tau^s|\xi|\left[\frac{\lambda+2\mu}{2(\lambda+\mu)}-\frac{\lambda+2\mu}{2\mu}h|\xi|-\frac{\lambda+3\mu}{2(\lambda+2\mu)}\Lambda|\xi|-\frac{(\lambda+\mu)^2}{2\mu(\lambda+2\mu)}\Lambda h\xi^2\right]\right\} \tag{48}$$

$$B = \frac{i\bar{q}(\xi)}{2(\lambda+2\mu)\eta\xi}e^{-|\xi|h}\left\{\lambda+3\mu-(\lambda+\mu)(\Lambda+2h)|\xi|+\frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu-\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|+\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right]\right\} \tag{49}$$

$$C = -\frac{i\bar{q}(\xi)e^{-|\xi|h}[\mu-(\lambda+\mu)h|\xi|]}{2|\xi|\xi(\lambda+2\mu)} \tag{50}$$

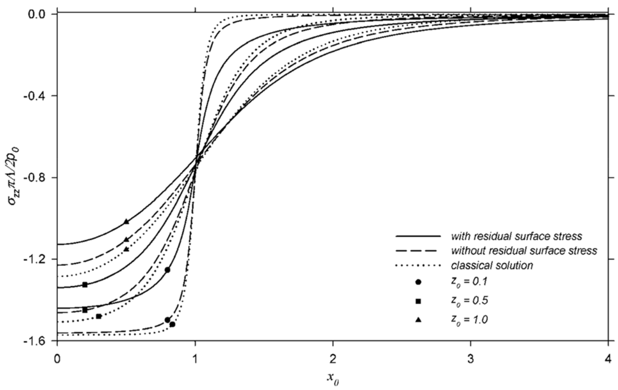
$$D = -\frac{i\bar{q}(\xi)e^{-|\xi|h}(\lambda+\mu)}{2\xi(\lambda+2\mu)} \tag{51}$$

$$E = -\frac{i\bar{q}(\xi)}{2(\lambda+2\mu)\eta|\xi|\xi}\left\{\begin{array}{l} e^{|\xi|h}\left\{[\mu+(\lambda+\mu)h|\xi|](1+\Lambda|\xi|)+\tau^s|\xi|\left[\frac{\lambda+2\mu}{2(\lambda+\mu)}+\frac{\lambda+2\mu}{2\mu}h|\xi|+\frac{\lambda+3\mu}{2(\lambda+2\mu)}\Lambda|\xi|+\frac{(\lambda+\mu)(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\Lambda h\xi^2\right]\right\} \\ -e^{-|\xi|h}\left\{[\mu-(\lambda+\mu)h|\xi|](1+\Lambda|\xi|)-\tau^s|\xi|\left[\frac{\lambda+2\mu}{2(\lambda+\mu)}-\frac{\lambda+2\mu}{2\mu}h|\xi|-\frac{\lambda+3\mu}{2(\lambda+2\mu)}\Lambda|\xi|-\frac{(\lambda+\mu)^2}{2\mu(\lambda+2\mu)}\Lambda h\xi^2\right]\right\} \end{array}\right\} \tag{52}$$

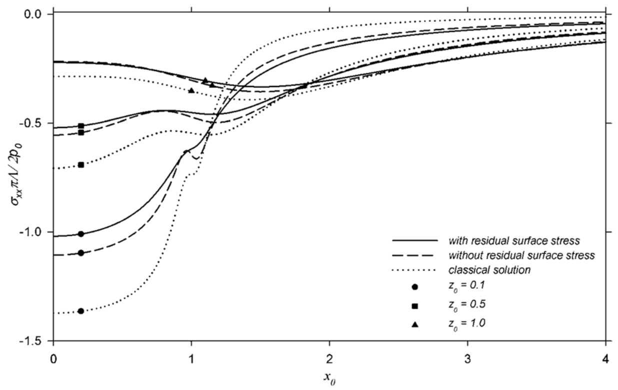
$$F = \frac{i\bar{q}(\xi)}{2(\lambda+2\mu)\eta\xi}\left\{\begin{array}{l} e^{|\xi|h}\left\{(\lambda+\mu)(1+\Lambda|\xi|)+\frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right]\right\} \\ +e^{-|\xi|h}\left\{\lambda+3\mu-(\lambda+\mu)(\Lambda+2h)|\xi|+\frac{\tau^s}{2\mu}|\xi|\left[\lambda+2\mu-\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|+\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right]\right\} \end{array}\right\} \tag{53}$$

4. Numerical results and discussion

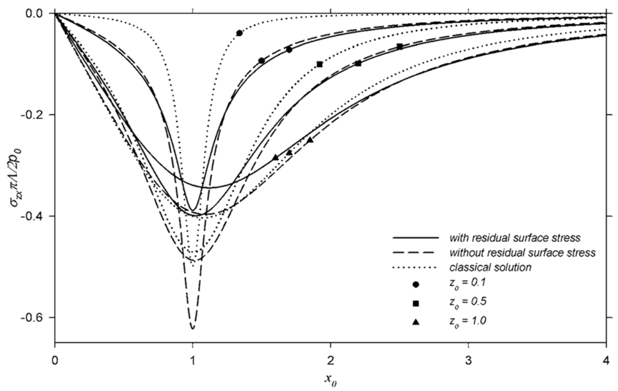
It is noted that the solutions for displacements and stresses given by Eqs. (7) - (11) are expressed in terms of semi-infinite integrals. A closed-form solution cannot be obtained due to the complexity of the integrands. Therefore, it is proposed to employ an accurate numerical scheme to evaluate these integrals. In this study, the integrals are evaluated by using globally adaptive numerical quadrature scheme based on 21-point Gauss-Kronrod rule (Piessens 1983). The surface elastic constants can be obtained by using atomistic simulations (Miller and Shenoy 2000, Shenoy 2005, Dingreville and Qu 2007). It is convenient to introduce the non-dimensional coordinates, $x_0 = x/\Lambda$ and $z_0 = z/\Lambda$, in the numerical study. The numerical results in the present study correspond to the case of an elastic layer subjected to a uniformly distributed load applied over a strip $-a \le x \le a$. In the numerical study, a hypothetical material with $\lambda/\mu = 2.226$ and $\Lambda = 1$ *nm* are used. In addition



(a)



(b)

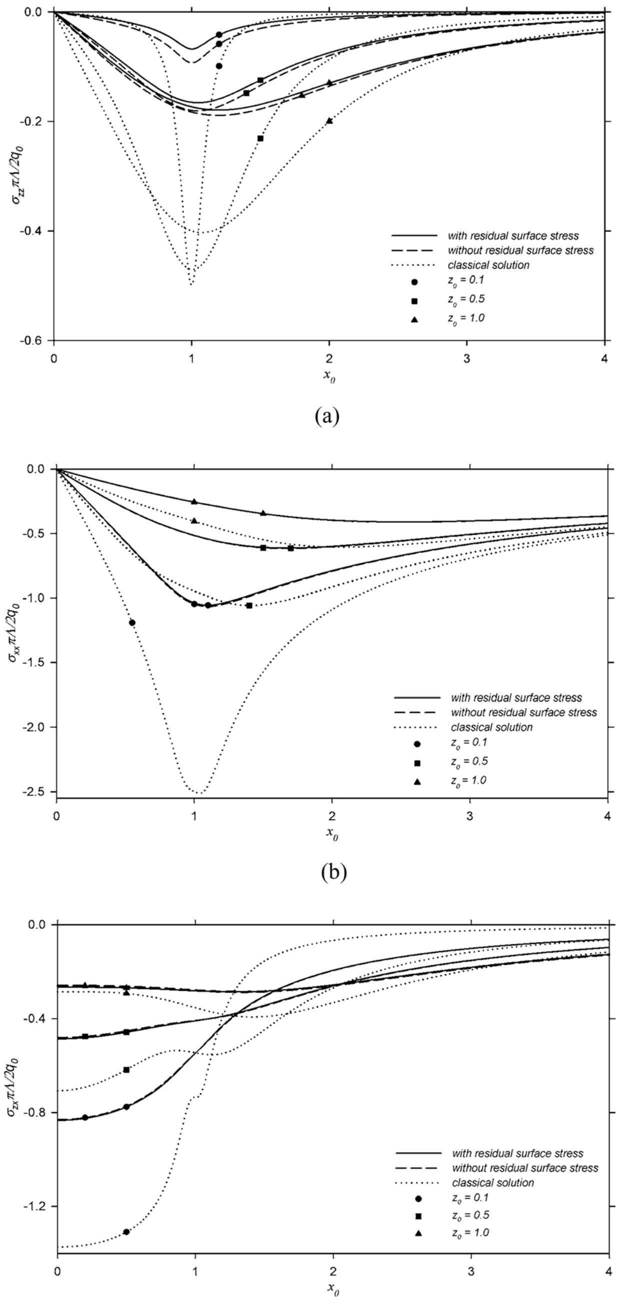


(c)

Fig. 2 Non-dimensional stress profiles of a half-plane under vertical surface load: (a) Vertical stress, (b) Horizontal stress, (c) Shear stress

$\tau^s = 5$ N/m is used to demonstrate the influence of residual surface stress.

Figs. 2 to 7 demonstrate the influence of surface elasticity and residual surface stress on the stress



(a)

(b)

(c)

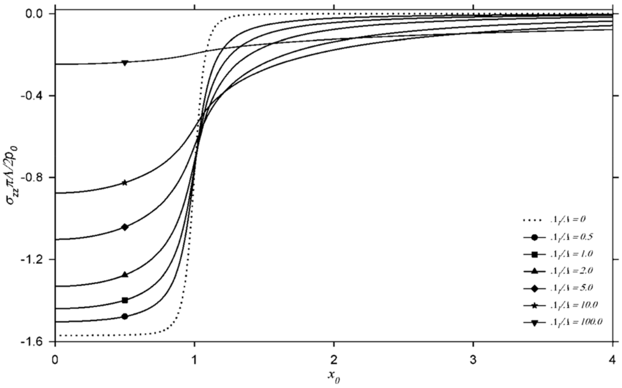
Fig. 3 Non-dimensional stress profiles of a half-plane under horizontal surface load: (a) Vertical stress, (b) Horizontal stress, (c) Shear stress

field of an elastic layer with very large value of t (a half-plane) under different loading cases. Figs. 2 and 3 show the variation of non-dimensional stresses along the x-direction of a half-plane at various depths under a uniform vertical strip load of magnitude p_0 and a horizontal strip load of magnitude q_0 respectively applied at the surface. A non-dimensional load width, $a_0 = a / \Lambda = 1$, is used in the numerical study. Only the solutions along the positive x-axis are presented due to the symmetry or anti-symmetry of the solutions about the z-axis.

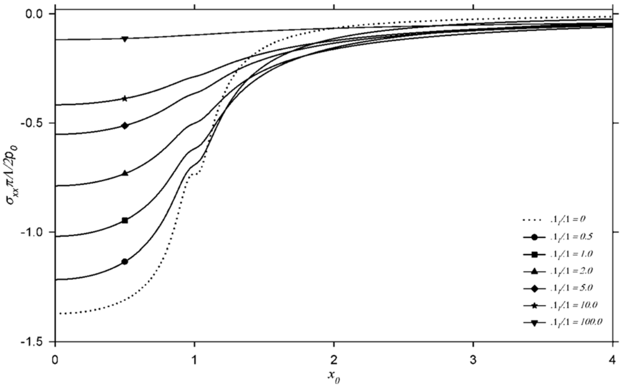
The influence of surface elasticity on an identical problem was previously examined by Zhao and Rajapakse (2009) by ignoring the out-of-plane component of surface stresses. The dotted lines denote the classical elasticity solutions corresponding to zero surface stress (i.e. $\kappa^s = \tau^s = 0$) and the dash lines denote the solutions that neglect the out-of-plane component of surface stresses (Zhao and Rajapakse 2009), which also disregard the influence of residual surface stress (τ^s) as previously discussed. It is evident from the figures that the influence of residual surface stress is more significant in the case of vertical strip load when compared to the horizontal strip load case. On the contrary, the influence of surface elasticity is more evident in the case of horizontal loading. It is also found that for the horizontal loading the influence of residual surface stress is negligible on horizontal normal and shear stresses but more evident on vertical normal stress, whereas in the case of vertical strip load all stress components depend significantly on the residual surface stress. This behavior can be described from the fact that the residual surface stress appears in the equilibrium equation of the vertical normal stress, Eq. (13), but apparently vanishes in the shear stress equation, Eq. (14), due to the assumption that the residual surface stress is constant. As expected, the influence of residual surface stress becomes significant only in a local region near the surface (i.e. $z_0 < 2.0$ for the vertical loading and $z_0 < 1.0$ for the horizontal loading) and would diminish with the distance from the free surface. In addition, the influence of the residual surface stress becomes negligible when $x_0 / a_0 > 4$.

To investigate the influence of the surface material parameter Λ and the residual surface stress τ^s, the non-dimensional stress profiles along the x-direction of a half-plane due to a uniform vertical strip load p_0 are shown in Fig. 4 for different values of Λ and in Fig. 5 for different values of τ^s respectively. Note that in Figs. 4 and 5 stresses are calculated at $z_0 = 0.1$. In Fig. 4, the non-dimensional stresses are presented for a hypothetical material with the surface material parameter Λ_1 being varied from 0 to 100Λ, whereas the residual surface stress parameter (τ^s) is unchanged. It can be seen from the figure that the free surface is stiffer with increasing values of Λ_1 resulting in the reduction of the stresses in the layer. The influence of the residual surface stress in Fig. 5 shows a similar trend to Fig. 4. It can be seen from Fig. 5 that all bulk stress components decrease when residual surface stress (τ^s) increases from 0 to 100 *N/m.*

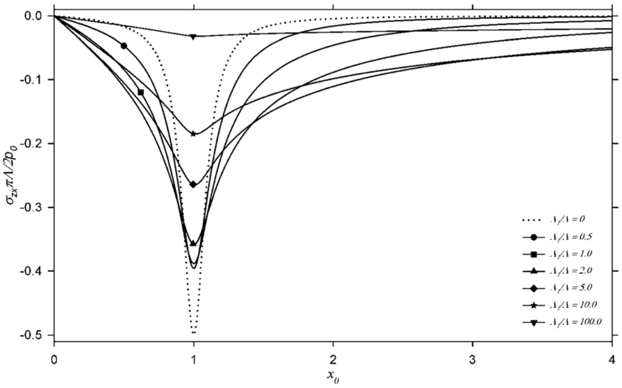
Figs. 6 and 7 show the variation of non-dimensional stresses along the z-axis of an elastic half-plane subjected to an internal vertical strip load p_0 and an internal horizontal strip load q_0 over a region $2a$ (with $a_0 = 1$) at various depths. A non-dimensional quantity, $h_0 = h / \Lambda$, is used in the numerical analysis. The influence of surface elasticity of an identical problem was recently considered by Intarit *et al.* (2010) without the out-of-plane component of surface stresses and the influence of residual surface stress (τ^s). Numerical results shown in Figs. 6 and 7 indicate that the stresses increase when approaching the plane of applied loading. A discontinuity in both vertical and horizontal stresses is observed at the level where the vertical strip load is applied, whereas for the case of a horizontal strip load the shear stress is discontinuous at the loading plane. It is found that the residual surface stress shows more significant influence on the stress field in the case of a vertical strip loading, especially at points closer to the free surface ($z_0 < 2$) when compared to the case of a horizontal strip loading. It should be noted that σ_{zz} in Fig. 6 is no longer zero at the surface due to the presence of the residual surface stress τ^s.



(a)



(b)



(c)

Fig. 4 Non-dimensional stress profiles at $z_0 = 0.1$ under vertical surface load for different material constants: (a) Vertical stress, (b) Horizontal stress, (c) Shear stress

To investigate the influence of layer thickness, the profiles of non-dimensional stresses in elastic layers of different thicknesses bonded to a rigid base and subjected to a uniformly distributed

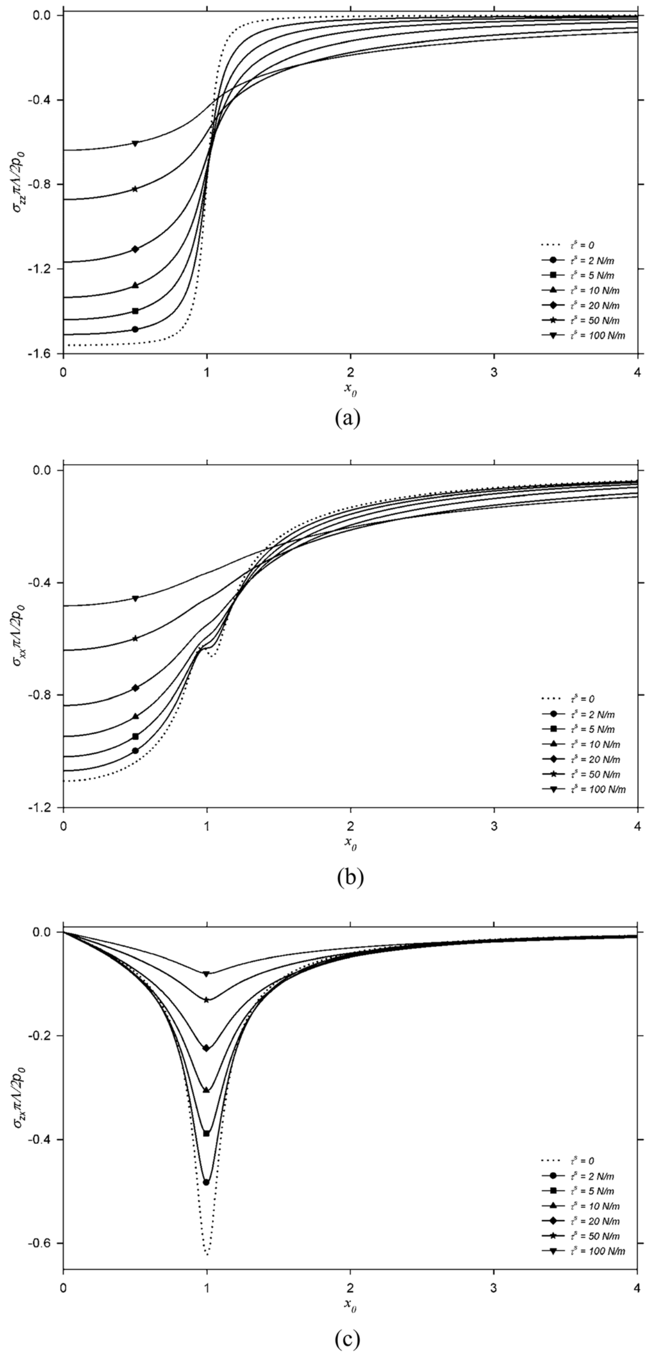
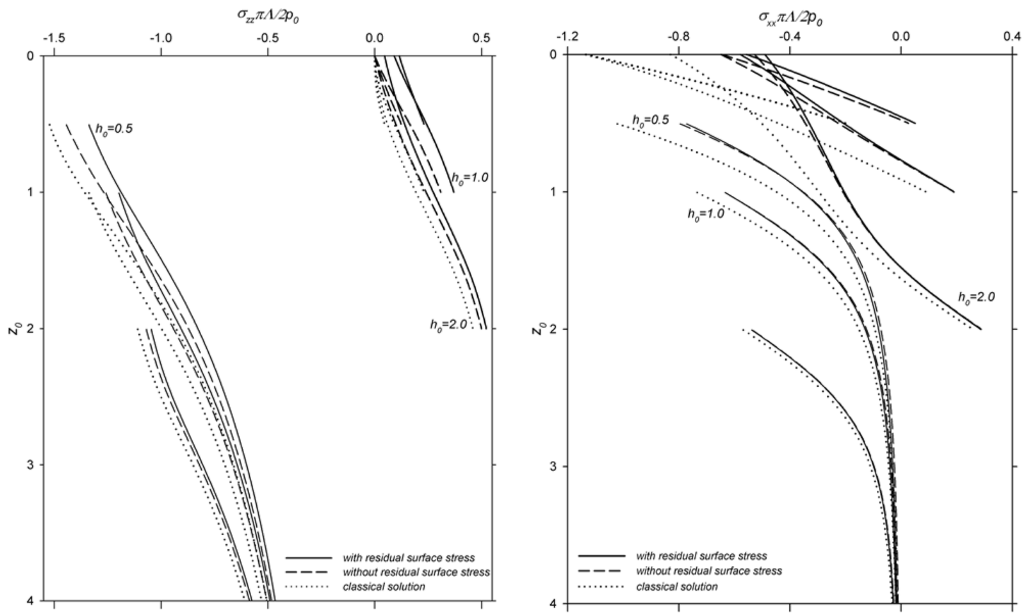


Fig. 5 Non-dimensional stress profiles at $z_0 = 0.1$ under vertical surface load for different residual surface stresses: (a) Vertical stress, (b) Horizontal stress, (c) Shear stress

vertical load p_0 and a horizontal load q_0 over a region $2a$ (with $a_0 = 1$) at the free surface are presented in Figs. 8 and 9 respectively. In this case, it is convenient to define the non-dimensional



(a) (b)

Fig. 6 Non-dimensional stress profiles along the *z*-axis of a half-plane under internal vertical load: (a) Vertical stress, (b) Horizontal stress

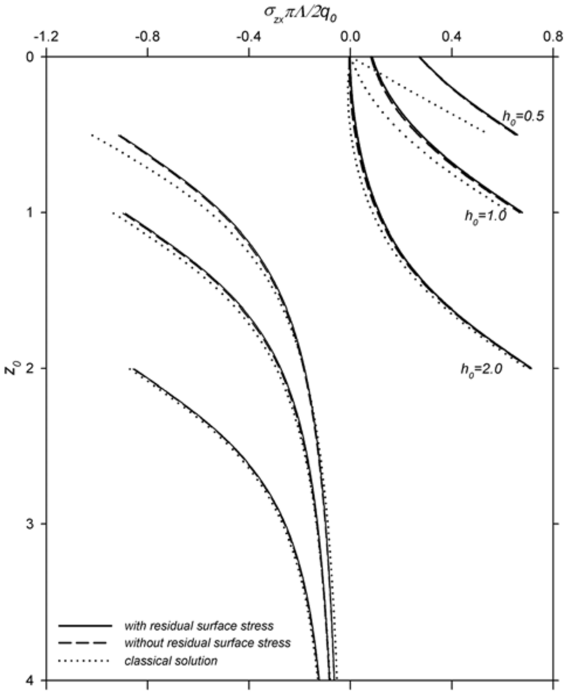
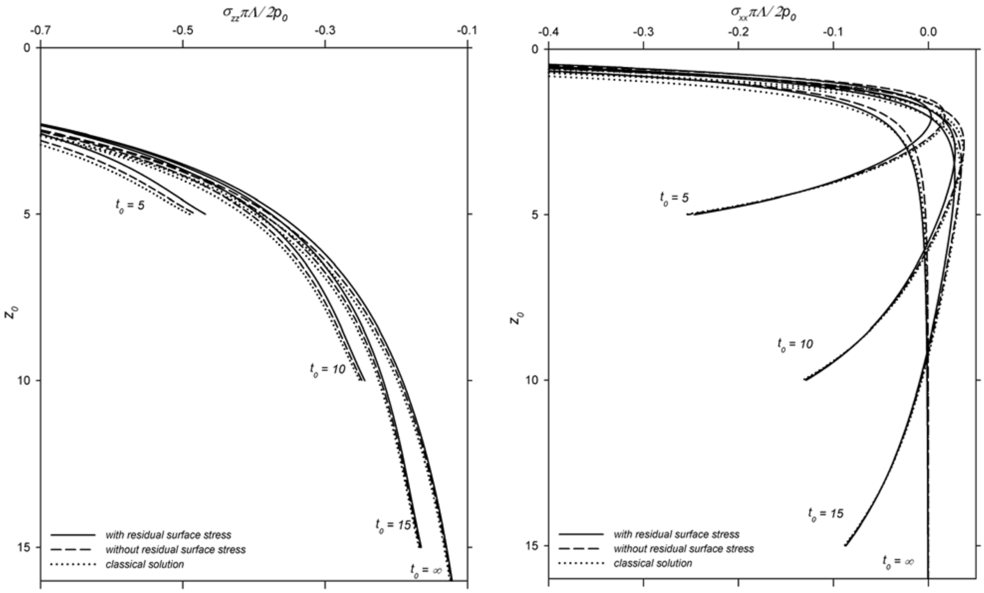


Fig. 7 Non-dimensional shear stress profiles along the *z*-axis of a half-plane under internal horizontal load



(a) (b)

Fig. 8 Non-dimensional stress profiles along the *z*-axis of a finite layer under vertical surface load: (a) Vertical stress, (b) Horizontal stress

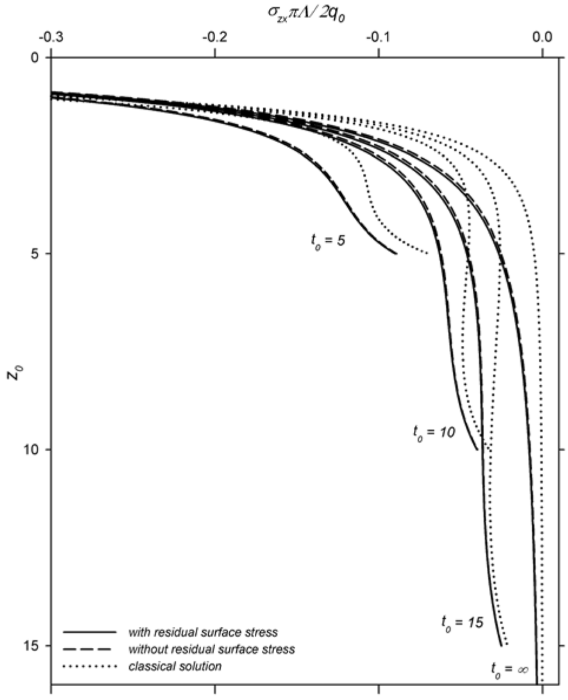


Fig. 9 Non-dimensional shear stress along the z-axis of a finite layer under horizontal surface load

layer thickness, $t_0 = t / \Lambda$. Once again, the residual surface stress shows more significant influence in the case of a vertical strip load when compared to a horizontal strip load. It is clear from these figures that the thickness of a layer has a significant influence on the stress field for both classical and non-classical cases. The stresses in both cases are mainly compressive and decrease with increasing layer thickness, except for the horizontal stresses under vertical strip load in Fig. 8(b), in which tensile stresses are also noted for layers with finite thickness. Numerical results shown in Figs. 2-9 confirm the fact that the influence of the residual surface stress should not be ignored in the analysis of the problems involving nanoscale layers or soft elastic materials.

5. Conclusions

A theoretical formulation based on the Gurtin-Murdoch continuum theory of elastic material surfaces is presented to study the elastic response of a nanoscale layer. An important aspect of the present study is the consideration of the out-of-plane term of the surface displacement gradient in the formulation. It is shown that the elastic field can be solved explicitly by using Fourier integral transform techniques. The final solution is expressed in terms of semi-infinite integrals that can be accurately computed by employing a numerical quadrature scheme. It is found from the analytical solution and numerical results that the effects of the surface energy on the elastic field are characterized by both the characteristic length parameter Λ that is related to the surface and bulk elastic moduli and the residual surface stress (τ^s). As expected, the influence of the surface elasticity and the residual surface stress becomes more significant in the vicinity of the layer surface. Numerical results also indicate that increasing Λ and τ^s result in a decrease in bulk stresses.

Acknowledgements

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Appendix

The expressions of A_i to H_i ($i = p, q$) and I appearing in Eqs. (30) - (34) are given as follows:

$$A_p = \left\{ \begin{array}{l} (\lambda+3\mu)\left[\left(\frac{\lambda+\mu}{\lambda+2\mu}h+\Lambda\right)|\xi|\cosh\left(\xi(2t-h)\right)+\left(1+\frac{\lambda+\mu}{\lambda+2\mu}\Lambda h\xi^2\right)\sinh\left(|\xi|(2t-h)\right)\right] \\ -(\lambda+\mu)\left[\left(2t+\frac{\lambda+3\mu}{\lambda+\mu}\Lambda-\frac{\lambda+\mu}{\lambda+2\mu}\left(2t(t-h)\Lambda\xi^2+h\right)\right)|\xi|\cosh(\xi h)+\left(\frac{\lambda+3\mu}{\lambda+\mu}+\frac{\lambda+\mu}{\lambda+2\mu}\left(2t(t-h)-\Lambda h\right)\xi^2+2\Lambda t\xi^2\right)\sinh\left(|\xi|h\right)\right] \\ -\frac{\tau^s}{2\mu}|\xi|\left\{(\lambda+3\mu)\left[\left(\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda+h\right)|\xi|\cosh\left(\xi(2t-h)\right)+\frac{\lambda+2\mu}{\lambda+\mu}\sinh\left(|\xi|(2t-h)\right)+\frac{(\lambda+\mu)^2}{2(\lambda+2\mu)^2}\left(e^{|\xi|(2t-h)}-\frac{\lambda+3\mu}{\lambda+\mu}e^{-|\xi|(2t-h)}\right)\Lambda h\xi^2\right]\right. \\ +(\lambda+\mu)\left[\left(h-\frac{2(\lambda+2\mu)}{\lambda+\mu}t-\frac{\lambda+3\mu}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|\right)\Lambda\right)|\xi|\cosh(\xi h)-\left(\frac{(\lambda+2\mu)(\lambda+3\mu)}{(\lambda+\mu)^2}+2t(t-h)\xi^2\right)\sin(|\xi|h)\right] \\ \left.+\frac{(\lambda+\mu)^3}{(\lambda+2\mu)^2}\Lambda\xi^2\left[\frac{(\lambda+3\mu)^2}{2(\lambda+\mu)^2}h-\left(t(t-h)|\xi|\right)e^{|\xi|h}-\frac{\lambda+3\mu}{2(\lambda+\mu)}\left(2t(t-h)|\xi|+h\right)e^{-|\xi|h}\right]\right\} \end{array} \right\} \frac{\overline{p}(\xi)}{2\xi^2} \tag{A.1}$$

$$B_p = \left\{ \begin{array}{l} (\lambda+\mu)(\lambda+3\mu)\left[2\Lambda|\xi|\left(\cosh\left(\xi(2t-h)\right)-\cosh(\xi h)\right)+\left(2h|\xi|+\frac{\lambda+3\mu}{\lambda+\mu}\right)e^{|\xi|(2t-h)}-e^{-|\xi|(2t-h)}-\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|\right)e^{|\xi|h}\right] \\ +(\lambda+\mu)^2\left[4(t-h)\left(\Lambda\sinh\left(|\xi|h\right)-te^{|\xi|h}\right)\xi^2+\left(\frac{\lambda+3\mu}{\lambda+\mu}-2(t-h)|\xi|\right)e^{-|\xi|h}\right] \\ -\frac{\tau^s}{2\mu}|\xi|\left\{(\lambda+3\mu)\left[\frac{2(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)}\Lambda|\xi|\cosh\left(\xi(2t-h)\right)+2(\lambda+2\mu)\left(\sinh\left(|\xi|(2t-h)\right)+\sinh\left(|\xi|h\right)\right)\right]\right. \\ -2(\lambda+\mu)\left[2(\lambda+2\mu)(t-h)+\frac{(\lambda+3\mu)^2}{\lambda+2\mu}\Lambda\right]|\xi|\cosh(\xi h)+\frac{2(\lambda+\mu)^2(\lambda+3\mu)}{\lambda+2\mu}\Lambda h\xi^2 e^{|\xi|(2t-h)} \\ \left.-\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda\xi^2\left[\left((\lambda+3\mu)t+2(\lambda+\mu)t(t-h)|\xi|\right)e^{|\xi|h}-(\lambda+3\mu)(t-h)e^{-|\xi|h}\right]\right\} \end{array} \right\} \frac{\overline{p}(\xi)}{4(\lambda+2\mu)|\xi|} \tag{A.2}$$

$$C_p = -\left\{ \begin{array}{l} (\lambda+3\mu)\left[\left(\frac{\lambda+\mu}{\lambda+2\mu}h+\Lambda\right)|\xi|\cosh\left(\xi(2t-h)\right)+\left(1+\frac{\lambda+\mu}{\lambda+2\mu}\Lambda h\xi^2\right)\sinh\left(|\xi|(2t-h)\right)\right] \\ -(\lambda+\mu)\left[\left(2t+\frac{\lambda+3\mu}{\lambda+\mu}\Lambda+\frac{\lambda+\mu}{\lambda+2\mu}\left(2t(t-h)\Lambda\xi^2-h\right)\right)|\xi|\cosh(\xi h)+\left(\frac{\lambda+3\mu}{\lambda+\mu}+\frac{\lambda+\mu}{\lambda+2\mu}\left(2t(t-h)-\Lambda h\right)\xi^2+2\Lambda t\xi^2\right)\sinh\left(|\xi|h\right)\right] \\ +\frac{\tau^s}{2\mu}|\xi|\left\{(\lambda+3\mu)\left[\left(\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda+h\right)|\xi|\cosh\left(\xi(2t-h)\right)+\frac{\lambda+2\mu}{\lambda+\mu}\sinh\left(|\xi|(2t-h)\right)+\frac{(\lambda+\mu)^2}{2(\lambda+2\mu)^2}\left(\frac{\lambda+3\mu}{\lambda+\mu}e^{|\xi|(2t-h)}-e^{|\xi|(2t-h)}\right)\Lambda h\xi^2\right]\right. \\ +(\lambda+\mu)\left[\left(h|\xi|-\frac{2(\lambda+2\mu)}{(\lambda+\mu)}t|\xi|-\frac{\lambda+3\mu}{\lambda+\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|-\frac{\lambda+2\mu}{\lambda+\mu}\right)\right)\cosh(\xi h)-\left(2(t-h)+\frac{2(\lambda+3\mu)}{\lambda+2\mu}\Lambda\right)t\xi^2\sin(|\xi|h)\right] \\ \left.-\frac{(\lambda+\mu)^3}{(\lambda+2\mu)^2}\Lambda\xi^2\left[\frac{\lambda+3\mu}{\lambda+\mu}\left(t(t-h)|\xi|-\frac{\lambda+3\mu}{2(\lambda+\mu)}h\right)e^{|\xi|h}+\left(\frac{(\lambda+3\mu)^2}{2(\lambda+\mu)^2}h-t(t-h)|\xi|\right)e^{-|\xi|h}\right]\right\} \end{array} \right\} \frac{\overline{p}(\xi)}{2\xi^2} \tag{A.3}$$

$$D_p = \left\{ \begin{array}{l} (\lambda+\mu)(\lambda+3\mu)\left[2\Lambda|\xi|\left(\cosh\left(\xi(2t-h)\right)-\cosh(\xi h)\right)+e^{|\xi|(2t-h)}+\left(2h|\xi|-\frac{\lambda+3\mu}{\lambda+\mu}\right)e^{-|\xi|(2t-h)}+\left(\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|\right)e^{-|\xi|h}\right] \\ -(\lambda+\mu)^2\left[4(t-h)\left(\Lambda\sinh\left(|\xi|h\right)-te^{-|\xi|h}\right)\xi^2+\left(\frac{\lambda+3\mu}{\lambda+\mu}+2(t-h)|\xi|\right)e^{|\xi|h}\right] \\ +\frac{\tau^s}{2\mu}|\xi|\left\{(\lambda+3\mu)\left[\frac{2(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)}\Lambda|\xi|\cosh\left(\xi(2t-h)\right)+2(\lambda+2\mu)\sinh\left(|\xi|(2t-h)\right)\right]\right. \\ -2(\lambda+\mu)\left[2(\lambda+2\mu)(t-h)+\frac{(\lambda+3\mu)^2}{\lambda+2\mu}\Lambda\right]|\xi|\cosh(\xi h)-\frac{2(\lambda+\mu)^2(\lambda+3\mu)}{\lambda+2\mu}\Lambda h\xi^2 e^{-|\xi|(2t-h)} \\ \left.-\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda\xi^2\left[\left((\lambda+3\mu)(t-h)+2(\lambda+\mu)t(t-h)|\xi|\right)e^{|\xi|h}-\left((\lambda+3\mu)t-2(\lambda+\mu)t(t-h)|\xi|\right)e^{-|\xi|h}\right]\right\} \end{array} \right\} \frac{\overline{p}(\xi)}{4(\lambda+2\mu)|\xi|} \tag{A.4}$$

$$E_p=\left\{\begin{array}{l}
(\lambda+3\mu)(1+\Lambda|\xi|)\left[\left(1-\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{|\xi|(2t+h)}+\left(1+\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{|\xi|(2t-h)}\right]+(\lambda+\mu)(1-\Lambda|\xi|)\left[\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|+\frac{\lambda+\mu}{\lambda+2\mu}(2t(t+h)|\xi|+h)|\xi|\right]e^{-|\xi|h}\\
+\left[\frac{\lambda^2+4\lambda\mu+7\mu^2}{\lambda+\mu}-(\lambda+3\mu)\left(\frac{\lambda+3\mu}{\lambda+2\mu}h+\Lambda\right)|\xi|-(\lambda+\mu)\left(2(1-\Lambda|\xi|)-\frac{2(\lambda+3\mu)}{\lambda+2\mu}t|\xi|\right)t|\xi|+\frac{(\lambda+\mu)^2}{\lambda+2\mu}\left(2th(1-2|\xi|t)-\Lambda(2t(t+h)|\xi|-h)\right)\xi^2\right]e^{|\xi|h}\\
+\frac{\tau^s}{2\mu}|\xi|\Bigg\{(\lambda+3\mu)\left[\left(\frac{\lambda+2\mu}{\lambda+\mu}+\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|\right)\left(1-\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{|\xi|(2t+h)}-\left(\frac{\lambda+2\mu}{\lambda+\mu}+h|\xi|+\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|+\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2}\Lambda h\xi^2\right)e^{|\xi|(2t-h)}\right]\\
+(\lambda+\mu)\left[(2t(t+h)|\xi|-h)|\xi|+\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|\right)-\frac{\lambda+3\mu}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|-\frac{\lambda+3\mu}{\lambda+2\mu}h|\xi|+\frac{2(\lambda+\mu)}{\lambda+2\mu}t^2\xi^2\right)\Lambda|\xi|-\frac{2(\lambda+\mu)^2}{(\lambda+2\mu)^2}(1-2|\xi|t)\Lambda th|\xi|^3\right]e^{|\xi|h}\\
-(\lambda+\mu)\left[(2t(t-h)|\xi|+h)|\xi|+\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|\right)-\frac{\lambda+3\mu}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|+\frac{\lambda+\mu}{\lambda+2\mu}(2t(t-h)|\xi|+h)|\xi|\right)\Lambda|\xi|\right]e^{-|\xi|h}\Bigg\}
\end{array}\right\}\frac{\overline{p}(\xi)}{4\xi^2} \tag{A.5}$$

$$F_p=\left\{\begin{array}{l}
(\lambda+\mu)(\lambda+3\mu)\left[(1+\Lambda|\xi|)e^{|\xi|(2t+h)}+\left(\frac{\lambda+3\mu}{\lambda+\mu}+(\Lambda+2h)|\xi|\right)e^{|\xi|(2t-h)}\right]\\
+(\lambda+\mu)^2\left[1-\frac{\lambda+3\mu}{\lambda+\mu}(\Lambda+2t)|\xi|+(2\Lambda(t+h)+4th)\xi^2\right]e^{|\xi|h}+(\lambda+\mu)(1-\Lambda|\xi|)\left[\lambda+3\mu-2(\lambda+\mu)(t-h)|\xi|\right]e^{-|\xi|h}\\
+\frac{\tau^s}{2\mu}|\xi|\Bigg\{(\lambda+3\mu)\left[\left(\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right)e^{|\xi|(2t+h)}-\left(\lambda+2\mu+\frac{(\lambda+\mu)^2}{\lambda+2\mu}\left(\frac{(\lambda+3\mu)}{\lambda+\mu}+2h|\xi|\right)\Lambda|\xi|\right)e^{|\xi|(2t-h)}\right]\\
+(\lambda+\mu)\left[(\lambda+2\mu)\left(\frac{\lambda+3\mu}{\lambda+\mu}-2(t+h)|\xi|\right)-\frac{\lambda+3\mu}{\lambda+2\mu}(\lambda+3\mu-2(\lambda+\mu)t|\xi|)\Lambda|\xi|-\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda th|\xi|^3\right]e^{|\xi|h}\\
-(\lambda+\mu)\left[(\lambda+2\mu)\left(\frac{\lambda+3\mu}{\lambda+\mu}-2(t-h)|\xi|\right)-\frac{\lambda+3\mu}{\lambda+2\mu}(\lambda+3\mu-2(\lambda+\mu)(t-h)|\xi|)\Lambda|\xi|\right]e^{-|\xi|h}\Bigg\}
\end{array}\right\}\frac{\overline{p}(\xi)}{4(\lambda+2\mu)|\xi|} \tag{A.6}$$

$$G_p=\left\{\begin{array}{l}
(\lambda+3\mu)(1-\Lambda|\xi|)\left[\left(1+\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{-|\xi|(2t+h)}+\left(1-\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{-|\xi|(2t-h)}\right]+(\lambda+\mu)(1+\Lambda|\xi|)\left[\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\frac{\lambda+\mu}{\lambda+2\mu}(2t(t-h)|\xi|-h)|\xi|\right]e^{|\xi|h}\\
+\left[\frac{\lambda^2+4\lambda\mu+7\mu^2}{\lambda+\mu}+(\lambda+3\mu)\left(\frac{\lambda+3\mu}{\lambda+2\mu}h+\Lambda\right)|\xi|+(\lambda+\mu)\left(2(1+\Lambda|\xi|)+\frac{2(\lambda+3\mu)}{\lambda+2\mu}t|\xi|\right)t|\xi|+\frac{(\lambda+\mu)^2}{\lambda+2\mu}\left(2th(1+2|\xi|t)+\Lambda(2t(t+h)|\xi|+h)\right)\xi^2\right]e^{-|\xi|h}\\
+\frac{\tau^s}{2\mu}|\xi|\Bigg\{(\lambda+3\mu)\left[\left(\frac{\lambda+2\mu}{\lambda+\mu}-h|\xi|-\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|+\frac{(\lambda+\mu)^2}{(\lambda+2\mu)^2}\Lambda h\xi^2\right)e^{-|\xi|(2t-h)}-\left(\frac{\lambda+2\mu}{\lambda+\mu}-\frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|\right)\left(1+\frac{\lambda+\mu}{\lambda+2\mu}h|\xi|\right)e^{-|\xi|(2t+h)}\right]\\
+(\lambda+\mu)\left[(2t(t-h)|\xi|-h)|\xi|+\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|\right)+\frac{\lambda+3\mu}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\frac{\lambda+\mu}{\lambda+2\mu}(2t(t-h)|\xi|-h)|\xi|\right)\Lambda|\xi|\right]e^{|\xi|h}\\
-(\lambda+\mu)\left[(2t(t+h)|\xi|+h)|\xi|+\frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|\right)+\frac{\lambda+3\mu}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\frac{\lambda+3\mu}{\lambda+2\mu}h|\xi|+\frac{2(\lambda+\mu)}{\lambda+2\mu}t^2\xi^2\right)\Lambda|\xi|+\frac{2(\lambda+\mu)}{(\lambda+2\mu)^2}(1+2t|\xi|)\Lambda th|\xi|^3\right]e^{-|\xi|h}\Bigg\}
\end{array}\right\}\frac{\overline{p}(\xi)}{4\xi^2} \tag{A.7}$$

$$H_p=-\left\{\begin{array}{l}
(\lambda+\mu)(\lambda+3\mu)\left[(1-\Lambda|\xi|)e^{-|\xi|(2t+h)}+\left(\frac{\lambda+3\mu}{\lambda+\mu}-(\Lambda+2h)|\xi|\right)e^{-|\xi|(2t-h)}\right]\\
+(\lambda+\mu)(1+\Lambda|\xi|)\left[\lambda+3\mu-2(\lambda+\mu)(t-h)|\xi|\right]e^{|\xi|h}+(\lambda+\mu)^2\left[1+\frac{\lambda+3\mu}{\lambda+\mu}(\Lambda+2t)|\xi|+(2\Lambda(t+h)+4th)\xi^2\right]e^{-|\xi|h}\\
-\frac{\tau^s}{2\mu}|\xi|\Bigg\{(\lambda+3\mu)\left[\left(\lambda+2\mu-\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right)e^{-|\xi|(2t+h)}-\left(\lambda+2\mu-\frac{(\lambda+\mu)^2}{\lambda+2\mu}\left(\frac{\lambda+3\mu}{\lambda+\mu}-2h|\xi|\right)\Lambda|\xi|\right)e^{-|\xi|(2t-h)}\right]\\
-(\lambda+\mu)\left[(\lambda+2\mu)\left(\frac{\lambda+3\mu}{\lambda+\mu}+2(t-h)|\xi|\right)+\frac{\lambda+3\mu}{\lambda+2\mu}(\lambda+3\mu+2(\lambda+\mu)(t-h)|\xi|)\Lambda|\xi|\right]e^{|\xi|h}\\
+(\lambda+\mu)\left[(\lambda+2\mu)\left(\frac{\lambda+3\mu}{\lambda+\mu}+2(t+h)|\xi|\right)+\frac{\lambda+3\mu}{\lambda+2\mu}(\lambda+3\mu+2(\lambda+\mu)t|\xi|)\Lambda|\xi|-\frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda th|\xi|^3\right]e^{-|\xi|h}\Bigg\}
\end{array}\right\}\frac{\overline{p}(\xi)}{4(\lambda+2\mu)|\xi|} \tag{A.8}$$

$$A_q = \left\{ \begin{array}{l} (\lambda+3\mu)\Big[\big(\mu-(\lambda+\mu)\Lambda h\xi^2\big)\cosh\big(\xi(2t-h)\big)+\big(\mu\Lambda-(\lambda+\mu)h\big)\xi\sinh\big(\xi(2t-h)\big)\Big] \\ -(\lambda+\mu)\Big[\Big(\dfrac{\mu(\lambda+3\mu)}{\lambda+\mu}-\mu\Lambda t\xi^2+(\lambda+\mu)\big(t(t-h)-h\Lambda\big)\xi^2\Big)\cosh(\xi h)-\Big(\mu t-\dfrac{\mu(\lambda+3\mu)}{\lambda+\mu}\Lambda-(\lambda+\mu)\big(\Lambda t(t-h)\xi^2-h\big)\Big)\xi\sinh(\xi h)\Big] \\ +\dfrac{\tau^s}{2}|\xi|\Big\{(\lambda+3\mu)\Big[\Big(-\dfrac{\lambda+2\mu}{\lambda+\mu}+\dfrac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|\Big)\cosh\big(\xi(2t-h)\big)+\dfrac{\lambda+2\mu}{\mu}h\xi\sinh\big(\xi(2t-h)\big)+\dfrac{(\lambda+\mu)^2}{2\mu(\lambda+2\mu)}\Big(e^{|\xi|(2t-h)}+\dfrac{\lambda+3\mu}{\lambda+\mu}e^{-|\xi|(2t-h)}\Big)\Lambda h\xi^2\Big] \\ +(\lambda+3\mu)\Big[\dfrac{\lambda+2\mu}{\lambda+\mu}-\dfrac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|+\dfrac{2(\lambda+\mu)(\lambda+2\mu)}{\mu(\lambda+3\mu)}t(t-h)\xi^2\Big]\cosh(\xi h)-(\lambda+2\mu)\Big[2t+\dfrac{\lambda+\mu}{\mu}h-\dfrac{2(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2}\Lambda t|\xi|\Big]\xi\sin(\xi h) \\ +(\lambda+\mu)\Big[\dfrac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}t(t-h)|\xi|-\dfrac{(\lambda+3\mu)^2}{2\mu(\lambda+2\mu)}h\Big]\Lambda\xi^2e^{|\xi|h}-\dfrac{(\lambda+\mu)^2(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\big(t(t-h)|\xi|+h\big)\Lambda\xi^2e^{-|\xi|h}\Big\} \end{array} \right\} \frac{\overline{q}(\xi)}{2(\lambda+2\mu)|\xi|\xi} \tag{A.9}$$

$$B_q = \left\{ \begin{array}{l} (\lambda+\mu)(\lambda+3\mu)\Big[-2\Lambda\xi\sinh\big(\xi(2t-h)\big)+\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}-2h|\xi|\Big)e^{|\xi|(2t-h)}-e^{-|\xi|(2t-h)}-\dfrac{4(\lambda+\mu)}{\lambda+3\mu}\Lambda(t-h)\xi^2\cosh(\xi h) \\ +2\Lambda\xi\sinh(\xi h)-\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}-2t|\xi|+\dfrac{4(\lambda+\mu)}{\lambda+3\mu}t(t-h)\xi^2\Big)e^{|\xi|h}+\Big(1+\dfrac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\Big)e^{-|\xi|h}\Big] \\ +\dfrac{\tau^s}{\mu}|\xi|\Big\{(\lambda+3\mu)\Big[\Big(\lambda+2\mu-\dfrac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\Big)\big(\cosh\big(\xi(2t-h)\big)-\cosh(\xi h)\big)+\dfrac{(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2e^{|\xi|(2t-h)}\Big] \\ +2(\lambda+\mu)(\lambda+2\mu)(t-h)\xi\sinh(\xi h)-\dfrac{(\lambda+\mu)^2(\lambda+3\mu)}{(\lambda+2\mu)}\Big[\Big(1-\dfrac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\Big)te^{|\xi|h}+(t-h)e^{-|\xi|h}\Big]\Lambda\xi^2\Big\} \end{array} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)\xi} \tag{A.10}$$

$$C_q = -\left\{ \begin{array}{l} (\lambda+3\mu)\Big[\big(\mu-(\lambda+\mu)\Lambda h\xi^2\big)\cosh\big(\xi(2t-h)\big)+\big(\mu\Lambda-(\lambda+\mu)h\big)\xi\sinh\big(\xi(2t-h)\big)\Big] \\ -(\lambda+\mu)\Big[\Big(\dfrac{\mu(\lambda+3\mu)}{\lambda+3\mu}-\mu\Lambda t\xi^2-(\lambda+\mu)\big(t(t-h)+h\Lambda\big)\xi^2\Big)\cosh(\xi h)-\Big(\mu t-\dfrac{\mu(\lambda+3\mu)}{\lambda+\mu}\Lambda-(\lambda+\mu)\big(\Lambda t(t-h)\xi^2-h\big)\Big)\xi\sinh(\xi h)\Big] \\ +\dfrac{\tau^s}{2}|\xi|\Big\{(\lambda+3\mu)\Big[\Big(\dfrac{\lambda+2\mu}{\lambda+\mu}+\dfrac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|\Big)\cosh\big(\xi(2t-h)\big)-\dfrac{\lambda+2\mu}{\mu}h\xi\sinh\big(\xi(2t-h)\big)-\dfrac{(\lambda+\mu)^2}{2\mu(\lambda+2\mu)}\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}e^{(|\xi|(2t-h))}-e^{-(|\xi|(2t-h))}\Big)\Lambda h\xi^2\Big] \\ +(\lambda+3\mu)\Big[\dfrac{\lambda+2\mu}{\lambda+\mu}-\dfrac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|-\dfrac{(\lambda+\mu)(\lambda+2\mu)}{\mu(\lambda+3\mu)}h|\xi|\Big]\cosh(\xi h)-(\lambda+2\mu)\Big[2t+\dfrac{2(\lambda+\mu)}{\mu}t(t-h)|\xi|-\dfrac{2(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2}\Lambda t|\xi|\Big]\xi\sin(\xi h) \\ +(\lambda+\mu)\Big[\Big(\dfrac{(\lambda+\mu)(\lambda+3\mu)}{\mu(\lambda+2\mu)}t(t-h)|\xi|-\dfrac{(\lambda+3\mu)^2}{2\mu(\lambda+2\mu)}h\Big)e^{|\xi|h}-\Big(\dfrac{(\lambda+3\mu)^2}{2\mu(\lambda+2\mu)}h-\dfrac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}t(t-h)|\xi|\Big)e^{-|\xi|h}\Big]\Lambda\xi^2\Big\} \end{array} \right\} \frac{\overline{q}(\xi)}{2(\lambda+2\mu)|\xi|\xi} \tag{A.11}$$

$$D_q = \left\{ \begin{array}{l} (\lambda+\mu)(\lambda+3\mu)\Big[-2\Lambda\xi\sinh\big(\xi(2t-h)\big)-e^{|\xi|(2t-h)}+\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}+2h|\xi|\Big)e^{-|\xi|(2t-h)}-\dfrac{4(\lambda+\mu)}{\lambda+3\mu}\Lambda(t-h)\xi^2\cosh(\xi h) \\ +2\Lambda\xi\sinh(\xi h)+\Big(1-\dfrac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\Big)e^{|\xi|h}-\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\dfrac{4(\lambda+\mu)}{\lambda+3\mu}t(t-h)\xi^2\Big)e^{-|\xi|h}\Big] \\ -\dfrac{\tau^s}{\mu}|\xi|\Big\{(\lambda+3\mu)\Big[\Big(\lambda+2\mu+\dfrac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\Big)\big(\cosh\big(\xi(2t-h)\big)-\cosh(\xi h)\big)+\dfrac{(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2e^{-|\xi|(2t-h)}\Big] \\ +(\lambda+\mu)\Big[2(\lambda+2\mu)(t-h)+\dfrac{2(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda t|\xi|\Big]\xi\sinh(\xi h)-\dfrac{(\lambda+\mu)^3}{\lambda+2\mu}\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}he^{|\xi|h}+2t(t-h)|\xi|e^{-|\xi|h}\Big)\Lambda\xi^2\Big\} \end{array} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)\xi} \tag{A.12}$$

$$E_q = -\left\{ \begin{array}{l} (\lambda+3\mu)(1+\Lambda|\xi|)\Big[\big(\mu+(\lambda+\mu)h|\xi|\big)e^{|\xi|(2t+h)}-\big(\mu-(\lambda+\mu)h|\xi|\big)e^{|\xi|(2t-h)}\Big]+\Big[\mu(\lambda+3\mu)+(\lambda+\mu)\big((\lambda+\mu)(2t(t-h)|\xi|-h)-2\mu t\big)|\xi|\Big](1-\Lambda|\xi|)e^{|\xi|h} \\ +(\lambda+\mu)\Big[\dfrac{\mu(3\lambda^2+12\lambda\mu+13\mu^2)}{(\lambda+\mu)^2}+\dfrac{\lambda+3\mu}{\lambda+\mu}\big(\mu\Lambda+(\lambda+3\mu)h\big)|\xi|+2\mu(1-\Lambda|\xi|)t|\xi|+2(\lambda+3\mu)t^2\xi^2+(\lambda+\mu)\Big(\Lambda\big(2t(t+h)|\xi|+h\big)+2th(1+2t|\xi|)\Big)\xi^2\Big]e^{-|\xi|h} \\ +\dfrac{\tau^s}{2}|\xi|\Big\{(\lambda+3\mu)\Big[\Big(\dfrac{\lambda+2\mu}{\lambda+\mu}+\dfrac{(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\Big)\Big(1+\dfrac{\lambda+\mu}{\mu}h|\xi|\Big)e^{|\xi|(2t+h)}+\Big(\dfrac{\lambda+2\mu}{\lambda+\mu}-\dfrac{\lambda+2\mu}{\mu}h|\xi|-\dfrac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi|-\dfrac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}\Lambda h\xi^2\Big)e^{|\xi|(2t-h)}\Big] \\ -(\lambda+\mu)(\lambda+2\mu)\Big\{\Big[\dfrac{\lambda+3\mu}{(\lambda+\mu)^2}+\dfrac{2}{\lambda+\mu}t|\xi|+\dfrac{1}{\mu}\big(2t(t-h)|\xi|+h\big)|\xi|-\dfrac{\lambda+3\mu}{(\lambda+2\mu)^2}\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\dfrac{\lambda+\mu}{\mu}\big(2t(t-h)|\xi|+h\big)|\xi|\Big)\Lambda|\xi|\Big]e^{-|\xi|h} \\ +\Big[\dfrac{\lambda+3\mu}{(\lambda+\mu)^2}+\dfrac{2}{\lambda+\mu}t|\xi|+\dfrac{1}{\mu}\big(2t(t+h)|\xi|-h\big)|\xi|+\dfrac{\lambda+3\mu}{(\lambda+2\mu)^2}\Big(\dfrac{\lambda+3\mu}{\lambda+\mu}+2t|\xi|+\dfrac{\lambda+3\mu}{\mu}h|\xi|+\dfrac{2(\lambda+\mu)}{\mu}t^2\xi^2-\dfrac{2(\lambda+\mu)^2}{\mu(\lambda+3\mu)}(1-2t|\xi|)th\xi^2\Big)\Lambda|\xi|\Big]e^{|\xi|h}\Big\}\Big\} \end{array} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)|\xi|\xi} \tag{A.13}$$

$$F_q = \left\{ \begin{aligned} &(\lambda+\mu)(\lambda+3\mu)\left[\left(1+\Lambda|\xi|\right)e^{|\xi|(2t+h)} + \left(\frac{\lambda+3\mu}{\lambda+\mu} - (\Lambda+2h)|\xi|\right)e^{|\xi|(2t-h)} \right. \\ &\left. + \left(\frac{\lambda+\mu}{\lambda+3\mu}\left(1+4th\xi^2+2(t+h)\Lambda\xi^2\right) + (\Lambda+2t)|\xi|\right)e^{|\xi|h} + \left(1-\Lambda|\xi|\right)\left(1+\frac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\right)e^{-|\xi|h} \right] \\ &+\frac{\tau^s}{2\mu}|\xi|\left\{ (\lambda+3\mu)\left[\left(\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right)e^{|\xi|(2t+h)} + \left(\lambda+2\mu-\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi| + \frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right)e^{|\xi|(2t-h)} \right] \right. \\ &-(\lambda+\mu)\left[\frac{(\lambda+2\mu)(\lambda+3\mu)}{\lambda+\mu} + 2(\lambda+2\mu)(t+h)|\xi| + \frac{(\lambda+3\mu)^2}{\lambda+2\mu}\left(1+\frac{2(\lambda+\mu)}{\lambda+3\mu}t|\xi| + \frac{2(\lambda+\mu)^2}{(\lambda+3\mu)^2}th\xi^2\right)\Lambda|\xi| \right]e^{|\xi|h} \\ &\left. -(\lambda+\mu)\left[\frac{(\lambda+2\mu)(\lambda+3\mu)}{\lambda+\mu} + 2(\lambda+2\mu)(t-h)|\xi| - \frac{(\lambda+3\mu)^2}{\lambda+2\mu}\left(1+\frac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\right)\Lambda|\xi| \right]e^{-|\xi|h} \right\} \end{aligned} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)\xi} \tag{A.14}$$

$$G_q = \left\{ \begin{aligned} &(\lambda+3\mu)(1-\Lambda|\xi|)\left[\left(\mu-(\lambda+\mu)h|\xi|\right)e^{-|\xi|(2t+h)} - \left(\mu+(\lambda+\mu)h|\xi|\right)e^{-|\xi|(2t-h)}\right] + \left[\mu(\lambda+3\mu)+(\lambda+\mu)\left((\lambda+\mu)\left(2t(t-h)|\xi|-h\right)-2\mu t\right)|\xi|\right](1+\Lambda|\xi|)e^{|\xi|h} \\ &+(\lambda+\mu)\left[\frac{\mu\left(3\lambda^2+12\lambda\mu+13\mu^2\right)}{(\lambda+\mu)^2} - \frac{\lambda+3\mu}{\lambda+\mu}\left(\mu\Lambda+(\lambda+3\mu)h\right)|\xi| + 2\mu(1+\Lambda|\xi|)t|\xi| + 2(\lambda+3\mu)t^2\xi^2 - (\lambda+\mu)\left(\Lambda\left(2t(t+h)|\xi|+h\right)+2th(1+2t|\xi|)\right)\xi^2 \right]e^{-|\xi|h} \\ &-\frac{\tau^s}{2}|\xi|\left\{ (\lambda+3\mu)\left[\left(\frac{\lambda+2\mu}{\lambda+\mu} - \frac{(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right)\left(1-\frac{\lambda+\mu}{\mu}h|\xi|\right)e^{-|\xi|(2t+h)} + \left(\frac{\lambda+2\mu}{\lambda+\mu} + \frac{\lambda+2\mu}{\mu}h|\xi| + \frac{\lambda+3\mu}{\lambda+2\mu}\Lambda|\xi| - \frac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}\Lambda h\xi^2\right)e^{-|\xi|(2t-h)} \right] \right. \\ &-(\lambda+\mu)(\lambda+2\mu)\left\{ \left[\frac{\lambda+3\mu}{(\lambda+\mu)^2} - \frac{2}{\lambda+\mu}t|\xi| + \frac{1}{\mu}\left(2t(t-h)|\xi|-h\right)|\xi| + \frac{\lambda+3\mu}{(\lambda+2\mu)^2}\left(\frac{\lambda+3\mu}{\lambda+\mu} - 2t|\xi| + \frac{\lambda+\mu}{\mu}\left(2t(t-h)|\xi|-h\right)|\xi|\right)\Lambda|\xi| \right]e^{|\xi|h} \right. \\ &\left.\left. + \left[\frac{\lambda+3\mu}{(\lambda+\mu)^2} - \frac{2}{\lambda+\mu}t|\xi| + \frac{1}{\mu}\left(2t(t+h)|\xi|+h\right)|\xi| - \frac{\lambda+3\mu}{(\lambda+2\mu)^2}\left(\frac{\lambda+3\mu}{\lambda+\mu} - 2t|\xi| - \frac{\lambda+3\mu}{\mu}h|\xi| + \frac{2(\lambda+\mu)}{\mu}t^2\xi^2 - \frac{2(\lambda+\mu)^2}{\mu(\lambda+3\mu)}(1+2t|\xi|)th\xi^2\right)\Lambda|\xi| \right]e^{-|\xi|h} \right\}\right\} \end{aligned} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)|\xi|\xi} \tag{A.15}$$

$$H_q = \left\{ \begin{aligned} &(\lambda+\mu)(\lambda+3\mu)\left[\left(1-\Lambda|\xi|\right)e^{-|\xi|(2t+h)} + \left(\frac{\lambda+3\mu}{\lambda+\mu} + (\Lambda+2h)|\xi|\right)e^{-|\xi|(2t-h)} \right. \\ &\left. + \left(1+\Lambda|\xi|\right)\left(1-\frac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\right)e^{|\xi|h} + \left(\frac{\lambda+\mu}{\lambda+3\mu}\left(1+4th\xi^2+2(t+h)\Lambda\xi^2\right) - (\Lambda+2t)|\xi|\right)e^{-|\xi|h} \right] \\ &-\frac{\tau^s}{2\mu}|\xi|\left\{ (\lambda+3\mu)\left[\left(\lambda+2\mu-\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi|\right)e^{-|\xi|(2t+h)} + \left(\lambda+2\mu+\frac{(\lambda+\mu)(\lambda+3\mu)}{\lambda+2\mu}\Lambda|\xi| + \frac{2(\lambda+\mu)^2}{\lambda+2\mu}\Lambda h\xi^2\right)e^{-|\xi|(2t-h)} \right] \right. \\ &-(\lambda+\mu)\left[\frac{(\lambda+2\mu)(\lambda+3\mu)}{\lambda+\mu} - 2(\lambda+2\mu)(t-h)|\xi| + \frac{(\lambda+3\mu)^2}{\lambda+2\mu}\left(1-\frac{2(\lambda+\mu)}{\lambda+3\mu}(t-h)|\xi|\right)\Lambda|\xi| \right]e^{|\xi|h} \\ &\left. -(\lambda+\mu)\left[\frac{(\lambda+2\mu)(\lambda+3\mu)}{\lambda+\mu} - 2(\lambda+2\mu)(t+h)|\xi| - \frac{(\lambda+3\mu)^2}{\lambda+2\mu}\left(1-\frac{2(\lambda+\mu)}{\lambda+3\mu}t|\xi| + \frac{2(\lambda+\mu)^2}{(\lambda+3\mu)^2}th\xi^2\right)\Lambda|\xi| \right]e^{-|\xi|h} \right\} \end{aligned} \right\} \frac{\overline{q}(\xi)}{4(\lambda+2\mu)\xi} \tag{A.16}$$

$$\begin{aligned} I &= (\lambda+3\mu)\left[\cosh(2t\xi)+\Lambda\xi\sinh(2t\xi)\right] + 2t\xi^2(\lambda+\mu)(\Lambda+t) + \frac{\lambda^2+4\lambda\mu+5\mu^2}{\lambda+\mu} \\ &+\tau^s\left\{ (\lambda+3\mu)\left[\frac{(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\Lambda\xi^2\cosh(2t\xi) + \frac{(\lambda+2\mu)}{2\mu(\lambda+\mu)}\xi\sinh(2t\xi) - \frac{(\lambda+3\mu)}{2\mu(\lambda+2\mu)}\Lambda\xi^2 \right] - \frac{(\lambda+2\mu)}{\mu}t\xi^2 - \frac{(\lambda+\mu)^2}{\mu(\lambda+2\mu)}\Lambda t^2\xi^4 \right\} \end{aligned} \tag{A.17}$$