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Investigation of dynamic response of "bridge girder-telpher-load" crane system due to telpher motion

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Abstract. The moving load causes the occurrence of vibrations in civil engineering structures such as bridges, railway lines, bridge cranes and others. A novel engineering method for separation of the variables in the differential equation of the elastic line of Bernoulli-Euler beam has been developed. The method can be utilized in engineering structures, leading to "a beam under moving load model" with generalized boundary conditions. This method has been implemented for analytical study of the dynamic response of the metal structure of a single girder bridge crane due to the telpher movement along the bridge girder. The modeled system includes: a crane bridge girder; a telpher, moving with a constant horizontal velocity; a load, elastically fixed to the telpher. The forced vibrations with their own frequencies and with a forced frequency, due to the telpher movement, have been analyzed. The loading resulting from the telpher uniform movement along the bridge girder is cyclical, which is a prerequisite for nucleation and propagation of fatigue cracks. The concept of "dynamic coefficient" has been introduced, which is defined as a ratio of the dynamic deflection of the bridge girder due to forced vibrations, to the static one. This ratio has been compared with the known from the literature empirical dynamic coefficient, which is due to the telpher track unevenness. The introduced dynamic coefficient shows larger values and has to be taken into account for engineering calculations of the bridge crane metal structure. In order to verify the degree of approximation, the obtained results have been compared with FEM outcomes. An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving force. The comparisons show a good agreement.

Keywords: engineering structures; dynamic response; forced vibrations; bridge crane

1. Introduction

The presence of movable load causes the occurrence of vibrations in civil engineering structures such as bridges, railway lines, bridge cranes and others. The computational scheme of these objects most commonly leads to a beam model under moving load. The occurrence of this engineering problem is connected with the construction and exploitation of railroad installations. Three studies mark the beginning of a solution to this problem. The first mathematical model of the elastic curve of Bernoulli-Euler beam, subjected to a load, moving with a constant horizontal

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velocity, is obtained by Willis (1849) for the case of a freely supported beam. The beam mass has been neglected and the system "beam-moving load" is modeled to such with one degree of freedom, having a constant mass and changeable elasticity. The opposite model of behaviour of the elastic curve of a freely supported beam under the influence of a moving load with a constant velocity is when its mass is neglected, respectively only the distributed beam mass is taken into account. An analytical solution of this problem is obtained by Krilov (1905) for the case of a constant force. An analytical solution for the case of a harmonic exciting force is proposed by Timoshenko (1922). Afterwards, three fundamental works devoted to this problem have been published: by Inglis (1934), Hillerborg (1951) (who proposed an analytical solution through Fourier's method of simple supported beams) and Fryba (1999).

During the last few decades many solutions have been made for beam models, subjected to a moving load, in different engineering applications: beam on elastic foundation-linear-elastic foundation (Kien and Ha 2011, Chonan 1978, Amiri and Onyango 2010, Awodola 2007, Dimitrovova 2016), viscoelastic foundation (Zehsaz *et al.* 2009, Sun and Luo 2008, Karami-Khorramabadi and Nezamabadi 2012, Luo *et al.* 2016), nonlinear elastic foundation (Ding *et al.* 2012, Hryniewicz 2011), elastic foundation, modeled through springs with different stiffness (Thambiratnam and Zhuge 1996); bilinear elastic foundation (Casrto Jorge *et al.* 2015); freely supported beam (Yang *et al.* 1997, Nikkhoo and Amankhani 2012, Michaltos 2002, Michaltos *et al.* 1996); inclined beam (Wu 2005); complex beam (Yau 2004); continuous beam (Prager and Save 1963, Zheng *et al.* 1998, Kerr 1972); beam on elastic supports (Mehri *et al.* 2009, Piccardo and Tubino 2012); beam with generalized boundary conditions (Hilal and Mohsen 2000); contilever beam (Lin and Chang 2006); curved beam traversed by off-center moving loads (Rostam *et al.* 2015). In view of the model of stressed and strained state, Bernoulli-Euler beams prevail (Dimitrovova 2016, Hilal and Mohsen 2000, Xia *et al.* 2006, Javanmard *et al.* 2013, Liu *et al.* 2013) over beams of Timoshenko (Luo *et al.* 2016, Azam *et al.* 2013, Chonan 1975). The analytical approach for solution prevails over the finite element one (Lin and Trethewey 1990, 1993). Some of the models refer to previously stressed beams through axial compressive load (Omorofe 2013, Zibdeh and Rackwitz 1995).

The models in which the mass of the moving load is not taken into account are prevailing. The effect of inertia from the passing load is numerically obtained in (Yang *et al.* 1997) through Newmark-β method, and is afterwards included in the force function of the load. In (Michaltos 2002) the mass and the moment of inertia of the moving load are taken into account, as well as the effect of inertia from the rotation of the beam's sections. However, the effect of inertia from the load mass directly on the beam transverse vibrations is not taken into account, but only through the weight force and the force of inertia from the horizontal acceleration of the load.

In the majority of publications, the case of the load's constant velocity is examined. A study of the influence of the acceleration has been made by Michaltos (2002) and Hilal and Mohsen (2000). The models with constant magnitude of the moving load are prevailing. In (Hilal and Mohsen, 2000) a model with moving force, altering by magnitude by a harmonic law has been made. In (Awodola 2007) the moving force changes exponentially and a numerical approach is applied-the finite difference method. In some of the models a dynamic absorber moving on the beam's axis has been implemented (Samani and Pellicano 2009, Soares *et al.* 2010). Models of more complicated objects have been made-a bridge system, examined in a resonance state (Xia *et al.* 2006) and a vehicle model (Esmailzadeh and Jalili 2003).

It is noteworthy that the models of two-supported beams mainly refer to a freely supported beam, which has a logical explanation: the frequency equation is most simple for this case and

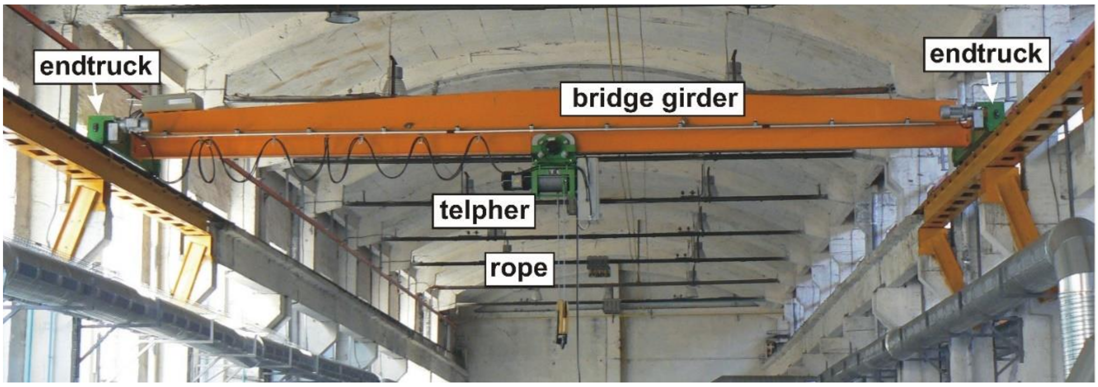

Fig. 1 Single girder bridge crane

therefore, the general solution in infinite sums can be obtained (Timoshenko 1972).

In the engineering practice the technical solutions for fixture of the ends of a beam, subjected to bending, lead to a model with elastic angular supports. A such model of Bernoulli-Euler beam under moving load was developed by Maximov (2014). The elastic angle supports restrict the rotations of the end cross-sections for beam bending, depending on the stiffness of the supports. For instance, in many constructional solutions the bridge girder (principal beam) of a bridge crane is connected in both its ends for the vertical internal faces of the endtrucks (runway beams) through plates and coupling flanges with fitted bolt connections (Fig. 1). The elasticity of the angular supports in a vertical plane is a function of the endtrucks torsion stiffness. In terms only of bending, the bridge girder is double statically indeterminated: hyperstatic quantities are the elastic moments in the two additional angular supports.

The moving on the bridge girder telpher with the elastically suspended load is the moving load. This system (bridge girder-telpher-load) departs from the scope of the known modeled tasks, as it assumes two generalized coordinates: deflection of the elastic line of the bridge girder (depending on the time and the abscissa defining a particular cross-section) and the elastic elongation of the "telpher-load" system. In the known methods for calculating the bridge girder of the bridge crane, the dynamic effect of movement on the bridge girder of the "hoist-load" system has been taken into account with a coefficient of dynamism, which is an empirical function of the telpher nominal velocity (Kolarov *et al.* 1986). This dynamic coefficient takes into account all unevenness (which, of course, have a stochastic nature) of the telpher route, which in turn are the reason for dynamic loading. In fact, the mobile "telpher-elastic suspended load" system causes forced vibrations of the bridge girder, which are superimposed on the free vibrations resulting from the random effects of the route unevenness. While the second type of vibrations rapidly subsides due to material hysteresis mostly, the forced vibrations exist during the whole telpher movement.

The main objective of this study is to evaluate those forced vibrations of the "bridge girder-telpher-load" system due to telpher movement along the bridge girder, respectively, to assess the dynamic effect on the bridge girder.

In this study, the bridge girder is modeled as Bernoulli-Euler beam with angular elastic supports and with constant cross-section, respectively, with constant mass per unit length. The "telpher-load" system is moved with constant horizontal velocity along the bridge girder. The connection between the telpher and the load is linear elastic. All masses (bridge girder, telpher, load) are taken into account.

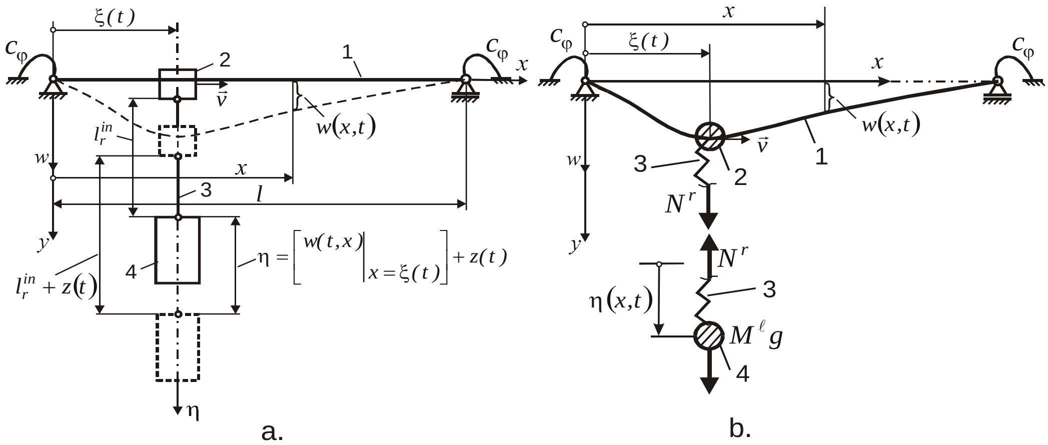

Fig. 2 Mechanical model of the “bridge girder-telpher-load” system: (a) general view; (b) after “release from connections”; 1-bridge girder, 2-telpher, 3-elastic rope, 4-load

In order to solve this problem, a method for a separation of the variables (time, abscissa) in the differential equation of the elastic line of Bernoulli-Euler beam has been developed. Unlike the Fourier’s method, the proposed method divides the variables before drawing up the differential equation. For this purpose, the elastic line of the beam is modeled in advance depending on the boundary conditions through the infinite trigonometric series method.

The developed method can be utilized in many engineering structures, leading to “a beam under moving load model”.

2. Mathematical model of the “bridge girder-telpher-load” system dynamic response

The mechanical model is depicted in Fig. 2(a). The “bridge girder-telpher-load” system has two degrees of freedom: the $w(t, x)$ dynamic deflection of the beam elastic line; the $z(t)$ elastic elongation of the “telpher-load” connection (elastic rope). The principle “release from the connections” has been applied. The behavior of the bridge girder-telfer system is described first, followed by the load behavior (Fig. 2(b)). The system equations in closed form are as follows

$$\begin{aligned}
&EJ\frac{\partial^4 w(x,t)}{\partial x^4}+\left\{\rho F+m^h H[x-\xi(t)]\right\}\frac{\partial^2 w(x,t)}{\partial t^2}+\\
&+\varsigma\rho F\frac{\partial w(x,t)}{\partial t}=N^r[\xi(t)]H[x-\xi(t)]\\
&M^\ell\frac{\partial^2\eta(x,t)}{\partial t^2}=M^\ell g-N^r[\xi(t)]\\
&N^r[\xi(t)]=c^r z(t)\\
&\eta(x,t)=w(x,t)\big|_{x=\xi(t)}+z(t)\\
&\xi(t)=vt
\end{aligned} \tag{1}$$

where EJ is the beam bending stiffness; ρ is the beam density; F is the beam cross-section area; m^h

is the telpher mass; H is heaviside function; ς is damping coefficient of distributed linear damping; N^r is the elastic force in the rope; M^{ℓ} is the load mass; g is gravity acceleration; $z(t)$ is the rope elongation; v is the telpher horizontal velocity; c^r is the rope stiffness ($\ell_r^{in} = const$); ℓ_r^{in} is the initial rope length.

The initial and boundary conditions are

$$w(x,0) = w_0(x);\ \eta(x,0) = w_0(x) + \frac{M^{\ell} g}{c^r} \tag{2}$$

$$\begin{aligned} & w(0) = w(\ell) = 0; \\ & EJ\frac{\partial^2 w(0)}{\partial x^2} - c_{\varphi}\frac{\partial w(0)}{\partial x} = 0; \\ & EJ\frac{\partial^2 w(\ell)}{\partial x^2} + c_{\varphi}\frac{\partial w(\ell)}{\partial x} = 0 \end{aligned} \tag{3}$$

The system (1) describing the "bridge girder-telpher-load" system dynamic response, cannot be integrated analytically. Two approaches are possible: numerical solution; development and implementation of appropriate engineering approach. The numerical solution requires a particular geometry and configuration of the mechanical system and does not always allow a thorough analysis to be conducted. In this study, the second approach has been adopted.

3. Essence of the proposed approach

In this section, an engineering approach for modeling the elastic line of Bernoulli-Euler beam with elastic angular supports is proposed. The method is based on the application of the infinite trigonometric series. A straight beam with elastic angular supports with stiffness equal to c_{φ}, limiting the bending rotations of the end cross-sections (Fig. 2), is considered. The beam elastic line lies in the xw plane. The $w(t, x)$ dynamic deflection is presented as

$$w(t,x) = \varphi(t) y(x) \tag{4}$$

where $\varphi(t)$ is normal coordinate and $y(x)$ is normal function.

In order to define the $y(x)$ normal function, the $w(t, x)$ dynamic deflection is considered in a static regime, i.e.,: $w=w(x)$.

The deflection $w(x)$ must satisfy the condition $w(0) = w(\ell) = 0$, but $w'(0) \neq 0$, $w'(\ell) \neq 0$, $w''(0) \neq 0$, $w''(\ell) \neq 0$, as between $w'(0)$ and $w''(0)$, respectively between $w'(\ell)$ and $w''(\ell)$, a correlation exists: for a particular angle of rotation of the end cross-section, there is a specific elastic moment. The expression for deflection $w(x)$ of the beam elastic line is offered in the form

$$\begin{aligned} w(x) = & \sum_{n=1}^{n=\infty} A_n \left(1 - \cos\frac{2n\pi x}{\ell}\right) + \\ & + \sum_{n=1,3,5,\ldots}^{n=\infty} B_n \sin\frac{n\pi x}{\ell} \end{aligned} \tag{5}$$

Each of the functions in the sums apparently satisfies the first group of boundary conditions

$$w(0) = w(\ell) = 0$$

The derivatives up to second order of (5) are

$$w' = \frac{2\pi}{\ell}\sum_{n=1}^{n=\infty} nA_n \sin\frac{2n\pi x}{\ell} + \\ + \frac{\pi}{\ell}\sum_{n=1,3,5,...}^{n=\infty} nB_n \cos\frac{n\pi x}{\ell}$$

$$w'' = \frac{4\pi^2}{\ell^2}\sum_{n=1}^{n=\infty} n^2 A_n \cos\frac{2n\pi x}{\ell} - \\ - \frac{\pi^2}{\ell^2}\sum_{n=1,3,5,...}^{n=\infty} n^2 B_n \sin\frac{n\pi x}{\ell}$$

The dependence between the coefficients A_n on the one hand and the coefficients B_n on the other hand, is determined by the second group of boundary conditions

$$w'(0) = \frac{n\pi B_n}{\ell} \tag{6}$$

The turns of the end cross-sections, Eq. (6), cause elastic moments

$$M(0) = -EJw''(0) \; M(\ell) = -EJw''(\ell) \tag{7}$$

where

$$w''(0) = \frac{4n^2\pi^2}{\ell^2} A_n \; w''(\ell) = \frac{4n^2\pi^2}{\ell^2} A_n \tag{8}$$

The elastic moments are defined as

$$M(0) = c_\varphi w'(0) \; M(\ell) = c_\varphi w'(\ell) \tag{9}$$

After substitution of (6)-(8) in (9), taking into account that $M(0)$ and $M(\ell)$ are opposite and solving toward B_n

$$B_n = \frac{kn\pi}{4} A_n \tag{10}$$

where

$$k = \frac{16EJ}{\ell c_\varphi} \tag{11}$$

In view of (10), the deflection $w(x)$ obtains the following form

$$w(x) = \sum_{n=1}^{n=\infty} A_n \left(1 - \cos\frac{2n\pi x}{\ell}\right) + \\ + \frac{k\pi}{4} \sum_{n=1,3,5,...}^{n=\infty} nA_n \sin\frac{n\pi x}{\ell} \tag{12}$$

The unknown coefficients A_n can be determined, for example through the principle of possible displacements for equilibrium position depending on the particular load.

The normal $\varphi(t)$ coordinate in Eq. (4) is actually the A_n coefficient in Eq. (12), when n=1 and the deflection depends on the time, i.e.,: $w=w(x, t)$. Two arguments exist in favor of this assumption (n=1):

• In the engineering structures the elastic curve of a two-supported beam usually corresponds to its basic eigentone under free vibrations;

• The practice shows that the bending stresses in a two-supported beam are biggest when the load is equally distant from both supports.

The normal $y(x)$ function is obtained from (12) after substitution of A_n=1 and n=1

$$y(x) = \left(1 - \cos\frac{2\pi x}{\ell}\right) + \frac{k\pi}{4}\sin\frac{\pi x}{\ell} \tag{13}$$

When k=0 (beam with both ends fixed), the second addend in (13) is removed: $y(x) = \left(1 - \cos\frac{2\pi x}{l}\right)$. When $k=\infty$ (simply supported beam), the normal function is $y(x) = \sin\frac{\pi x}{l}$. Any further transformations were made with the assumption that k is a finite number greater than zero.

4. Forced vibrations of the "bridge girder-telpher- load" system

The developed approach has been used in order to study the dynamic response of the "bridge girder-telpher-load" system due to telpher motion. The equations of motions of this system can be presented as

$$\begin{aligned} &\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\varphi}} - \frac{\partial}{\partial \varphi}\left(E_k - E_p\right) = Q_\varphi \\ &\frac{d}{dt}\frac{\partial E_k}{\partial \dot{z}} - \frac{\partial}{\partial z}\left(E_k - E_p\right) = Q_z \\ &\xi = vt \end{aligned} \tag{14}$$

where

$$E_k = E_k^b + E_k^h + E_k^\ell \tag{15}$$

is the system kinetic energy: E_k^b, E_k^h and E_k^ℓ are respectively kinetic energies of the beam, telpher and load (the kinetic energy of the rope is neglected); E_p is the system potential energy; Q_φ and Q_z are generalized forces; $\vec{v}$ is the telpher horizontal velocity, $v=const$.

The kinetic energy of the bridge girder is

$$E_k^b = \frac{\rho F}{2}\dot{\varphi}^2 \int_0^\ell y^2(x)dx = \frac{1}{2}\overline{M}^b \dot{\varphi}^2 \tag{16}$$

where

$$\overline{M}^b = \rho Fl\left(\frac{3}{2} + \frac{4}{3}k + \frac{\pi^2 k^2}{32}\right) \tag{17}$$

is the beam reduced mass, where the k coefficient is determined by Eq. (11).

The kinetic energies of the telpher and the load are obtained only from their vertical velocities. Their horizontal velocities are $\dot{\xi}(t) = v$.

The telpher kinetic energy is

$$E_k^h = \frac{1}{2}m^h\left[\dot{\varphi}(t)F(t) + \varphi(t)\dot{F}(t)\right]^2 \tag{18}$$

where

$$F(t) = y(x)\big|_{x=vt} = 1 - \cos\frac{2\pi vt}{l} + \frac{k\pi}{4}\sin\frac{\pi vt}{l} \tag{19}$$

The load kinetic energy is

$$E_k^\ell = \frac{1}{2}M^\ell v_\ell^2 \tag{20}$$

where $\vec{v}_\ell$ is load vertical velocity at constant l_r^{in} initial rope length.

The magnitude of $\vec{v}_\ell$ is

$$v_\ell = \dot{\eta} = F(t)\dot{\varphi}(t) + \dot{F}(t)\varphi(t) + \dot{z}(t) \tag{21}$$

After substitution of (21) in (20)

$$E_k^\ell = \frac{1}{2}M^\ell\left[F(t)\dot{\varphi}(t) + \dot{F}(t)\varphi(t) + \dot{z}(t)\right]^2 \tag{22}$$

Function (19) converts Eq. (14) in a system nonlinear differential equations. In the present study, $F(t)$ has been replaced by its average integral value

$$\overline{F} = \frac{v}{l}\int_0^{\frac{l}{v}} F(t)dt = 1 + \frac{k}{2} \tag{23}$$

Thus, the nonlinear Eq. (14) are approximated by a system of linear differential equations of the second order, which has an analytical solution. Simultaneously a certain error is introduced: when the telpher is in the beam middle, the computed natural frequencies are higher than the actual ones and vice versa: the computed frequencies will be smaller when the telpher is positioned at the beam ends.

The system potential energy is

$$E_p = E_p^b + E_p^{elas} + E_p^r \tag{24}$$

where E_p^b, E_p^{elas} and E_p^r are components respectively from the bridge girder, the elastic angular supports and the elastic rope.

The sum of the first two components is

$$\begin{aligned} E_p^b + E_p^{elas} &= \frac{EJ}{2}\varphi^2 \int_0^{\ell} [y''(x)]^2 dx + c_\varphi [y'(0)]^2 \varphi^2 = \\ &= \frac{EJ\pi^4}{l^3}\left(4 + \frac{5}{3}k + \frac{\pi^2 k^2}{64}\right)\varphi^2 \end{aligned} \tag{25}$$

The elastic rope potential energy is

$$E_p^r = \frac{1}{2} c^r z^2 \tag{26}$$

The external potential forces of the "bridge girder-telpher-load" system are: Q^{ℓ} - the load weight; $q=\rho Fg$ - the distributed load of the beam weight, where $\vec{g}$ - the gravitational acceleration; G^h - the telpher weight. An increase of the w deflection, equal to δw, is assigned which leads to a distortion of the beam elastic line (the z coordinate remains constant)

$$\delta w = \delta\varphi . y(x) \tag{27}$$

The applied points of the forces G^h and Q^{ℓ} perform displacements, respectively

$$\delta w_{G^h} = \delta w_{Q^{\ell}} = \delta\varphi . F(t) \tag{28}$$

The virtual works of the G^h and Q^{ℓ} forces, and the distributed load q, are respectively

$$\delta A\left(G^h + Q^{\ell}\right) = \left(G^h + Q^{\ell}\right) F(t) \delta\varphi \tag{29}$$

$$\delta A(q) = \int_0^{\ell} q \partial w dx = q\ell\left(1 + \frac{k}{2}\right)\delta\varphi \tag{30}$$

The virtual work of the generalized force Q_φ is

$$\delta A\left(Q_\varphi\right) = Q_\varphi . \delta\varphi \tag{31}$$

From the virtual work of the forces G^h and Q^{ℓ}, and the distributed load q

$$Q_{\varphi} = \left(G^{h} + Q^{\ell}\right)F(t) + ql\left(1 + \frac{k}{2}\right) \tag{32}$$

follows for Q_{φ}, in accordance with the principle of virtual displacements.

A virtual δz increase of the z coordinate is assigned (the φ coordinate remains constant). The virtual work of the Q^{ℓ} force is

$$\delta A\left(Q^{\ell}\right) = Q^{\ell}\delta z \tag{33}$$

The virtual work of the generalized force Q_z is

$$\delta A\left(Q_z\right) = Q_z \delta z \tag{34}$$

From (33) and (34)

$$Q_z = Q^{\ell} \tag{35}$$

follows.

After substitution of (35), (32), (24)-(26), (23), (22), (18), (16) and (15) in (14),

$$\left| \begin{array}{l} \overline{M}_{\varphi}\ddot{\varphi} + \overline{M}_{z}\ddot{z} + A_{\varphi}\varphi = F(t).Q_{\Sigma} + G_{\Sigma} \\ M^{\ell}\ddot{\varphi} + M^{\ell}\ddot{z} + c^{r}z = Q^{\ell} \end{array} \right. \tag{36}$$

is obtained for the differential equations of the “bridge girder-telpher-load” system, where

$$\overline{M}_{\varphi} = \overline{M}^{b} + m^{h}\overline{F}^{2} + M^{l}\overline{F}^{2}$$

$$\overline{M}_{z} = M^{l}\overline{F}$$

$$A_{\varphi} = \frac{2EJ\pi^{4}}{\ell^{3}}\left(4 + \frac{5}{3}k + \frac{\pi^{2}k^{2}}{64}\right)$$

$$Q_{\Sigma} = G^{h} + Q^{l}$$

$$G_{\Sigma} = q\ell\left(1 + \frac{k}{2}\right)$$

After transformations, the system (36) obtains the form

$$\left| \begin{array}{l} \left(1 - \frac{\overline{M}_{z}}{\overline{M}_{\varphi}}\right)\ddot{z} - \frac{A_{\varphi}}{M_{\varphi}}\varphi + \frac{z}{d} = g - F(t)\frac{Q_{\Sigma}}{\overline{M}_{\varphi}} - \frac{G_{\Sigma}}{\overline{M}_{\varphi}} \\ \left(\frac{\overline{M}_{\varphi}}{\overline{M}_{z}} - 1\right)\ddot{\varphi} + \frac{A_{\varphi}}{\overline{M}_{z}}\varphi - \frac{z}{d} = F(t)\frac{Q_{\Sigma}}{\overline{M}_{z}} + \frac{G_{\Sigma}}{\overline{M}_{z}} - g \end{array} \right. \tag{37}$$

where $d = \dfrac{M^{l}}{c^{r}}$ is the reciprocal value of the square of the natural frequency of the "rope-load" system.

After summing the two Eqs. (37)

$$\left(\frac{\overline{M}_{\varphi}}{\overline{M}_{z}}-1\right)\ddot{\varphi}+\left(1-\frac{\overline{M}_{z}}{\overline{M}_{\varphi}}\right)\ddot{z}+A_{\varphi}\left(\frac{1}{\overline{M}_{z}}-\frac{1}{\overline{M}_{\varphi}}\right)\varphi = \\ =\left[F(t)Q_{\Sigma}+G_{\Sigma}\right]\left(\frac{1}{\overline{M}_{z}}-\frac{1}{\overline{M}_{\varphi}}\right) \tag{38}$$

Both sides of the second of the Eq. (37) is differentiated twice and the obtained equation is solved with respect to $\ddot{z}$

$$\ddot{z} = d\left(\frac{\overline{M}_{\varphi}}{\overline{M}_{z}}-1\right)\frac{d^{4}\varphi}{dt^{4}}+d\,\frac{A_{\varphi}}{\overline{M}_{z}}\ddot{\varphi}-\ddot{F}(t).\frac{Q_{\Sigma}}{\overline{M}_{z}}.d \tag{39}$$

After substitution of (39) in (38), the differential equation of the $\varphi(t)$ normal coordinate is obtained

$$\frac{d^{4}\varphi}{dt^{4}}+2ab\ddot{\varphi}+ac\varphi = F(t)aQ_{\Sigma}+aG_{\Sigma}+adQ_{\Sigma}\ddot{F}(t) \tag{40}$$

where

$$a = \frac{1}{\left(\overline{M}_{\varphi}-\overline{M}_{z}\right)d}$$

$$2b = \overline{M}_{\varphi}+A_{\varphi}d$$

$$c = A_{\varphi}$$

The roots of the characteristic equation are

$$r_{1,2} = \pm i\sqrt{ab-\sqrt{a^{2}b^{2}-ac}}$$

$$r_{3,4} = \pm i\sqrt{ab+\sqrt{a^{2}b^{2}-ac}}$$

For physically acceptable values of the parameters of the "bridge girder-telpher-load" mechanical system the condition

$$ab^{2}-c>0$$

should always be fulfilled.

$$ab-\sqrt{a^{2}b^{2}-ac} = \omega_{1}^{2}; \\ ab+\sqrt{a^{2}b^{2}-ac} = \omega_{2}^{2} \tag{41}$$

is set.

The total integral of the non-homogeneous differential Eq. (40) is

$$\begin{aligned}\varphi &= C_1 \cos \omega_1 t + C_2 \sin \omega_1 t + \\ &+ C_3 \cos \omega_2 t + C_4 \sin \omega_2 t + \varphi_1(t)\end{aligned} \tag{42}$$

The particular $\varphi_1(t)$ integral has been found in the form

$$\varphi_1(t) = A_1 + A_2 \cos 2\Omega t + A_3 \sin \Omega t \tag{43}$$

where $\Omega = \frac{\pi v}{\ell}$ is the forced frequency caused by the moving load and the constants are

$$A_1 = \frac{Q_\Sigma + G_\Sigma}{c}$$

$$A_3 = \frac{aQ_\Sigma\left(1 - \frac{\Omega^2}{\omega_\ell^2}\right)}{\Omega^4 - 2ab\Omega^2 + ac}$$

$$A_2 = \frac{aQ_\Sigma\left(1 - 4\frac{\Omega^2}{\omega_\ell^2}\right)}{\Omega^4 - 8ab\Omega^2 + ac}$$

$$\omega_\ell^2 = \frac{1}{d}$$

For determination of the C_i integration constants in the total integral (39), the second of the Eqs. (37) is solved with respect to z

$$z = \frac{\overline{m}}{\omega_\ell^2}\ddot{\varphi} + \frac{\overline{c}}{\omega_\ell^2}\varphi - F(t).\frac{Q_\Sigma}{\overline{M}_z\omega_\ell^2} - \frac{G_\Sigma - \overline{M}_z g}{\overline{M}_z\omega_\ell^2} \tag{44}$$

and its first integral of motion

$$\dot{z} = \frac{\overline{m}}{\omega_\ell^2}\dddot{\varphi} + \frac{\overline{c}}{\omega_\ell^2}\dot{\varphi} - \dot{F}(t)\frac{Q_\Sigma}{\overline{M}_z\omega_\ell^2}, \tag{45}$$

is obtained,
where

$$\overline{m} = \frac{\overline{M}_\varphi}{\overline{M}_\varphi} - 1$$

$$\bar{c} = \frac{A_\varphi}{\bar{M}_z}$$

It is assumed that in the t=0 moment the telpher is on the left support. Therefore, the initial conditions are

$$\varphi(0) = 0;\ \dot{\varphi}(0) = 0;$$
$$z(0) = \frac{Q^\ell}{c^r} = \frac{g}{\omega_l^2};\ \dot{z}(0) = 0 \tag{46}$$

From (42) and (44)-(46)

$$C_1 = \frac{-A_1\bar{m}\omega_2^2 - A_2\bar{m}\left(\omega_2^2 - 4\Omega^2\right) + \frac{G_\Sigma - \bar{M}_z g}{\bar{M}_z} + \frac{\omega_1^2 Q^l}{c^r}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{47}$$

$$C_2 = \frac{-A_3\bar{m}\Omega\left(\omega_2^2 - 4\Omega^2\right) + \frac{k\pi\Omega Q_\Sigma}{4\bar{M}_z}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)\omega_2} \tag{48}$$

$$C_3 = \frac{A_1\bar{m}\omega_1^2 - A_2\bar{m}\left(\omega_1^2 - 4\Omega^2\right) + \frac{G_\Sigma - \bar{M}_z g}{\bar{M}_z} + \frac{\omega_1^2 Q^l}{c^r}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{49}$$

$$C_4 = \frac{A_3\bar{m}\Omega\left(\omega_1^2 - 4\Omega^2\right) - \frac{k\pi\Omega Q_\Sigma}{4\bar{M}_z}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)\omega_2} \tag{50}$$

follows for the C_i integration constants.

The constants C_1 and C_3 are presented as

$$C_i = C_i^{free} + C_i^{forced},\ i = 1,3 \tag{51}$$

where

$$C_i^{free} = \frac{\omega_1^2 Q^l}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)c^r} = \frac{g}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)},\ i = 1,3 \tag{52}$$

$$C_1^{forced} = \frac{-A_1\bar{m}\omega_2^2 - A_2\bar{m}\left(\omega_2^2 - 4\Omega^2\right) + \frac{G_\Sigma - \bar{M}_z g}{\bar{M}_z}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{53}$$

$$C_3^{forced} = \frac{A_1\bar{m}\omega_1^2 + A_2\bar{m}\left(\omega_1^2 - 4\Omega^2\right) - \frac{G_\Sigma - \bar{M}_z g}{\bar{M}_z}}{\bar{m}\left(\omega_2^2 - \omega_1^2\right)} \tag{54}$$

The constants C_i^{free}, $i=1,3$, define the amplitudes of free vibrations of the mechanical system. These vibrations are caused by the initial conditions-the third condition from (46). These vibrations subside quickly, mainly due to the material hysteresis. The constants C_i^{forced}, $i=1,3$, define the amplitudes of free vibrations caused by the horizontal velocity $\vec{v}$ of the moving telpher with a load.

From (4), (13), (42), (43), (48), (50), (53) and (54)

$$w(t,x)=\begin{pmatrix} C_1^{forced}\cos\omega_1 t+C_2\sin\omega_1 t+ \\ +C_3^{forced}\cos\omega_2 t+C_4\sin\omega_2 t+ \\ +A_1+A_2\cos 2\Omega t+A_3\sin\Omega t \end{pmatrix}.\left(1-\cos\frac{2\pi x}{l}+\frac{k\pi}{4}\sin\frac{\pi x}{l}\right) \tag{55}$$

follows for the bridge girder bending forced vibrations, caused by the moving telpher with elastically mounted load.

The forced vibrations (55) are superposition of vibrations with own frequencies ω_1 and ω_2, and forced frequency $\Omega=\pi v/l$.

5. Dynamic coefficient

The static $w(x)$ deflection is obtained from the analogous to (4) dependence

$$w(x)=\varphi_{st}.y(x) \tag{56}$$

where φ_{st} is obtained from the first Eq. of (26) after substitution of $\ddot{\varphi}=\ddot{z}=0$, replacement of $F(t)$ with $y(x)$ and solving and in regard to φ

$$\varphi_{st}=\frac{Q_\Sigma y(x)+G_\Sigma}{A_\varphi} \tag{57}$$

In order to assess the dynamic effect from the telpher movement, a dynamic coefficient k_d is introduced. The latter is defined as a ratio of the dynamic deflection (55) and the static one (56). Taking into account the normal function (13) of the bridge girder elastic line, the dynamic coefficient k_d has largest relevance for the girder middle cross-section

$$k_d=\left.\frac{w(t,x)}{w(x)}\right|_{x=\frac{l}{2}} \tag{58}$$

The dynamic coefficient k_d shows how many times the static loading is increased due to the “telpher-load” system horizontal velocity. The definition of this ratio by means of (58) has the following advantage. The approximation that is imported into the decision by means of the proposed method of separation of the variables, slightly affects the dynamic effect of the moving “telpher-load” system, since the error introduced by defining the normal function is the same for the numerator and the denominator of dependence (58).

6. Numerical results and discussions

The numerical results are shown for a single girder bridge crane with bridge girder length $\ell = 20m$, maximum lifting capacity Q=50 *kN* and coefficient k=11.33 (see Eq. (11)), respectively stiffness of the angular supports c_φ=24489156.8 *Nm*/*rad*. The sizes of the bridge girder cross-section (Fig. 3) are: u=0.015 *m*; v=0.02 *m*; a=0.06 *m*; b=0.395 *m*; h=0.595 *m*; F=0.0324 m^2; J_y=0.001734 m^4. The telpher mass is m^h=1000 *kg*. The bridge girder density and the rope stiffness are respectively ρ=7850 kg/m^3 and c^r=107 *N*/*m*.

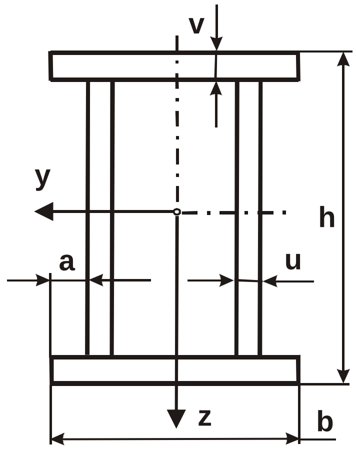

Fig. 3 Cross-section of the bridge girder

Fig. 4 visualizes the components of the function of forced vibrations (55) caused by the horizontal telpher velocity v=0.5 *m*/*s* as well as free vibrations and static deflection of the bridge girder middle cross-section. The time interval corresponds to the telpher location in vicinity of the middle section: in the interval ±2 *m* from the middle cross-section. Apparently, the contribution of the forced vibrations with own frequencies for obtaining the maximum dynamic deflection is greatest (Fig. 4).

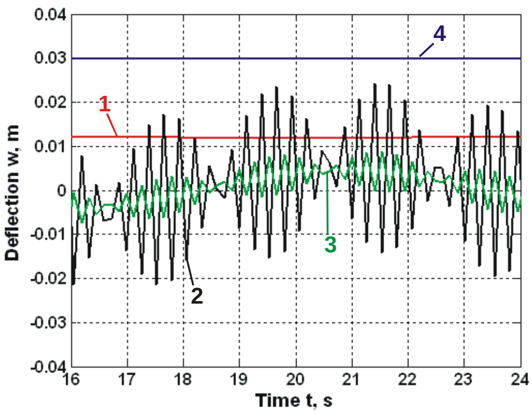

Fig. 4 Forced vibrations, free vibration and static deflection: 1 - forced vibrations with forced frequency Ω; 2 - forced vibrations with natural frequencies ω_i; 3 - free vibrations; 4 - static deflection

The resulting forced vibrations of the same cross-section are illustrated in Fig. 5. It is noteworthy that the maximum dynamic deflection of the bridge girder middle cross-section corresponds to the telpher position, which does not coincide exactly with this cross-section (respectively, for a moment of time *t*=20 *s*). The maximum deflection occurs when the telpher has already passed the beam middle. As it can be seen from Fig. 5, the bending stress at a critical point from the bridge girder middle cross-section will change at an asymmetric cycle, similar to the dynamic deflection.

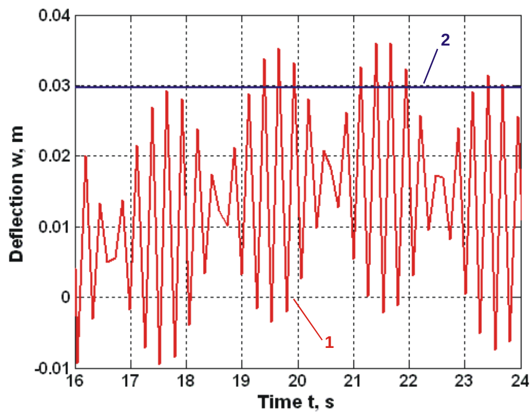

Fig. 5 Resultant forced vibrations: 1 - forced vibrations; 2 - static deflection

Figs. 6 and 7 show the effect of the coefficient *k*, defining the angular supports stiffness of the bridge girder, on the forced vibrations. *k*=0 and *k*=∞ define respectively a fixed beam and a freely supported beam. With the increase of *k*, the dynamic deflection of the bridge girder middle cross-section from forced vibration with worked Ω frequency increases (Fig. 6), while the amplitude swing decreases. For a freely supported beam this component has an emphasized static character. The forced vibrations with natural frequencies (Fig. 7) are symmetrical for each value of *k*, as the amplitude is smallest for *k*=0.

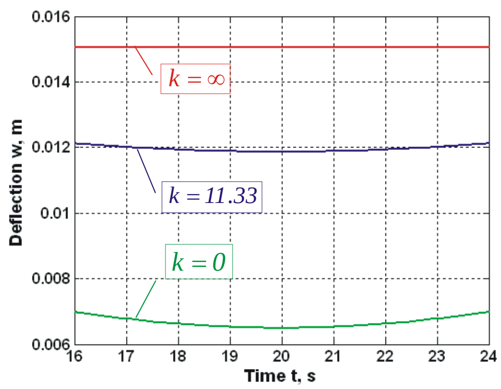

Fig. 6 Effect of the coefficient *k* on the forced vibrations with forced frequency Ω

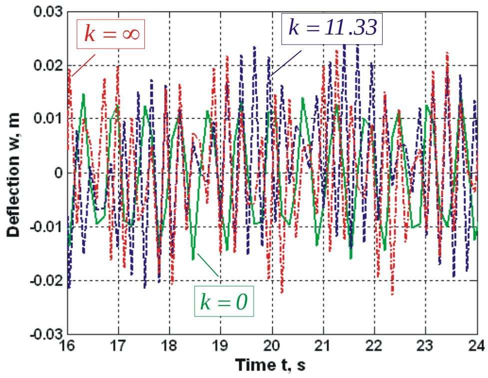

Fig. 7 Effect of the coefficient k on the forced vibrations with natural frequencies ω_i

Fig. 8 shows the effect of the coefficient k on the dynamic coefficient k_d, defined for the beam middle cross-section. The coefficient k_d is largest for fixed beam, k=0 (Fig. 8(c)), and significantly exceeds the dynamic coefficient k_d=1.04+0.06v due to the telpher motion, shown in (Kolarov *et al.* 1986). With the increase of k, the dynamic coefficient k_d sharply decreases. Of course, the cases where k=∞ (Fig. 8(a)) and k=0 (Fig. 8(c)) have a theoretical significance only. For practical real values of k (see Fig. 8(b)), the dynamic coefficient k_d calculated in

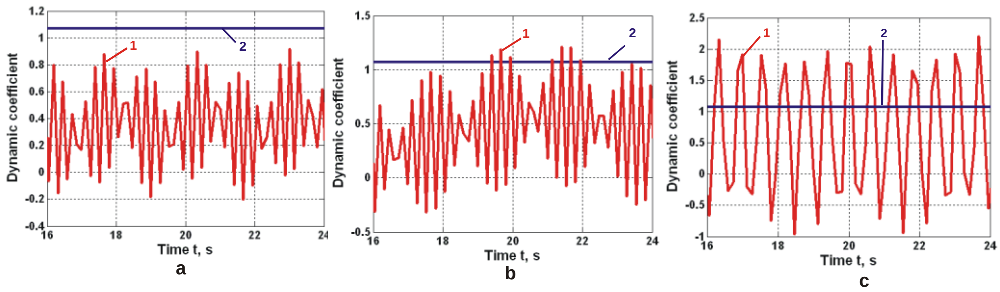

Fig. 8 Effect of the coefficient k on the dynamic coefficient k_d: 1 - dynamic coefficient according to (58); 2 - dynamic coefficient according to [47]; (a) k=∞; (b) k=11.33; (c) k=0

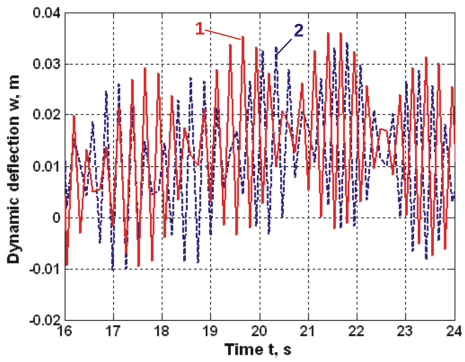

Fig. 9 Comparison of the forced vibrations for the middle beam section: (a) $F(t)=\bar{F}$; (b) for the actual value of the $F(t)$ function

accordance with (58) shows higher values compared to these in (Kolarov *et al.* 1986) and it actually reveals the mechanism of the bridge girder loading: the loading resulting from the telpher uniform movement along the bridge girder is cyclical. It is therefore a prerequisite for nucleation and propagation of fatigue cracks. It is noteworthy that the maximum value of the dynamic coefficient is not obtained in a position of the telpher in the beam middle, and when the telpher is located shortly before or after the beam middle cross-section.

In order to evaluate the error from linearization of the Eq. (14) by means of the substitution $F(t) = \overline{F}$, the forced vibration for the middle beam section is shown in Fig. 9, where the telpher is poisoned in the beam middle, for two cases: 1). $F(t) = \overline{F}$; 2). for the actual value of the function $F(t)$, namely $F(t)\big|_{t=\frac{\ell}{2v}} = \frac{k\pi}{4}$. The maximum dynamic deflections for both cases are respectively 0.034 *m* and 0.036 *m*. The approximation (the second case) leads to a larger deflection, as the error is approximately 5%.

7. Verification of the proposed method

The proposed method is an approximation of the exact solution of the Eq. (1). The degree of approximation can be evaluated through a comparison between the proposed method solution and the exact solution, if the latter exists. However, in this case the exact solution is unknown. For this reason the developed method was compared with another approximate method, for instance FEM. Implicit dynamic analysis was carried out using ABAQUS v. 6.12.1. An iterative approach was developed using “restart options” and reading the outcomes from the (i-1)-th analysis as initial conditions in the i-th analysis. Thus, the mass and load motion were simulated, using appropriate time curves. The FEM model for the i-th analysis is depicted in Fig. 10.

Because of the peculiarities of ABAQUS v. 6.12.1 (beam elements cannot be transferred form one ABAQUS/Standard analysis to another), the crane bridge girder is modeled by means of plane stress elements. The model consists of 1920 linear quadrilateral elements of type CPS4R and 2169 nodes. The elastic angular supports at each end of the beam are modeled by a pair of parallel axial springs. The stiffness c_a of each spring is calculated with the formula

$$c_a = \frac{2c_\varphi}{h^2}$$

where c_φ is the angular stiffness of the support and h is the distance between the two axial springs.

The rope is modeled by two axial springs having stiffness equal to the rope stiffness.

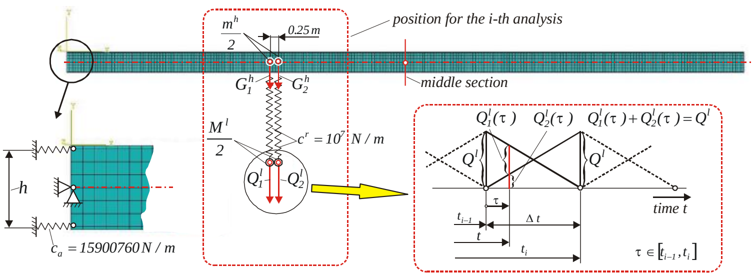

Fig. 10 FEM model for the i-th analysis

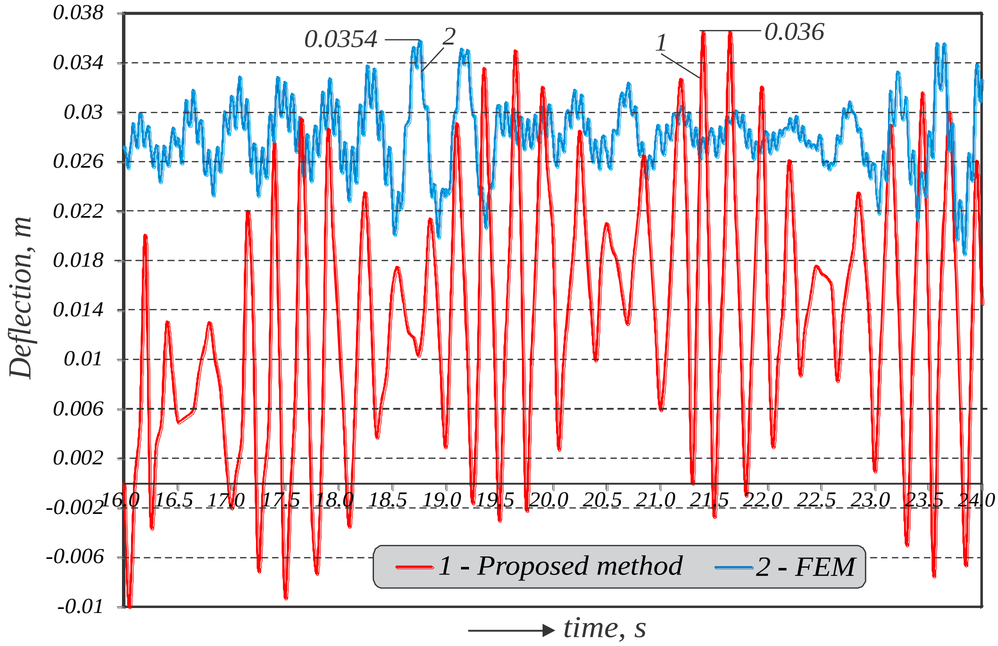

Fig. 11 Deflection of the middle section centre: 1. proposed method; 2. FEM

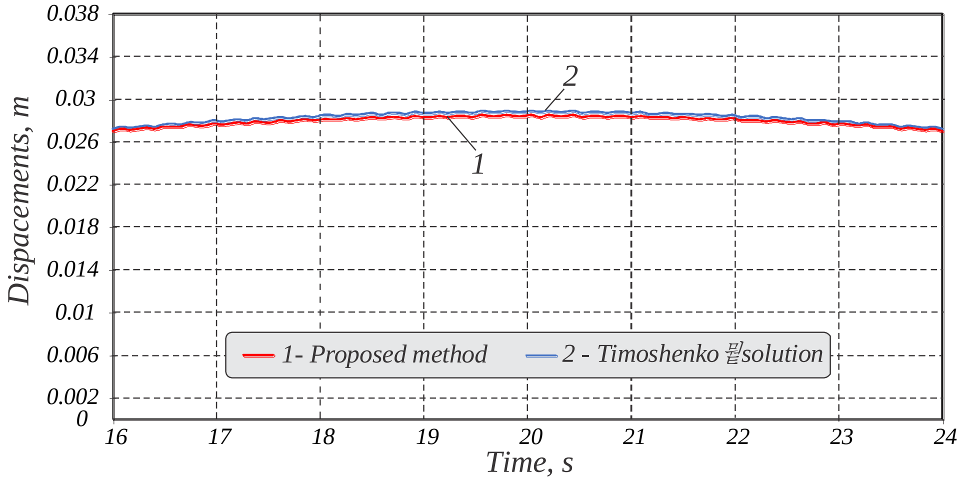

Fig. 12 Deflection of the middle section centre of simply supported beam: 1. proposed method; 2. Timoshenko

FEM outcomes for the displacement of the middle cross-section centre (point *A*) of the beam are depicted in Fig. 11. The comparison of the obtained FEM results with those from the proposed method shows good agreement with respect to the maximum deflection: 0.0354 m versus 0.0360 m.

An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving constant force. Tymoshenko's solution in an infinite series is (Timoshenko 1972)

$$w = D_1 \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi x}{\ell}}{i^2\left(i^2\pi^2 a^2 - v^2\ell^2\right)} \sin i\Omega t - \\ - D_2 \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi x}{\ell}}{i^3\left(i^2\pi^2 a^2 - v^2\ell^2\right)} \sin \frac{i^2\pi^2 at}{\ell^2} \tag{59}$$

where

$$D_1 = \frac{2gP\ell^3}{\gamma F\pi^2}$$

$$D_2 = \frac{2gP\ell^4 v}{\gamma F\pi^3 a}$$

$$a = \sqrt{\frac{EJg}{\gamma F}}$$

$$\gamma = \rho g$$

$$\Omega = \frac{\pi v}{\ell}$$

P is the moving constant force.

For the simply supported beam, subjected to a moving constant force *P*, the proposed method gives the following solution

$$w = D \sin\frac{\pi x}{\ell}\left[\frac{\sin \Omega t}{\omega^2 - \Omega^2} - \frac{\Omega \sin \omega t}{\omega\left(\omega^2 - \Omega^2\right)}\right] \tag{60}$$

where

$$D = \frac{2P}{\rho F\ell}$$

$$\omega = \frac{\pi^2}{\ell^2}\sqrt{\frac{EJ}{\rho F}}$$

The solution (60) is obtained using the methodology proposed in Section 4, referring to a simply supported beam. In order to obtain the conditions of Tymoshenko's task, the mechanical model, depicted in Fig. 2 is modified in the following manner:

- The normal function (see Eq. (13)) is $y(x) = \sin \frac{\pi x}{l}$, respectively the elastic angular supports are removed;
- The distributed load *q* of the beam weight is neglected;
- The masses of the telpher 2 and the load 4 are neglected;

• The rope 3 is assumed to be rigid (non-deformable);
• The moving constant force P has a magnitude: $P = G^h + Q^\ell$.

Using the numeric data from Section 6, the graphs of Eqs. (59) and (60) for $x = \frac{\ell}{2}$ (middle cross-section) are shown in Fig. 12. The comparison shows a very good agreement between the solutions.

8. Conclusions

• A method for separation of the variables (time, abscissa) in the differential equation of the elastic line of Bernoulli-Euler beam has been developed. Unlike Fourier's method, the proposed method divides the variables before drawing up the differential equation. For this purpose, the elastic line of the beam is modeled in advance depending on the boundary conditions through the infinite trigonometric series method. The developed method can be utilized in many engineering applications, leading to "a beam under moving load model".

• The forced vibrations of the "bridge girder-telpher-load" system of single girder bridge crane, due to telpher motion along bridge girder, have been established and analyzed through the developed method. A conclusion has been made that the normal stress at a critical point form the bridge girder middle cross-section will change at an asymmetric cycle, similar to the dynamic deflection.

• The concept of "dynamic coefficient" has been introduced, which is a ratio of the dynamic deflection of the principal beam, due to the forced vibrations, to the static one. This ratio has been compared with the known from literature empirical dynamic coefficient during the telpher movement, due to the track unevenness. The dynamic coefficient k_d, calculated in accordance with Eq. (58), shows larger values than that in (Kolarov *et al.* 1986), and it actually reveals the mechanism of the bridge girder loading: the loading resulting from the telpher uniform movement along the bridge girder is cyclical. Therefore, the telpher movement along the bridge girder is a prerequisite for nucleation and propagation of fatigue cracks. The introduced dynamic coefficient has to be taken into account for engineering calculations of the bridge crane metal structure.

• In order to verify the degree of approximation, the obtained results have been compared with FEM outcomes. An additional comparison has been made with the exact solution, proposed by Timoshenko, for the case of simply supported beam subjected to a moving force. The comparisons show a good agreement.

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