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The effect of internal axial forces of a cantilever beam with a lumped mass at its free end

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Abstract. When a cantilever beam with a lumped mass at its free end undergoes free transverse vibration, internal axial forces are produced in the beam. Such internal axial forces have an effect on free transverse vibration of the beam. This effect is studied in this paper. The equations of motion for the beam in terms of the generalized coordinates including the effect are derived. The method for determining free transverse vibration of the beam including the effect is presented. In numerical simulations, the results of free transverse vibration of the free end of the beam including and not including the effect are obtained. Based on comparison between the results obtained, the conclusions concerning the effect are given.

Keywords: cantilever beam; lumped mass; free transverse vibration; internal axial force; curve motion

1. Introduction

Let us consider a cantilever beam with a lumped mass at its free end in free transverse vibration as shown in Fig. 1. Since the lumped mass moves along a curve B_0B_1 instead of a straight line B_0B_2 during the vibration, the lumped mass has an axial acceleration component, and thus there exist internal axial forces in the beam. Such internal axial forces, which are produced by curve motion of the lumped mass, have a certain effect on free transverse vibration of the beam. It is noted that although the analyses of free transverse vibration of cantilever beams with a lumped mass at the free end are given in many vibration references (Ni 1989, Geradin and Rixen 1997, Thomson and Dahleh 1997, Mobley 1999, Meirovitch 2001, Dukkipati and Srinivas 2004, Svetlitsky 2005, Rao 2007, Kelly 2012, Magrab 2012, Mao and Chen 2016, Béri *et al*. 2017), the effect of internal axial forces produced by curve motion of the lumped mass on free transverse vibration of the beams is not considered in these references. In the present paper, this effect is studied with two objectives in mind. First is to show how to determine free transverse vibration of the beams including the effect. The second objective is to show that internal axial forces produced by curve motion of the lumped mass have a certain effect on free transverse vibration of the beam. This paper is organized as follows. In Sec. 2, taking into account the effect of the internal axial forces, the equations of motion for the beam in terms of the generalized coordinates are derived.

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Then, the method for determining free transverse vibration of the beam including the effect is presented. In Sec. 3, a numerical simulation example is given, with comparison between the results including and not including the effect. Finally, in Sec. 4, conclusions are given.

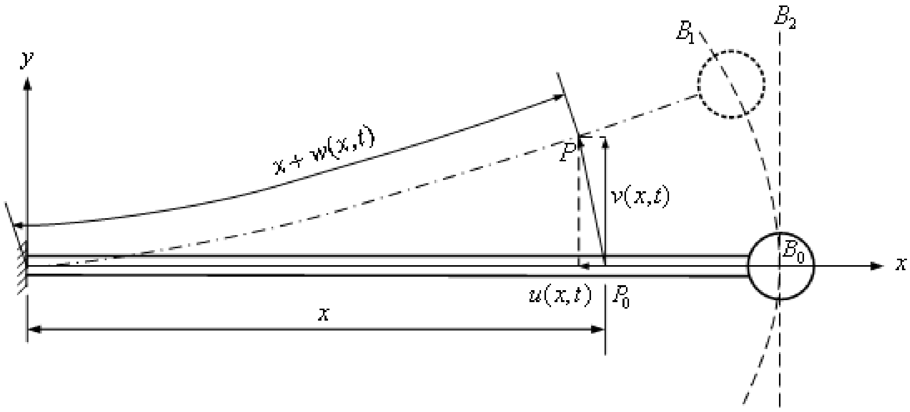

Fig. 1 Cantilever beam with a lumped mass at its free end

2. Derivation of the equations of motion

We consider a uniform cantilever beam with a lumped mass at its free end in free transverse vibration shown in Fig. 1, where $u(x,t)$ and $v(x,t)$ denote the longitudinal and transverse deflections of the beam, respectively. It is noticeable that during free transverse vibration of the beam, there is not only the transverse deflection but also the longitudinal deflection (transverse and longitudinal deflections are coupled). In order to derive the equation of motion for the beam using Newtonian approach, the free-body diagram of a beam differential element at an arbitrary instant is shown in Fig. 2, in which N is the axial force, Q the shearing force and M the bending moment, θ the rotation of the cross section. According to sign conventions, N, Q, M, θ, $u(x,t)$ and $v(x,t)$ are regarded as positive in this figure.

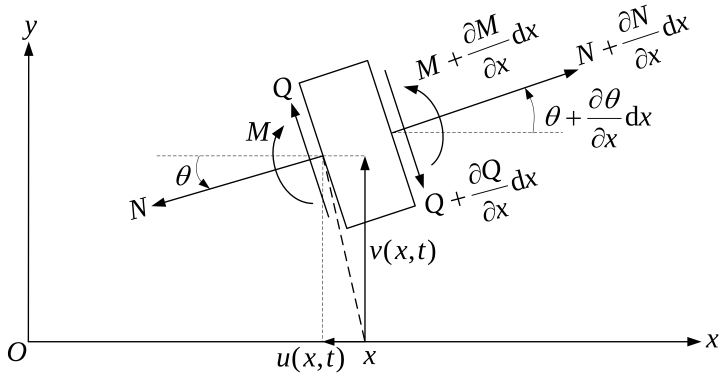

Fig. 2 Free-body diagram of a beam differential element at an arbitrary instant

From Fig. 2, the equation of motion for the beam differential element in the y-direction is obtained by using Newton's second law as

$$(\rho A\mathrm{d}x)\frac{\partial^2 v(x,t)}{\partial^2 t} = -N\sin\theta + Q\cos\theta + (N + \frac{\partial N}{\partial x}\mathrm{d}x)\sin(\theta + \frac{\partial \theta}{\partial x}\mathrm{d}x) - (Q + \frac{\partial Q}{\partial x}\mathrm{d}x)\cos(\theta + \frac{\partial \theta}{\partial x}\mathrm{d}x) \qquad (1)$$

where ρ is the mass density of the beam, A is the cross-sectional area of the beam. Only small deformation of the beam is considered here, so we have

$$\sin\theta \approx \theta \qquad (2)$$

$$\cos\theta \approx 1 \qquad (3)$$

$$\sin(\theta + \frac{\partial \theta}{\partial x}\mathrm{d}x) \approx \theta + \frac{\partial \theta}{\partial x}\mathrm{d}x \qquad (4)$$

$$\cos(\theta + \frac{\partial \theta}{\partial x}\mathrm{d}x) \approx 1 \qquad (5)$$

$$\theta \approx \frac{\partial v}{\partial x} \qquad (6)$$

Substituting Eqs. (2)-(5) into Eq. (1), ignoring second-terms in dx and dividing the resulting equation by $\mathrm{d}x$ leads to

$$\rho A\frac{\partial^2 v}{\partial t^2} = -\frac{\partial Q}{\partial x} + \theta\frac{\partial N}{\partial x} + N\frac{\partial \theta}{\partial x} \qquad (7)$$

Referring to Fig. 2, neglecting the rotary inertia of the beam differential element (the Euler-Bernoulli beam assumption), we can obtain the sum of the moments about the left end of the element as follows

$$\left(M + \frac{\partial M}{\partial x}\mathrm{d}x\right) - M - \left(Q + \frac{\partial Q}{\partial x}\mathrm{d}x\right)\mathrm{d}x = 0 \qquad (8)$$

Ignoring second-term in $\mathrm{d}x$ and canceling appropriate terms, Eq. (8) is reduced to

$$Q = \frac{\partial M}{\partial x} \qquad (9)$$

Substituting Eq. (9) and the Euler-Bernoulli formula $M = EI\frac{\partial^2 v}{\partial x^2}$ into Eq. (7) leads to

$$\rho A\frac{\partial^2 v}{\partial t^2} = -EI\frac{\partial^4 v}{\partial x^4} + \underline{\frac{\partial (N\theta)}{\partial x}} \qquad (10)$$

The underlined term in this equation represents the effect of the internal axial forces on free transverse vibration of the beam. Note that in the conventional analyses of free transverse vibration of the beam given in many vibration references (Ni 1989, Geradin and Rixen 1997, Thomson and

Dahleh 1997, Mobley 1999, Meirovitch 2001, Dukkipati and Srinivas 2004, Svetlitsky 2005, Rao 2007, Kelly 2012, Magrab 2012, Mao and Chen 2016), the underlined term $\underline{\frac{\partial(N\theta)}{\partial x}}$ is omitted. Therefore, in these conventional analyses, the effect of the internal axial forces on free transverse vibration of the beam is not taken into account. In the present paper, the effect of the internal axial forces on free transverse vibration of the beam is taken into consideration.

The internal axial force N can be expressed as

$$N = EA\frac{\partial w(x,t)}{\partial x} \tag{11}$$

where $w(x,t)$ (see Fig. 1) is the extension of the beam. Substituting Eqs. (11) and (6) into Eq. (10), we can obtain equation

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + \rho A\frac{\partial^2 v(x,t)}{\partial t^2} - EA\frac{\partial v(x,t)}{\partial x}\cdot\frac{\partial^2 w(x,t)}{\partial x^2} - EA\frac{\partial^2 v(x,t)}{\partial x^2}\cdot\frac{\partial w(x,t)}{\partial x} = 0 \tag{12}$$

This is the equation of free transverse vibration $v(x,t)$ of the beam. Similarly, from Fig. 2, we can also obtain the equation of free longitudinal vibration $u(x,t)$ of the beam as

$$\rho A\frac{\partial^2 u(x,t)}{\partial t^2} + EA\frac{\partial^2 w(x,t)}{\partial x^2} + EI\frac{\partial^4 v(x,t)}{\partial x^4}\cdot\frac{\partial v(x,t)}{\partial x} + EI\frac{\partial^3 v(x,t)}{\partial x^3}\cdot\frac{\partial^2 v(x,t)}{\partial x^2} = 0 \tag{13}$$

Considering the foreshortening of the beam due to bending, $u(x,t)$ can be expressed as (Li *et al*. 2010, Choi and Yoo 2017)

$$u(x,t) = \frac{1}{2}\int_0^x\left(\frac{\partial v(\xi,t)}{\partial \xi}\right)^2 \mathrm{d}\xi - w(x,t) \tag{14}$$

where the first term of the right side of this equation is the foreshortening of the beam due to bending. Substituting Eq. (14) into Eq. (13) results in

$$\begin{aligned}\rho A\frac{\partial^2 w(x,t)}{\partial t^2} - EA\frac{\partial^2 w(x,t)}{\partial x^2} - EI\frac{\partial^4 v(x,t)}{\partial x^4}\cdot\frac{\partial v(x,t)}{\partial x} - EI\frac{\partial^3 v(x,t)}{\partial x^3}\cdot\frac{\partial^2 v(x,t)}{\partial x^2}\\ - \rho A\int_0^x\frac{\partial}{\partial t}\left(\frac{\partial v(\xi,t)}{\partial \xi}\cdot\frac{\partial^2 v(\xi,t)}{\partial \xi\,\partial t}\right)\mathrm{d}\xi = 0\end{aligned} \tag{15}$$

This equation can be regard as the equation of free axial vibration $w(x,t)$ of the beam.

Eqs. (12) and (15) form the set of free transverse and axial vibration equations of motion for the beam including the effect of the internal axial forces. Noting that these two equations have both variables $v(x,t)$ and $w(x,t)$, we know that free transverse and axial vibrations of the beam are coupled.

Since the set of Eqs. (12) and (15) is continuous and nonlinear, a discretization process using the assumed-mode method (Cha and Hu 2017, Li and Song 2014) is employed here. According to this method, $v(x,t)$ and $w(x,t)$ can be expressed as

$$v(x,t) = \sum_{i=1}^{n}\phi_{1i}(x)\,q_{1i}(t) \tag{16}$$

$$w(x,t)=\sum_{i=1}^{n}\phi_{2i}(x)\,q_{2i}(t) \tag{17}$$

where $\phi_{1i}(x)$ and $q_{1i}(t)$ are the assumed-mode functions and generalized coordinates of transverse vibration of the beam, respectively, $\phi_{2i}(x)$ and $q_{2i}(t)$ are the assumed-mode functions and generalized coordinates of axial vibration of the beam, respectively, n is the number of the assumed-modes selected. If the mode shapes of transverse vibration of uniform cantilever beams with a lumped mass at its free end are chosen as $\phi_{1i}(x)$, then $\phi_{1i}(x)$ can be expressed as (Ni 1989)

$$\phi_{1i}(x)=\cos\beta_i x-\cosh\beta_i x+\gamma_i(\sin\beta_i x-\sinh\beta_i x),\ \ i=1,2,\cdots,n \tag{18}$$

where

$$\gamma_i=-\frac{\cos\beta_i l+\cosh\beta_i l}{\sin\beta_i l+\sinh\beta_i l},\quad i=1,2,\cdots,n \tag{19}$$

and l is the length of the beam, β_i are the roots of the frequency equation

$$\rho A(1+\cos\beta l\cosh\beta l)=m\beta(\sin\beta l\cosh\beta l-\cos\beta l\sinh\beta l) \tag{20}$$

in which m is the lumped mass at the free end.

If the mode shapes of axial vibration of uniform cantilever beams with a lumped mass at its free end are chosen as $\phi_{2i}(x)$, then $\phi_{2i}(x)$ can be expressed as (Ni 1989)

$$\phi_{2i}(x)=\sin\frac{b_i x}{l},\ \ i=1,2,\cdots,n \tag{21}$$

where b_i are the roots of the frequency equation

$$b\tan b=\frac{\rho A l}{m} \tag{22}$$

Substituting Eqs. (16) and (17) into Eq. (12), multiplying the resulting equation by $\phi_{1k}(x)(k=1,2,\cdots,n)$ and integrating over the length of the beam leads to

$$\rho A\sum_{i=1}^{n}a_{ki}\ddot{q}_{1i}(t)+EI\sum_{i=1}^{n}c_{ki}q_{1i}(t)-EA\sum_{i=1}^{n}\sum_{j=1}^{n}h_{kij}q_{1i}(t)q_{2j}(t)=0\qquad k=1,2,\cdots,n \tag{23}$$

where

$$a_{ki}=\int_0^l\phi_{1k}(x)\phi_{1i}(x)\mathrm{d}x\qquad k,i=1,2,\cdots,n \tag{24}$$

$$c_{ki}=\int_0^l\phi_{1k}(x)\phi_{1i}^{(4)}(x)\mathrm{d}x\qquad k,i=1,2,\cdots,n \tag{25}$$

$$h_{kij}=\int_0^l\phi_{1k}(x)[\phi_{1i}'(x)\phi_{2j}''(x)+\phi_{1i}''(x)\phi_{2j}'(x)]\mathrm{d}x\qquad k,i,j=1,2,\cdots,n \tag{26}$$

Similarly, substituting Eqs. (16) and (17) into Eq. (15), multiplying the resulting equation by

$\phi_{2k}(x)(k=1,2,\cdots,n)$ and integrating over the length of the beam leads to

$$\begin{aligned}&\rho A\sum_{i=1}^{n} f_{ki}\ddot{q}_{2i}(t) - EA\sum_{i=1}^{n} g_{ki}q_{2i}(t) - EI\sum_{i=1}^{n}\sum_{j=1}^{n} r_{kij}q_{1i}(t)q_{1j}(t)\\ &-\rho A\sum_{i=1}^{n}\sum_{j=1}^{n} s_{kij}[\dot{q}_{1i}(t)\dot{q}_{1j}(t)+q_{1i}(t)\ddot{q}_{1j}(t)]=0 \qquad k=1,2,\cdots,n\end{aligned} \tag{27}$$

where

$$f_{ki}=\int_0^l \phi_{2k}(x)\phi_{2i}(x)\mathrm{d}x \qquad k,i=1,2,\cdots,n \tag{28}$$

$$g_{ki}=\int_0^l \phi_{2k}(x)\phi''_{2i}(x)\mathrm{d}x \qquad k,i=1,2,\cdots,n \tag{29}$$

$$r_{kij}=\int_0^l \phi_{2k}(x)[\phi_{1i}^{(4)}(x)\phi'_{1j}(x)+\phi_{1i}^{(3)}(x)\phi''_{1j}(x)]\mathrm{d}x \qquad k,i,j=1,2,\cdots,n \tag{30}$$

$$s_{kij}=\int_0^l \phi_{2k}(x)[\int_0^x \phi'_{1i}(\xi)\phi'_{1j}(\xi)d\xi]\mathrm{d}x \qquad k,i,j=1,2,\cdots,n \tag{31}$$

Eqs. (23) and (27) are the equations of motion for the beam in terms of the generalized coordinates including the effect of the internal axial forces. The set of these equations can be numerically integrated by using the standard ode45 solver (Butt 2009) to determine the numerical solutions of $q_{1i}(t)$ and $q_{2i}(t)$ if the initial values $q_{1i}(0)$, $q_{2i}(0)$, $\dot{q}_{1i}(0)$ and $\dot{q}_{2i}(0)$ $(i=1,2,\cdots,n)$ are known. Now consider how to determine these values. Letting $t=0$ in Eq. (16) yields

$$v(x,0)=\sum_{i=1}^{n}\phi_{1i}(x)q_{1i}(0) \tag{32}$$

Multiplying this equation by $\phi_{1k}(x)(k=1,2,\cdots,n)$ and integrating over the length of the beam leads to

$$d_k=\sum_{i=1}^{n} a_{ki}q_{1i}(0) \qquad k=1,2,\cdots,n \tag{33}$$

where a_{ki} is defined by Eq. (24), and d_k defined by

$$d_k=\int_0^l v(x,0)\phi_{1k}(x)\mathrm{d}x \qquad k=1,2,\cdots,n \tag{34}$$

By solving the set of the linear algebraic Eq. (33), one can determine the values of $q_{1i}(0)\,(i=1,2,\cdots,n)$. Similarly, one can determine the values of $\dot{q}_{1i}(0)$, $q_{2i}(0)$ and $\dot{q}_{2i}(0)\,(i=1,2,\cdots,n)$ by solving the sets of the linear algebraic equations (35), (36) and (37), respectively.

$$D_k=\sum_{i=1}^{n} a_{ki}\dot{q}_{1i}(0) \qquad k=1,2,\cdots,n \tag{35}$$

$$e_k = \sum_{i=1}^{n} f_{ki} q_{2i}(0) \qquad k = 1, 2, \cdots, n \tag{36}$$

$$E_k = \sum_{i=1}^{n} f_{ki} \dot{q}_{2i}(0) \qquad k = 1, 2, \cdots, n \tag{37}$$

where f_{ki}, D_k, e_k and E_k are defined by Eqs. (28), (38), (39) and (40), respectively.

$$D_k = \int_0^l \dot{v}(x,0)\phi_{1k}(x)\mathrm{d}x \qquad k = 1, 2, \cdots, n \tag{38}$$

$$e_k = \int_0^l w(x,0)\phi_{2k}(x)\mathrm{d}x \qquad k = 1, 2, \cdots, n \tag{39}$$

$$E_k = \int_0^l \dot{w}(x,0)\phi_{2k}(x)\mathrm{d}x \qquad k = 1, 2, \cdots, n \tag{40}$$

Having determined the numerical solutions of $q_{1i}(t)$ $(i = 1, 2, \cdots, n)$, free transverse vibration of the beam including the effect of the internal axial forces can be determined from Eq. (16). Based on the above formulas, the method for determining free transverse vibration of the beam including the effect of the internal axial forces can be summarized as follows:

1) Determine the functions $\phi_{1i}(x)$ and $\phi_{2i}(x)$ $(i = 1, 2, \cdots, n)$ using Eqs. (18) and (21), respectively.

2) Determine the values of $a_{ki}, c_{ki}, h_{kij}, f_{ki}, g_{ki}, r_{kij}, s_{kij}, d_k, D_k, e_k$ and E_k $(k, i, j = 1, 2, \cdots, n)$ from Eqs. (24)-(26), (28)-(31), (34), (38)-(40), respectively.

3) Determine the values of $q_{1i}(0)$, $\dot{q}_{1i}(0)$, $q_{2i}(0)$ and $\dot{q}_{2i}(0)$ $(i = 1, 2, \cdots, n)$ by solving the sets of the linear algebraic equations (33), (35)-(37), respectively.

4) Determine the numerical solutions of $q_{1i}(t)$ and $q_{2i}(t)$ $(i = 1, 2, \cdots, n)$ by numerically integrating the set of ordinary differential equations (23) and (27) using the standard ode45 solver (Butt 2009).

5) Finally, determine free transverse vibration of the beam including the effect of the internal axial forces using Eq. (16).

3. Numerical simulation

To illustrate the effect of internal axial forces produced by curve motion of a lumped mass at the free end of a uniform cantilever beam, the following example is considered.

Example: the parameters of a uniform cantilever beam with a lumped mass at its free end considered here as shown in Fig. 1 are as follows: length $l = 0.5\mathrm{m}$, cross-sectional area $A = 1\times10^{-5}\mathrm{m}^2$, second moments of area $I = 8.3333\times10^{-13}\mathrm{m}^4$, density $\rho = 7.866\times10^3\mathrm{kg/m}^3$, Young modulus $E = 2.01\times10^{11}\mathrm{N/m}^2$, lumped mass $m = k\rho Al$ (where $k = 0, 1, \cdots, 5$ represents the ratio of the lumped mass to the mass of the beam). The initial conditions of the beam are $v(x,0) = \dfrac{l\phi_{11}(x)}{8\phi_{11}(l)}$ (where $\phi_{11}(x)$ is given by Eq. (18)), $\dot{v}(x,0) = 0$, $w(x,0) = 0$ and $\dot{w}(x,0) = 0$.

Determine free transverse vibration response of the free end of the beam.

Selecting the number of the assumed-modes, n equals to 2, and using the method summarized in section 2, the results of free transverse vibration of the free end of the beam including the effect of internal axial forces produced by curve motion of the lumped mass for the cases of $k = 0, 1, \cdots, 5$ are obtained as the solid lines in Figs. 3 to 8, respectively. In these figures, the dashed lines represent the results obtained by using the conventional method in which the effect of internal axial forces produced by curve motion of the lumped mass is not included. From the comparison between the solid line and the dashed line of Fig.3, it is seen that in the case of $k = 0$, the results of free transverse vibration response of the beam including and not including this effect are in excellent agreement, so we can conclude that if there is not a lumped mass at the free end, free transverse vibration response of the beam can also be accurately determined by using the conventional method. This can be easily understood. In the absence of a lumped mass at the free end, the magnitude of the internal axial force at arbitrary cross section of the beam is extreme small, and consequently the effect of the internal axial forces can be neglected. Comparing the solid lines with the dashed lines of Figs. 4 to 8, it can be seen that in the presence of a lumped mass at the free end, the frequencies of free transverse vibration of the beam including the effect of internal axial forces produced by curve motion of the lumped mass are higher than these not including the effect. For instance, in Fig. 7, the frequency of free transverse vibration of the beam including the effect is 0.8602 Hz, and that not including the effect is 0.7816 Hz. This implies that internal axial forces produced by curve motion of the lumped mass have an effect of increasing the frequencies of free transverse vibration of the beam. This effect can be understood as follows: the geometric nonlinearity of bending vibration of the beam leads to the curve motion of the lumped mass shown in Fig. 1, and consequently there exist tensile forces (internal axial forces) in the beam. Such tensile forces have an influence of increasing the frequencies of transverse vibration of the beam.

From Figs. 4 to 8, it is observed that the difference between free transverse vibration responses of the beam including and not including this effect increases with the lumped mass. We can see from Figs. 6 to 8 that when $k \geq 3$, the above-mentioned difference becomes significant. This indicates that in the presence of a relatively large lumped mass at the free end (in the case of $k \geq 3$), the inclusion of the effect of internal axial forces produced by curve motion of the lumped mass in determining free transverse vibration of the beam is necessary to obtain an accurate solution.

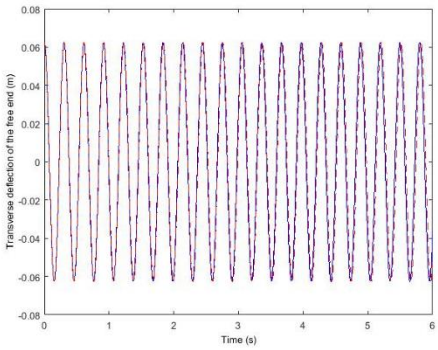

Fig. 3 Free transverse vibration response of the free end of the beam for the case of $k = 0$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

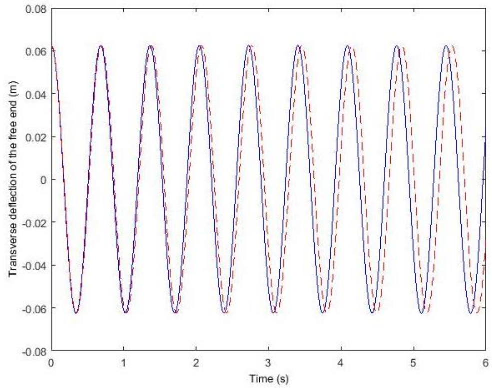

Fig. 4 Free transverse vibration response of the free end of the beam for the case of $k=1$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

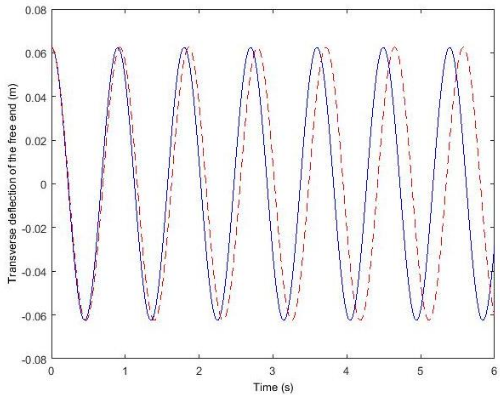

Fig. 5 Free transverse vibration response of the free end of the beam for the case of $k=2$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

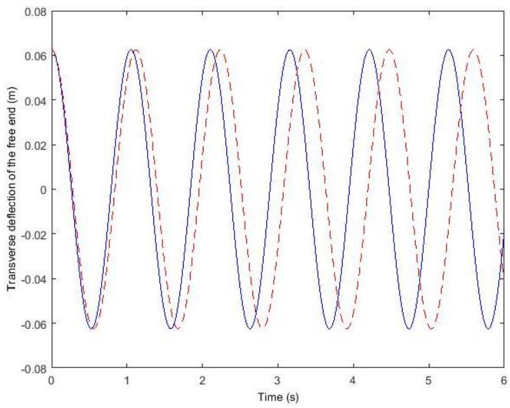

Fig. 6 Free transverse vibration response of the free end of the beam for the case of $k=3$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

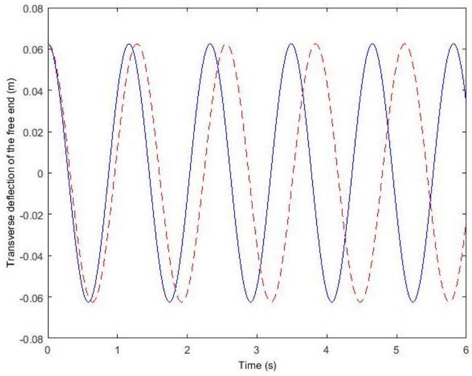

Fig. 7 Free transverse vibration response of the free end of the beam for the case of $k = 4$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

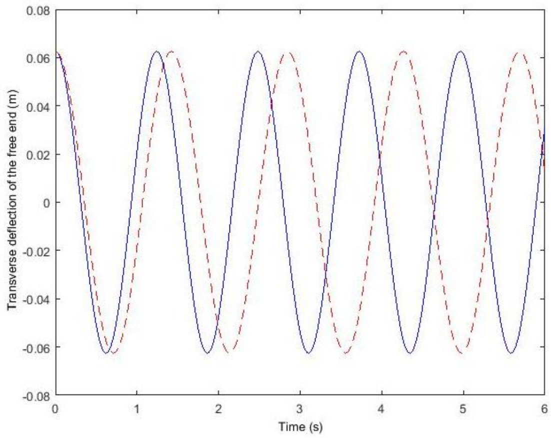

Fig. 8 Free transverse vibration response of the free end of the beam for the case of $k = 5$: (—) the result including the effect of internal axial forces produced by curve motion of the lumped mass, (– –) the result not including the effect

4. Conclusions

This paper deals with the effect of internal axial forces produced by curve motion of a lumped mass at the free end of a uniform cantilever beam in free transverse vibration. The method for determining free transverse vibration of the beam including this effect is presented. Based on the numerical simulation results given in this paper, following conclusions are obtained:

• If there is not a lumped mass at the free end, free transverse vibration of the beam can be accurately determined by using the conventional method.

• In the presence of a lumped mass at the free end, the frequencies of free transverse vibration of the beam including the effect of internal axial forces produced by curve motion of the lumped mass are higher than these not including the effect. This indicates that internal axial forces produced by curve motion of the lumped mass have an effect of increasing the frequencies of free transverse vibration of the beam.

• The difference between free transverse vibration responses of the beam including and not including the effect of internal axial forces produced by curve motion of the lumped mass increases as the lumped mass increases.

• In the presence of a relatively large lumped mass at the free end, the inclusion of the effect of internal axial forces produced by curve motion of the lumped mass in determining free transverse vibration of the beam is necessary in order to obtain an accurate solution.

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