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Nonlinear oscillations of a composite microbeam reinforced with carbon nanotube based on the modified couple stress theory

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Abstract. This paper presents nonlinear oscillations of a carbon nanotube reinforced composite beam subjected to lateral harmonic load with damping effect based on the modified couple stress theory. As reinforcing phase, three different types of single walled carbon nanotubes distribution are considered through the thickness in polymeric matrix. The non-linear strain-displacement relationship is considered in the von Kármán nonlinearity. The governing nonlinear dynamic equation is derived with using of Hamilton's principle. The Galerkin's decomposition technique is utilized to discretize the governing nonlinear partial differential equation to nonlinear ordinary differential equation and then is solved by using of multiple time scale method. The frequency response equation and the forced vibration response of the system are obtained. Effects of patterns of reinforcement, volume fraction, excitation force and the length scale parameter on the nonlinear responses of the carbon nanotube reinforced composite beam are investigated.

Keywords: carbon nanotubes; composite beams; modified couple stress theory; nonlinear oscillations

1. Introduction

Carbon nanotubes (CNTs) are a type of reinforcements which have high strength, Young's Modulus, strength-to-weight, high performance and low density. CNTs have used many engineering applications, such as structures, reactor vessels, space vehicles biomedical devices, automotive, electronic devices, civil, machine, marine engineering applications. CNTs are discovered by Sumio Iijima (1991) and using CNTs in engineering applicants has increasing day by day.

Because of its higher strength and flexible properties, Carbon nanotubes experience large displacements and rotations which means nonlinear behavior. Thus, the nonlinear analysis of Carbon nanotubes and its structural behavior are very important for understanding for design and using in the engineering applications. In the open literature, many investigations have been presented about dynamic, stability and static behavior of CNTs in last years. Some studies of them

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dfor CNTs reinforced beams are summarized as: Shen (2009) investigated nonlinear static analysis of functionally graded nanocomposite plates reinforced by SWCNTs under thermal effect. Ke *et al*. (2010) investigated nonlinear vibration functionally graded nanobeams reinforced by CNTs by using Timoshenko beam theory and von Kármán geometric nonlinearity. Yas and Samadi (2012) investigated vibration and stability of CNT nanocomposite beams resting on elastic foundation by using differential quadrature method. Wattanasakulpong and Ungbhakorn (2013) analyzed buckling, static and dynamics of CNT reinforced beams resting on elastic foundation by Navier solution. Rafiee *et al*. (2014) studied non-linear dynamic stability of piezoelectric CNT reinforced composite plates. Fernandes *et al*. (2016) presented nonlinear dynamic responses of microbeams by using finite strain and velocity gradient theories. Shi *et al*. (2017) presented free vibration of functionally graded CNTs beams with different boundary conditions. Tornabene *et al*. (2017) investigated vibration analysis of CNT/polymer/fiber laminated nanocomposite structures by using generalized differential quadrature method. Tagrara *et al*. (2015) solved static, free vibration and buckling analysis of CNTs composite beams embedded in elastic foundation by using high order beam theory. Thang *et al*. (2017) studied nonlinear buckling of functionally graded CNTs plates by analytically. Heidari and Arvin (2019) analyzed nonlinear free vibrations of functionally graded rotating beams reinforced by CNT by using Timoshenko beam theory and Von-Karman nonlinearity. Fernandes *et al*. (2016) presented nonlinear dynamic responses of microbeams by using finite strain and velocity gradient theories. Chu *et al*. (2020) presented a review study about nonlinear absorption properties of carbon nanotubes. Guo and Zhang (2016) investigated nonlinear vibration of a composite plate reinforced by CNTs subjected to combined forceds by using the Galerkin method. Huang *et al*. (2021a, 2021b), Zerrouki *et al*. (2021), Heidari *et al*. (2021), Bendenia *et al*. (2020), Akbaş (2013, 2014, 2016, 2017, 2018a, 2018b, 2018c, 2018d, 2019a, 2019b, 2019c, 2019d, 2019e, 2019f, 2019g, 2020a, 2020b, 2020c, 2020d, 2021a, 2021b, 2021c, 2022), Arshid *et al*. (2021), Rouabhia *et al*. (2020), Asghar *et al*. (2020), Kumar *et al*. (2021), Kocatürk and Akbaş (2010, 2011, 2013), Al-Furjan *et al*. (2020a, 2020b, 2020c, 2020d, 2021a, 2021b), Bourada *et al*. (2020), Kırlangıç and Akbaş (2020, 2021), Bousahla *et al*. (2020), Akbaş and Kocatürk (2012, 2013), Alimirzaei *et al*. (2019, Alimoradzadeh *et al*.(2019, 2020, 2021) and Alimoradzadeh and Akbaş (2021, 2022a, 2022b) investigated stability and vibration analysis of macro and nano scaled composite structure with different mechanical cases. Wu *et al*. (2018) investigated nonlinear free vibration of multi walled CNTs resting on foundation. Babu Arumugam *et al*. (2019) obtained finite element solution of dynamic responses of Carbon nanotubes reinforced composite (CNTRC) beams. Ponnusami *et al*. (2020) examined nonlinear static nad stability of CNTs by using variational asymptotic method. Ton-That (2020) examined nonlinear free vibrations functionally graded CNTs plates by using Four-Node Quadrilateral Element. Van Do (2020) investigated free vibration and dynamic transient responses of CNT reinforced plates by using Bézier extraction based isogeometric analysis method coupled and higher-order shear deformation theory. Ghayesh (2019) investigated nonlinear dynamic responses of functionally graded composite beams with viscoelastic model. Alimoradzadeh and Akbaş (2022c) analyzed sub and super harmonic analysis of CNTRC beams by using of multiple time scale method.

Nonlinear oscillation analysis of composite beams by reinforced carbon nanotubes has not been investigated broadly. Primary objective of this investigation is to analyze nonlinear oscillations of CNTRC under lateral harmonic load with damping effect based on the modified couple stress theory by using Galerkin's decomposition technique with using of multiple time scale method. Effects of patterns of reinforcement, volume fraction, excitation force and the length scale parameter on the frequency-response curves and phase trajectory of the carbon nanotube

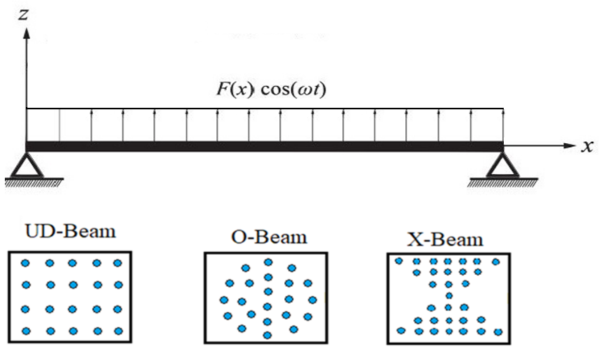

Fig. 1 A simply supported beam made of CNTRC under to lateral distributed harmonic excitation load with three different patterns of CNTs

reinforced composite beam are investigated.

2. Problem formulation

Fig. 1 shows a simply supported beam reinforced CNTs with length L, thickness h and width b, in x, y and z direction is considered as shown in Fig. 1. It is assumed that the simply supported beam is subjected to supersonic air flow. In this study, three different patterns of CNTs reinforcement over the beam are considered as uniform distribution (UD), and functionally distribution O and X as shown in Fig. 1.

It is assumed that, the CNTs are embedded in an isotropic polymer matrix without abrupt interface through whole region of the beam. In order to represents the effective material properties of carbon nanotube-reinforced composite (CNTRC), the rule of mixture model can be used. Based on the rule of mixture model, modulus of Young's modulus E, shear modulus G, Poisson's ratio ϑ and density ρ of the CNTRC beams can be defined as below (Wattanasakulpong and Ungbhakorn 2013, Shen 2009)

$$E_{11} = \eta_1 V_{CNT} E_{11}^{CNT} + V_p E^p \tag{1}$$

$$\frac{\eta_2}{E_{22}} = \frac{V_{CNT}}{E_{22}^{CNT}} + \frac{V_p}{E^p} \tag{2}$$

$$\frac{\eta_3}{G_{12}} = \frac{V_{CNT}}{G_{12}^{CNT}} + \frac{V_p}{G^p} \tag{3}$$

$$V_{CNT} + V_p = 1 \tag{4}$$

$$v = V_{CNT} v^{CNT} + V_p v^p \tag{5}$$

$$\rho = V_{CNT} \rho^{CNT} + V_p \rho^p \tag{6}$$

where superscripts *CNT* and *p* respectively symbolize the related material properties of carbon nanotube and polymer matrix. η_1, η_2, η_3 can be indicated the efficiency parameters of CNT. Also, V_{CNT} and V_p define the volume fractions for CNT and polymer matrix, respectively.Volume fractions of CNTs as a function of thickness direction for different patterns of CNTs

Table 1 For different distributions of CNTs Volume fractions of CNTs dependent thickness direction (Wattanasakulpong and Ungbhakorn 2013)

Patterns of CNTs	V_{CNT}
UD	V_{CNT}^*
FG-O	$2V_{CNT}^*\left(1-2\frac{\lvert z\rvert}{h}\right)$
FG-X	$4V_{CNT}^*\frac{\lvert z\rvert}{h}$

(Wattanasakulpong and Ungbhakorn 2013) are presented in Table 1. In this table, V_{CNT}^* is the given volume fraction of CNTs. In this study, the efficiency parameters of CNTs for three different values of V_{CNT}^* are considered as (Yas and Samadi 2012)

$$\eta_1 = 1.2833, \eta_2 = \eta_3 = 1.055 \text{ for } V_{CNT}^* = 0.12 \tag{7a}$$

$$\eta_1 = 1.3414, \eta_2 = \eta_3 = 1.7101 \text{ for } V_{CNT}^* = 0.17 \tag{7b}$$

$$\eta_1 = 1.3238, \eta_2 = \eta_3 = 1.738 \text{ for } V_{CNT}^* = 0.28 \tag{7c}$$

The normal stress and nonlinear strain-displacement component relationship can be defined by using of Von-Karman strain nonlinearity as follows

$$\sigma_{xx} = \frac{E_{11}(z)}{1-\vartheta^2(z)}\varepsilon_{xx} \tag{8a}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{8b}$$

where u and w represent axial and lateral displacement of the midplane along x and z direction, respectively.

Based on the modified couple stress theory, the strain energy of the beam is given as follows (Yang *et al*. 2002)

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right)dV \quad i, j, k \in [x, y, z] \tag{9}$$

where, ε_{ij} and χ_{ij} denote the components of the strain tensor and the symmetric part of the curvature tensor, respectively. Also in Eq. (1) σ_{ij} and m_{ij} denotes the stress tensor and the deviatoric part of couple stress tensor respectively and can be define as below (Yang *et al*. 2002)

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{10a}$$

$$m_{ij} = 2\mu l^2 x_{ij} \tag{10b}$$

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial\theta_i}{\partial x_j} + \frac{\partial\theta_j}{\partial x_i}\right) \tag{10c}$$

where, l is the material length scale parameter δ_{ij} is the Kronecker delta, and θ is the rotation vector, λ and μ are lame's constants that can be expressed as below

$$\lambda(z) = \frac{E(z)\vartheta(z)}{(1+\vartheta(z))(1-2\vartheta(z))} \tag{11b}$$

$$\mu(z) = G_{12}(z) \tag{11c}$$

$$\theta_x = \theta_z = 0\,,\, \theta_y = -\frac{\partial w}{\partial x} \tag{11d}$$

Using of Eqs. (10)-(11) leads to the non-zero components of the symmetric curvature tensor and the couple stress tensor as follows (Yang *et al*. 2002)

$$\chi_{xy} = \chi_{yx} = -\frac{1}{2}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \tag{12a}$$

$$\mathrm{m_{xy}} = \mathrm{m_{yx}} = -G_{12}\mathrm{l}^2\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2} \tag{12b}$$

Substituting Eqs. (8,10,11,12), into Eq. (9) leads to

$$U_s = \frac{1}{2}\int_0^L \left[\mathrm{A}_{11}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{1}{2}\left(\frac{\partial \mathrm{w}}{\partial \mathrm{x}}\right)^2\right)^2 - 2\mathrm{B}_{11}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}\left(\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{1}{2}\left(\frac{\partial \mathrm{w}}{\partial \mathrm{x}}\right)^2\right) + (\mathrm{D}_{11} + \Gamma)\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}^2}\right)^2\right] dx \tag{13}$$

where

$$\mathrm{A}_{11}, \mathrm{B}_{11}, \mathrm{D}_{11} = \int_{\mathrm{A}} \frac{\mathrm{E}_{11}(\mathrm{z})}{1-\vartheta^2(\mathrm{z})}(1, \mathrm{z}, \mathrm{z}^2)\mathrm{dA} \tag{14a}$$

$$\int_A G_{12}(z) l^2 dA = \Gamma \tag{14b}$$

The CNT reinforced composite beam is subjected to external forces includes lateral harmonic force $\boldsymbol{F_w}$ and damping force $\boldsymbol{F_D}$ due to medium. The virtual work done by external forces and the kinetic energy of the beam can be defined as follows (Ramezani 2012)

$$\mathrm{W}^{EXT} = \int_0^L [(F_D + F_w) w(x,t)] dx \tag{15}$$

where

$$F_D = -C_d \frac{\partial w}{\partial t} \tag{16a}$$

$$F_w = F(x)\cos(\Omega t) \tag{16b}$$

In above equations $F(x)$ and Ω represents transverse external load and the frequency of the excitation force respectively. Also, $\boldsymbol{C_d}$ is the coefficient of the viscous damping due to viscous medium.

The kinetic energy (K) of the beam can be expressed as below

$$K = \frac{1}{2}\int_0^L \left\{ I_0\left[\left(\frac{\partial \mathrm{u}}{\partial \mathrm{t}}\right)^2 + \left(\frac{\partial \mathrm{w}}{\partial \mathrm{t}}\right)^2\right] + I_2\left(\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{t}}\right) - 2I_1\left(\frac{\partial \mathrm{u}}{\partial \mathrm{t}}\frac{\partial^2 \mathrm{w}}{\partial \mathrm{x}\,\partial \mathrm{t}}\right)\right\} dx^2 \tag{17}$$

where

$$I_0, I_1, I_2 = \int_A \rho(z)(1, z, z^2) dA \tag{18}$$

where ρ indicates the mass density. The nonlinear partial differential equation governing the motion can be derived by using of Hamilton's principle which is expressed as below

$$\delta \int_{t_1}^{t_2} [K - U_s + \mathrm{W}^{EXT}] dt = 0 \tag{19}$$

where δ denotes the variational symbol. Substituting Eqs. (13), (15), (16) and (17) into Eq. (19) leads to nonlinear governing equation of the CNT composite beams of the CNT composite beams as follows

$$\frac{\partial}{\partial x}\left[A_{11}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)-B_{11}\frac{\partial^2 w}{\partial x^2}\right]=I_0\frac{\partial^2 u}{\partial t^2}-I_1\frac{\partial^3 w}{\partial x\,\partial t^2} \tag{20}$$

$$I_0\frac{\partial^2 w}{\partial t^2}+\frac{\partial}{\partial x}\left[I_1\frac{\partial^2 u}{\partial t^2}-I_2\frac{\partial^3 w}{\partial x\,\partial t^2}\right]+C_d\frac{\partial w}{\partial t}++\frac{\partial^2}{\partial x^2}\left[(D_{11}+\Gamma)\frac{\partial^2 w}{\partial x^2}-B_{11}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right]-$$
$$\frac{\partial}{\partial x}\left[A_{11}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)-B_{11}\frac{\partial^2 w}{\partial x^2}\right]\frac{\partial w}{\partial x}-\left[A_{11}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)-B_{11}\frac{\partial^2 w}{\partial x^2}\right]\frac{\partial^2 w}{\partial x^2}=F_0\cos(\Omega t) \tag{21}$$

In the case of Euler-Bernoulli beam theory, the axial inertia and the rotational inertia of the beam cross section can be neglected. By ignoring the axial inertia, the rotational inertia and the external force due free oscillation analysis the Eqs. (20) and (21) takes the following form

$$\frac{\partial}{\partial x}\left[A_{11}\left(\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)-B_{11}\frac{\partial^2 w}{\partial x^2}\right]=0 \tag{22}$$

Eq. (22) can be reformulated as below

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left[-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{B_{11}}{A_{11}}\frac{\partial^2 w}{\partial x^2}\right] \tag{23}$$

Integrating Eq. (23) along x-axis yields

$$\frac{\partial u}{\partial x}=-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{B_{11}}{A_{11}}\frac{\partial^2 w}{\partial x^2}-\frac{N_0(t)}{A_{11}} \tag{24}$$

The integration of Eq. (24) leads to

$$u=\int_0^x-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2dx+\frac{B_{11}}{A_{11}}\frac{\partial w}{\partial x}-\frac{N_0 x}{A_{11}}+N_1(t) \tag{25}$$

It is assumed that the beam has immovable support. Hence, the following boundary condition can be considered

$$u(0,t)=u(L,t)=0 \tag{26}$$

Substituting Eq. (26) into Eq. (25) yields

$$N_0=-\frac{A_{11}}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2dx+\frac{B_{11}}{L}\left[\frac{\partial w(L,t)}{\partial x}-\frac{\partial w(0,t)}{\partial x}\right] \tag{27}$$

$$N_1(t)=-\frac{B_{11}}{A_{11}}\frac{\partial w(0,t)}{\partial x} \tag{28}$$

Finally, by substituting Eqs. (22) and (24) into Eq. (21), one can obtain the following nonlinear partial differential equation governing the forced vibration of the CNT composite beam

$$I_0\frac{\partial^2 w}{\partial t^2}+C_d\frac{\partial w}{\partial t}+\left[\left(D_{11}-\frac{{B_{11}}^2}{A_{11}}\right)+\Gamma\right]\frac{\partial^4 w}{\partial x^4}+N_0\frac{\partial^2 w}{\partial x^2}=F(x)\cos(\Omega t) \tag{29}$$

where $N_0(t)$ is expressed in Eq. (27). In order to derive the governing ordinary differential equation of motion from the partial one mentioned in Eq. (29), the Galerkin's method, is utilized. Based on the Galerkin's method, the solution of the governing equation can be defined as below

$$(x,t)=\sum_{n=1}^{\infty}\psi_n(x)\,.\,q_n(t) \tag{30}$$

where $\psi_n(x)$and $q_n(t)$ are the n-th mode shape functions (admissible function) and n-th is the modal coefficient respectively. Since the dominant mode in the beam is the first mode, the solution of eq. (29) can be express as follows

$$w(x,t)=\psi(x).\,q(t) \tag{31}$$

For the simply supported beam boundary conditions without axial movement at both ends, the mode shape function can be express as follows (Şimşek 2014)

$$\psi(x) = \sin\left(\frac{\pi x}{L}\right) \tag{32}$$

where satisfies the following kinematic boundary conditions for simply supported beam (Ansari *et al*. (2010)

$$w(0,t) = \frac{\partial^2 w(0,t)}{\partial x^2} = 0 \tag{33}$$

$$w(L,t) = \frac{\partial^2 w(L,t)}{\partial x^2} = 0 \tag{34}$$

Substituting Eq. (32) in to Eq. (29) leads to

$$\ddot{q}[I_0\psi] + C_d\psi\dot{q} + a_1 q + a_2 q^2 + a_3 q^3 = F_0\cos(\Omega t) \tag{35}$$

where, $\dot{q}$ and $\ddot{q}$ is the first and the second derivative of ***q*(*t*)** with respect to time, respectively. Also, the coefficientsa$_0$, a_1 , a_2 and a_3 are

$$a_1 = \psi_{xxxx}\left[\left(D_{11} - \frac{B_{11}{}^2}{A_{11}}\right) + \Gamma\right],$$

$$a_2 = \frac{B_{11}}{L}\psi_{xx}[\psi_x(L) - \psi_x(0)],\quad a_3 = \psi_{xx}\left[-\frac{A_{11}}{2L}\int_0^L \psi_x{}^2 dx\right] \tag{36}$$

where ψ_x, $\psi_{xx}$and ψ_{xxxx} are the first, the second, and the fourth derivative of $\psi(x)$ with respect to x , respectively.

Considering the transverse external load as $F(x) = f_0.\psi(x)$,multiplying both side of Eq.(35) with a mode shape function $\psi(x)$and integrating the result equation over domain (0,L) leads to the nonlinear ordinary differential equation governing the motion of the microscale CNTR composite beam as follows

$$\ddot{q} + 2\hat{\mu}\dot{q} + \omega_0^2 q + \hat{\eta}_2 q^2 + \hat{\eta}_3 q^3 = f\cos(\Omega t) \tag{37}$$

where

$$\mu = \frac{1}{2}\left(\frac{\int_0^L C_d\psi^2 dx}{\int_0^L I_0\psi^2(x)dx}\right), \tag{38a}$$

$$\omega_0^2 = \frac{\int_0^L a_1\psi(x)dx}{\int_0^L I_0\psi^2(x)dx}, \tag{38b}$$

$$\hat{\eta}_2 = \frac{\int_0^L a_2\psi(x)dx}{\int_0^L I_0\psi^2(x)dx}, \tag{38c}$$

$$\hat{\eta}_3 = \frac{\int_0^L a_3\psi(x)dx}{\int_0^L I_0\psi^2(x)dx} \tag{38d}$$

$$f = \frac{\int_0^L f_0.\psi^2(x)dx}{\int_0^L I_0\psi^2(x)dx} \tag{38e}$$

where, f_0 is the amplitude of the lateral external load. The nonlinear ordinary differential equation is solved by using the method of multiple scales. In order to obtain solution assumption, following assumption is used.

$$\hat{\mu} = \epsilon\mu \tag{39a}$$

$$\hat{\eta}_2 = \epsilon\eta_2 \tag{39b}$$

$$\hat{\eta}_3 = \epsilon\eta_3 \tag{39c}$$

$$\hat{f} = \epsilon^2 f \tag{39d}$$

ϵ indicates bookkeeping parameter. Inserting Eq. (39) to Eq. (37) yields

$$\ddot{q} + 2\epsilon^2\mu\dot{q} + \omega_0^2 q + \epsilon\eta_2 q^2 + \epsilon^2\eta_3 q^3 = \epsilon^2 f\cos(\Omega t) \tag{40}$$

In method of multiple scales, time variable is defined as following

$$T_n = \epsilon^n t\,, n = 0, 1,2,3, .. \tag{41}$$

With processing chain rule for Eq. (39), the following form is obtained

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 + \cdots \tag{42a}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0 D_2) + 2\epsilon^3(D_1 D_2) + \cdots \tag{42b}$$

where

$$D_i = \frac{\partial}{\partial T_i}, \quad i = 0,1,2,3 \tag{43}$$

In the solution of Eq. (40), the method of multiple scales obtained as follows (Nayfeh *et al.* 1980, Shafiei and Setoodeh 2017)

$$q = q_0(T_0, T_1, T_2) + \epsilon q_1(T_0, T_1, T_2) + \epsilon^2 q_2(T_0, T_1, T_2) + \ldots \tag{44}$$

Substituting Eq. (44) into Eq. (40) together with using Eqs. (42) and then equating coefficient of similar power of $\boldsymbol{\epsilon}$ to zero yields

$$(D_0^2 + \omega_0^2)q_0 = 0 \tag{45}$$

$$(D_0^2 + \omega_0^2)q_1 = -2D_0 D_1 q_0 - \eta_2 q_0^2 \tag{46}$$

$$(D_0^2 + \omega_0^2)q_2 = -2D_0 D_1 q_1 - (D_1^2 + 2D_0 D_2)q_0 - 2\mu D_0 q_0 - 2\eta_2 q_0 q_1 - \eta_3 q_0^3 + f\cos(\Omega t) \tag{47}$$

Solution of Eq. (45) is obtained as follows

$$q_0 = A(T_1, T_2)\exp(i\omega_0 T_0) + \Lambda(T_1, T_2)\exp(-i\omega_0 T_0) \tag{48}$$

where $A(T_1, T_2)$ is an unknown complex function and will be determined by eliminating the secular terms from q_1, CC denotes the complex conjugated of the previous terms and Λ is the complex conjugate of A. Substituting Eq. (48) into Eq. (46) yields

$$(D_0^2 + \omega_0^2)q_1 = -2i\omega_0 D_1 A\, e^{i\omega_0 T_0} - \eta_2\left[A^2 e^{2i\omega_0 T_0} + A\overline{A}\right] + cc \tag{49}$$

where *cc* stands for the complex conjugate of the preceding terms. To eliminate the secular terms from q_1 equating the coefficients of $\exp(\mp i\omega_0 T_0)$ to zero as follows

$$D_1 A(T_1, T_2) = 0 \tag{50}$$

Therefore, *A* only is a function of T_2. With considering the Eq. (50), the particular solution of Eq. (49) can be define as below

$$q_1 = \frac{\eta_2}{3\omega_0^2}\left[A^2 e^{2i\omega_0 T_0} + \bar{A}^2 e^{-2i\omega_0 T_0} - 6A\overline{A}\right] \tag{51}$$

In primary resonance, it is assumed that the excitation frequency Ω is near to linear frequency ω_0 of the system ($\Omega \approx \omega_0$) as below

$$\Omega = \omega_0 + \epsilon^2\sigma \tag{52}$$

where σ is the detuning parameter and used to illustrate the nearness of Ω to ω_0. Substituting Eqs. (48), (51) and (52) into Eq. (47) and recalling that $D_1A = 0$, yields

$$(D_0^2 + \omega_0^2)q_2 = -A^3\left[\frac{2\eta_2^2}{3\omega_0^2} + \eta_3\right]e^{3i\omega_0 T_0} + \Big[-2i\omega_0(D_2A + \mu A) + A^2\bar{A}\left(\frac{10\eta_2^2}{3\omega_0^2} - 3\eta_3\right) + \frac{f}{2}e^{i\sigma T_2}\Big]e^{i\omega_0 T_0} + CC \tag{53}$$

To eliminate the secular terms from q_2 equating the coefficients of $\exp(\mp i\omega_0 T_0)$ in Eq. (53) to zero as follows

$$-2i\omega_0(D_2A + \mu A) + A^2\bar{A}\left(\frac{10\eta_2^2}{3\omega_0^2} - 3\eta_3\right) + \frac{f}{2}e^{i\sigma T_2} = 0 \tag{54}$$

Considering $A(T_2)$ in the polar form as follows (Nayfeh *et al.* 1980)

$$A(T_2) = \frac{1}{2}a\,exp(i\beta) \tag{55}$$

where, $a(T_2)$ and $\beta(T_2)$ indicate real functions of T_1. Inserting Eq. (55) into Eq. (54) and separating the results in to its real and imaginary parts leads to

$$\acute{a} + \mu a = \frac{f}{2\omega_0}\sin(\bar{\theta}) \tag{56a}$$

$$a(\sigma - \bar{\theta}^{/}) - \frac{9\omega_0^2\eta_3 - 10\eta_2^2}{24\omega_0^3}a^3 = -\frac{f}{2\omega_0}\cos(\bar{\theta}) \tag{56b}$$

where, (´) is the first derivative with respect to T_2. Also, $\bar{\boldsymbol{\theta}}$ is defined as below

$$\bar{\theta} = \sigma T_2 - \beta \tag{57}$$

In the case of steady state motion of the system the amplitude $\boldsymbol{a}$ and the phase of the system $\boldsymbol{\theta}$ are not charge at a singular point (Nayfeh *et al.* 1980)

$$\acute{a} = \bar{\theta}^{/} = 0 \tag{58}$$

Substituting Eq. (58) into Eqs. (52a) and (52b) leads to

$$\mu = \frac{f}{2a\omega_0}\sin(\bar{\theta}) \tag{59a}$$

$$\sigma + \frac{10\eta_2^2 - 9\omega_0^2\eta_3}{24\omega_0^3}a^2 = -\frac{f}{2a\omega_0}\cos(\bar{\theta}) \tag{59b}$$

Squaring and adding Eqs. (59a) and (59b) leads to the frequency response equation as follows

$$\sigma = \frac{9\omega_0^2\eta_3 - 10\eta_2^2}{24\omega_0^3}a^2 \pm \sqrt{\frac{f^2}{4\omega_0^2a^2} - \mu^2} \tag{60}$$

Substituting Eq. (60) into Eq. (52) yields

$$\Omega = \omega_0 + \epsilon^2\left[\frac{9\omega_0^2\eta_3 - 10\eta_2^2}{24\omega_0^3}a^2 \pm \sqrt{\frac{f^2}{4\omega_0^2a^2} - \mu^2}\right] \tag{61}$$

Substituting Eqs. (48), (51) and (55) into Eq. (44) leads to

$$q = a\cos\hat{\theta}(t) - \epsilon a^2 \frac{\eta_2}{2\omega_0^2}\left[1 - \frac{1}{3}\cos 2\hat{\theta}(t)\right] + O(\epsilon^2) \tag{62}$$

where

$$\hat{\theta} = \omega_0 t + \beta \tag{63}$$

Using of Eqs. (52), (57), (62) and (63) leads to the approximate solution as below

$$q = a\cos(\Omega t - \theta) - \epsilon a^2 \frac{\eta_2}{2\omega_0^2}\left[1 - \frac{1}{3}\cos(2\Omega t - 2\theta))\right] + O(\epsilon^2) \tag{64}$$

3. Numerical results

In numerical results, the material and geometry parameters are used as follows; (Wattanasakulpong and Ungbhakorn 2013, Yas and Samadi 2012): $E_{11}^{CNT} = 600$ GPa, $E_{22}^{CNT} = 10$ GPa, $G_{12}^{CNT} = 17.2$ GPa, $\upsilon^{CNT} = 0.19$, $\rho^{CNT} = 1400$ kg/m^3, $E^p = 2.5$ GPa, $\upsilon^p = 0.30$, and $\rho^p = 1190$ kg/m^3, L=300 μm; h=2 μm, b=h, length scale parameter$(l) = 0.5$ μm, amplitude of the load $f_0 = 0.012$ N maximum amplitude (Λ)= 1.0 μm, damping coefficient $C_d = 0.005$ Pa. s

In order to accuracy of present method, a comparison study is performed. For this purpose, the fundamental frequencies of a microscale simply supported beam made of pure polymer are calculated with different slenderness ratio and compared with those of Kong *et al.* (2008) corresponding to the Euler-Bernoulli beam theory in Table 2. In the obtaining of the vibration frequency from this study, the eigenvalue process is implemented in Eq. (38b). It is found from Table 2, the current results are in good harmony with the related results of Kong *et al.* (2008). It is worth noting that, Kong *et al.* (2008) neglected the contribution of the Poisson's which leads to the small difference between the results presented in Table 2.

Moreover, the results of Table 3 presents the linear oscillations at $l = 0$ and $V_{CNT}^* = 0.17$. To validate the obtained results from the current work, this table compared the linear natural frequency of this research with the results presented by Shafiei and Setoodeh (2017). As can be seen in Table 3, the current results are in good harmony with the related results of Shafiei and Setoodeh (2017).

Figs. 2 and 3 presents the frequency response curves and the phase trajectory of the system for the classical theory and the MCST for X beam, $V_{CNT}^* = 012$, $f_0 = 0.013$ N, respectively. As can be

Table 2 Comparative results for fundamental frequencies of a simply supported fully polymer microscale beam. $l = 0.5$ μm

$\frac{L}{h}$	Natural Frequency (MHz)	
	Kong *et al* (2008)	Present
30.0	2.604	2.702
50.0	0.938	0.973
60.0	0.651	0.676
70.0	0.478	0.496
90.0	0.289	0.300
100.0	0.234	0.243

Table 3 Comparative results for fundamental frequencies (MHz) of a simply supported CNTR Composite beam for different patterns of CNTs for $l = 0$

	UD-Beam	
L/h	Shafiei and Setoodeh (2017)	Present
30.0	17.56	17.56
50.0	6.32	6.32
60.0	4.39	4.39
70.0	3.23	3.23
90.0	1.95	1.95
100.0	1.58	1.58
	X-Beam	
L/h	Shafiei and Setoodeh (2017)	Present
30.0	19.58	19.58
50.0	7.05	7.05
60.0	4.90	4.90
70.0	3.60	3.60
90.0	2.18	2.18
100.0	1.76	1.76
	O-Beam	
L/h	Shafiei and Setoodeh (2017)	Present
30.0	15.34	15.34
50.0	5.52	5.52
60.0	3.84	3.84
70.0	2.82	2.82
90.0	1.70	1.70
100.0	1.38	1.38

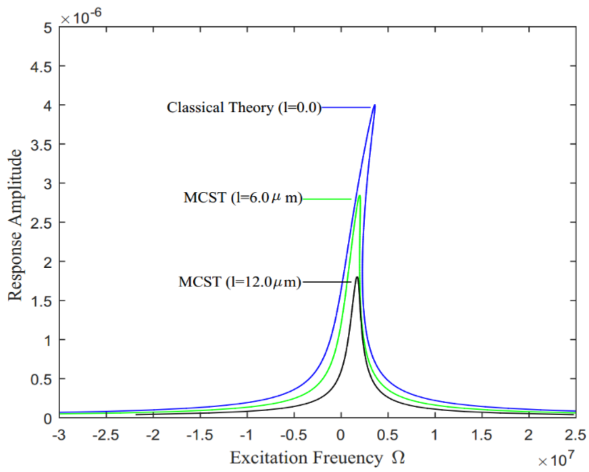

Fig. 2 The effects of the Length scale parameter on the frequency response curves of the X Beam for $V_{CNT}^{*} = 012$, $f_0 = 0.013$ N

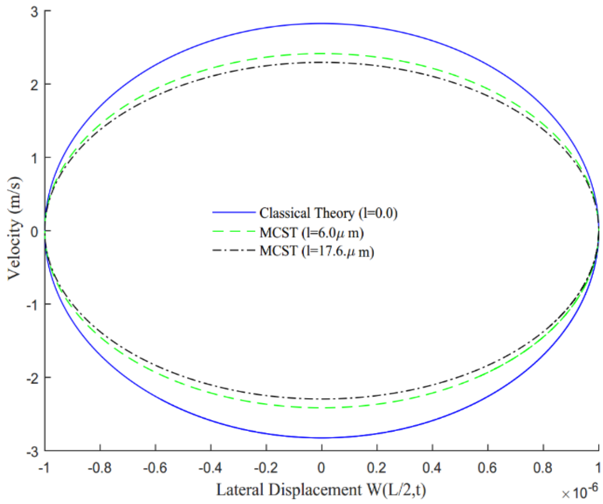

Fig. 3 The effects of the Length scale parameter on the phase trajectory of the X Beam for $V_{CNT}^{*} = 012$, $f_0 = 0.013$ N

seen in Figs. 2 and 3, the frequency response curves consist of two branch. The upper branch represents the stable solution and the lower branch represents stable and unstable solutions. The deviation of the curves illustrates the type of nonlinearity. As can be observed form these figures, the curve is deviated to the right which exhibits the hardening type behavior of the system. The MCST predicts weaker hardening behavior and lower resonance frequency. Moreover, whole response region become narrow and the height of the jump phenomena (sudden change in the amplitude and phase of the response with small change in the excitation frequency due to nonlinear nature of the system) decreases with increase in the material length scale parameter. The nonlinear vibration is stable with finite limit cycle and the classical theory predicts higher vibration velocity compare the MCST (see Fig. 3).

The frequency-response curves and the phase trajectory of the system for different pattern of reinforcement are presented in Figs. 4 and 5, respectively. The Results indicate that, as the pattern of reinforcement changes as the order X Beam, UD and O distributions, whole response region become wider and the frequency response curves bend to the left which means that the hardening behavior of the system become weaker. With changing the pattern of reinforcement as the order X Beam, UD and O distributions, the peak amplitude and the nonlinear resonant frequency and the height of the jump increase. In addition, the results demonstrated that, with changing the pattern of reinforcement as the order X Beam, UD and O distributions, the phase trajectory expand outward and the velocity of the nonlinear oscillation increase.

Figs. 6 and 7 show the frequency response curves and the phase trajectory of the CNTR composite microscale beam with X pattern of reinforcement for different values of V_{CNT}^{*}, respectively. As can be seen, the frequency response curves bends to the right which illustrates increase in the hardening behavior of the system increase in V_{CNT}^{*}. The results illustrate that, whole response region become narrow and at the same time the peak amplitude of the nonlinear oscillation and the nonlinear resonance frequency decreases increase in V_{CNT}^{*}. Moreover, with increasing V_{CNT}^{*}, the height of the jump decreases, the phase trajectory shrink inward and velocity of the nonlinear oscillation decreases.

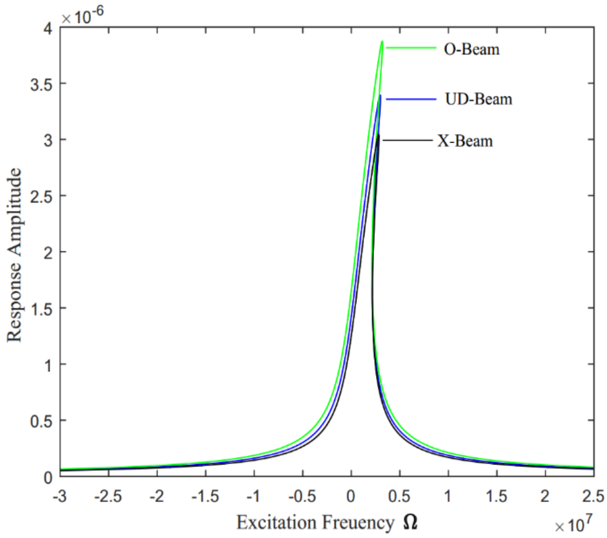

Fig. 4 The effects of the pattern of reinforcement on the frequency response curves for $V^*_{CNT} = 017$

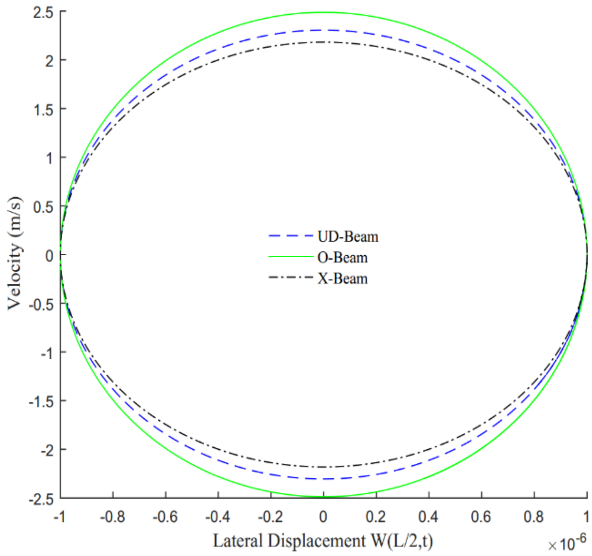

Fig. 5 The effects of the pattern of reinforcement on the phase trajectory for $V^*_{CNT} = 017$

Figs. 8 and 9 demonstrates the frequency response curves and the phase trajectory of the microscale CNTR composite beam for X distribution for different values of forcing amplitude, respectively. The presented results from Figs. 8 and 9 indicate that with the whole response region become wider and the peak amplitude of the nonlinear oscillation and the nonlinear resonance frequency increases increasing the amplitude of the force. Also, as increase in the amplitude of the load increase the height of the jump, the phase trajectory expands out ward and the velocity of the nonlinear oscillation increases while the system remains in stable situation (finite limit cycle).

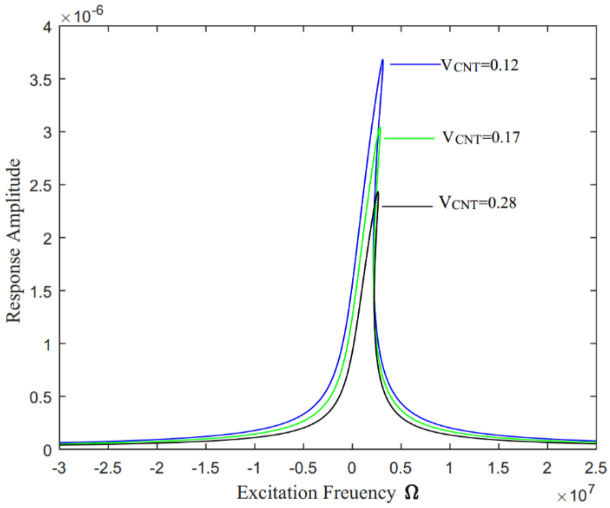

Fig. 6 The effects of V_{CNT}^{*} on the frequency response curves for X beam.

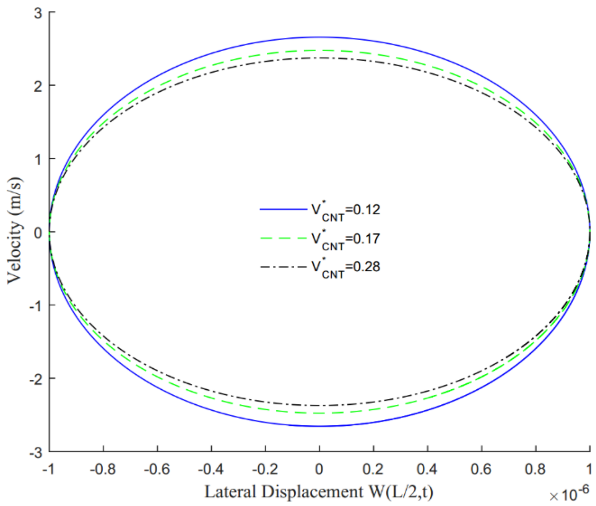

Fig. 7 The effects of V_{CNT}^{*} on the phase trajectory for X beam.

however, the hardening behavior of the system remain steady.

4. Conclusions

In this paper, nonlinear oscillation of a CNTRC microscale beam subjected to lateral harmonic load and damping force due to viscous medium are investigated based on modified couple stress theory the Euler-Bernoulli beam theory, von Kármán type of geometrical nonlinearity. The Galerkin's decomposition technique is utilized to discretize the governing nonlinear partial differential equation to nonlinear ordinary differential equation and then is solved by using of

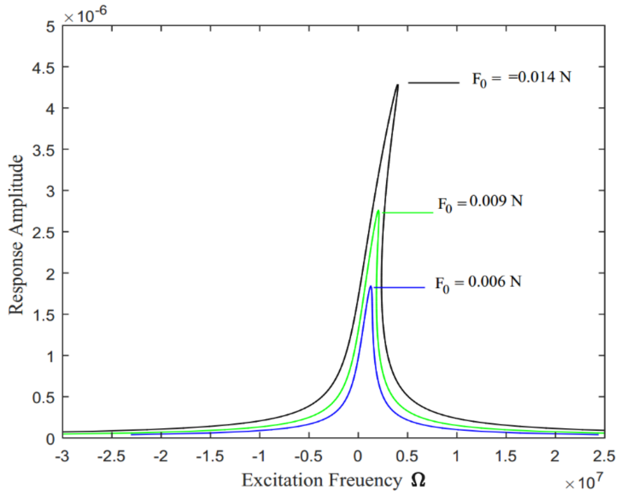

Fig. 8 The effects of the amplitude of the frequency response curves for X beam and $V_{CNT}^{*} = 012$

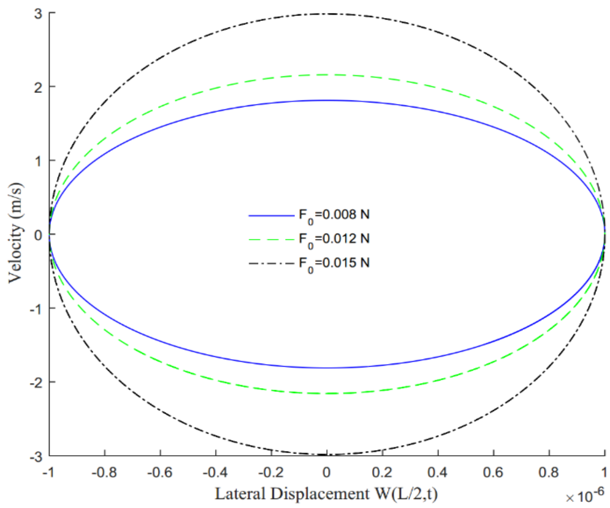

Fig. 9 The effects of the amplitude of the Load on the phase trajectory for X beam and $V_{CNT}^{*} = 012$

multiple time scale method. In numerical studies, effects of patterns of reinforcement, volume fraction, excitation force and the length scale parameter on the nonlinear responses of the carbon nanotube reinforced composite beam are investigated. In the obtained results, the important consequences are presented as;

- The MCST predicts weaker hardening behavior and lower resonance frequency. The classical theory predicts higher vibration velocity compare the MCST.
- The distribution of CNTs play an important role on nonlinear dynamics of CNTRC beam. With changing the pattern of reinforcement in order; X, UD and O, the hardening behavior of the system become weaker.
- Increase in volume fraction of CNTs, the hardening behavior of the system increase.

Nonlinear vibration responses of the CNTRC beam change considerably with volume fraction of CNTs.

• With the increasing in the material length scale parameter, the hardening behavior of the system decreases and the nonlinear vibration responses of the CNTRC microscale beam change considerably.

• With the increasing in the amplitude of the excitation force, hardening behavior of the system remains steady, the whole response region become wider and the peak amplitude of the nonlinear oscillation and the nonlinear resonance frequency increases.

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