Coupled Systems Mechanics, *Vol. 6, No. 4 (2017) 487-499*
DOI: https://doi.org/10.12989/csm.2017.6.4.487

Modeling radon diffusion equation in soil pore matrix by using uncertainty based orthogonal polynomials in Galerkin's method

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(Received August 10, 2017, Revised October 16, 2017, Accepted October 23, 2017)

Abstract. This paper investigates the approximate solution bounds of radon diffusion equation in soil pore matrix coupled with uncertainty. These problems have been modeled by few researchers by considering the parameters as crisp, which may not give the correct essence of the uncertainty. Here, the interval uncertainties are handled by parametric form and solution of the relevant uncertain diffusion equation is found by using Galerkin's Method. The shape functions are taken as the linear combination of orthogonal polynomials which are generated based on the parametric form of the interval uncertainty. Uncertain bounds are computed and results are compared in special cases viz. with the crisp solution.

Keywords: radon; orthogonal; coupled; polynomials; crisp; uncertainty; interval

1. Introduction

Radon is an inert gas with chemical symbol Rn^{222}, and atomic number 86. It is a radioactive, colorless, orderless, tasteless noble gas, occurring naturally as a decay product of uranium. Recent research has shown that radon is the second leading cause of lung cancer. Radon in the soil, groundwater, or building materials is emitted and diffused in the working and living species and then disintegrates into its decay products. So, there is a need to trace the variability of radon levels in different soils. Many experimental researches for soil radon transport have been modelled by diffusion equation through various mediums. There exist variety of physical factors on which radon generation depends viz. radium concentration, porosity and diffusion coefficients which are usually measured experimentally. As such, one may obtain uncertain values or bounds of the parameters rather than exact values. So, the equation describing diffusion of radon in soil pore matrix coupling with uncertain parameters (as intervals) is solved in this work by using the Galerkin's Method with shape functions taken as the linear combination of orthogonal polynomials in uncertain environment.

First, we discuss related important literature based on radon diffusion. The determination of 222 Rn exhalation and effective 226 Ra activity in soil samples are explained by Escobaret *et al.*

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http://www.techno-press.org/?journal=csm&subpage=8 ISSN: 2234-2184 (Print), 2234-2192 (Online)

(1999). Ren (2001) explained the regional variations of indoor, outdoor 222Rn concentrations. Kozak *et al.* (2003) developed one-dimensional flow and transport model to describe the movement of two fluid phases, gas and water within a porous medium.

The release of radon isotopes under conditions of combined diffusion and flow from a fractured, semi-infinite medium such as soil is analysed by Schery *et al.* (1988). Nazaroff (1992) described the mechanism of radon generated within the upper few meters of the earth surface by the radioactive decay of radium and nature of the diffusion and advection coefficients. Validation of radon transport in soil, measurements of combined advective and diffusive radon transport under well-defined and controlled conditions studied by Van der Spoel *et al.* (1998). Dimbylow *et al.* (1985) modeled radon diffusion equation describing the flow of radon from soil through cracks in concrete slabs using numerical methods. As regards, mathematical model describing steady state diffusion of radon-222 daughters, based on a uniform distribution of radon has been developed by Wrenn *et al.* (1969), solutions are generated for four coupled, non-homogenous differential equations satisfying the boundary conditions. Coupling between CO_2, water vapor, temperature, and radon and their fluxes in an idealized equilibrium boundary layer over land have been investigated by Betts *et al.* (2004). On the other hand, Dulaiova *et al.* (2010) have studied coupled radon, methane and nitrate sensors for large-scale assessment of groundwater discharge.

In general, while doing experiment the values of involved parameters may deviate significantly from the actual values. Such insufficient information may be considered as uncertain viz. intervals or fuzzy numbers. So, there is a need of solving differential equations when coupled with interval parameters. As such, we include few literature related to interval uncertainty in differential equations. Introduction to interval computations given by Alefeld and Herzberger (1983). Moore *et al.* (2009) analysed the basic concepts of interval numbers (Interval Analysis). The basic concepts of fuzzy differential equations, fuzzy fractional differential equations and its applications described by Chakraverty *et al.* (2016). Stefanini and Bede (2009) proposed Hukuhara differentiability of interval differential equations. An interval difference method for solving the Poisson equation based on the conventional central-difference has been given by Hoffmann and Marciniak (2013). Nickel (1986) described interval methods for the numerical solution of ODE's. A new technique to solve n-th order linear uncertain interval differential equations with uncertain initial conditions using the interval midpoint, a new approach to solve nth order fuzzy differential equations has been proposed by Tapaswini and Chakraverty (2014, 2017).

As such the present section gives the introduction related to the problem. Interval arithmetic and parametric concepts have been discussed in sections 2 and 3 respectively. Then the Galerkin's method to solve uncertain boundary value problem is explained in section 4. Section 5 presents the radon diffusion mechanism with crisp parameters. The radon diffusion mechanism with uncertainty has been described in section 6. Section 7 includes results and discussions. Finally, conclusions are drawn in the last section.

2. Interval arithmetic

An interval may be denoted as $\tilde{p} = [\underline{p}, \overline{p}]$, where $\underline{p}$ and $\overline{p}$ represent the lower and upper bounds of interval $\tilde{p}$. Any two intervals are said to be equal if their corresponding end points are equal (Alefeld *et al.* 1983, Moore *et al.* 2009). The basic interval arithmetic operations are as follows)

$$\tilde{p}+\tilde{q}=[\underline{p}+\underline{q},\ \overline{p}+\overline{q}]$$
$$\tilde{p}-\tilde{q}=[\underline{p}-\overline{q}\ \ \overline{p}-\underline{q}]$$
$$\tilde{p}\times\tilde{q}=[\min(\underline{p}\times\overline{q},\ \underline{p}\times\underline{q},\ \overline{p}\times\underline{q},\ \overline{p}\times\overline{q}),\ \max(\underline{p}\times\overline{q},\ \underline{p}\times\underline{q},\ \overline{p}\times\underline{q},\ \overline{p}\times\overline{q})\ ]$$

$$\frac{\tilde{p}}{\tilde{q}}=\left[\min\left(\frac{\underline{p}}{\underline{q}},\frac{\underline{p}}{\overline{q}},\frac{\overline{p}}{\underline{q}},\frac{\overline{p}}{\overline{q}}\right),\ \max\left(\frac{\underline{p}}{\underline{q}},\frac{\underline{p}}{\overline{q}},\frac{\overline{p}}{\underline{q}},\frac{\overline{p}}{\overline{q}}\right)\right],\ \underline{q},\ \overline{q}\neq 0$$

$$\Delta p=\frac{(\overline{p}-\underline{p})}{2}\qquad \text{(Radios of interval p)}$$

$$\tilde{p}_c=\frac{(\overline{p}+\underline{p})}{2}\quad \text{(Center value of interval p)}$$

3. Parametric approach

Parametric approach is used here to represent an interval in crisp form. In this approach, the interval $\tilde{Z}=[\underline{Z},\ \overline{Z}]$ may be written as (Behera and Chakraverty 2015, Tapaswini and Chakraverty 2013)

$\tilde{Z}=\beta(\overline{Z}-\underline{Z})+\underline{Z}$, where $0\leq\beta\leq 1$ is a parameter.

It can also be written as

$$\tilde{Z}=2\beta\Delta Z+\underline{Z},\ \Delta Z=\frac{(\overline{Z}-\underline{Z})}{2},$$

The lower and upper bounds of the solution can then be obtained by substituting $\beta=0$ and 1 respectively as follows

$$\tilde{Z}=\underline{Z}\quad\text{when }\ \beta=0,$$

$$\tilde{Z}=\overline{Z}\quad\text{when }\ \beta=1.$$

4. Galerkin's method for boundary value problems coupled with uncertainty

Let us consider a second order uncertain boundary value problem in [a, b] (Rodriguez 1992)

$$\tilde{a}_1\tilde{y}''+\tilde{a}_2\tilde{y}=q(x) \tag{1}$$

subject to the uncertain boundary conditions in terms of intervals as

$$\tilde{y}(a) = \tilde{a}_3, \quad \tilde{y}(b) = \tilde{a}_4$$

where $\tilde{a}_i = [\underline{a}_{i,} \overline{a}_i]$ are interval values for i=1,2,3 ,4.

The above involved interval values can be represented as follow by using parametric concept $\tilde{a}_i = 2\alpha_i \Delta a_i + \underline{a}_i = k_{\alpha_i}$, where $\Delta a_i = \dfrac{(\overline{a}_i - \underline{a}_i)}{2}$ and $0 \le \alpha_i \le 1$ for i=1, 2, 3, 4. Here k_{α_i} is the crisp representation of $\tilde{a}_i$, k_{α_i} represents crisp values for a fixed $\alpha_i \in [0,1]$.

We assume an approximate solution satisfying the boundary conditions and involving unknown constants $k_0, k_1, k_2 \ldots, k_n$ of Eq. (1) as

$$\tilde{y}(x) = \sum_{i=0}^{n} k_i \phi_i(x) \tag{2}$$

where $\phi_i(x)$, are linearly independent orthogonal polynomials with interval uncertainty. It may be worth mentioning that the functions ϕ_i's actually involve interval uncertainties represented by k_{α_i}. As such one may use the interval computations given in section II.

Generating Orthogonal Polynomials (where the interval uncertainties of involved parameters represented in terms of k_{α_i}):

Let us assume a function $f(x)$ (which involves uncertainty to be represented in parametric form), that satisfies the boundary conditions of Eq. (1). We start with functions, $f_0 = 1,\ f_1 = x, f_2 = x^2, \cdots$ for the approximation of orthogonal polynomials. Assume that $\phi_0(x),\ \phi_1(x)$, $\phi_2(x)$, $\cdots$ are linearly independent orthogonal polynomials with interval uncertainty generated by using the polynomials $L_0(x), L_1(x), L_2(x) \cdots$ of the form,

$L_0(x) = f(x) f_0,$

$L_1(x) = f(x) f_1,$

$L_2(x) = f(x) f_2,$

………………

By using Gram Schmidt orthogonalisation procedure (Bhat and Chakraverty 2004),

$\phi_0(x) = L_0(x),$

$\phi_1(x) = L_1(x) - \gamma_{10}\phi_0(x),$

$\phi_2(x) = L_2(x) - \gamma_{20}\phi_0(x) - \gamma_{21}\phi_1(x),$

……………………………

where, $\gamma_{10} = \dfrac{\langle f_0(x), \phi_0(x)\rangle}{\langle \phi_0(x), \phi_0(x)\rangle}$, $\gamma_{20} = \dfrac{\langle f_1(x), \phi_0(x)\rangle}{\langle \phi_0(x), \phi_0(x)\rangle}$, $\gamma_{21} = \dfrac{\langle f_2(x), \phi_1(x)\rangle}{\langle \phi_1(x), \phi_1(x)\rangle}$ etc.

Here $\langle\ ,\ \rangle$ are inner product defined for any two functions as below

$$\langle \phi_i(x),\ \phi_j(x)\rangle = \int_a^b \phi_i(x)\phi_j(x)dx \tag{3}$$

From Eq. (1) and Eq. (2) we may find the residual '*R*' as

$$R(x;k_{\alpha_i};k_{0,}\,k_1\cdots k_n) = k_{\alpha_1}\sum_{i=0}^{n} k_i\phi_i''(x) + k_{\alpha_2}\sum_{i=0}^{n} k_i\phi_i(x) - q(x) \tag{4}$$

Here the residual R is orthogonalized to the (n+1) functions $\phi_0,\ \phi_1,\cdots\phi_n$.
This gives

$$\int_a^b R(x;k_{\alpha_i};k_{0,}\,k_1\cdots k_n)\phi_j(x)dx = 0, \quad j = 0,1,2,\cdots n \tag{5}$$

$$\Rightarrow \sum_{i=0}^{n}\left[\int_a^b \left\{k_{\alpha_1}k_i\phi_i''(x)\phi_j(x) + k_{\alpha_2}k_i\phi_i(x)\phi_j(x) - q(x)\phi_j(x)\right\}dx\right] = 0 \tag{6}$$

Eq. (5) is (n+1) simultaneous equations in (n+1) unknowns, which can be solved by any standard method. Finally, by substituting the evaluated constants $k_0,\ k_1, k_2 \ldots k_n$ in Eq. (2) we may get the approximate solutions for the uncertain boundary value problem (Eq. (1)) by varying $0 \le \alpha_i \le 1$ for i=1,2,3,4.

5. Radon diffusion mechanism

In pore matrix such as soil, radon is continuously released to the pore volume of the matrix due to the emanation from the grains containing ^{226}Ra. Let us consider that radon diffusion occurs in vertical direction i.e., in x direction after emanating from soil grain to pore space. Let C(x) be the steady state concentration in the soil pore space. Soil properties and radioactivity distributions are assumed to be homogeneous. Then the profile C(x) satisfies the following steady state diffusion equation (Savovic *et al.* 2011, Hafez and Awad 2016)

$$D\frac{\partial^2 C(x)}{\partial x^2} - \lambda C(x) + \lambda C_\infty = 0, \tag{7}$$

where, C(x)= Radon concentration (Bq kg^{-1}) in the soil,
D= The diffusion coefficient of radon in the soil matrix (m^{2}s^{-1}),
λ = The radon decay constant (s^{-1}),
C_∞= The radon concentration when $x\rightarrow-\infty$.

The first term and second term of Eq. (7) represents the loss of radon in the pore space of the soil matrix by the process of diffusion and radioactive decay respectively, while the third term represents the production of radon due to emanation from soil grain to pore volume. The boundary conditions, are supposed to be

$C(x=0)=C_0,$

$C(x=-L)=C_\infty.$

Analytical solution of this equation may easily be obtained as

$$C(x)=(C_0-C_\infty)e^{\sqrt{\frac{\lambda}{D}}x}+C_\infty, \tag{8}$$

The above is for the radon diffusion equation without uncertainty which is well known. Our target is here to investigate the same when the involved parameters are uncertain in terms of intervals.

6. Radon diffusion mechanism coupling with interval uncertainty

The general diffusion equation is obtained by a limiting process of the rate of change of radon activity in an infinitesimal pore volume, arising as a result of the difference between the generation rate of radon and losses. Different models have been developed based on these transport mechanisms to study the anomalous behavior of soil radon. These models have been employed to estimate process driven parameters from the measured data of soil radon. Estimation of parameters (such as (D, C_∞)) may deviate significantly from the true values. So, there is a need to handle radon diffusion equation coupling with uncertainty.

As such the uncertain second order diffusion equation may be written as follow

$$\tilde{D}\frac{\partial^2\tilde{C}(x)}{\partial x^2}-\lambda\tilde{C}(x)+\lambda\tilde{C}_\infty=0, \tag{9}$$

subject to the uncertain boundary conditions

$C(x=0)=C_0,\ \ \tilde{C}(x=-L)=\tilde{C}_\infty.$

By using parametric form, $\tilde{D}$ and $\tilde{C}_\infty$ can be represented as

$\tilde{D}=2\beta_1\,\Delta D+\underline{D}=K_{\beta_1},\ \ \Delta D=\frac{(\overline{D}-\underline{D})}{2}$

$\tilde{C}_\infty=2\beta_2\,\Delta C_\infty+\underline{C_\infty}=K_{\beta_2},\ \ \Delta C_\infty=\frac{(\overline{C}_\infty-\underline{C_\infty})}{2}$ for all $\beta_1,\beta_2\in[0,1].$

Here K_{β_1}, K_{β_2} are the controlling parameters for the interval uncertainty's of $\tilde{D}$ and $\tilde{C}_\infty$. For fixed values of $\beta_1,\beta_2\in[0,1]$, K_{β_1} and K_{β_2} represents crisp values.

An approximate uncertain solution of Eq. (9) is assumed as

$$\tilde{C}(x)=\sum_{i=0}^{n}A_i\phi_i(x) \tag{10}$$

Here $\phi_0(x)$, $\phi_1(x)$ are orthogonal polynomials with interval uncertainty, which satisfies the given boundary conditions (involved interval values may be represented in terms of K_{β_1} and K_{β_2}) and $A_0, A_1,\cdots A_n$ are real constants.

Generating Orthogonal Polynomials (for uncertain radon diffusion equation):

Let us choose $f(x)=(C_0-K_{\beta_2})e^x+K_{\beta_2}$, which satisfies the boundary conditions involving the parameters in terms of K_{β_2}. We start with two functions, $f_0=1,\ f_1=x$ for two term approximation. Assume that $\phi_0(x),\ \phi_1(x)$ are two orthogonal polynomials (involved uncertainty represented by K_{β_2}) generated by using polynomials $L_0(x),\ L_1(x)$ of the form,

$$L_0(x)=f(x)f_0=(C_0-K_{\beta_2})e^x+K_{\beta_2},$$

$$L_1(x)=f(x)f_1=(C_0-K_{\beta_2})xe^x+xK_{\beta_2}$$

By using Gram Schmidt orthogonalisation procedure we have

$$\left.\begin{aligned}\phi_0(x)&=L_0(x),\\ \phi_1(x)&=L_1(x)-\delta\phi_0(x)\end{aligned}\right\} \tag{11}$$

where,

$$\Rightarrow \delta=\frac{(C_0-K_{\beta_2})^2\left(\frac{-1}{4}+\frac{Le^{-2L}}{2}+\frac{e^{-2L}}{4}\right)-\frac{K_{\beta_2}^2L^2}{2}+2(C_0-K_{\beta_2})K_{\beta_2}(-1+Le^{-L}+e^{-L})}{(C_0-K_{\beta_2})^2\left(\frac{1}{2}-\frac{e^{-2L}}{2}\right)+K_{\beta_2}^2L+2(C_0-K_{\beta_2}^2)K_{\beta_2}^2(1-e^{-L})}$$

So, the orthogonal polynomials $\phi_0(x)$ and $\phi_1(x)$ can be represented as

$$\left.\begin{aligned}\phi_0(x)&=(C_0-K_{\beta_2})e^x+K_{\beta_2},\\ \phi_1(x)&=(x-\delta)\left((C_0-K_{\beta_2})e^x+K_{\beta_2}\right).\end{aligned}\right\} \tag{12}$$

Galerkin's Method to solve uncertain radon diffusion equation by using orthogonal polynomials:

We consider two term approximation based on interval uncertainty to approximate the solution of the said diffusion Eq. (9) as

$$\tilde{C}(x)=A_0\phi_0(x)+A_1\phi_1(x), \tag{13}$$

Now, from Eq. (9) and Eq. (13) we have

$$K_{\beta_1}(A_0\phi_0''(x)+A_1\phi_1''(x))-\lambda(A_0\phi_0(x)+A_1\phi_1(x))+\lambda K_{\beta_2}=0 \tag{14}$$

From Eq. (14) the residual '*R*' can be represented as

$$R(x;A_0,A_1,K_{\beta_1},K_{\beta_2})=K_{\beta_1}(A_0\phi_0''(x)+A_1\phi_1''(x))-\lambda(A_0\phi_0(x)+A_1\phi_1(x))+\lambda K_{\beta_2}$$

Here the residual *R* is orthogonalized to the functions $\phi_0(x),\phi_1(x)$.

This gives

$$\int_{-L}^{0}R(x;A_0,A_1,K_{\beta_1},K_{\beta_2})\phi_0(x)dx=0 \tag{15}$$

$$\int_{-L}^{0} R(x;A_0,A_1,K_{\beta_1},K_{\beta_2})\phi_1(x)dx=0 \tag{16}$$

From Eq. (15)

$$\int_{-L}^{0}(K_{\beta_1}(A_0\phi_0''(x)+A_1\phi_1''(x))-\lambda(A_0\phi_0(x)+A_1\phi_1(x))+\lambda K_{\beta_2})\phi_0(x)dx=0 \tag{17}$$

Here $\phi_1(x)$, $\phi_0(x)$ are orthogonal to each other $\Rightarrow \int_{-L}^{0}\phi_1(x)\phi_0(x)dx=0$

Now from Eq. (15)

$$A_0a_1+A_1b_1=c_1 \tag{18}$$

$$a_1=\left(\frac{M^2}{2}+MK_{\beta_2}-\frac{M^2e^{-2L}}{2}-MK_{\beta_2}e^{-L}\right)-\lambda\left(M^2\left(\frac{1}{2}-\frac{e^{-2L}}{2}\right)+K_{\beta_2}^2L+2MK_{\beta_2}(1-e^{-L})\right),$$

$$b_1=K_{\beta_1}\left(M^2\left(\frac{-1}{4}+\frac{Le^{-2L}}{2}+\frac{e^{-2L}}{4}\right)+(2-\delta)M\left(\left(\frac{1}{2}-\frac{e^{-2L}}{2}\right)+K_{\beta_2}\left(1-e^{-L}\right)\right)+K_{\beta_2}M(-1+Le^{-L}+e^{-L})\right),$$

$$c_1=-\lambda K_{\beta_2}\left(M-Me^{-L}+K_{\beta_2}L\right).$$

where,

$$M=(C_0-K_{\beta_2})$$

Similarly, from Eq. (16) we have

$$A_0a_2+A_1b_2=c_2 \tag{19}$$

where,

$$a_2=K_{\beta_1}\left(M^2\left(\frac{-1}{4}+Le^{-2L}+\frac{e^{-2L}}{4}\right)+K_{\beta_2}M(-1+Le^{-L}+e^{-L})-\delta M^2\left(\frac{1}{2}-\frac{e^{-2L}}{2}\right)-\delta MK_{\beta_2}(1-e^{-L})\right),$$

$$b_2=K_{\beta_1}\left(\frac{M(M+e^{\frac{36}{25}}K_{\beta_2})L^2(25L^2+50\delta^2+44\delta-144)}{144e^{\frac{72}{25}}}\right)-\lambda\left(\frac{1}{144}\left(\frac{M}{e^{\frac{36}{25}}}+K_{\beta_2}\right)^2L^2(25L^2+2\delta(25\delta+72))\right),$$

$$c_2=-\lambda K_{\beta_2}\left(M(\delta+1)+e^{-L}M(L+\delta+1)-\frac{K_{\beta_2}L(L+2\delta)}{2}\right)$$

By solving Eq. (18) and Eq. (19) we may get

$$A_1=\frac{(c_1a_2-c_2a_1)}{(b_1a_2-b_2a_1)} \quad \text{and} \quad A_0=\frac{(c_1-b_1A_1)}{a_1}$$

So, final two term solution of Eq. (9) is

$$\Rightarrow \tilde{C}(x)=\left((C_0-K_{\beta_2})e^x+K_{\beta_2}\right)\left(A_0+A_1(x-\delta)\right) \tag{20}$$

One my note that a_1, b_1, c_1 and a_2, b_2, c_2 contain the parameters in terms of K_{β_1} and K_{β_2}, where $\beta_1,\beta_2\in[0,1]$ which control the uncertainty. Eq. (20) represents the uncertain (interval) solutions of the diffusion Eq. (9). The uncertain band of the diffusion equation (Eq. (9)) may be obtained by varying the values of $\beta_1, \beta_2 \in [0,1]$

7. Results and discussions

Here, the results are presented based on radon diffusion equation (Eq. (9)) solved by Galerkin's Method using uncertainty based orthogonal polynomials. A soil pore matrix is considered with depth (L=10m), in which the radon diffusion occurs in vertical direction x. It is assumed that the initial radon concentration in soil pore matrix at $x=0$ as $C_0 = 10(Bq/m^3)$ and radon concentration at $x=-L$ is supposed to be exposed to high radon concentration $C_\infty = 1000(Bq/m^3)$. The value $D = 2.1\times10^{-6}\,(m^2/s)$ was used for the radon diffusion coefficient in soil and the decay constant (λ) of radon taken as $2.1\times10^{-6}s^{-1}$.

Table 1 lists the numerical values of the involved crisp and interval parameters when the radon diffusion equation is coupled with interval uncertainty.

Table 1 Numerical values for involved parameters of uncertain based diffusion equation (Savovic *et al.* 2011)

Parameter	Crisp Value	Interval Value
C_∞	$1000(Bq/m^3)$	$[980, 1020](Bq/m^3)$
D	$2.1\times10^{-6}(m^2/s)$	$[1.5\times10^{-6}, 2.7\times10^{-6}]$
C_0	$10(Bq/m^3)$	$10(Bq/m^3)$
λ	$2.1\times10^{-6}s^{-1}$	$2.1\times10^{-6}s^{-1}$

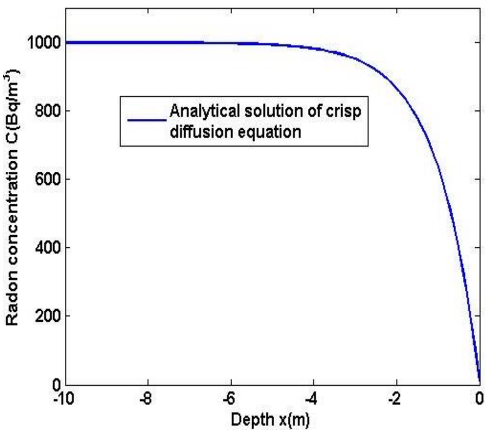

Fig. 1 Analytical solution of crisp diffusion equation

Fig. 1 presents the analytical solution of the crisp radon diffusion Eq. (7). Fig. 2 depicts the comparison of analytical solution of the crisp diffusion equation (Eq. (7)) with the center solution obtained by solving the uncertain diffusion equation (Eq. (9)) by using Galerkin's Method (when

$\beta_1 = 0.5, \beta_2 = 0.5$). The lower, center and upper radon concentration of Eq. (9) solved by using Galerkin's Method with orthogonal polynomials for a fixed value of $\beta_1 = 0$ and $\beta_2 = 0:0.5:1$ are presented in Fig. 3. Similarly, Figs. 4 and 5 represent the lower, center and upper radon concentrations of Eq. (9) solved by using Galerkin's Method with orthogonal polynomials for a fixed values of $\beta_1 = 0.5, 1$ and $\beta_2 = 0:0.5:1$. Finally, the upper and lower bounds of radon concentration along with the center radon concentration of the uncertain diffusion equation for all the possible combinations by varying $\beta_1, \beta_2 \in [0,1]$ illustrated in Fig. 6.

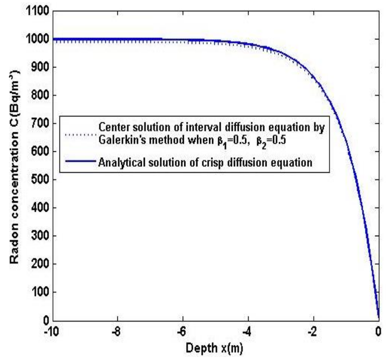

Fig. 2 Comparison of center solution of interval diffusion equation obtained by Galerkins Method when $\beta_1 = 0.5, \beta_2 = 0.5$ with analytical solution of crisp diffusion equation

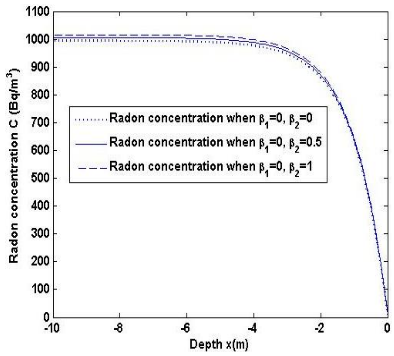

Fig. 3 Lower, center, upper radon concentration when $\beta_1 = 0, \ \beta_2 = 0:0.5:1$

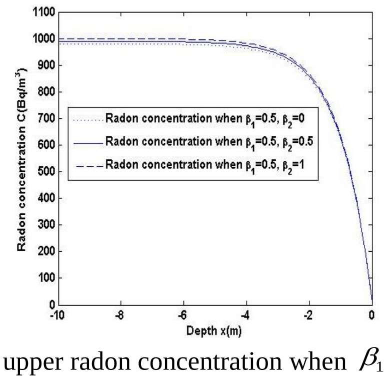

Fig. 4 Lower, center, upper radon concentration when $\beta_1 = 0.5, \ \beta_2 = 0:0.5:1$

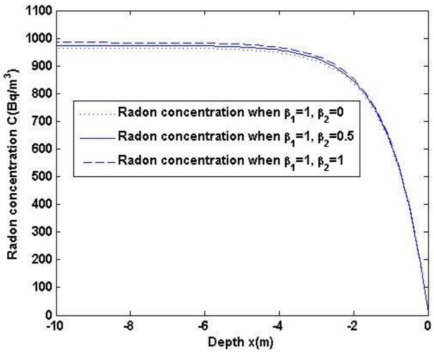

Fig. 5 Lower, center, upper radon concentration when $\beta_1 = 1,\ \beta_2 = 0:0.5:1$

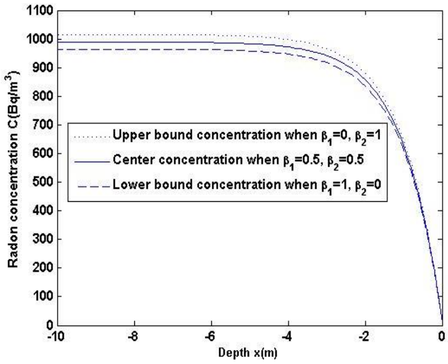

Fig. 6 Lower, upper bounds and center radon concentration from all the possible combinations $\beta_1 = 0:0.5:1,\quad \beta_2 = 0:0.5:1$

Table 2 presents the different radon concentrations for a typical fixed depths ($x=-1, x=-5, x=-10$) by varying $0 \le \beta_1, \beta_2 \le 1$.

Table 2 Radon concentration for different combinations of $\beta_1, \beta_2 \in [0,1]$ at fixed depths $(x=-1, x=-5, x=-10)$

Parametric values		Concentration		
β_1	β_2	$x=-1$	$x=-5$	$x=-10$
0	0	633.2081	989.1446	995.7145
	0.4	638.3467	997.2189	1003.8
	0.8	643.4854	1005.3	1012
0.4	0	625.1409	976.5602	983.0687
	0.4	630.2139	984.5316	991.0938
	0.8	635.2870	992.5030	999.1188
0.8	0	617.2813	964.2931	970.7332
	0.4	622.2905	972.1642	978.6574
	0.8	627.2997	980.0353	986.5817

From the above presented Table, one may observe the increase of radon concentration with respect to the depth of the soil and the effect of diffusion coefficient (for the lower values of diffusion coefficient concentration giving high).

8. Conclusions

In this paper, we presented a new approach to solve radon diffusion equation when coupled with uncertainty (Eq. (9)). Here the approximate solution of the same is assumed first as a linear combination of orthogonal polynomials with interval uncertainty. The involved parameters with interval uncertainty are represented by using the parametric concept. Then the uncertain radon diffusion equation has been solved by using Galerkin's Method. Finally, we depicted different uncertainty bands of radon diffusion equation (Eq. (9)) by varying $\beta_1, \beta_2 \in [0,1]$. For the validation, the results are compared with the known analytical solution and are found to be in good agreement.

Acknowledgments

The authors are thankful to Board of Research in Nuclear Sciences (BRNS), Mumbai, India (Grant Number: 36(4)/40/46/2014-BRNS) for the support and funding to carry out the Present research work.

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