Coupled Systems Mechanics, *Vol. 3, No. 3 (2014) 291-304*
DOI: http://dx.doi.org/10.12989/csm.2014.3.3.291

Analysis of body sliding along cable

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(Received September 5, 2014, Revised September 15, 2014, Accepted September 20, 2014)

Abstract. Paper discusess a dynamic engineering problem of a mass attached to a pendulum sliding along a cable. In this problem the pendulum mass and the cable are coupled together in a model described by a system of differential algebraic equations (DAE). In the paper we have presented formulation of the system of differential equations that models the problem and determination of the initial conditions. The developed model is general in a sense of free choice of support location, elastic cable properties, pendulum length and inclusion of braking forces. Examples illustrate and validate the model.

Keywords: elastic rope/cable; sliding mass; sliding pendulum; differential-algebraic equations

1. Introduction

Sliding mass problem describes a situation where a mass slides along a cable or a rope. There are numerous examples in engineering practice: cable cars, elevators, cranes, various temporary structures (e.g., in bridge construction), amusement parks, military applications, etc. After careful consideration of the problem it is clear that center of the mass is usually dislocated from the axis of the cable. Body can be considered rigid so the coupled system can be modeled as a mass attached to a pendulum sliding along the cable.

At first sliding mass problem resembles moving mass or force problem that authors have analyzed before because the resulting system of differential equations could seem similar, see Kožar and Torić Malić (2013) and Torić Malić and Kožar (2012). However, there is a substantial difference. In the moving mass or force problem dynamic structural equations remain valid and only the right hand side changes, i.e. the forcing function takes specific time dependent form. In the sliding mass problem there is no equilibrium without the sliding mass, i.e. cable and mass are coupled into one system (actually, cable imposes nonlinear constraints onto dynamic equations of mass movement). The resulting system of equations describing the interaction between the mass and the cable is a differential algebraic system (DAE) of equations (e.g., see Biegler 2000).

Traditionally, cable problems are usually analyzed using finite elements, e.g., see Ibrahimbegovic (1992). Moving mass problem requires special finite elements like the one described by Zhou *et al*. (2004). In that paper the analytical formulation of the sliding mass problem is briefly touched as a mean for testing of the special finite element formulation. The

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http://www.techno-press.org/?journal=csm&subpage=7 ISSN: 2234-2184 (Print), 2234-2192 (Online)

problem is briefly touched as a mean for testing of the special finite element formulation. The formulation given there is for the inextensible cable with supports on the equal height and in a form unsuitable for general differential equation solvers.

None of the finite element formulations for the mass hanging on a pendulum would realistically describe the engineering problem of a sliding body. In this paper we have developed the analytical model of a mass attached to a pendulum sliding along the massless rope or cable. Since we would like to solve a real engineering problem it is important to take into account cable elastic properties (EA). Furthermore, we allow for general placement of supports and addition of various forms of braking force. Formulation is suitable for differential equation solvers and equation based languages like Modelica (Fritzson 2011).

Terms ‘cable’ and ‘rope‘ are used interchangeably throughout the paper because we would like to stress that cable is massless in the model.

In the paper, first we describe the mathematical model of a sliding mass. Second, we expose on determination of initial conditions. After that a mass attached to a sliding pendulum is introduced and the required- modifications of the equations are presented. In the sequel relevant examples are presented as an illustration and confirmation of the mathematical model. Finally, in the conclusion we discuss and summarize the model.

2. Sliding mass model

Assumptions for the model: rope/cable is straight, i.e., the self-weight of the cable is neglected. This makes model suitable for analysis of light ropes (compared to the weight of the sliding mass) and thin steel cables with very small sag.

We start from the dynamic balance equation of a mass sliding along on the rope. Note that in the case of a sliding mass force is constant along the rope (dynamic equilibrium) and in the case of a hanging mass forces on the left and on the right of the mass are different (static equilibrium).

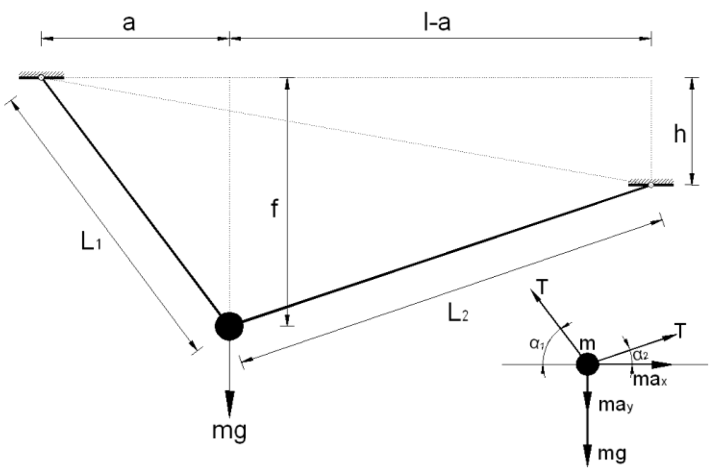


Fig. 1 Sliding mass on the cable and dynamic equilibrium of the sliding mass

With nomenclature from Fig. 1 we have

$$-\mathrm{T}\cos(\alpha_1) + \mathrm{T}\cos(\alpha_2) - \mathrm{R} = \mathrm{m}\ddot{\mathrm{a}} \tag{1}$$
$$-\mathrm{T}\sin(\alpha_1) - \mathrm{T}\sin(\alpha_2) + \mathrm{mg} = \mathrm{m}\ddot{\mathrm{f}}$$

where R is resistance force (due to friction, air resistance or braking) and T is cable force.
This is a system of two second order differential equations with three unknowns: tension force T, horizontal position and vertical position f. Angles are easily calculated from geometric relations

$$\cos(\alpha_1) = \frac{\mathrm{a}}{\mathrm{L}_1} \quad ; \quad \cos(\alpha_2) = \frac{\mathrm{l}-\mathrm{a}}{\mathrm{L}_2} \tag{2}$$
$$\sin(\alpha_1) = \frac{\mathrm{f}}{\mathrm{L}_1} \quad ; \quad \sin(\alpha_2) = \frac{\mathrm{f}-\mathrm{h}}{\mathrm{L}_2}$$
$$\tan(\beta) = \frac{\mathrm{h}}{\mathrm{l}}$$

The third equation is actually an algebraic constraint related to the length of the rope. In the case of an inextensible rope/cable we have

$$\mathrm{L}_1 + \mathrm{L}_2 = \mathrm{L} \tag{3}$$

and in the case of an extensible (elastic) rope/cable

$$\mathrm{L}_1 + \mathrm{L}_2 = \mathrm{L} + \Delta\mathrm{L} \tag{4}$$

We now have a system of differential - algebraic equations (DAE). Application of the in-extensibility conditions leads to a more complicated DAE system of index 3 whose solution is sensitive to errors and more important, initial conditions are very difficult to determine because additional differential equations with non-physical parameters appear in the system. In this paper we are adopting the elastic cable conditions that gives DAE system of index 1. It is more realistic and in the limit it can approximate the inextensible cable. An example with more detailed explanation of DAE index can be found in Kožar and Ožbolt (2010).

In order to solve the DAE system it has to be transformed into a system of ordinary differential equations (ODE). Two second-order differential equations are transformed into four first-order differential equations with the substitution

$$\mathrm{u} = \frac{\mathrm{da}}{\mathrm{dt}} = \dot{\mathrm{a}} \quad ; \quad \mathrm{v} = \frac{\mathrm{df}}{\mathrm{dt}} = \ddot{\mathrm{f}} \tag{5}$$

The crucial step is an introduction of the elastic constraint that is written as

$$\mathrm{L}_1 + \mathrm{L}_2 = \mathrm{L} + \frac{\mathrm{TL}}{\mathrm{EA}} \tag{6}$$

with EA being the elastic characteristic of the rope/cable and T is cable force.
After one derivation the elastic constraint becomes the differential equation

$$\dot{\mathrm{T}} = \frac{\mathrm{EA}}{\mathrm{L}}(\dot{\mathrm{L}_1} + \dot{\mathrm{L}_2}) \tag{7}$$

Here we have to express derivatives of rope parts with already defined variables. From geometric relations we have

$$a^2 + f^2 = {L_1}^2 \quad ; \quad (l-a)^2 + (f-h)^2 = {L_2}^2 \tag{8}$$

After some manipulation it follows

$$\dot{L_1} = \frac{au+fv}{L_1} \quad ; \quad \dot{L_2} = \frac{(f-h)v-(l-a)u}{L_2} \tag{9}$$

In the end we have system of five first-order differential equations

$$\dot{a} = u, \dot{u} = \frac{T}{m}\left(\frac{l-a}{L-\sqrt{(a^2+f^2)}} - \frac{a}{\sqrt{(a^2+f^2)}}\right) \tag{10}$$

$$\dot{f} = v, \dot{v} = g - \frac{T}{m}\left(\frac{f-h}{L-\sqrt{(a^2+f^2)}} - \frac{f}{\sqrt{(a^2+f^2)}}\right)$$

$$\dot{T} = \frac{EA}{L}\left(\frac{au+fv}{L-\sqrt{(a^2+f^2)}} - \frac{(f-h)v-(l-a)u}{\sqrt{(a^2+f^2)}}\right)$$

This system can be further reduced to four differential equations by substitution of the expression for T into the second and the fourth equation.

<u>Note</u>

Even without that reduction the system can be solved with any differential equation solver, in our case with Mathematica (see Mathematica 9 Documentation 2014). The main difference is in initial conditions where the system of five equations requires initial force T_0. However, T_0 has to be calculated anyway (see Initial conditions below) as well as T (since it is of interest). In the system with five differential equations T is obtained as a part of the result whereas in the four DE system it has to be determined from Eq. (6) which requires calculation of L_1 and L_2 from Eq. (8) (otherwise not of direct interest). In the end the difference in calculation effort between the system with five and the system with four differential equations is not so pronounced.

3. Initial conditions

Determination of initial conditions requires calculation of a_0, u_0, f_0 and v_0 (and T_0 regardless of the system of DE we use). We start with given (assumed) $a = a_0$ and $b_0 = l - a_0$ and calculate the rest

$$f_0 = a_0 \tan(\alpha_1) \tag{11}$$

but α_1 cannot be determined directly. Instead, we use Eqs. (8) and (6) to obtain

$$\sqrt{{a_0}^2 + {f_0}^2} + \sqrt{{b_0}^2 + (f_0 - h)^2} - L + \frac{T_0 L}{EA} = 0 \tag{12}$$

The second initial condition equations is obtained from the equilibrium at time t=0 (see Fig.1(b)). At the beginning of the simulation the mass is not moving and vertical acceleration a_y=0 (horizontal acceleration $a_x \neq 0$ and can be calculated). We have

$$mg - T_0 \sin(\alpha_1) - T_0 \sin(\alpha_2) = ma_y = 0 \tag{13a}$$

Finally, writing mg=G (mass weight) we get the second initial conditions equation

$$G - T_0 \frac{f_0}{\sqrt{a_0{}^2+f_0{}^2}} + T_0 \frac{f_0 - h}{\sqrt{b_0{}^2+(f_0-h)^2}} = 0 \tag{13b}$$

From the system of nonlinear algebraic Eqs. (12) and (13(b)) we get the initial values for f_0 and T_0. There are at most four solutions of the above system of equations with one positive and one negative f_0 and T_0. This can be seen from the graphic representation of the system of equations in Fig. 2.

In Fig. 2(a) the blue line is for Eq. (12) and the red line is for Eq. (13(b)). We see the intersections that are present solutions for tension and compression force in the rope. Fig. 2(b) is for Eq. (12) for different initial rope lengths L; for shorter L the contour "goes" upward. From Fig. 2(a) positive f_0 and T_0 are the solutions of interest (displacement f goes down and rope is in tension). From Fig. 2(b). we see that for short initial cable length L there is only one real solution to the system of equations.

Values u and v represent horizontal and vertical mass velocity and initially they are both zero, i.e., u_0=0 and v_0=0. With this initial conditions are completely and uniquely determined.

<u>Note</u>

In the case of an inextensible rope/cable (EA-> ∞) Eq. (12) is a function only of f_0, i.e., instead of a system of two nonlinear Eqs. (12) and (13(b)) we have two uncoupled equations. First, Eq. (12) is solved for f_0 and with f_0 known Eq. (13(b)) is solved for T_0 or not solved at all if we work with the system of four differential equations. One could try this approach even for extensible rope assuming that EA is large enough. Indeed, differences in results are negligible but only for rope lengths $L < \sqrt{(l^2+h^2)}$, i.e., rope does not have any initial tension. For ropes with any initial tension initial conditions have to be determined from the system Eqs. (12) and (13(b)) because f_0 and T_0 from the uncoupled equations results in divergence of the solution algorithm of the system of differential equations. Fig. 2(b) confirms great sensitivity of Eq. (12) to the rope length L.

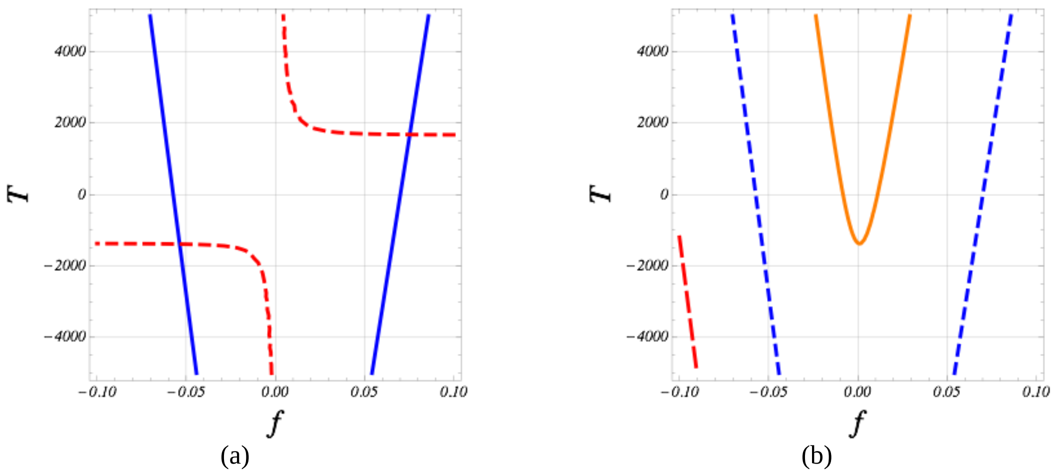


Fig. 2 (a) Solution domain of the system of Eqs. (12) and (13(b)), (b) dependance of Eq. (12) on the rope length L

4. Addition of pendulum

In practice it is rare that we analyze mass sliding along a rope/cable, it is usually a rigid object eccentrically attached to the rope/cable. Considering practical application one can conclude that the attached object can best be described as a pendulum. In the sequel we will replace the sliding mass with a sliding pendulum and obtain a realistic description of a practical situation where a realistic object is sliding along a rope/cable.

In Fig. 3 we see the change from Fig. 1, it is only a pendulum that is added to the system. Only one additional unknown is required: pendulum angle θ.

Mass position has changed and Eq. (1) have to be adapted accordingly. New mass center is now

$$a_P = a + L_P \sin(\theta), f_P = f + L_P \cos(\theta) \tag{14}$$

where L_P is pendulum length and θ is pendulum position (angle). Derivative of those two coordinates is required in Eq. (1)

$$\frac{d^2 a_P}{dt^2} = \ddot{a} + L_P\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2 \sin(\theta)\right) \tag{15}$$

$$\frac{d^2 f_P}{dt^2} = \ddot{f} + L_P\left(-\dot{\theta}^2 \cos(\theta) - \ddot{\theta}\sin(\theta)\right)$$

where ‘a’ and ‘f’ are from Fig.1.

Replacement of ‘a’ and ‘f’ with ‘a_P’ and ‘f_P’ results in additional terms that depend only on L_P and θ. Modified Eq. (10) are now

$$\dot{a} = u, \dot{u} = \frac{EA}{m} \cdot \frac{L_1+L_2}{L-1} \cdot \left(\frac{a-l}{L_2} - \frac{a}{L_1}\right) - L_P(\dot{p}\cos(\theta) - p^2 \sin(\theta)) \tag{16}$$

$$\dot{f} = v, \dot{v} = g - \frac{EA}{m} \cdot \frac{L_1 + L_2}{L - 1}\left(\frac{f-h}{L_2} - \frac{f}{L_1}\right) + L_P(\dot{p}\sin(\theta) + p^2 \cos(\theta))$$

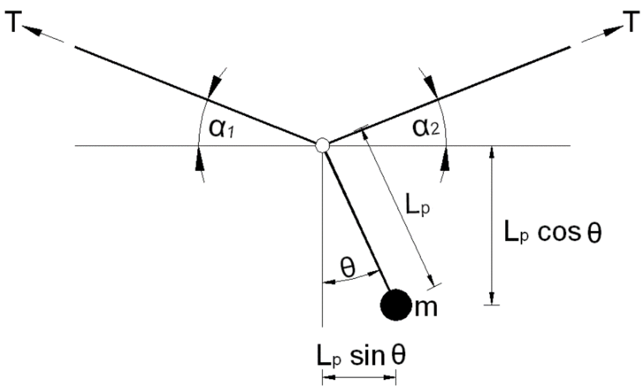


Fig. 3 Coordinates describing pendulum sliding along the rope/cable

In the above equations the expression for rope tension 'T' has been substituted at the appropriate places and thus removed from this system of nonlinear differential equations.

Additional unknown θ requires an additional equation

$$L_P\ddot{\theta} = \ddot{f_P}\sin(\theta) - \ddot{a_P}\cos(\theta) - g\sin(\theta) \tag{17}$$

which is actually an angular balance equation. The right hand side represents rotating moment due to mass weight and due to accelerations in the direction of coordinates 'a' and 'f'.

Introducing θ' = p and using similar expressions for a' and f' we have the required differential equation for θ

$$\dot{p} = \frac{\dot{v}\sin(\theta)}{2L_p} - \frac{\dot{u}\cos(\theta)}{2L_p} - \frac{g\sin(\theta)}{2L_p} \tag{18}$$

that is nonlinear as the other equations in the system.

5. Initial conditions with pendulum

Initial conditions for 'a' and 'f' do not change with the presence of pendulum, Eq. (13) remain valid. Initial condition for θ_0 can be chosen at will (initial angle of the pendulum) and always $p_0=0$ (initial angular velocity of the pendulum).

6. Examples

6.1 Examples of sliding mass

There are two examples differing only in cable length, one with cable longer and the other with cable shorter than the straight line between the supports (603.0 m). Note: unlike the shorter one, the longer one can also be approximately solved using the inextensible cable theory.

Geometric properties of the cable (distances between supports) correspond to a real engineering structure in adrenalin park in Croatia and are:

l = 600.0 m

h = 60.0 m

and material properties (assuming Φ12 mm 6 x 19 steel IWRC cable) are

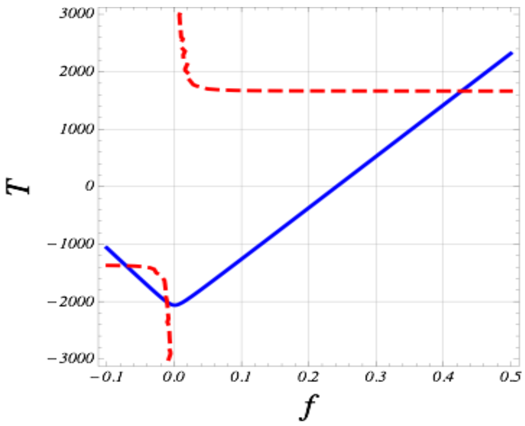
L = 603.2 m and L = 602.8 m

EA = 6. E6 N,

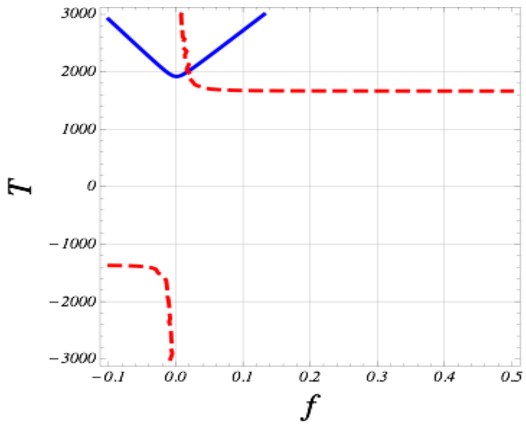
and mass m = 150.0 kg.

Graphical presentation of the initial conditions solution in Fig. 4 demonstrates that for the long cable both tension and compression are possible as initial cable force (of course, only the tension force is physically admissible). For short cable only tension is possible since cable has to be tensioned first to be placed in the position.

Total analysis time t=60.0 seconds in which time mass reaches the end of the cable and reflects back. In reality nobody would like reflecting back from the support but it is useful for the model stability analysis to trace the solution that far. Neither damping nor braking forces are considered in those examples.

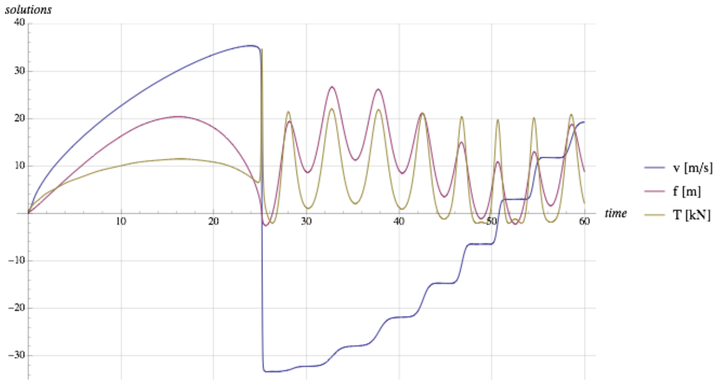


(a) longer cables

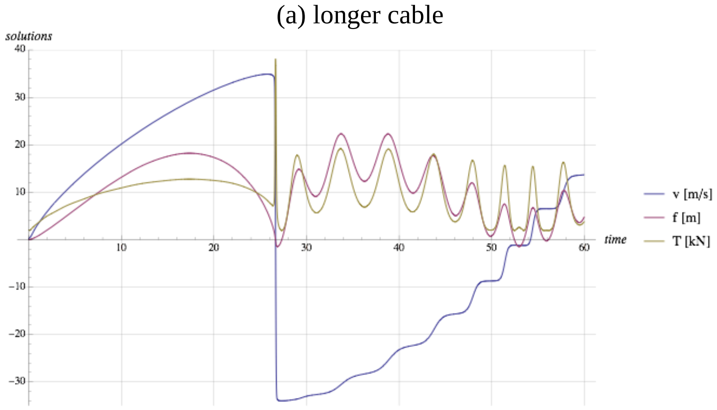


(b) shorter cables

Fig. 4 Initial conditions for all examples



(a) longer cable



(b) shorter cable

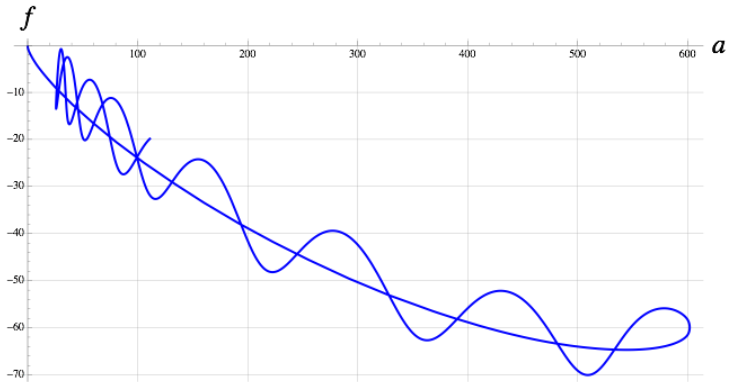
Fig. 5 Solutions in time

Fig. 5 shows solutions for velocities da/dt, vertical displacements ‘f’ and cable forces ‘T’ for longer and shorter cable respectively.

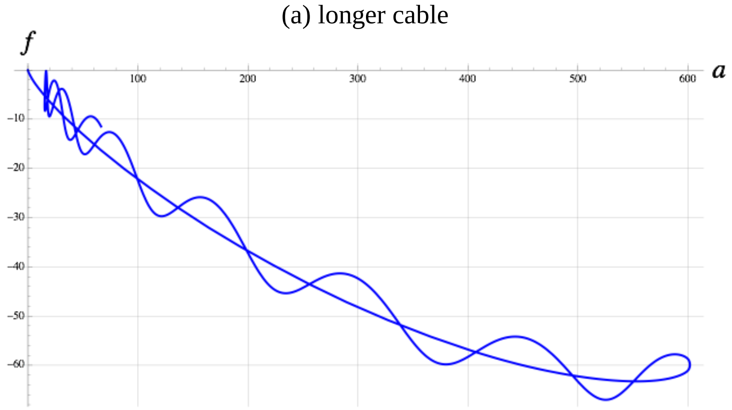
Comparison of results from Fig. 5 indicates that higher tension cable (shorter cable) has always somewhat smaller vertical displacement and in the reflecting phase slightly smaller forces. It is interesting to observe shorter travel time for longer cable that is result of larger speed (127 km/h vs. 125 km/h). Higher speed results from larger deflection in longer cable which allows for greater contribution of gravity.

We could also plot displacements ‘a’ and ‘f’ against each other to trace the mass path, see Fig. 6 We see that in the first phase of the sliding mass path along the cable is half of an inclined ellipse and that ellipse is flatter for higher tension (shorter cable).

From Fig. 6 it is evident why there are oscillations in the reflecting phase. At the reflection mass circles around the support and starts returning from above and after some time due to gravity mass falls down again. That has started oscillations that persist (since there is no damping in the model). Numerical experiments revealed that high tension in the cable reduces the oscillations considerably (the ellipse is flatter and at reflection the mass circles with smaller radius around the support).



(a) longer cable



(b) shorter cable

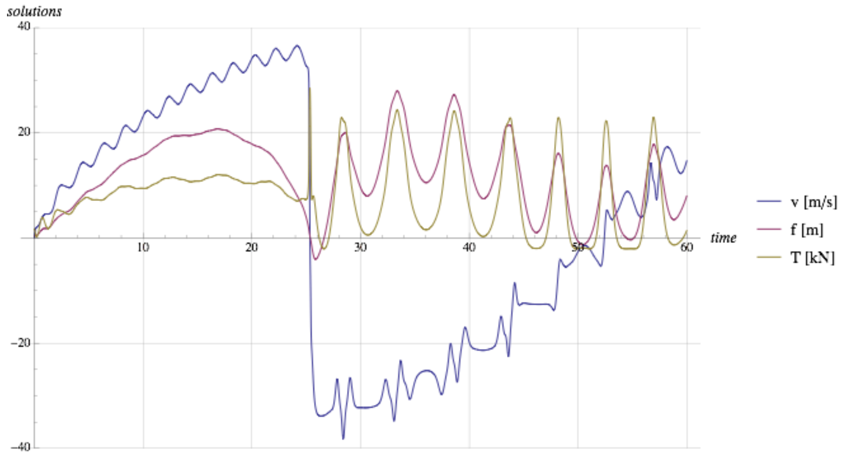
Fig. 6 Mass path in time

6.2 Examples of sliding pendulum

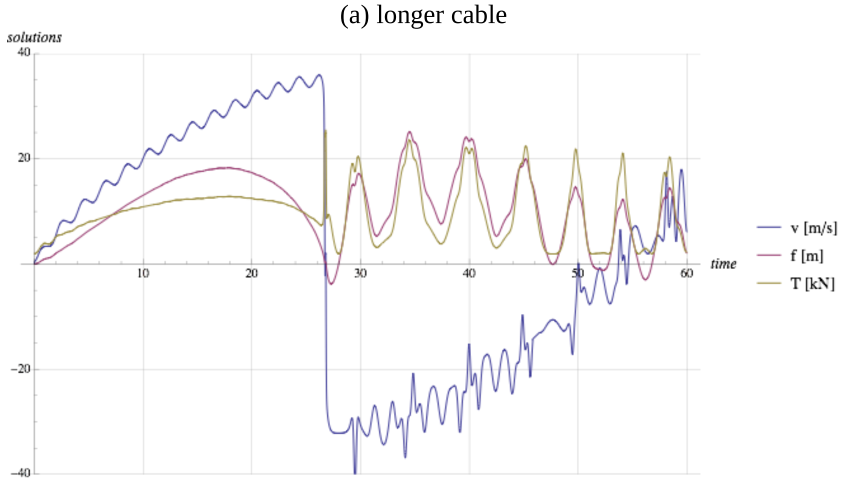
Geometric and material properties are the same as in the above examples, only the pendulum is added, with length L_P = 1.0 m (approximately man hanging from the rope).

Solutions are presented in Fig. 7 where we see similarity in the first phase of analysis and more pronounced oscillations after reflecting. Additional numerical experiments confirmed that in the case of smaller L_P differences diminish approaching the solution without the pendulum.

In Fig. 7 two points are traced in time: position on cable where pendulum is attached (here would mass be if there was no pendulum) and tip of the pendulum (center of the mass). Maximum possible difference between the lines is the pendulum length L_P. It is interesting to note that the point on the cable (red) oscillates more then the mass on the pendulum (blue) which moves more smoothly.

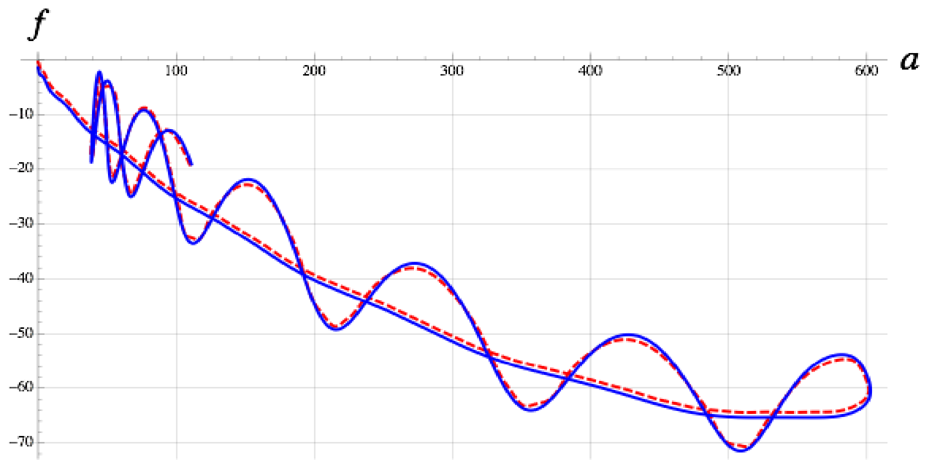


(a) longer cable

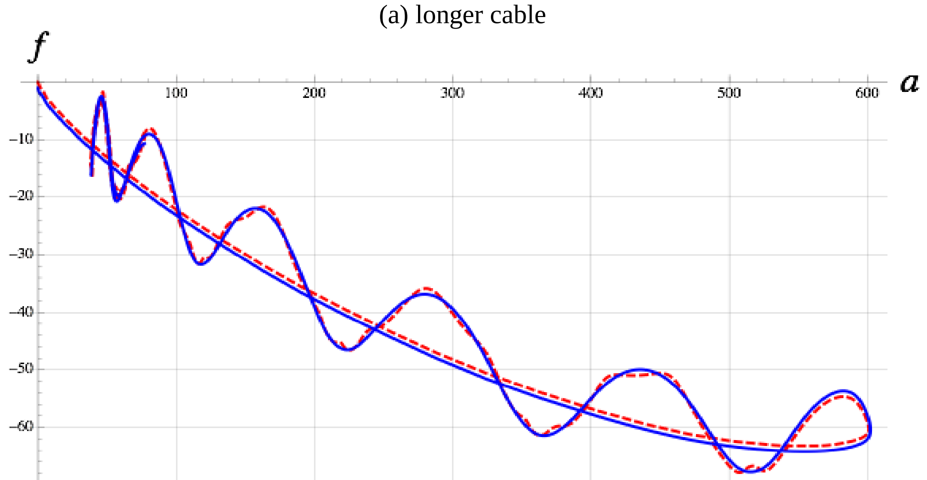


(b) shorter cable

Fig. 7 Solutions in time with pendulum



(a) longer cable

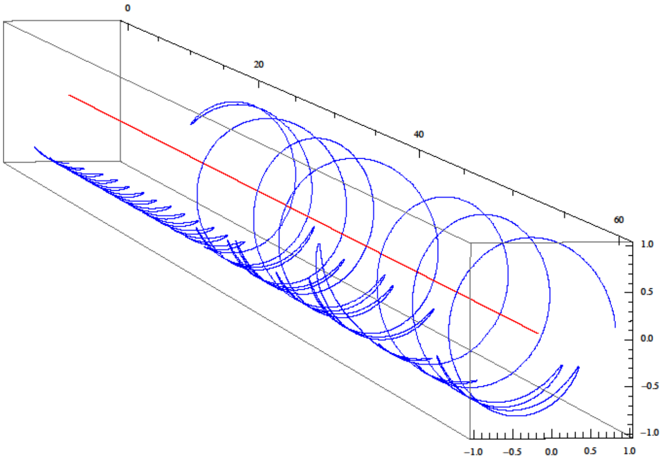


(b) shorter cable

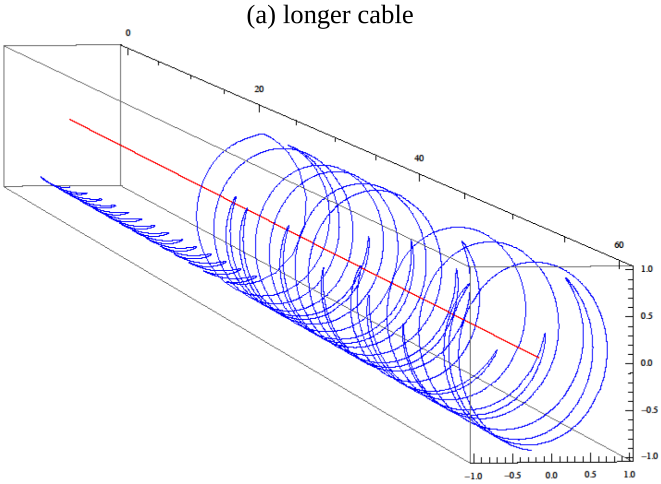
Fig. 8 Mass path in time, position on the cable (dashed red) and pendulum mass (solid blue)

From Fig. 8 we see that the circling of the mass at reflection is almost completely dominated by the pendulum length L_P. Additional numerical experiments revealed that even very high tension does not influence the cable behavior significantly.

Pendulum angle θ is another interesting result from this model and is presented in Fig. 9. In the first phase of analysis (prior to reflecting) the pendulum is only swinging back and forth, approximately from -0.7 rad to 0.4 rad. After reflecting and circling about the support it continues to make full circles, back and forth on the longer cable and only in one direction (multiple circles) on shorter cable. We have to recall that there is neither damping nor braking in this example.



(a) longer cable



(b) shorter cable

Fig.9. Pendulum angle in time

7. Conclusions

It has been shown that in the case of a sliding mass, cable and mass are coupled system where cable imposes constraints onto the dynamic equations of mass movement. The resulting model is described with a system of differential algebraic equations (DAE) of index 3 in the case of an inextensible cable. DAE of index 3 is difficult and tedious to solve. This paper analyzes a more general case of an extensible cable. It turns out that an elastic constraint imposed in this case has an additional benefit that the resulting DAE is of index 1. Thus a more general case has a simpler solution. Addition of cable elastic properties even allowed us to remove that additional unknown from the system of equations. In the end there is no significant difference in the computational effort whether the additional equation is removed from the system or not. In both cases initial conditions are determined by solving a system of two nonlinear algebraic equations.

In case of a sliding body it is more realistic to analyze a mass attached to a pendulum then only a mass sliding along a cable. Mathematical model of sliding mass has been extended to accommodate the sliding pendulum.

Validity of the adopted approach is demonstrated through examples with and without the pendulum and with longer (softer) and shorter (stiffer) cable. In the case of sliding mass difference in cable stiffness (determined by cable length) influences the solution significantly. In the case with pendulum it has been demonstrated that its length is the key parameter of the solution.

Also, the three dimensional version of the model is under development.

Acknowledgments

The research described in this paper has been partially supported by University of Rijeka through Project no. 13.05.1.1.02 and by European Union through Grant IPA2007/HR/16IPO/001-040513. The authors are thankful for this support.

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Appendix

Notation

l	horizontal distance between cable supports ('span')
h	vertical distance between cable supports
L	cable length
L_P	pendulum length
EA	cable elastic properties (modulus of elasticity * cable cross section area)
m	mass of the sliding body
a	horizontal position of mass or pendulum attachment
f	vertical position of mass or pendulum attachment
a_P	horizontal position of the center of the mass attached to the pendulum
f_P	vertical position of the center of the mass attached to the pendulum
L_1	cable length left of the mass or pendulum attachment
L_2	cable length right of the mass or pendulum attachment
T	tension in the cable
θ	angle of the pendulum