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Fuzzy finite element method for solving uncertain heat conduction problems

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Abstract. In this article we have presented a unique representation for interval arithmetic. The traditional interval arithmetic is transformed into crisp by symbolic parameterization. Then the proposed interval arithmetic is extended for fuzzy numbers and this fuzzy arithmetic is used as a tool for uncertain finite element method. In general, the fuzzy finite element converts the governing differential equations into fuzzy algebraic equations. Fuzzy algebraic equations either give a fuzzy eigenvalue problem or a fuzzy system of linear equations. The proposed methods have been used to solve a test problem namely heat conduction problem along with fuzzy finite element method to see the efficacy and powerfulness of the methodology. As such a coupled set of fuzzy linear equations are obtained. These coupled fuzzy linear equations have been solved by two techniques such as by fuzzy iteration method and fuzzy eigenvalue method. Obtained results are compared and it has seen that the proposed methods are reliable and may be applicable to other heat conduction problems too.

Keywords: finite element method; uncertainty; interval arithmetic; fuzzy number; fuzzy finite element method

1. Introduction

Uncertainty plays a vital role in various fields of engineering and science. These uncertainties occur due to incomplete data, impreciseness, vagueness, experimental error and different operating conditions influenced by the system. Different authors proposed various methods to handle uncertainty. They have used some probabilistic or statistical method as a tool to operate uncertain parameters. But in this process we need more number of observed data or experimental results to analyse the problem. Practically it may not be possible something to get a large number of data because it needs more number of experiments to perform. So instead of probabilistic or statistical we may use interval or fuzzy parameters to handle uncertainty which require less number of data. In general traditional interval/fuzzy arithmetic are complicated to manage the problem rigorously.

Accordingly, we have proposed here new techniques to handle such difficulty which are simple and efficient. The main aim of this paper is not the particular problem considered rather we shall concentrate on how the proposed methods can be applied to various heat conduction problems.

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Modelling of heat conduction problems may be represented by different types of differential equations. The governing differential equations although are solved earlier by various authors using exact methods (Carlslaw and Jaeger 1986, Liu *et al*. 1986 and Bondarev 1997, Monte 2000) and numerical methods (Peterson 1999, Iijima 2004, Liu and Cheng 2006). Also some probabilistic and statistical methods have also been introduced to calculate variability of involved parameters through the heat and mass transfer model. To handle such variability, several probabilistic methods have introduced. The Monte Carlo method is used to solve heat and mass propagation problem. It essentially involves a large number of process samples which are obtained by numerically solving the problem for artificially generated random parameter samples. Monte Carlo methods have been used to analyse thermal food processes with variable parameters (Wang *et al*. 1991, Varga *et al*. 2000, Caro-Corrales *et al*. 2002, Demir *et al*. 2003, Halder *et al*. 2007, Laguerre and Flick 2010). Deng and Liu (2002) implemented Monte Carlo method to solve the direct bio heat transfer problems. They have demonstrated the bio heat transfer problem with transient or space-dependent boundary conditions, blood perfusion, metabolic rate, and volumetric heat source for tissue.

On the other hand various numerical techniques are proposed viz. finite difference method (FDM), finite volume method (FVM) and finite element method (FEM) (Magnus *et al*. 2011, Muhieddine *et al*. 2009). Magnus *et al*. (2011) used finite difference method in his paper to model and solve the governing ground water flow rates, flow direction and hydraulic heads through an aquifer. Muhieddine *et al*. (2009) described one dimensional phase change problem. They have used vertex centred finite volume method to solve the problem. Edward and Robert (1966) used FEM to solve heat conduction problem and analyse it. A non-iterative, finite element-based inverse method for estimating surface heat flux histories on thermally conducting bodies is developed by (Ling *et al*. 2003). They considered both linear and non-linear problems, and sequentially minimizes the least square error norm between corresponding sets of measured and computed temperatures. Further Onate *et al*. (2006) used Galarkin FEM for convective-diffusive problems with sharp gradients using finite calculus. In view of the above literatures, it reveals that the traditional finite element method may easily be used where the parameters or the values are exact that is in crisp form. But in actual practice the values may be in a region of possibility or we can say the values are uncertain. These uncertain parameters give uncertain model predictions. Although the uncertainty may be reduced by appropriate experiments but still it may also give the variability in the parameters. Then finite element perturbation method is used by Nicolaï and De Baerdemaeker (1993) and Nicolaï *et al*. (2000) for heat conduction problem considering uncertain physical parameters. Further Nicolaï *et al*. (1999a,b) found the temperature in heat conduction problem for randomly varying parameters with respect to time. They (Nicolaï *et al*. (1999a,b)) have used a variance propagation technique to calculate the mean and covariance of the temperatures.

In particular, it is very difficult to get a large number of experimental data so we need an alternative method in which we may handle the uncertainty considering few experimental data. In this context Zadeh (1965) proposed an alternate idea i.e., fuzzy number to handle uncertain values. Hence we need the help of interval/fuzzy analysis for handling these types of data. The direct implementation of interval or fuzzy becomes more complex and the computation is also a difficult task. So to avoid such difficulty different authors tried new techniques to handle such difficulty. Dong and Shah (1987) proposed vertex method for computing functions of fuzzy variables. Dong and Wong (1987) used Fuzzy Weighted Average Method (FWAM). Yang *et al*. (1993) discussed the calculation of functions with fuzzy numbers. They developed methods which require less computation than the FWAM. Klir (1997) revised fuzzy arithmetic by considering the relevant requisite constraints. Hanss (2002) gave a transformation method based on the concept of α-cut where the fuzzy arithmetic is reduced to the interval computation.

Here we have taken the parameters in fuzzy and then it is converted into interval. The interval values are transformed into crisp form using a proposed transformation. Then we present the traditional finite element procedure (Neumaier 1990, Kulpa 1998 and Muhanna and Mullen 2001) for solving the problem by taking these parameters in interval. Next interval/fuzzy finite element technique is described for the said problem. Here we operate interval/fuzzy parameters through finite element method using proposed arithmetic. Fuzzy finite element method results into a set of algebraic equations. These set of algebraic equations will be a fuzzy system of simultaneous equations in this case. Matinfar *et al*. (2008) used householder decomposition method to solve fuzzy linear equations and they considered only the right hand side column vector as fuzzy and solved some example problems. For the fuzzy coefficient matrix A, Panahi *et al*. (2008) obtained lower triangular and upper triangular matrix separately. Senthilkumar and Rajendran (2011) considered symmetric coefficient matrix to solve Fuzzy Finite Linear System (FFLS) of equations. Here they decomposed the coefficient matrix by using Cholesky method. However Vijayalakshmi and Sattanathan (2011) introduced ST decomposition procedure to solve fully fuzzy system of linear equations. In this paper a simple arithmetic is presented to handle fuzzy system of linear equations. Further, Bart *et al*. (2011) investigated the uncertain solution of heat conduction problem and gave a good comparison between response surface method and other methods. Finite element method in the present problem turns into a system of coupled equations. These equations are difficult to solve directly. So we have used two different techniques to solve and these are iteration and eigenvalue methods respectively. These methods are broadly discussed in the fifth section. We have used a new form of fuzzy finite element method to solve the uncertain heat conduction problem. Finally we have given numerical example results and compared different methods in special cases.

2. Interval and fuzzy arithmetic

In view of the above literature we have seen that the parameters involved in systems are vague or uncertain. Considering these parameters as interval/fuzzy we may use the well-known interval/fuzzy arithmetic.

Let us take the values in interval form, and then the same may be written as

$$[\underline{x}, \bar{x}] = \{x : x \in \Re, \underline{x} \le x \le \bar{x}\}$$

Where $\underline{x}$ is the left value and $\bar{x}$ is the right value of the interval respectively. Let $m = \dfrac{\underline{x} + \bar{x}}{2}$ is the mid value and $w = \bar{x} - \underline{x}$ is the width of the interval $[\underline{x}, \bar{x}]$.

If $[\underline{x}, \bar{x}]$ and $[\underline{y}, \bar{y}]$ be two intervals then interval arithmetic may be written as

i. $[\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]$

ii. $[\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]$

iii. $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\}]$

iv. $[\underline{x}, \bar{x}] \div [\underline{y}, \bar{y}] = [\min\{\underline{x} \div \underline{y}, \underline{x} \div \bar{y}, \bar{x} \div \underline{y}, \bar{x} \div \bar{y}\}, \max\{\underline{x} \div \underline{y}, \underline{x} \div \bar{y}, \bar{x} \div \underline{y}, \bar{x} \div \bar{y}\}]$.

Fuzzy number

Let us consider a fuzzy set A then this fuzzy set A consists of a set of order pair. This order pair contains an element along with its membership function. The assigned membership function μ_A over a fuzzy set A is dened as $\mu_A : X \to [0,1]$, where [0,1] denotes the interval of real numbers from 0 to 1. Such a function is called a membership function. A fuzzy number is a convex, normalized fuzzy set $A \subseteq R$ which is piecewise continuous function. There are different types of fuzzy numbers depending upon the membership functions. Here we have taken only Triangular Fuzzy Number (TFN). In TFN the membership function is normalized at a unique point. If we consider symmetric TFN then the corresponding point which normalized the membership function is called the center.

A fuzzy number $\tilde{A} = [a^L, a^N, a^R]$ is said to be triangular fuzzy number when the membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le a^L \\ \dfrac{x-a^L}{a^N-a^L}, & a^L \le x \le a^N \\ \dfrac{a^R-x}{a^R-a^N}, & a^N \le x \le a^R \\ 0, & x \ge a^R \end{cases}$$

The Triangular Fuzzy Number $\tilde{A} = [a^L, a^N, a^R]$ may be transformed into interval form by using α-cut as follow.

$$\tilde{A} = [a^L, a^N, a^R] = [a^L + (a^N - a^L)\alpha, a^R - (a^R - a^N)\alpha], \quad \alpha \in [0,1]$$

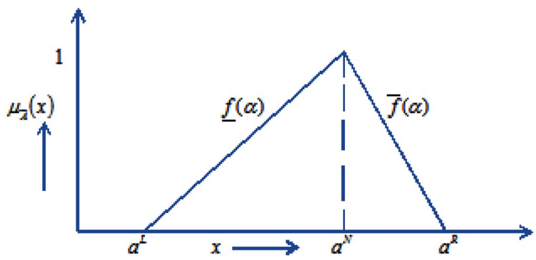

Fig. 1 Triangular Fuzzy Number (TFN)

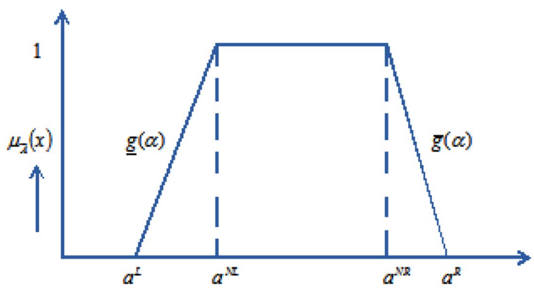

Fig. 2 Trapezoidal Fuzzy Number (TRFN)

A fuzzy number $\tilde{A} = [a^L, a^{NL}, a^{NR}, a^R]$ is said to be trapezoidal fuzzy number when the membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le a^L \\ \dfrac{x - a^L}{a^{NL} - a^L}, & a^L \le x \le a^{NL} \\ 1, & a^{NL} \le x \le a^{NR} \\ \dfrac{a^R - x}{a^R - a^{NR}}, & a^{NR} \le x \le a^R \\ 0, & x \ge a^R \end{cases}$$

Again, the trapezoidal fuzzy number in interval form may be represented as

$$\tilde{A} = [a^L, a^{NL}, a^{NR}, a^R] = [a^L + (a^{NL} - a^L)\alpha, a^R - (a^R - a^{NR})\alpha], \quad \alpha \in [0,1]$$

2.1 Proposed interval arithmetic

The traditional interval arithmetic is complicated and difficult to handle. When more number of computations is involved then the process becomes difficult to handle and the uncertainty rises. It is also difficult to formulate the methods in general. So a general form for this interval arithmetic which may properly handle the interval values as well as minimizes the uncertainty is presented below.

Let us consider $[\underline{a}, \bar{b}]$ be an arbitrary interval then $[\underline{a}, \bar{b}]$ can be written as $[\underline{a}, \bar{b}] = a + \dfrac{w}{n} = l$, where w is the width of the interval and $n \in [1,\infty)$.

If all the values of the interval are in R^+ or R^- then the arithmetic rules become

i. $[\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\min\{\lim_{n\to\infty} l_1 + \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 + \lim_{n\to 1} l_2\}, \max\{\lim_{n\to\infty} l_1 + \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 + \lim_{n\to 1} l_2\}]$

ii. $[\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\min\{\lim_{n\to\infty} l_1 - \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 - \lim_{n\to\infty} l_2\}, \max\{\lim_{n\to\infty} l_1 - \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 - \lim_{n\to\infty} l_2\}]$

iii. $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min\{\lim_{n\to\infty} l_1 \times \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 \times \lim_{n\to 1} l_2\}, \max\{\lim_{n\to\infty} l_1 \times \lim_{n\to\infty} l_2, \lim_{n\to 1} l_1 \times \lim_{n\to 1} l_2\}]$

iv. $[\underline{x}, \bar{x}] \div [\underline{y}, \bar{y}] = [\min\{\lim_{n\to\infty} l_1 \div \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 \div \lim_{n\to\infty} l_2\}, \max\{\lim_{n\to\infty} l_1 \div \lim_{n\to 1} l_2, \lim_{n\to 1} l_1 \div \lim_{n\to\infty} l_2\}]$

2.2 Proposed fuzzy arithmetic

Considering the above proposed interval arithmetic we may extend it for the fuzzy values. The fuzzy number is transformed into interval form then the interval arithmetic may be applied. The following paragraph describes about the fuzzy arithmetic.

The above defined triangular fuzzy number $\tilde{A} = [a^L, a^N, a^R]$ in previous section may be transformed into interval form by using α- cut as follow.

$$\tilde{A} = [a^L, a^N, a^R] = [a^L + (a^N - a^L)\alpha, a^R - (a^R - a^N)\alpha] = [\underline{f}(\alpha), \bar{f}(\alpha)], \quad \alpha \in [0,1]$$

Here $\underline{f}(\alpha)$ and $\bar{f}(\alpha)$ are monotonic increasing and decreasing function respectively. Using these functions one may modify the proposed interval arithmetic.

If the fuzzy numbers are taken in interval form then using the modified interval arithmetic, the arithmetic rules may be written as

1. $[\underline{x}(\alpha), \bar{x}(\alpha)] + [\underline{y}(\alpha), \bar{y}(\alpha)]$

$$= [\min\{\lim_{n\to\infty} m_1 + \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 + \lim_{n\to 1} m_2\}, \max\{\lim_{n\to\infty} m_1 + \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 + \lim_{n\to 1} m_2\}]$$

2. $[\underline{x}(\alpha), \bar{x}(\alpha)] - [\underline{y}(\alpha), \bar{y}(\alpha)]$

$$= [\min\{\lim_{n\to\infty} m_1 - \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 - \lim_{n\to\infty} m_2\}, \max\{\lim_{n\to\infty} m_1 - \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 - \lim_{n\to\infty} m_2\}]$$

3. $[\underline{x}(\alpha), \bar{x}(\alpha)] \times [\underline{y}(\alpha), \bar{y}(\alpha)]$

$$= [\min\{\lim_{n\to\infty} m_1 \times \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 \times \lim_{n\to 1} m_2\}, \max\{\lim_{n\to\infty} m_1 \times \lim_{n\to\infty} m_2, \lim_{n\to 1} m_1 \times \lim_{n\to 1} m_2\}]$$

4. $[\underline{x}(\alpha), \bar{x}(\alpha)] \div [\underline{y}(\alpha), \bar{y}(\alpha)]$

$$= [\min\{\lim_{n\to\infty} m_1 \div \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 \div \lim_{n\to\infty} m_2\}, \max\{\lim_{n\to\infty} m_1 \div \lim_{n\to 1} m_2, \lim_{n\to 1} m_1 \div \lim_{n\to\infty} m_2\}]$$

where for any arbitrary interval

$$[\underline{f}(\alpha), \bar{f}(\alpha)] = \left\{\underline{f}(\alpha) + \frac{\bar{f}(\alpha) - \underline{f}(\alpha)}{n} = m \,|\, \underline{f}(\alpha) \le m \le \bar{f}(\alpha), n \in [1, \infty)\right\}.$$

Similarly we may transform trapezoidal fuzzy numbers into interval form using alpha cut and arithmetic rules are defined by the same way as the triangular fuzzy numbers. This interval and fuzzy arithmetic may be used as tool to handle the uncertainty involved in system of uncertain (interval/fuzzy) linear equations. This system of linear equations is discussed in the next section and given a new method to solve interval/fuzzy system of linear equations.

3. System of interval/fuzzy linear equations

In mathematics often we come across the system of linear equations. In general for a practical problem, the coefficients involved in the coefficient matrix are uncertain. To handle such uncertainty we need some method to solve interval/fuzzy system of linear equations. Some authors have proposed new methods (as described in the introduction) which can handle these situations. But methods are depending on problem to problem, so we need a general method to solve uncertain system of linear equations which may apply to various problems. In view of these we have proposed a method to solve uncertain system of linear equations. Before that first we discussed a well-known method viz. the vertex method.

3.1 Vertex method

In this context, Dong and Shah (1987) introduced an excellent method known as vertex method for interval system of linear equations. Here combinations of the extremes of interval parameters are used. For N interval parameters this corresponds to 2^N simulations. Out of all the computations the lowest and highest values are selected. This method is very simple to implement and can be computationally efficient but it may become tedious if the number of vertices increases. The number of computation increases exponentially with the increase in number of interval parameters. Further, the vertex method is accurate only when the conditions of continuity are satisfied. Hence, it may not find optima corresponding to parameter combinations which are not on the vertex of the parameter hypercube. Now for all combinations one may use corresponding finite element method to investigate the solution hypercube.

3.2 Proposed method for system of interval/fuzzy linear equations

The uncertain parameters which are in intervals may be changed into symbolic crisp form by applying the above proposed transformation and the new representation of uncertain parameters involves mathematical limits. Now the crisp values run through the intervals. Using the concept of limits we can compute the interval arithmetic. This arithmetic may easily be computed for solving system of linear interval equations.

Let us consider the linear system as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= y_n \end{aligned} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$, $1 \le i \le n$; $1 \le j \le n$ is a $n \times n$ interval matrix and y_i, $1 \le i \le n$ are intervals, then the system is called interval system of linear equations.

The compact form of Eq. (1) may be presented in the following way

$$\sum_{j=1}^{n} [a_{ij}, b_{ij}]\{x_j\} = [c_j, d_j], \quad i = 1, 2, \ldots, n \tag{2}$$

By applying proposed method, Eq. (2) may be converted to the following crisp form

$$\sum_{j=1}^{n} (a_{ij} + \alpha_{ij})x_j = c_j + \beta_j, \quad i = 1, 2, 3, \ldots, n \tag{3}$$

where

$$\alpha_{ij} = \frac{b_{ij} - a_{ij}}{k} \text{ and } \beta_j = \frac{d_j - c_j}{k}, k \in [1, \infty)$$

Now the crisp system of linear Eq. (6) may easily be solved symbolically. Then the symbolic solutions are operated through the corresponding mathematical limit and the following solution vector may be obtained.

$$\begin{gathered}([\min\{\lim_{n\to 1} x_1, \lim_{n\to\infty} x_1\}, \max(\lim_{n\to 1} x_1, \lim_{n\to\infty} x_1)], [\min\{\lim_{n\to 1} x_2, \lim_{n\to\infty} x_2\}, \max(\lim_{n\to 1} x_2, \lim_{n\to\infty} x_2)] \\ \ldots, [\min\{\lim_{n\to 1} x_n, \lim_{n\to\infty} x_n\}, \max(\lim_{n\to 1} x_n, \lim_{n\to\infty} x_n)])^T.\end{gathered}$$

This concept may be extended to handle fuzzy values by using α-cut approach. Here the interval system of linear equations may be applied for the transformed fuzzy numbers into interval form. Then fuzzy system of linear equations may be solved and the solution vector for the corresponding system can be written as

$$([\lim_{n\to\infty} x_1(\alpha), \lim_{n\to 1} x_1(\alpha)], [\lim_{n\to\infty} x_2(\alpha), \lim_{n\to 1} x_2(\alpha)], \ldots, [\lim_{n\to\infty} x_n(\alpha), \lim_{n\to 1} x_n(\alpha)])^T.$$

It is found that number of computations becomes less as compare to the vertex method and this method is simple as well as efficient. This method may be applied to solve the uncertain system of linear equations which generally occurs when heat conduction problems are solved using interval/ fuzzy finite element method. In the next sections we will discuss about the finite element method for classical (crisp) and interval/fuzzy parameters respectively w.r.to heat conduction problem.

4. Finite element formulation and heat equation

The principle of conservation of energy says that the sum of input energy and energy generated is equal to the sum of increase in energy and the output energy. This principle may be applied to heat transfer. Generally we classify heat conduction in two ways; these are steady and unsteady state heat conduction. Consider a time dependent heat transfer problem in a three dimensional anisotropic solid Ω bounded by a surface Γ. Then the governing differential equation for this problem is given by

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+Q = \rho c\frac{\partial T}{\partial t} \tag{4}$$

where

q_x, q_y and q_z are components of heat flow rate per unit area in Cartesian coordinates, Q is the internal heat generation per unit volume, ρ is the density and c is the specific heat capacity. The finite element formulation for the above governing differential equation for one dimension is well known but are given below for the sake of completeness.

$$[C]\left\{\frac{dT}{dt}\right\}+[K]\{T\} = \{F_Q\}+\{F_g\} \tag{5}$$

If N is the shape function then the terms defined above in Eq. (2) are as follows

$$\begin{aligned}
[C] &= \int_{\Omega^{(e)}} \rho c\{N\}[N]d\Omega \\
[K_c] &= \int_{\Omega^{(e)}} [B]^T[k][B]d\Omega \\
[K_h] &= \int_{\Omega^{(e)}} \rho c\{N\}[N]d\Omega \\
\{F_Q\} &= \int_e Q\{N\}d\Omega \\
\{F_g\} &= -\int_\Gamma q.\hat{n}\{N\}d\Gamma
\end{aligned} \tag{6}$$

The general form of the governing equation for three dimensional time dependent heat conduction problem, may be written as

$$K_x \frac{\partial^2 T}{\partial x^2} + K_y \frac{\partial^2 T}{\partial y^2} + K_z \frac{\partial^2 T}{\partial z^2} + Q = c\rho \frac{\partial T}{\partial t} \tag{7}$$

The domain which satisfies the above equation is discretized into some finite numbers of elements. For every element we find the approximated function and then we get stiffness matrices for each element. For finding the stiffness matrices of complete domain we need to assemble each element. There are two different stiffness matrices (for each element) used for the said problem and these are capacitance and conductance matrices.

Considering one dimensional heat conduction problem (for example), the capacitance and conductance matrix for each element may be represented as $[C^{(e)}] = (c\rho AL/6)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $[K^{(e)}] = (kA/\mathrm{L})\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ respectively, where k is the thermal conductivity, c is specific heat, ρ is density, A is area of cross section and L is the length. The global stiffness matrices for capacitance and conductance for n elements are given as

$$[C] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & \vdots \\ 0 & 1 & 4 & 1 & 0 & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \ddots & 4 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix}_{(n+1)\times(n+1)} \quad \text{and } [K] = \frac{kA}{L}\begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \vdots \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & 0 & 0 & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}_{(n+1)\times(n+1)} \quad \text{respectively}$$

It may be noted that the above capacitance and conductance matrices are easy to generate for the crisp problem. But when we take the uncertainty in term of interval or fuzzy then these will involve the imprecise parameters. Then the corresponding equations will be difficult to handle. As said earlier, researchers proposed few methods to solve these but these are done for some special cases. Here the newly proposed method of uncertain value is generated and used in a simple problem of heat conduction. The main aim of this paper is not the example problem taken but the proposed new method which can be used for other complicated problem of heat conduction.

Consider the parameter vectors involved in Eqs. (4), (5) and (7) are interval, then the solution T would also be in interval and we get the approximate solution set T in a form of hypercube of dimension n. The main objective is to construct the smallest hypercube around the temperature T by using interval/fuzzy finite element method. Taking the help of above discussed proposed method using finite element methods we may construct the smallest hypercube around T. Next considering the proposed interval/fuzzy finite element method a simple heat conduction problem may be solved which is discussed in the succeeding sections.

5. Time dependent heat conduction problem

Heat conduction may be divided into two category i.e. steady and unsteady state. In steady state heat conduction problem we don't consider the time dependent term whereas unsteady state heat conduction is time dependent. These problems can be solved numerically by using numerical method such as finite difference and finite element methods etc. Here finite element method is considered to solve the example problem. Again the governing differential equations for heat conduction problems involve uncertainty. Hence to solve uncertain governing differential equations we have presented a proposed interval/fuzzy finite element method. After using interval/fuzzy finite element method we will get a system of interval/fuzzy ordinary differential equations. We have considered two approaches to solve this system of interval/fuzzy ordinary differential equations. These approaches are discussed in the following subsections.

5.1 Iterative method

Consider a general first order time dependent differential equation as follows

$$[\tilde{C}]\{\dot{\tilde{T}}\} + [\tilde{K}]\{\tilde{T}\} = [\tilde{R}(t)] \tag{8}$$

where $\tilde{C}, \tilde{K}, \tilde{R}$ are fuzzy.

This can be solved by the method of iteration. The procedure relies on deriving the recursion formula that relates the value $\{\tilde{T}\}$ at one instant of time t to the values of $\{\tilde{T}\}$ at time $t + \Delta t$, where Δt is the time step. The recursion formula make it possible for the solution to be marched out in time, starting from the initial condition at time $t = 0$ sec. and continuing step by step until reaching the desired duration.

Let $t_{n+1} = t_n + \Delta t$, where $n = 0, 1, 2, ..., N$.

Now applying finite difference approximation with a time step $t = 1$ sec., Eq. (8) becomes

$$[\tilde{C}]\frac{\{\tilde{T}\}_{n+1} - \{\tilde{T}\}_n}{\Delta t} + [\tilde{K}]\{\tilde{T}\}_n = [\tilde{R}(t)]$$

$$or,\ \{\tilde{T}\}_{n+1} = \Delta t[\tilde{R}(t)][\tilde{C}]^{-1} - \Delta t[\tilde{K}][\tilde{C}]^{-1}\{\tilde{T}\}_n + \{\tilde{T}\}_n \tag{9}$$

Using this algorithm we may found the value of $\{\tilde{T}\}$ which converges to the desired values depending upon the relative error of temperatures.

5.2 Eigenvalue method

Eigenvalue method is widely used for second order matrix equations encountered in structural dynamic problems. The same method is extracted for first order differential equations viz. Eq. (8) also discussed below.

So, let us consider the first order differential Eq. (8)

Substituting $\{\tilde{T}\} = \{\tilde{P}\}e^{-\lambda t}$ in Eq. (7) and taking $\tilde{R}(t) = 0$ we get

$$[[\tilde{K}] - \lambda[\tilde{C}]]\{\tilde{T}\} = 0 \tag{10}$$

where $\{\tilde{P}\}$ is a modal vector and λ is a modal decay constant.

This is similar to eigenvalue problem. For $n \times n$ matrices there will be n values of λ_i (eigenvalues) and n values of $\{\tilde{P}\}_i$ (eigenvectors).

For two different eigenvectors $\{\tilde{P}\}_i$ and $\{\tilde{P}\}_j$ corresponding to two eigenvalues $\tilde{\lambda}_i$ and $\tilde{\lambda}_j$ respectively, we have the orthogonal condition as

$$\begin{aligned} \{\tilde{P}\}_i[\tilde{K}]\{\tilde{P}\}_j &\neq 0, \;\; i \neq j \\ \{\tilde{P}\}_i[\tilde{C}]\{\tilde{P}\}_j &\neq 0, \;\; i \neq j \end{aligned} \tag{11}$$

For $i = j$ we have

$$\begin{aligned} \{\tilde{P}\}_i[\tilde{K}]\{\tilde{P}\}_i &= K_{ii}^* \\ \{\tilde{P}\}_i[\tilde{C}]\{\tilde{P}\}_i &= C_{ii}^* \end{aligned} \tag{12}$$

where K_{ii}^* and C_{ii}^* are constants.

Now a transformation $\{\tilde{T}\} = [\tilde{P}]\{\tilde{y}\}$ in Eq. (7) is used where P is a modal matrix containing the eigenvectors as columns and it becomes

$$[\tilde{C}][\tilde{P}]\{\dot{\tilde{y}}\} + [\tilde{K}][\tilde{P}]\{\tilde{y}\} = \{\tilde{R}\} \tag{13}$$

Multiplying $[P]^T$ both the sides of Eq. (13), we get

$$[\tilde{P}]^T[\tilde{C}][\tilde{P}]\{\dot{\tilde{y}}\} + [\tilde{P}]^T[\tilde{K}][\tilde{P}]\{\tilde{y}\} = [\tilde{P}]^T\{\tilde{R}\} \tag{14}$$

In view of Eqs. (11) and (12) in (14) we will get a system of first order ordinary differential equations which may easily be solved by any standard methods.

6. Test problem

Let us consider a cylindrical rod whose right end of the rod is provided at a constant temperature 30ºC. At time zero, the entire rod is at a temperature of 30ºC when a heat source is applied to the left end, bringing the temperature of the left end immediately to 80ºC and maintaining that temperature indefinitely. Corresponding data for this problem is provided in Table 1.

It is very well known from Fick's law of diffusion that the diffusive flux goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient. We can easily say that the diffusive flux tries to attain an equilibrium position. Here we have taken the said propagation problem where the temperature of the rod propagates and want to stable. By using the iteration scheme we have found that the temperatures become stable after some iterations depending upon the error. The crisp results of iteration method for the said problem are given in the Table 2.

Table 1 Fuzzy parameters for Test problem

Parameters	Notations	Normalized value	Fuzzy value
Diameter	D	12 mm	12 mm
Length	l	100 mm	100 mm
Density	d	2700 kg/m^3	2700 kg/m^3
Thermal conductivity	k	200 W/m–ºC	[195, 200, 205] W/m–ºC
Specific heat capacity	c	900 J/kg–ºC	[895, 900, 905] J/kg–ºC

Table 2 Temperatures at different level of relative errors (with crisp variables)

Temperature	Error (0.3%)	Error (0.1%)	Error (0.03%)
T_2	66.0145	66.9731	67.3288
T_3	52.8949	54.2506	54.7536
T_4	41.0097	41.9683	42.3240
Number of iterations	30	42	56

Table 3 Temperatures (with uncertain fuzzy variables) at different level of relative errors

Temperature		Error (0.3%)	No. of iterations	Error (0.1%)	No. of iterations	Error (0.03%)	No. of iterations
T_2	Left	66.1115	31	67.0332	43	67.3705	55
	Right	66.1259	30	67.0480	42	67.3835	57
	Center	66.1187	30	67.0406	42	67.377	56
T_3	Left	53.0411	31	54.3445	43	54.8188	55
	Right	53.0587	30	54.3628	42	54.8399	57
	Center	53.0499	30	54.35365	42	54.82935	56
T_4	Left	41.1164	31	42.0381	43	42.3710	55
	Right	41.1264	30	42.0485	42	42.3884	57
	Centre	41.1214	30	42.0433	42	42.3797	56

The iterations are taken with a time interval of 1 sec. each. The temperatures converge if we take 30, 42, 56 iterations for relative errors 0.3%, 0.1% and 0.03% respectively. Here temperature starts from initial value which is given in the said problem and then transported through the rod and maintain a constant temperature after a certain time. At different level of relative errors one can see that the temperatures at the nodes (T_2, T_3, T_4) differ with a little margin. But in real practice the parameters used for the said problem are not in crisp form rather in interval/fuzzy. So considering the fuzzy value of involved parameters we may evaluate the nodal temperature. Considering different relative errors the resultant temperature distribution are shown in Table 2. When value of c and k both are taken as fuzzy, we observe that the corresponding temperatures changes slightly with the variations of relative errors. Again the uncertainty of temperatures at the stage of stability becomes narrow. If we compare the results of Table 3 with Table 2, we get the center value of Table 3 very close to the value of the Table 2. The time taken for stability is also same if we compare with the time taken in Table 2.

In comparison with the vertex method, proposed method may be a better alternative. The number of computations in vertex method is 2^N, where N is the number of fuzzy parameters used and in our approach only two computations are needed. Again the presence of limits becomes easy to handle. By this we do not have to check all the combinations for the uncertain parameters except for the two combinations.

Next, applying the above defined eigenvalue method we may also solve the said problem. The modeled differential equation gives a coupled set of differential equations which may sometimes be difficult to solve. So we have used eigenvalue method which transforms the coupled equations into uncoupled equations. Now these uncoupled equations give a system of first order linear differential equation which has been solved by using the usual method. Again we have included in the differential equation the fuzzy parameters and using proposed method the temperatures are found.

Table 4. Temperatures (for uncertain fuzzy parameters) with 20 sec. of intervals

Temperature		T_2	T_3	T_4
20 sec.	Left	63.4823	49.3429	38.4935
	Right	63.7662	49.7382	38.7727
	Center	63.62425	49.54055	38.6331
40 sec.	Left	66.7242	53.9212	41.7246
	Right	66.8555	54.1027	41.8541
	Center	66.78985	54.01195	41.78935
60 sec.	Left	67.3286	54.7762	42.329
	Right	67.396	54.8672	42.3946
	Center	67.3623	54.8217	42.3618
80 sec.	Left	67.4414	54.9359	42.4418
	Right	67.4907	55.0011	42.4892
	Center	67.46605	54.9685	42.4655
100 sec.	Left	67.4625	54.9657	42.4629
	Right	67.5072	55.0246	42.5058
	Center	67.48485	54.99515	42.48435

Table 5 Comparison of temperatures between Exact and Centre

Temperature		T_2	T_3	T_4
20 sec.	Center	63.62425	49.54055	38.6331
	Exact	63.6291	49.5359	38.6391
40 sec.	Center	66.78985	54.01195	41.78935
	Exact	66.7939	54.0055	41.7947
60 sec.	Center	67.3623	54.8217	42.3618
	Exact	67.3655	54.8139	42.3663
80 sec.	Center	67.46605	54.9685	42.4655
	Exact	67.4689	54.9602	42.4697
100 sec.	Center	67.48485	54.99515	42.48435
	Exact	67.4876	54.9866	42.4884

For different nodes of the rod the temperatures are given in Table 4. Fuzzy parameters for the said problem are same as used for Table 3. The solutions of the first ordered fuzzy linear differential equations depend on time. Here we have considered the temperatures at time 20, 40, 60, 80 and 100 sec. respectively. As usual, we observed that the nodal temperatures get closer if the systems spend more time. As we increase the time the uncertainty width of the solution temperature becomes small which suggest that stability condition getting better and better by spending more time.

Next, we have given a comparison between the center solutions of fuzzy with the crisp solution in Table 5. The center solutions of uncertain fuzzy values and the exact solutions of crisp values at 20, 40, 60, 80 and 100 sec. are presented in Table 5. We can easily see that the center and exact values are very close which means the uncertain fuzzy values are giving good results. The fuzzy results are given in Figs. 3 to 5. Here five different membership functions are given for corresponding time 20,

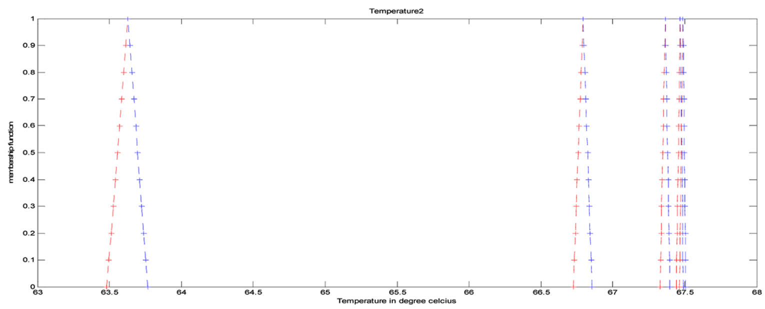

Fig. 3 Membership functions of temperature T_2 at a time difference of 20 sec

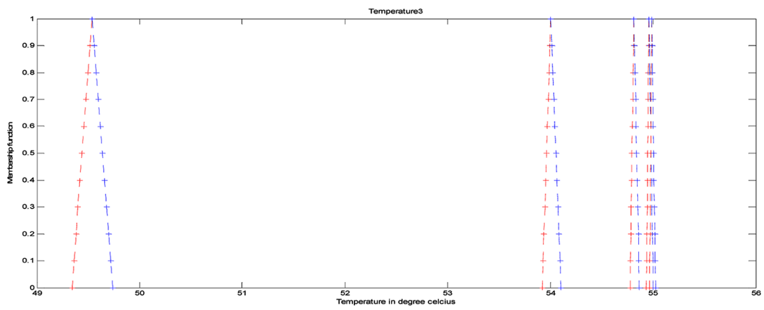

Fig. 4 Membership functions of temperature T_3 at a time difference of 20 sec

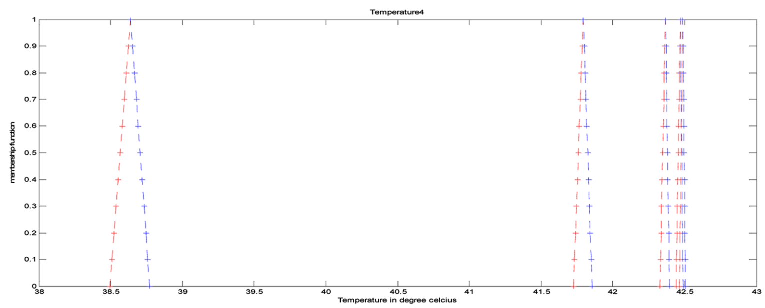

Fig. 5 Membership functions of temperature T_4 at a time difference of 20 sec

40, 60, 80 and 100 sec. The membership functions getting closer and closer if we move with time. For vertex method we need more complication to solve differential equations. More number of fuzzy parameters will need large number of calculations. To find the bound of uncertainty in vertex method we have to calculate maximum and minimum values out of all the possible solutions. Whereas, we need only two computations to find the uncertain bound of solutions using the proposed method. Also the differential equation is much easier to handle by the eigenvalue method.

7. Conclusions

The main aim of this paper is to provide an alternative non probabilistic method to operate uncertain parameters involved in the system. The traditional interval arithmetic is transformed into a unique way and a new method is proposed for interval arithmetic. This interval arithmetic is then extended for triangular fuzzy numbers and is used in finite element method. It may also be noted that the above interval arithmetic may also be extended for trapezoidal fuzzy numbers in the same way using alpha cut techniques. The resulting fuzzy finite element method is used to solve a test problem as discussed. Corresponding differential equation for the said problem is converted into fuzzy algebraic equations by the use of fuzzy finite element method. Hence we get a coupled set of fuzzy linear equations. Due to the occurrence of difficulty to manage coupled set of fuzzy linear equations we have proposed two techniques viz. fuzzy iteration method and fuzzy eigenvalue method. Obtained results are compared in both the techniques and it is found that the proposed methods have great utility. It has been seen that the proposed method is better in comparison with the well-known vertex method. We observed that the number of computations and the time taken is less in comparison with the vertex methods. Further the symbolic form of the proposed methods may give better approximate results for various points in the interval as well as in the bound form for different problems.

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