

Resonance of a rectangular plate influenced by sequential moving masses

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(Received August 19, 2015, Revised March 2, 2016, Accepted March 5, 2016)

Abstract. In this work, an improved semi-analytical technique is adopted to track the dynamic response of thin rectangular plates excited by sequential traveling masses. This technique exploits a so-called indirect definition of inertial interaction between the moving masses and the plate and leads to a reduction, in the equations of motion, of the number of time-varying coefficients linked to the changing position of the masses. By employing this optimized method, the resonance of the plate can be obtained according to a parametric study of relevant maximum dynamic amplification factor. For the case of evenly spaced, equal masses travelling along a straight line, the resonance velocity of the masses themselves is also approximately predicted via a fast methodology based on the fundamental frequency of the system only.

Keywords: thin rectangular plate; resonance; traveling masses; semi-analytical procedures; optimized method

1. Introduction

The problem of numerically predicting the dynamics of bridges under vehicular loads has so far attracted numerous researchers, who attacked the problem by studying the vibrations of a continuum under moving loads (Eftekhari Azam *et al.* 2013, Frýba 1999, Maximov 2014, Mohebpour *et al.* 2013, Nikkhoo *et al.* 2014, Ouyang 2011, Pimoradian *et al.* 2014, Roshandel *et al.* 2014). The so-called moving force/mass/oscillator/system assumptions for the vehicular load pattern are nowadays well-known.

In the moving force approach, the vehicular loads are assumed to be constant forces with time-varying positions (Bajer and Dyniewicz 2012, Frýba 1999, Martínez-Rodrigo *et al.* 2013a,b). The moving mass approach allows instead for moving masses with their own inertial properties, which interact with the base structure (Bilello and Bergman 2004, Bilello *et al.* 2004, Roshandel *et*

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al. 2014, Shadnam *et al.* 2001, Nikkhoo 2014, Zarfam *et al.* 2013). With both the aforementioned methods, the suspension systems of automobiles and trains are somehow neglected. Nevertheless, by accounting for the inertial effects of the vehicular loads it was highlighted in (Green and Cebon 1994, Stancioiu *et al.* 2009) that a rather accurate estimation of fundamental features of the vibrations of the base structure can be attained, using the moving mass approach.

As far as the structural behavior is concerned, bridges carrying moving loads are typically modelled as vibrating beams (Frýba 1999, Lotfollahi-Yaghin *et al.* 2015, Nikkhoo *et al.* 2014, Nikkhoo *et al.* 2015, Ouyang 2011, Roshandel *et al.* 2014, Wang and Ming-Yuan 2014). However, the computational model for a slab-type bridge would be more realistic if a plate kinematics were adopted, see Nikkhoo *et al.* (2014), Martinez-Rodrigo *et al.* (2010a,b), Ebrahimzadeh Hassanabadi *et al.* (2014) and Kumar *et al.* (2015).

Focusing specifically on the structural resonance induced by moving loads, in (Nikkhoo *et al.* 2014, Wang *et al.* 2003, Khoraskani *et al.* 2016, Yang *et al.* 2004, Dimitrovová 2014) procedures for engineering practitioners were proposed. It is in fact well-known that the resonance caused by successive vehicular loads can give rise to undesired consequences, as for the structural serviceability and maintenance (Nikkhoo *et al.* 2014, Martínez-Rodrigo *et al.* 2010a,b). Accounting also for the importance to determine the bridge life-time dynamic amplification factor (González *et al.* 2008), it would be of interest developing fast numerical procedures to catch the resonance of continua carrying multiple moving masses. Such procedures would also help calibrate the dynamic characteristics of controlling devices like tuned mass dampers (Wang *et al.* 2003).

A system composed of a plate and a series of fixed, attached masses has vibration frequency components different from those of the bare plate alone. A major complexity in the problem arises if the position of the masses varies in time, since in this case the system features time-varying frequency components. Accordingly, for the dynamic vehicle/bridge combined system, the frequency components are linked to the properties of the moving loads, such as weight, spacing and velocity, as well as to the base structure characteristics. For given values of spacing and mass, the moving mass-plate system resonance occurs at specific velocities, herein called resonance velocities.

In this work, two fast methods are employed to predict the aforementioned resonance velocity of sequential moving masses traversing a thin rectangular plate. With the first method given in Section 2, the response of the plate is obtained via a series expansion, but the masses/plate interaction is mathematically formulated in a different way compared to conventional approaches. Through the definition of ad-hoc auxiliary functions, the number of time-varying coefficients in the equations describing the system vibrations is remarkably reduced, leading to a highly efficient algorithm to scrutinize the maximum *dynamic amplification factor* (DAF) of the system. An even faster method is then proposed in Section 3, wherein the resonance velocity of equal, evenly spaced masses is obtained from the fundamental natural frequency of the plate only. Hence, this approach does not require a time-consuming parametric exploration of the DAF of the system to get an approximation for the resonance velocity.

In Section 4, results are presented for a plate simply supported along two opposite edges and free along the others, crossed by a series of masses evenly spaced and travelling along a rectilinear trajectory. It is shown that the mentioned former approach can provide remarkable analysis speedups with respect to conventional methods, whilst the latter one leads to (approximate) estimations of the resonance velocity featuring a discrepancy typically smaller than 2% in comparison with those obtained through the series expansion method.

To slightly simplify the notation throughout the whole paper, time derivatives will be always denoted with superposed dots, no matter whether total or partial derivatives have to be considered. The frame, and the dependence of the functions to be differentiated in time will anyhow allow to clearly distinguish the two cases.

2. Tracking the dynamic response of the plate to multiple moving masses

To date, no analytical solutions have been provided for the resonance velocity of a sequence of moving masses traversing a beam or a plate. For a plate, the resonant state is usually identified via a numerical analysis of its dynamic response, and the resonance velocity of the moving masses is obtained through the back-bone curves of DAF.

In Section 2.1, the classical (semi-analytical) modal expansion approach to the problem is briefly discussed. Next, in Section 2.2 we discuss an improved modal expansion method, which leads to a remarkable reduction of the computational costs.

2.1 The customary modal analysis method

To obtain a solution using the modal expansion method, an eigenanalysis of the plate should be first dealt with.

A rectangular thin plate with a uniform thickness is considered. We introduce an orthonormal reference frame, featuring axes x and y located over the mid-plane of the plate, see Fig. 1. In the absence of attached masses and any external load, the dynamics of the plate is governed by the differential equation (Leissa 1973)

$$D\nabla^4 w(x, y, t) + \rho h \ddot{w}(x, y, t) = 0 \quad (1)$$

where: $w(x, y, t)$ is the out-of-plane displacement, or transverse deflection of the plate, which is a function of position and time t ; $\ddot{w} = \frac{\partial^2 w}{\partial t^2}$ is the local out-of-plane acceleration field; D is the flexural rigidity of the plate, given by $D = \frac{Eh^3}{12(1-\nu^2)}$, E being the Young's modulus of the plate material and ν its Poisson's ratio; h is the plate thickness; ρ is the material density; and ∇^4 stands for the biharmonic differential operator.

Free vibrations of the plate are linked to the natural mode shapes $W_j(x, y)$, with $j = 1, \dots, \infty$, and relevant circular frequencies ω_j according to (Leissa 1973)

$$D\nabla^4 W_j(x, y) = \rho h \omega_j^2 W_j(x, y) \quad (2)$$

Focusing on a single-span rectangular plate without additional masses and simply supported along two opposite edges, the analytical eigensolutions were comprehensively addressed in (Leissa 1973).

Let us consider now the simplified case of inertial loads evenly spaced by L , all featuring the same speed v and mass M , moving in the same direction (see Fig. 1). The nondimensional parameters $\lambda = L/a$, $\alpha = v/v_p$ (where $v_p = L/T_1$ and $T_1 = 2\pi/\omega_1$) and $\gamma = M/\rho h a b$ are introduced to respectively provide normalizations for spacing, velocity and mass values.

We assume the plate to be initially (i.e., at $t = 0$) at rest, and excited only by the masses moving along a rectilinear path parallel to the x axis. Hence, the position of the k -th mass

evolves in time according to: $X_k(t) = vt - (k - 1)L$ and $Y_k(t) = \beta b$, where $0 \leq \beta \leq 1$.

By considering the effects of the constant (as induced by the gravity acceleration g) and varying terms directly in the modal equations, according to (Nikkhoo *et al.* 2014) we obtain, in the case of a set of moving masses M_k , $k = 1, \dots, N$

$$w(x, y, t) = \sum_{j=1}^n a_j(t) W_j(x, y) \quad (3-1)$$

$$[\omega_j^2 a_j(t) + \ddot{a}_j(t)] \delta_{ij} = -\sum_{k=1}^N M_k \xi(X_k(t), Y_k(t)) \{g + \sum_{j=1}^n [W_j(X_k(t), Y_k(t)) a_j(t)]\} W_i(X_k(t), Y_k(t)) \quad (3-2)$$

where $a_j(t)$ are the mode amplifiers, and function $\xi(x, y)$ reads

$$\xi(x, y) = (H(x) - H(x - a))(H(y) - H(y - b)) \quad (4)$$

$H(\blacksquare)$ being the Heaviside step function, that is $H = 1$ if $\blacksquare \geq 0$ and $H = 0$ otherwise. Due to the moving placement of the masses on the plate, coordinates X_k and Y_k are time-varying and it results that

$$[W_j(X_k(t), Y_k(t)) a_j(t)]'' = A_{kj}(t) \ddot{a}_j(t) + B_{kj}(t) \dot{a}_j(t) + C_{kj}(t) a_j(t) \quad (5)$$

where

$$\begin{aligned} A_{kj}(t) &= W_j(X_k, Y_k) \\ B_{kj}(t) &= 2 \left(W_{j,x}(X_k, Y_k) \dot{X}_k(t) + W_{j,y}(X_k, Y_k) \dot{Y}_k(t) \right) \end{aligned} \quad (6)$$

$$\begin{aligned} C_{kj}(t) &= W_{j,xx}(X_k, Y_k) \dot{X}_k^2(t) + W_{j,yy}(X_k, Y_k) \dot{Y}_k^2(t) + 2W_{j,xy}(X_k, Y_k) \dot{X}_k(t) \dot{Y}_k(t) \\ &\quad + W_{j,x}(X_k, Y_k) \ddot{X}_k(t) + W_{j,y}(X_k, Y_k) \ddot{Y}_k(t) \end{aligned}$$

and $W_{j,x}$, $W_{j,y}$, $W_{j,xx}$, $W_{j,xy}$, $W_{j,yy}$ denote the partial derivatives of the vibration mode W_j with respect to the in-plane spatial coordinates x and y , computed at the current mass position $X_k(t), Y_k(t)$. For ease of notation, the dependence of the k -th mass coordinates on time has been partially dropped in Eq. (6). In matrix form, the governing equations can be written as, see (Nikkhoo *et al.* 2014)

$$\bar{\mathbf{M}}(t) \ddot{\mathbf{T}}(t) + \bar{\mathbf{C}}(t) \dot{\mathbf{T}}(t) + \bar{\mathbf{K}}(t) \mathbf{T}(t) = \bar{\mathbf{F}}(t) \quad (7)$$

where now

$$\begin{aligned} \bar{M}_{ij}(t) &= \delta_{ij} + \sum_{k=1}^N M_k W_i(X_k, Y_k) A_{kj}(X_k, Y_k) \\ \bar{C}_{ij}(t) &= \sum_{k=1}^N M_k W_i(X_k, Y_k) B_{kj}(X_k, Y_k) \\ \bar{K}_{ij}(t) &= \omega_j^2 \delta_{ij} + \sum_{k=1}^N M_k W_i(X_k, Y_k) C_{kj}(X_k, Y_k) \\ \bar{F}_i(t) &= - \sum_{k=1}^N M_k g W_i(X_k, Y_k) \end{aligned} \quad (8)$$

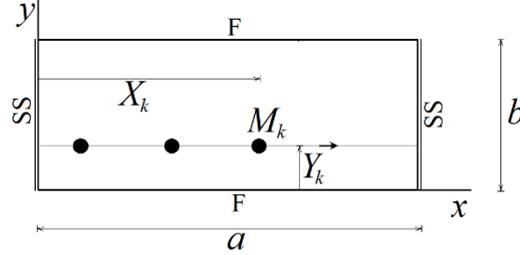


Fig. 1 Plane view of a rectangular plate, simply supported (SS) along two opposite edges and free to move (F) along the other two, carrying a series of moving masses

The entries of the matrices $\bar{\mathbf{M}}(t)$, $\bar{\mathbf{C}}(t)$ and $\bar{\mathbf{K}}(t)$ and of the vector $\bar{\mathbf{F}}(t)$ are all time-varying and of order n . Since they depend on the vibration modes and relevant derivatives, their computation and update over the time interval of the analysis would represent a major effort for the solution of Eq. (7) in the time domain.

The approach here discussed and based on Eq. (7) is a customary one for beams and plates, see e.g. (Ouyang 2011, Nikkhoo *et al.* 2014, Roshandel *et al.* 2014, Bajer and Dyniewicz 2012, Shadnam *et al.* 2001, Lotfollahi-Yaghin *et al.* 2015, Nikkhoo *et al.* 2015).

2.2 Improved modal analysis method

Even if the aforementioned semi-analytical solution, independently of the kind of boundary conditions, is amongst the fastest methods for the problem here attacked, an even faster algorithm is discussed hereafter. Recently, in (Ebrahimzadeh Hassanabadi *et al.* 2015) a so-called indirect form was employed to allow for the contact forces exchanged between each single moving mass and a circular thin plate. The method, which is still based on a series expansion, is enhanced through the use of ad-hoc auxiliary functions $\Theta_k(t)$, all gathered in a vector $\Theta(t)$, which actually account for the interaction forces between the plate and the moving masses. The definition of function $\Theta_k(t)$, relevant to the k -th mass, reads

$$\ddot{\Theta}_k(t) = -M_k \xi(X_k(t), Y_k(t)) \{g + \sum_{j=1}^n [W_j(X_k(t), Y_k(t)) a_j(t)]\} \quad , \quad k = 1, \dots, N \quad (9)$$

By extending the vector of unknowns according to

$$\boldsymbol{\tau}(t) = \begin{bmatrix} \mathbf{T}_{n \times 1}(t) \\ \boldsymbol{\Theta}_{N \times 1}(t) \end{bmatrix} \quad (10)$$

the vibrations of the supported masses-plate system turn out to be governed by the equations

$$\mathbf{m}(t) \ddot{\boldsymbol{\tau}}(t) + \mathbf{c}(t) \dot{\boldsymbol{\tau}}(t) + \mathbf{k}(t) \boldsymbol{\tau}(t) = \mathbf{f} \quad (11)$$

where now

$$\mathbf{m}(t) = \begin{bmatrix} \mathbf{I}_{n \times n} & \begin{bmatrix} -W_i(X_j(t), Y_j(t)) \xi(X_j(t), Y_j(t)) \end{bmatrix}_{n \times N} \\ \begin{bmatrix} M_i \xi(X_i(t), Y_i(t)) A_{ij}(t) \end{bmatrix}_{N \times n} & \mathbf{I}_{N \times N} \end{bmatrix}_{(n+N) \times (n+N)}$$

$$\begin{aligned}
\mathbf{c}(t) &= \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times N} \\ [M_i \xi(X_i(t), Y_i(t)) B_{ij}(t)]_{N \times n} & \mathbf{0}_{N \times N} \end{bmatrix}_{(n+N) \times (n+N)} \\
\mathbf{k}(t) &= \begin{bmatrix} \mathbf{K}_{n \times n} & \mathbf{0}_{n \times N} \\ [M_i \xi(X_i(t), Y_i(t)) C_{ij}(t)]_{N \times n} & \mathbf{0}_{N \times N} \end{bmatrix}_{(n+N) \times (n+N)} \\
\mathbf{f} &= \begin{bmatrix} \mathbf{0}_{n \times 1} \\ [-M_i g]_{N \times 1} \end{bmatrix}_{(n+N) \times 1}
\end{aligned} \tag{12}$$

are matrices and a vector of order $n + N$.

Although the number of unknowns in Eq. (11) gets increased from n to $n + N$, the main advantage of this formulation is that the number of time-varying coefficients is remarkably reduced in comparison with the conventional formulation in Eqs. (7) and (8); this is further discussed in the forthcoming results Section.

If w_{sm} denotes the maximum static deflection of the plate at its center point, which reads

$$w_{sm} = \max \left\{ Mg \sum_{i=1}^n \left[\frac{w_i(a/2, b/2)}{\omega_i^2} \sum_{k=1}^N \xi(X_k, Y_k) w_i(X_k, Y_k) \right] \right\} \tag{13}$$

we compute the normalized dynamic response of the plate as $W_N(x, y, t) = w(x, y, t)/w_{sm}$, and the DAF is given by the maximum of such normalized dynamic response. When the plate is traversed by a series of moving masses, its resonance corresponds to a peak value in the DAF spectrum.

3. A simplified approximate method

The calculation of the maximum in the DAF spectra with the standard method discussed in Section 2.1 may not be favorable, due to its computational costs; the modified method discussed in Section 2.2 provides enhancements in this regard. However, one may be interested in an approximate but fast method to compute the resonance velocity only. A very simple formulation was given in Afghani Khoraskani *et al.* (2016) to estimate such resonance velocity, based on a modification factor to be applied to the corresponding velocity of the sequential moving forces problem. In this Section, we propose a modification factor with accuracy higher than that by Afghani Khoraskani *et al.* (2016); a comparison of the results furnished by the two approaches is provided in Section 4.

If additional masses are attached to the plate, they can alter the free/forced vibration characteristics as the total mass of the system is increased, while the distributed structural stiffness remains the same. If the added masses hold fixed positions, the equation governing the free vibrations would be (as for the notation, see again Fig. 1)

$$D\nabla^4 w(x, y, t) + [\rho h + \sum_{k=1}^N M_k \delta(x - X_k) \delta(y - Y_k)] \ddot{w}(x, y, t) = 0 \tag{14}$$

which can be obtained, e.g., through the Hamilton's principle, and where the last term covers the inertial terms due to the masses spread over the plate. In Eq. (14): $\delta(\blacksquare)$ is the Dirac delta; N , as before, is the total number of masses simultaneously placed over the plate; and M_k is the k -th mass, whose in-plane position is $x = X_k, y = Y_k$.

By imposing $w(x, y, t) = \Psi(x, y) e^{i\omega t}$ in Eq. (14), where i is the imaginary unit, ω the

circular frequency and $\Psi(x, y)$ the natural shape function for the free vibration of the plate with attached masses, the following equation is arrived at

$$D\nabla^4\Psi(x, y) = [\rho h + \sum_{k=1}^N M_k \delta(x - X_k)\delta(y - Y_k)]\varpi^2\Psi(x, y) \quad (15)$$

If we now introduce a series expansion of $\Psi(x, y)$ in terms of the vibration modes $W_j(x, y)$ of the bare plate (accounting for the fact that Eq. (2) contains a self-adjoint differential operator), we can write

$$\Psi(x, y) = \sum_{j=1}^n a_j W_j(x, y) \quad (16)$$

where a_j denotes the modal amplitude as described before. By introducing this series expansion into Eq. (15), multiplying both sides of the resulting equation by $W_i(x, y)$ (where $i = 1, 2, \dots, n$ in the discrete model) and then integrating over the whole plate area A , the spatial dependency of the solution can be removed. To this end, the orthogonality of the (normalized) modes is exploited, providing

$$\iint_A \rho h W_i(x, y) W_j(x, y) dA = \delta_{ij} \quad (17)$$

where δ_{ij} is the Kronecker delta, that is $\delta = 1$ if $i = j$, and $\delta = 0$ otherwise.

The amplifiers a_j finally have to satisfy the following equation in matrix form

$$\mathbf{KT} = \varpi^2 \mathbf{MT} \quad (18)$$

where entries of the matrices and vector, all of order n (see Eq. (16)), respectively read

$$M_{ij} = \delta_{ij} + \sum_{k=1}^N M_k W_i(X_k, Y_k) W_j(X_k, Y_k) \quad (19)$$

$$K_{ij} = \omega_j^2 \delta_{ij}$$

As the non-zero solution is to be sought, the eigenvalue problem (18) should be dealt with the constraint

$$\det(\mathbf{M}^{-1}\mathbf{K} - \varpi_j^2 \mathbf{I}) = 0 \quad (20)$$

wherein \mathbf{I} is the identity matrix of order n .

For a given inertia value, each resonance state corresponds to a velocity of loads at which the dynamic amplification of the system response occurs; the relevant value of the velocity is the already mentioned resonance one v_{res} . Computing the resonance velocity through the moving forces approach, which leads to a value equal to v_p , could be remarkable error-prone as the inertial effects of the moving masses may be dominant.

An approximation to the resonance velocity of the masses can be obtained by accounting for the inertial effect of the masses on the fundamental frequency of the bare structure. Such approximation is given as

$$\alpha_{\text{res}} \approx \Omega / \omega_1 \quad (21)$$

where, according to the normalization provide in Section 2.1, $\alpha_{\text{res}} = v_{\text{res}}/v_p$, and Ω is the average value of $\varpi_1(x)$ over L , i.e.

$$\Omega = \frac{1}{L} \int_0^L \varpi_1(x) dx \quad (22)$$

4. Results

To get insights into the performance of the two approaches proposed in Sections 2.2 and 3, we consider a thin square plate characterized by $\frac{a}{b} = 1$ and $\frac{a}{h} = 33$. Exemplary solutions are reported next for $Y_k(t) = 0.2b$, hence for masses not travelling along an axis of symmetry of the plate; similar results can be achieved for any other value of Y_k . Relevant properties of the plate material are assumed to be: $E = 210$ GPa, $\nu = 0.3$, $\rho = 2400$ kg/m³. The formulations have been implemented in a Wolfram Mathematica code, and the simulations run on a PC featuring an Intel Core i7 CPU, with a 64 bit operating system. In such code, the solution of the equations of motion has been advanced in time through a standard constant acceleration, time integration procedure.

Former analyses via the standard eigenfunction expansion method showed that $n = 25$ provides accurate enough results, in terms of plate dynamics for well-known benchmarks (Nikkhoo *et al.* 2014, Nikkhoo and Rofooei 2012). Hence, the same maximum number of modes has been here adopted.

For the case under investigation, and for $\lambda = 0.9$, $\gamma = 0.3$, $\alpha = 1.0$ describing a specific excitation induced by 4 moving masses, Fig. 2 provides a contour plot of $W_N(x, y)$ at $t = 0.388T_1$. It is shown that convergence is rapidly attained over the whole plate, by adopting a few vibration mode contributions in the analysis. The analysis speedup as a function of n is also reported in Table 1; such speedup is given by $\text{speedup}(\%) = (1 - T_B/T_A) \times 100$, where T_A and T_B are the CPU times required to compute $W_N(x, y)$ with the two methods of Sections 2.1 and 2.2, respectively. Remarkably, thanks to the limited number of time-varying coefficients in the proposed formulation, the speedup increases with n up to a value of about 96% for the very accurate solution based on 25 modes. It thus results that the computational effort for the calculation of the state-space matrices within each time step of the customary formulation, is highly dominant.

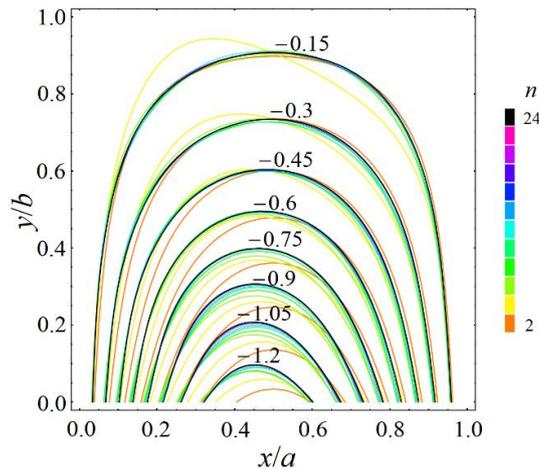


Fig. 2 Thin square plate ($\frac{a}{b} = 1$, $\frac{a}{h} = 33$), 4 moving masses, $\lambda = 0.9$, $\gamma = 0.3$, $\alpha = 1.0$: modal contributions to $W_N(x, y, 0.388T_1)$

Table 1 Thin square plate, 4 moving masses, $\lambda = 0.9$, $\gamma = 0.3$, $\alpha = 1.0$: effect on the speedup of the number n of modes adopted in the analysis

n	1	2	3	4	5	6	7	25
T_A (s)	2.3	5.9	14.1	20.6	35.0	61.7	70.7	977.5
speedup (%)	18	43	66	68	78	82	85	96

To show the effects of the moving masses on plate dynamics, results relevant to the full problem are now compared to those related to multiple moving forces. The corresponding formulation can be easily obtained from what reported in Section 2, by disregarding the inertial terms linked to the moving loads. In this latter case, the state-space matrices get explicitly known and the set of uncoupled equations governing the modal amplitudes can be solved independently. In this case, Eq. (3) yields a closed-form solution. As the vibration frequency of this system corresponds to that of the bare plate, resonance would be induced by a consecutive loading with periodicity equal to $iT_1, i = 1, 2, \dots$. The traveling forces, evenly spaced by L , would produce such periodicity of excitation if moving at a speed v_p/i . This is depicted in Fig. 3, where the peak values in the spectral plot correspond to the resonance states of vibration induced by 30 moving forces spaced so as to have $\lambda = 0.5$.

The output of the moving forces case is compared with the results obtained with the moving masses formulation in the spectral graph of Fig. 4, where curves are reported at varying value of the nondimensional mass γ ; hence, $\gamma = 0$ stands for the moving forces case. It is shown that the inertial effects lead to a remarkable shift of the system resonance towards lower α values, so towards lower mass velocity values. Moreover, the small bumps shown in this graph by the moving masses case only are due to the mode coupling in Eq. (3), caused by the additional inertial terms induced by the moving masses: owing to such coupling, when the second (or higher) vibration mode of the plate experiences resonance, the first one is indirectly excited and parametric resonance shows up see Kumar *et al.* (2015). Looking at Fig. 5, where the time evolution of the dynamic response is reported in terms of W_N at the center of the plate for $v = v_p$ (and where plot (a) corresponds to the moving forces case), it is once again shown that disregarding the (even small) additional inertial contributions to the system due to the loading can lead to considerable errors.

In view of this, a parametric exploration of the DAF plots is adopted to reveal the true system resonance velocity via the moving masses approach; spectral data like those depicted in Fig. 4 are accordingly generated. As already remarked, the fast formulation presented in Section 3 and, specifically, Eq. (21) remarkably reduces the computational costs of such exploration. A comparison of the nondimensional resonance velocity α_{res} obtained with the fast method and that obtained through Eq. (11) is reported in Table 2: it can be observed that, within the investigated intervals of γ and λ , the estimations provided by the two approaches are always very close to each other, with a maximum discrepancy amounting to about 2%.

Afghani Khoraskani *et al.* (2016) proposed a simpler approximate formula to predict α_{res} , according to

$$\alpha_{res} = \sqrt{\frac{1}{1+\kappa\gamma}} \quad (23)$$

wherein κ is the maximum number of objects that can be simultaneously placed on the structure and will cause the maximum static deflection at the center point of the base structure. The estimations obtained with this latter simplified formula are also included in Table 2 for comparison: it can be seen that the estimation of α_{res} provided by Eq. (21) leads to notably smaller errors.

Eventually, as far as the effects of mass placement over the plate are concerned, the variation of the natural frequency of the plate with attached masses as a function of X_1 , namely of the position of the first mass travelling across the plate, is depicted in Fig. 6 for $\lambda = 0.7$ and a varying nondimensional mass γ . It can be observed that the fundamental frequency of the system obviously varies with periodicity L , and that the larger the inertia of the masses the larger the variation of the fundamental frequency.

Table 2 Nondimensional resonance velocity α_{res} at varying values of γ and λ , 30 moving masses: comparison between the estimations provided by Eqs. (21) and (23), and the true values

		γ	0.05	0.10	0.15	0.20	0.25
$\lambda = 0.15$	α_{res} (true resonance)		0.848	0.739	0.661	0.602	0.557
	Eq. (21)	α_{res}	0.850	0.741	0.663	0.605	0.559
		Error (%)	0.2	0.3	0.3	0.5	0.4
	Eq. (23)	α_{res}	0.894	0.816	0.756	0.707	0.667
		Error (%)	5.4	10.4	14.4	17.4	19.7
	$\lambda = 0.25$	α_{res} (true resonance)		0.904	0.824	0.758	0.705
Eq. (21)		α_{res}	0.905	0.825	0.760	0.707	0.663
		Error (%)	0.1	0.1	0.3	0.3	0.3
Eq. (23)		α_{res}	0.933	0.877	0.830	0.791	0.756
		Error (%)	3.2	6.4	9.5	12.2	14.4
$\lambda = 0.50$		α_{res} (true resonance)		0.948	0.899	0.855	0.814
	Eq. (21)	α_{res}	0.951	0.905	0.862	0.824	0.789
		Error (%)	0.3	0.7	0.8	1.2	1.3
	Eq. (23)	α_{res}	0.953	0.913	0.877	0.845	0.816
		Error (%)	0.5	1.6	2.6	3.8	4.7
	$\lambda = 0.70$	α_{res} (true resonance)		0.963	0.925	0.897	0.852
Eq. (21)		α_{res}	0.966	0.930	0.898	0.868	0.840
		Error (%)	0.3	0.5	0.1	1.9	0.5
Eq. (23)		α_{res}	0.976	0.953	0.933	0.913	0.894
		Error (%)	1.3	3.0	4.0	7.2	6.9

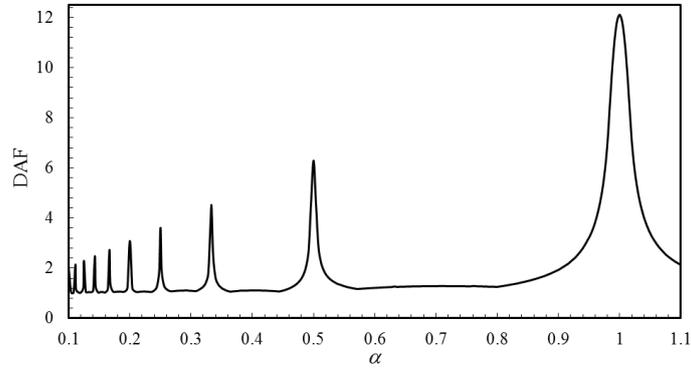


Fig. 3 Thin square plate, $N = 30$, $\lambda = 0.5$: resonance under a series of moving forces

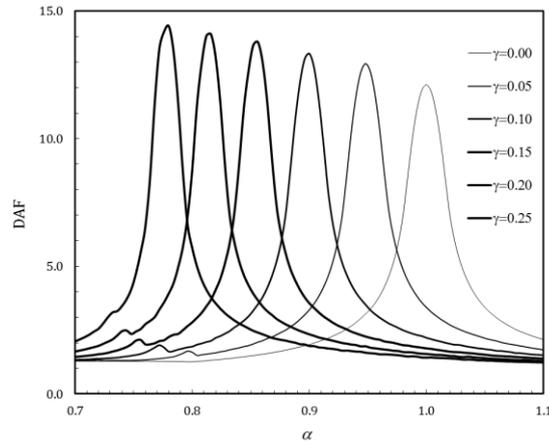
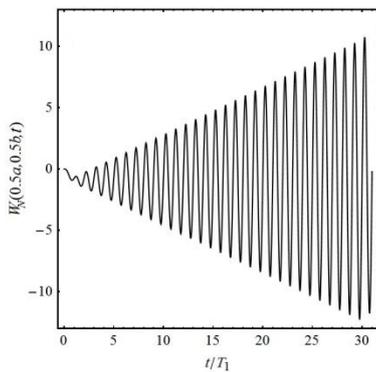
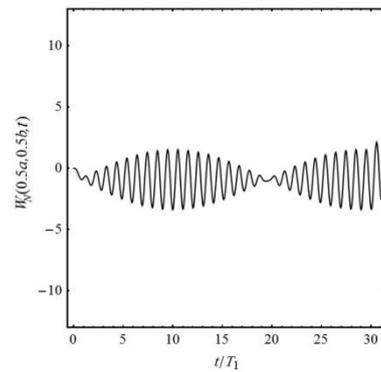


Fig. 4 Thin square plate, 30 moving masses, $\lambda = 0.5$: DAF spectra at varying value of the nondimensional mass γ .



(a) Moving force case $\gamma = 0.00$



(b) Moving mass case $\gamma = 0.05$

Fig. 5 Thin square plate, 30 moving masses, $\lambda = 0.5$, $\alpha = 1.0$: time history of W_N at the center of the plate

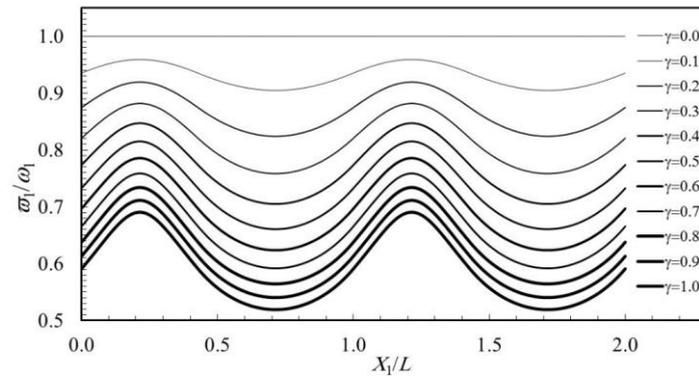


Fig. 6 Thin square plate, $\lambda = 0.7$: variation of the fundamental frequency of the plate with the position X_1 of the first attached mass, at varying value of the nondimensional mass γ

5. Conclusions

Fundamental properties of the vibrations of plates, like e.g. slab-type bridges, under vehicular loads have been studied with improved semi-analytical approaches. Specifically, the resonance velocity of sequential moving masses traversing the plate has been investigated. It has been shown that even small values of the inertia of the moving objects, in comparison to the mass of the bare structure, lead to contributions to system dynamics that cannot be properly accounted for via a moving forces approach.

As a parametric exploration of the dynamic amplification factor of the plate/attached masses system is required to determine the aforementioned resonance velocity, two methods based on an eigenfunction expansion have been proposed, with the goal of speeding up the analysis. In the first one, a technique based on auxiliary functions to account for the inertial interaction between the plate and the masses has been adopted to modify the customary series solution methods, already providing on its own a considerable speedup. In the second one, an approximate evaluation of the resonance velocity based on the fundamental frequency of the structure only has been adopted to avoid the time-consuming parametric survey of the DAF peaks, still preserving high accuracy.

Acknowledgements

S.M. wishes to acknowledge a partial financial support from Fondazione Cariplo through project *Safer Helmets*.

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