Coupled Systems Mechanics, *Vol. 9, No. 3 (2020) 219-235*
DOI: https://doi.org/10.12989/csm.2020.9.3.219

Natural convection of nanofluid flow between two vertical flat plates with imprecise parameter

U. Biswal[a], S. Chakraverty[*] and B.K. Ojha[b]

Department of Mathematics, National Institute of Technology Rourkela, Rourkela-769008, Odisha, India

(Received July 29, 2019, Revised November 3, 2019, Accepted December 14, 2019)

Abstract. Natural convection of nanofluid flow between two vertical flat plates has been analyzed in uncertain environment. A non-Newtonian fluid Sodium Alginate (SA) as base fluid and nanoparticles of Copper (Cu) are taken into consideration. In the present study, we have taken nanoparticle volume fraction as an uncertain parameter in terms of fuzzy number. Fuzzy uncertainties are controlled by r-cut and parametric concept. Homotopy Perturbation Method (HPM) has been used to solve the governing fuzzy coupled differential equations for the titled problem. For validation, present results are compared with existing results for some special cases viz. crisp case and they are found to be in good agreement.

Keywords: natural convection; nanofluid; coupled system; homotopy perturbation method; fuzzy number; uncertainty

1. Introduction

Natural convection problems have taken attention of researchers due to their various applications in physical problems of science and engineering such as heat exchangers, geothermal systems, petroleum reservoirs, nuclear waste repositories etc. In this regard, the study of convective heat transfer of nanofluid is a challenging problem. Convectional heat transfer fluids including oil, water and ethylene glycol mixture are poor heat transfer fluid. Heat transfer may be increased by using nano-sized particles in the base fluid and these fluids with added nanoparticles are termed as nanofluid (Choi and Eastman 1995).

Researchers have investigated Newtonian and non-Newtonian fluid flow through two infinite parallel vertical plates. Analysis of Sodium Alginate (SA) based nanofluid flow between two vertical parallel plates have been carried out by Hatami and Ganji (2014). They have used two numerical methods namely Differential Transform Method (DTM) and Least Square Method (LSM). Ziabakhsh and Domairry (2009) have analyzed the same problem but for real fluid by Homotopy Analysis Method (HAM). Few investigations have been carried out on the natural convection of nanofluid. Garoosi *et al*. (2015) have studied the natural convection of nanofluid in heat exchangers using the Buongiorno model. They have also discussed the effect of diameter of

*Corresponding author, Professor, E-mail: sne_chak@yahoo.com
[a]Ph.D. Student, E-mail: uddhababiswal789@gmail.com
[b]Professor, E-mail: bkojha@nitrkl.ac.in

http://www.techno-press.org/?journal=csm&subpage=8 ISSN: 2234-2184 (Print), 2234-2192 (Online)

nanoparticles on the heat transfer rate. Natural convection boundary layer flow along a vertical cone with variable temperature in presence of magnetohydrodynamics has been studied by Ellahi *et al*. (2015). Sheikholeslami and Seyednezhad (2018) have studied the nanofluid flow and natural convection in porous media under the influence of the electric field.

As mentioned above, a few literatures related to the natural convection of non-Newtonian nanofluid already exist in crisp form. Parameter like nanoparticle volume fraction on natural convection problem of nanofluid is important. However, it may be seen from various sources (Bakar *et al*. 2016) the value of volume fraction may vary between 0 and 0.2. Since the value of volume fraction depends upon the volume of fluid and that of added nanoparticle, so it may be taken as an uncertain parameter. As such we are motivated to handle the natural convection problem in uncertain environment. In this regard, volume fraction has been taken as an uncertain parameter in terms of fuzzy number in the present study.

This paper aims to investigate the non-Newtonian nanofluid flow between two vertical parallel plates in uncertain environment. Nanoparticle volume fraction has been taken as an uncertain parameter in terms of fuzzy number. There are various numerical methods to handle non-linear differential equations in crisp cases such as Finite Element Method, Finite Difference Method, Homotopy Perturbation Method, Galerkin's Method etc. In this regard, seismic analysis of concrete gravity dams considering soil-structure-fluid interaction has been studied by Mandal and Maity (2019) with the help of the finite element method. Ziaolhagh *et al*. (2016) have used the finite element method to study the free vibration analysis of a dynamical coupled system: flexible gravity dam-compressible rectangular reservoir. Natural convection of non-Newtonian nanofluid flow between two vertical parallel plates has been studied by Biswal *et al*. (2019). Biswal *et al*. have used HPM and Galerkin's method to handle the related non-linear coupled differential equations. Karunakar and Chakraverty (2017) have used HPM for a comparative study of linear and non-linear shallow water wave equations. It is worth mentioning that HPM was first developed by J. He in 1999 who has used HPM to solve various non-linear differential equations (He 1999, He 2005, He 2006). The above mentioned methods have been extended to handle fuzzy differential equations by different authors. As such, 2-D shallow water wave equations with fuzzy basin depth have been solved by Karunakar and Chakraverty (2018) using HPM. Nayak and Chakraverty (2012) have used fuzzy finite element method to solve uncertain heat conduction problems. Orthogonal polynomials based Galerkin's method has been used by Rao and Chakraverty (2017) to handle uncertain radon diffusion equation in soil pore matrix. Different efficient techniques for the solution of uncertain differential equations may be found in (Chakraverty and Nayak 2017). Basic concepts of uncertainty in terms of interval and a brief description of the interval finite element method to handle uncertain differential equations may be found in (Nayak and Chakraverty 2018). In view of the above, we have used the parametric concept to handle the fuzzy parameters and HPM has been used to solve the governing non-linear fuzzy differential equations.

Rest of the paper is organized as follows. In section 2, we have discussed some preliminaries related to the fuzzy number and parametric approach to convert the fuzzy differential equation to parametric form. The fuzzy governing equations of the considered problem have been presented in section 3. A brief idea of HPM has been included in section 4. In section 5, the related fuzzy differential equations have been solved by HPM with the help of the parametric approach of fuzzy parameters. In section 6, present results have been compared with the existing results for special case viz. crisp. Some fuzzy plots of velocity and temperature profiles are also presented in section 6. Finally, in section 7, conclusions have been drawn.

2. Preliminaries

In this section, we will discuss some basic concepts of fuzzy theory and some notations that we have used later in this article.

Fuzzy Set

A fuzzy set $\tilde{S}$ is a set consisting of ordered pairs of the elements λ of a universal set say U and their membership value, written as

$\tilde{S} = \{(\lambda, \mu(\lambda)) : \lambda \in U, m(\lambda) \in [0,1]\}$, where $m(\lambda)$ is a defined membership function for the fuzzy set $\tilde{S}$.

Fuzzy Number

Fuzzy number is a fuzzy set that is convex, normalized and defined on the real line *R*. Moreover, its membership function must be piecewise continuous. There are different types of fuzzy numbers based on membership function viz. Triangular, Gaussian, Quadratic, Exponential Fuzzy Number etc. Here, we have used Triangular Fuzzy Number (TFN) and the membership function of a TFN $\tilde{S} = [a,b,c]$ is defined as (Hanss 2005)

$$m(\lambda) = \begin{cases} 0, & \lambda \le a \\ \dfrac{\lambda - a}{b - a}, & a \le \lambda \le b \\ \dfrac{c - \lambda}{c - b}, & b \le \lambda \le c \\ 0, & \lambda \ge b \end{cases}$$

r-cut

By using r-cut, TFN $\tilde{S} = [a,b,c]$ may be converted into interval form as (Chakraverty *et al.* 2016, Chakraverty and Perera 2018)

$$\tilde{S} = [a,b,c] = [(b-a)r + a,\ c - (c-b)r],\ r \in [0,1] .$$

Parametric Approach

In general, an interval $I = [\underline{I}, \overline{I}]$ may be transformed into crisp form by the help of parametric concept as (Chakraverty *et al.* 2016, Chakraverty and Perera 2018)

$I = \beta(\overline{I} - \underline{I}) + \underline{I}$, where β is a parameter which lies in the closed interval $[0,1]$.

It can also be written as $I = 2\beta \Delta I + \underline{I}$, where $\Delta I = \dfrac{\overline{I} - \underline{I}}{2}$ is the radius of I .

3. Formulation of the problem

A schematic diagram of the problem is shown in Fig. 1, which consists of two vertical flat plates separated by a distance 2b apart. The walls at $x=-b$ and $x=+b$ are held at a constant temperatures T_1 and T_2 respectively, with $T_1>T_2$. Due to this difference in temperature, the fluid near the wall at $x=-b$ rises above whereas the fluid near $x=+b$ falls down. The fluid in between the plates is considered as a non-Newtonian SA based nanofluid containing nanoparticles of Cu. We assumed that the base fluid and the nanoparticles are in thermal equilibrium and no slips occur between them. Some physical properties of the nanofluid are given in Table 1 (Hatami and Ganji 2014).

The effective density ρ_{nf}, effective dynamic viscosity μ_{nf}, heat capacitance $(\rho C_p)_{nf}$ and thermal conductivity k_{nf} of nanofluid are given as (Hatami and Ganji 2014, Maxwell 1881, Brinkman 1952)

$$\rho_{nf} = \rho_f (1-\phi) + \rho_s \phi \tag{3.1}$$

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}} \tag{3.2}$$

$$(\rho C_p)_{nf} = (\rho C_p)_f (1-\phi) + (\rho C_p)_s \phi \tag{3.3}$$

$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f + 2\phi(k_s - k_f)}{k_s + 2k_f - \phi(k_s - k_f)} \tag{3.4}$$

where ϕ denotes the nanoparticle volume fraction.

Here C_p denotes specific heat.

Let us define similarity variables as (Rajagopal *et al.* 1985)

$$V = \frac{v}{V_0}, \eta = \frac{x}{b} \text{ and } \theta = \frac{T - T_m}{T_1 - T_2}. \tag{3.5}$$

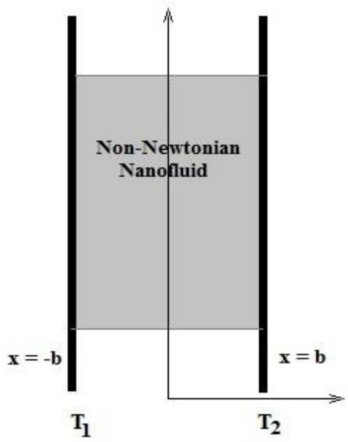


Fig. 1 Nanofluid flow between two vertical flat plates

Table 1 Some properties of non-Newtonian fluid and nanoparticles

Material	Symbol	Density (kg/m^3)	C_p (J/kgK)	Thermal conductivity, W/mK
Copper	Cu	8933	385	401
Sodium Alginate	SA	989	4175	0.6376

By using these assumptions and following nanofluid model proposed by Maxwell-Garnetts (1881), the Navier-Stokes and energy equations may be reduced to the coupled differential equations as (Hatami and Ganji 2014)

$$\frac{d^2V}{d\eta^2}+6\delta(1-\phi)^{2.5}\left(\frac{dV}{d\eta}\right)^2\frac{d^2V}{d\eta^2}+\theta=0, \tag{3.6}$$

$$\frac{d^2\theta}{d\eta^2}+Ec\Pr\left(\frac{(1-\phi)^{-2.5}}{A_1}\right)\left(\frac{dV}{d\eta}\right)^2+2\delta Ec\Pr\left(\frac{1}{A_1}\right)\left(\frac{dV}{d\eta}\right)^4=0. \tag{3.7}$$

where *Pr* and *Ec* stands for Prandtl and Eckert number respectively, δ for dimensionless non-Newtonian viscosity, A_1 for the ratio of thermal conductivity of nanofluid and that of real fluid and they have the forms

$$Ec=\frac{\rho_f V_0^2}{(\rho C_p)_f(T_1-T_2)},\Pr=\frac{\mu_f(\rho C_p)_f}{\rho_f K_f},\delta=\frac{6\beta_3 V_0^2}{\mu_f b^2}, \tag{3.8}$$

$$A_1=\frac{k_{nf}}{k_f}=\frac{k_s+2k_f+2\phi(k_s-k_f)}{k_s+2k_f-\phi(k_s-k_f)}. \tag{3.9}$$

The boundary conditions are assumed as

$$V(\eta)=\begin{cases}0, & \eta=1\\ 0, & \eta=-1\end{cases} \tag{3.10}$$

$$\theta(\eta)=\begin{cases}-0.5, & \eta=1\\ 0.5, & \eta=-1\end{cases} \tag{3.11}$$

It may be worth mentioning here that, a small change in the value of nanoparticle volume fraction may affect the velocity and temperature profile. So we are motivated to handle such a complex problem in uncertain environment by taking volume fraction as a fuzzy number.

As such fuzzy form of governing coupled Eqs. (3.6) and (3.7) may be written as

$$\frac{d^2\tilde{V}}{d\eta^2}+6\delta(1-\tilde{\phi})^{2.5}\left(\frac{d\tilde{V}}{d\eta}\right)^2\frac{d^2\tilde{V}}{d\eta^2}+\tilde{\theta}=0, \tag{3.12}$$

$$\frac{d^2\tilde{\theta}}{d\eta^2}+Ec\Pr\left(\frac{(1-\tilde{\phi})^{-2.5}}{\tilde{A}_1}\right)\left(\frac{d\tilde{V}}{d\eta}\right)^2+2\delta Ec\Pr\left(\frac{1}{\tilde{A}_1}\right)\left(\frac{d\tilde{V}}{d\eta}\right)^4=0. \tag{3.13}$$

Here "~" stands for the fuzzy form.

4. Homotopy Perturbation Method (HPM)

To delineate briefly the idea of HPM, let us consider the differential equation (He 1999, He 2006, He 2005)

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{4.1}$$

with given boundary condition

$$B\left(u, \frac{\partial u}{\partial r}\right) = 0, \qquad r \in \Gamma \tag{4.2}$$

where A is a differential operator that can be divided into two parts viz. linear (L) and non-linear (N), B stands for boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

By splitting A into linear and non-linear part, Eq. (4.1) may be written as

$$L(u) + N(u) - f(r) = 0, \qquad r \in \Omega \tag{4.3}$$

Now, we construct a homotopy $v(r,q) : \Omega \times [0.1] \to R$ satisfying

$$H(v,q) = (1-q)\left[L(v) - L(u_0)\right] + q\left[A(v) - f(r)\right] = 0 \tag{4.4}$$

where q is an embedding parameter lies between 0 and 1, u_0 is an initial approximation satisfying boundary condition Eq. (4.2).

From Eq. (4.4), one may observe that

when $q = 0$, $L(v) = L(u_0)$ and for $q = 1$, $A(v) - f(r)$=0

that is when q converges to 1, we may get approximate solution of Eq. (4.1).

As q is a small parameter, the solution of Eq. (4.4) can be expressed as a power series in q

$$v = v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \ldots$$

By setting q=1 results the best approximation of Eq. (4.1) that is

$$v = \lim_{q \to 1} (v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \ldots).$$

In the next session, we have applied HPM to solve the present problem when the nanoparticle volume fraction is uncertain in terms of fuzzy.

5. Application to the present problem

Firstly we have used r-cut to convert the coupled differential equations into interval form and then the parametric approach has been used to reduce it to parametric form.

For simplicity, Eqs. (3.12) and (3.13) can be written as

$$\tilde{V}'' + 6\delta(1-\tilde{\phi})^{2.5}\left(\tilde{V}'\right)^2 \tilde{V}'' + \tilde{\theta} = 0 \tag{5.1}$$

$$\tilde{\theta}'' + Ec\,\mathrm{Pr}\left(\frac{(1-\tilde{\phi})^{-2.5}}{\tilde{A}_1}\right)\left(\tilde{V}'\right)^2 + 2\delta Ec\,\mathrm{Pr}\left(\frac{1}{\tilde{A}_1}\right)\left(\tilde{V}'\right)^4 = 0. \tag{5.2}$$

By using r-cut for fuzzy form, Eqs. (5.1) and (5.2) may be converted into interval form as

$$\begin{gathered}\left[\underline{V}''(\eta,r),\, \overline{V}''(\eta,r)\right] + \\ 6\delta\left([1,\,1] - \left[\underline{\phi}(r),\, \overline{\phi}(r)\right]\right)^{2.5}\left(\left[\underline{V}'(\eta,r),\, \overline{V}'(X,r)\right]\right)^2\left(\left[\underline{V}''(\eta,r),\, \overline{V}''(\eta,r)\right]\right) + \left[\underline{\theta}(\eta,r), \overline{\theta}(\eta,r)\right] = 0\end{gathered} \tag{5.3}$$

and

$$\left[\underline{\theta}''(\eta,r),\ \overline{\theta}''(\eta,r)\right]+Ec\Pr\left(\frac{\left([1,\,1]-\left[\underline{\phi}(r),\ \overline{\phi}(r)\right]\right)^{-2.5}}{[\underline{A}_1(r),\ \overline{A}_1(r)]}\right)\left(\left[\underline{V}'(\eta,r),\ \overline{V}'(\eta,r)\right]\right)^2+ \tag{5.4}$$

$$2\delta Ec\Pr\left(\frac{1}{[\underline{A}_1(r),\ \overline{A}_1(r)]}\right)\left(\left[\underline{V}'(\eta,r),\ \overline{V}'(\eta,r)\right]\right)^4=0.$$

Next, by applying the parametric concept for the intervals involved in Eqs. (5.3) and (5.4), these coupled interval differential equations may be written in crisp form as

$$\begin{aligned}&\beta\left(\overline{V}''(\eta,r)-\underline{V}''(\eta,r)\right)+\underline{V}''(\eta,r)+\\&6\delta\left(1-\left\{\beta\left(\overline{\phi}(r)-\underline{\phi}(r)\right)+\underline{\phi}(r)\right\}\right)^{2.5}\left(\left\{\beta\left(\overline{V}'(\eta,r)-\underline{V}'(\eta,r)\right)+\underline{V}'(\eta,r)\right\}\right)^2\left(\beta\left(\overline{V}''(\eta,r)-\underline{V}''(\eta,r)\right)+\underline{V}''(\eta,r)\right)+\\&\beta\left(\overline{\theta}(\eta,r)-\underline{\theta}(\eta,r)\right)+\underline{\theta}(\eta,r)=0\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}&\beta\left(\overline{\theta}''(\eta,r)-\underline{\theta}''(\eta,r)\right)+\underline{\theta}''(\eta,r)+\\&Ec\Pr\left(\frac{\left(1-\left\{\beta\left(\overline{\phi}(r)-\underline{\phi}(r)\right)+\underline{\phi}(r)\right\}\right)^{-2.5}}{\beta\left(\overline{A}_1(r)-\underline{A}_1(r)\right)+\underline{A}_1(r)}\right)\left(\left\{\beta\left(\overline{V}'(\eta,r)-\underline{V}'(\eta,r)\right)+\underline{V}'(\eta,r)\right\}\right)^2+\\&2\delta\,Ec\Pr\left(\frac{1}{\beta\left(\overline{A}_1(r)-\underline{A}_1(r)\right)+\underline{A}_1(r)}\right)\left(\left\{\beta\left(\overline{V}'(\eta,r)-\underline{V}'(\eta,r)\right)+\underline{V}'(\eta,r)\right\}\right)^4=0.\end{aligned} \tag{5.6}$$

where β is a parameter lies between 0 and 1.

For simplicity, let us denote

$$\beta\left(\overline{V}''(\eta,r)-\underline{V}''(\eta,r)\right)+\underline{V}''(\eta,r)=V''(\eta,r,\beta)$$
$$\beta\left(\overline{\phi}(r)-\underline{\phi}(r)\right)+\underline{\phi}(r)=\phi(r,\beta)$$
$$\beta\left(\overline{\theta}(\eta,r)-\underline{\theta}(\eta,r)\right)+\underline{\theta}(\eta,r)=\theta(\eta,r,\beta)$$
$$\beta\left(\overline{\theta}''(\eta,r)-\underline{\theta}''(\eta,r)\right)+\underline{\theta}''(\eta,r)=\theta''(\eta,r,\beta)$$
$$\beta\left(\overline{A}_1(r)-\underline{A}_1(r)\right)+\underline{A}_1(r)=A_1(r,\beta)$$

By using these defined notations, Eqs. (5.5) and (5.6) can be written as

$$V''(\eta,r,\beta)+6\delta(1-\phi(r,\beta))^{2.5}\left(V'(\eta,r,\beta)\right)^2V''(\eta,r,\beta)+\theta(\eta,r,\beta)=0 \tag{5.7}$$

$$\theta''(\eta,r,\beta)+Ec\Pr\left(\frac{(1-\phi(r,\beta))^{-2.5}}{A_1(r,\beta)}\right)\left(V'(\eta,r,\beta)\right)^2+2\delta Ec\Pr\left(\frac{1}{A_1(r,\beta)}\right)\left(V'(\eta,r,\beta)\right)^4=0. \tag{5.8}$$

Similarly, boundary conditions may be expressed as

$$V(\eta,r,\beta)=\begin{cases}0, & \eta=1\\ 0, & \eta=-1\end{cases} \tag{5.9}$$

$$\theta(\eta, r, \beta) = \begin{cases} -0.5, & \eta = 1 \\ 0.5, & \eta = -1 \end{cases} \tag{5.10}$$

Now, we apply HPM to solve Eqs. (5.7) and (5.8). Homotopy for Eqs. (5.7) and (5.8) may be constructed as

$$\begin{aligned} H_1(q,\eta) = (1-q)\left[V''(\eta,r,\beta) + \theta(\eta,r,\beta)\right] + \\ q\left[V''(\eta,r,\beta) + 6\delta(1-\phi)^{2.5}\left(V'(\eta,r,\beta)\right)^2 V''(\eta,r,\beta) + \theta(\eta,r,\beta)\right] = 0, \end{aligned} \tag{5.11}$$

$$H_2(q,\eta) = (1-q)\left[\theta''(\eta,r,\beta)\right] +$$

$$q\left[\theta''(\eta,r,\beta) + Ec.\Pr.\left(\frac{(1-\phi(r,\beta))^{-2.5}}{A_1(r,\beta)}\right)\left(V'(\eta,r,\beta)\right)^2 + 2\delta\, Ec \Pr\left(\frac{1}{A_1(r,\beta)}\right)\left(V'(\eta,r,\beta)\right)^4\right] = 0. \tag{5.12}$$

According to this method,

$$V(\eta,r,\beta) = V_0(\eta,r,\beta) + qV_1(\eta,r,\beta) + q^2 V_2(\eta,r,\beta) + q^3 V_3(\eta,r,\beta) + \ldots \tag{5.13}$$

and

$$\theta(\eta,r,\beta) = \theta_0(\eta,r,\beta) + q\theta_1(\eta,r,\beta) + q^2\theta_2(\eta,r,\beta) + q^3\theta_3(\eta,r,\beta) + \ldots \tag{5.14}$$

are the assumed series solution of Eqs. (5.11) and (5.12).

Now our goal is to find the unknown functions $V_0(\eta,r,\beta), V_1(\eta,r,\beta), V_2(\eta,r,\beta), V_3(\eta,r,\beta), \ldots$ and $\theta_0(\eta,r,\beta), \theta_1(\eta,r,\beta), \theta_2(\eta,r,\beta), \theta_3(\eta,r,\beta), \ldots$.

Assuming solution $V(\eta,r,\beta)$ and $\theta(\eta,r,\beta)$ should satisfy corresponding differential equations, by substituting Eqs. (5.13) and (5.14) in Eqs. (5.11) and (5.12) and collecting co-efficient of various power of q we may get

$$\begin{aligned} &\left[V_0''(\eta,r,\beta) + \theta_0(\eta,r,\beta)\right]q^0 + \\ &\left[V_1''(\eta,r,\beta) + 6\delta(1-\phi(r,\beta))^{2.5}\left(V_0'(\eta,r,\beta)\right)^2 V_0''(\eta,r,\beta) + \theta_1(\eta,r,\beta)\right]q^1 + \\ &\left[V_2''(\eta,r,\beta) + 6\delta(1-\phi(r,\beta))^{2.5}\left(V_0'(\eta,r,\beta)\right)^2 V_1(\eta,r,\beta) + \right. \\ &\left. 12\delta(1-\phi(r,\beta))^{2.5}\left(V_0'(\eta,r,\beta)\right)\left(V_1'(\eta,r,\beta)\right)\left(V_0''(\eta,r,\beta)\right) + \theta_2(\eta,r,\beta)\right]q^2 + \ldots = 0 \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} &\left[\theta_0''(\eta,r,\beta)\right]q^0 + \\ &\left[\theta_1''(\eta,r,\beta) + Ec\Pr\frac{(1-\phi(r,\beta))^{-2.5}}{A_1(r,\beta)}\left(V_0'(\eta,r,\beta)\right)^2 + \frac{2\delta\, Ec\Pr}{A_1(r,\beta)}\left(V_0'(\eta,r,\beta)\right)^4\right]q^1 \\ &\left[\theta_2''(\eta,r,\beta) + 2Ec\Pr\frac{(1-\phi(r,\beta))^{-2.5}}{A_1(r,\beta)}\left(V_0'(\eta,r,\beta)\right)\left(V_1'(\eta,r,\beta)\right) + \right. \\ &\left. + \frac{8\delta\, Ec\Pr}{A_1(r,\beta)}\left(V_0'(\eta,r,\beta)\right)^3\left(V_1'(\eta,r,\beta)\right)\right]q^2 + \ldots = 0. \end{aligned} \tag{5.16}$$

Further, by equating coefficients of various power of q from Eqs. (5.15) and (5.16) to zero separately and using proper boundary condition we may get the functions $V_0(\eta,r,\beta), V_1(\eta,r,\beta), V_2(\eta,r,\beta),\ldots$ and $\theta_0(\eta,r,\beta), \theta_1(\eta,r,\beta), \theta_2(\eta,r,\beta),\ldots$ explicitly.

One may notice that coefficient of q^0 in Eq. (5.15) contains terms related to both V_0 and θ_0 whereas in Eq. (5.16) coefficient of q^0 contains only term related to θ_0. So first we are taking the coefficient of q^0 in Eq. (5.16). That is

$$\theta_0''(\eta,r,\beta)=0 \tag{5.17}$$

with boundary conditions

$$\theta_0(\eta,r,\beta)=\begin{cases}-0.5, & \eta=1\\ 0.5, & \eta=-1\end{cases}. \tag{5.18}$$

From Eq. (5.17) and (5.18) we may obtain

$$\theta_0(X,r,\beta)=\frac{-\eta}{2}. \tag{5.19}$$

Next we equate coefficient of q^0 in Eq. (5.15) to zero

$$V_0''(\eta,r,\beta)+\theta_0(\eta,r,\beta)=0$$

$$\Rightarrow V_0''(\eta,r,\beta)-\frac{\eta}{2}=0 \tag{5.20}$$

and boundary conditions will be

$$V(\eta,r,\beta)=\begin{cases}0, & \eta=1\\ 0, & \eta=-1\end{cases} \tag{5.21}$$

By solving Eq. (5.20) with boundary conditions Eq. (5.21) we may get

$$V_0(\eta,r,\beta)=\frac{1}{12}\eta^3-\frac{1}{12}\eta. \tag{5.22}$$

Similarly, by taking into consideration the coefficient of q^1 in Eq. (5.16) and using Eqs. (5.19), (5.21) and boundary conditions

$$\theta_1(\eta,r,\beta)=\begin{cases}0, & \eta=1\\ 0, & \eta=-1\end{cases} \tag{5.23}$$

the explicit form of $\theta_1(\eta,r,\beta)$ may be obtained as

$$\begin{aligned}\theta_1(\eta,r,\beta)=\frac{-Ec\Pr}{12^2A_1(r,\beta)}\Bigg\{&\left(1-\phi(r,\beta)\right)^{-2.5}\left(\frac{1}{56}\eta^8-\frac{1}{10}\eta^5+\frac{1}{2}\eta^2+\frac{1}{10}\eta-\frac{29}{56}\right)\\ &+\frac{\delta}{12^2}\left(\frac{1}{91}\eta^{14}-\frac{4}{55}\eta^{11}+\frac{3}{14}\eta^8-\frac{2}{5}\eta^5+\eta^2+\frac{2366}{5005}\eta-\frac{2453}{2002}\right)\Bigg\}.\end{aligned} \tag{5.24}$$

Again by using $\theta_0(\eta,r,\beta)$, $V_0(\eta,r,\beta)$ and $\theta_1(\eta,r,\beta)$ and coefficient of q^1 in Eq. (5.15) with boundary condition

$$V_1(\eta) = \begin{cases} 0, & \eta = 1 \\ 0, & \eta = -1 \end{cases} \tag{5.25}$$

we may get

$$V_1(\eta,r,\beta) = \frac{-Ec\Pr}{12^2 A_1(r,\beta)}\left\{(1-\phi(r,\beta))^{-2.5}\left(\frac{1}{5040}\eta^{10} - \frac{1}{420}\eta^7 + \frac{1}{24}\eta^4 + \frac{1}{60}\eta^3 - \frac{29}{112}\eta^2 - \frac{1}{70}\eta + \frac{547}{2520}\right)\right.$$
$$+\frac{\delta}{12^2}\left(\frac{1}{21840}\eta^{16} - \frac{1}{2145}\eta^{13} + \frac{1}{420}\eta^{10} - \frac{1}{105}\eta^7 + \frac{1}{12}\eta^4 + \frac{1183}{15015}\eta^3 - \frac{2453}{4004}\eta^2\right.$$
$$\left.\left.-\frac{1033}{15015}\eta + \frac{11507}{21840}\right)\right\} - 6\delta(1-\phi)^{2.5}\left(\frac{1}{1344}\eta^7 - \frac{1}{960}\eta^5 + \frac{1}{1728}\eta^3\right). \tag{5.26}$$

By continuing the above process for coefficient of different powers of q form Eqs. (5.15) and (5.16), we may have the explicit form of $\theta_2(\eta,r,\beta), V_2(\eta,r,\beta), \theta_3(\eta,r,\beta), V_3(\eta,r,\beta)$. According to the present method, the approximate solutions of Eqs. (5.7) and (5.8) may be given by

$$V(\eta,r,\beta) = \lim_{q\to 1}\left(V_0(\eta,r,\beta) + qV_1(\eta,r,\beta) + q^2V_2(\eta,r,\beta) + \ldots\right),$$
$$\theta(\eta,r,\beta) = \lim_{q\to 1}\left(\theta_0(\eta,r,\beta) + q\theta_1(\eta,r,\beta) + q^2\theta_2(\eta,r,\beta) + \ldots\right).$$

Now from Eqs. (5.19), (5.22), (5.24) and (5.26), two terms approximation solutions of Eqs. (5.7) and (5.8) may be given as

$$V(\eta,r,\beta) = \frac{1}{12}\eta^3 - \frac{1}{12}\eta - \frac{-Ec\Pr}{12^2 A_1(r,\beta)}\left\{(1-\phi(r,\beta))^{-2.5}\left(\frac{1}{5040}\eta^{10} - \frac{1}{420}\eta^7 + \frac{1}{24}\eta^4 + \frac{1}{60}\eta^3 - \frac{29}{112}\eta^2\right.\right.$$
$$\left.-\frac{1}{70}\eta + \frac{547}{2520}\right) + \frac{\delta}{12^2}\left(\frac{1}{21840}\eta^{16} - \frac{1}{2145}\eta^{13} + \frac{1}{420}\eta^{10} - \frac{1}{105}\eta^7 + \frac{1}{12}\eta^4 + \frac{1183}{15015}\eta^3 - \frac{2453}{4004}\eta^2\right.$$
$$\left.\left.-\frac{1033}{15015}\eta + \frac{11507}{21840}\right)\right\} - 6\delta(1-\phi(r,\beta))^{2.5}\left(\frac{1}{1344}\eta^7 - \frac{1}{960}\eta^5 + \frac{1}{1728}\eta^3\right), \tag{5.27}$$

and

$$\theta(\eta,r,\beta) = \frac{-\eta}{2} - \frac{Ec\Pr}{12^2 A_1(r,\beta)}\left\{(1-\phi(r,\beta))^{-2.5}\left(\frac{1}{56}\eta^8 - \frac{1}{10}\eta^5 + \frac{1}{2}\eta^2 + \frac{1}{10}\eta - \frac{29}{56}\right)\right.$$
$$\left.+\frac{\delta}{12^2}\left(\frac{1}{91}\eta^{14} - \frac{4}{55}\eta^{11} + \frac{3}{14}\eta^8 - \frac{2}{5}\eta^5 + \eta^2 + \frac{2366}{5005}\eta - \frac{2453}{2002}\right)\right\}. \tag{5.28}$$

Similarly, one may include more terms depending upon the required accuracy. Here the fuzzy solutions are controlled by the parameters r and β.

6. Results and discussion

In this section results obtained for velocity and temperature of the considered natural convection problem have been discussed. For validation, present results are compared with the existing results in some special cases viz. crisp case. We have assumed ϕ as a TFN viz. $\tilde{\phi} = [0, 0.1, 0.2]$. Fig. 2

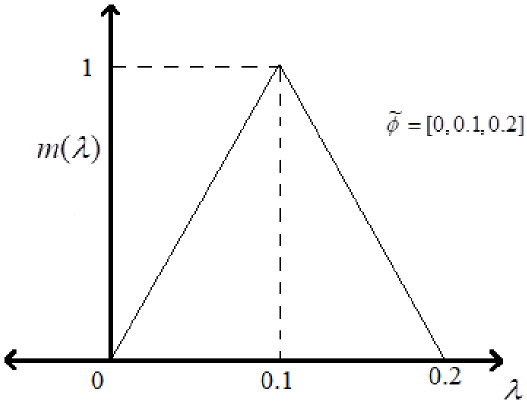


Fig. 2 TFN $\tilde{\phi} = [0, 0.1, 0.2]$

Table 2 Comparison of present result when *r*=*β*=0, *δ*=0.5, *Ec*=*Pr*=1 with existing crisp result for real fluid

η	Velocity $V(\eta,0,0)$ Present Result	Velocity $V(\eta,0,0)$ HAM (Ziabakhsh and Domairry 2009)	Temperature $\theta(\eta,0,0)$ Present Result	Temperature $\theta(\eta,0,0)$ HAM (Ziabakhsh and Domairry 2009)
-1.0	0.00000000	0.00000000	0.49987599	0.49999999
-0.9	0.01423539	0.01411679	0.45083302	0.45044180
-0.8	0.02416171	0.02391937	0.40157624	0.40073588
-0.7	0.03014722	0.02979069	0.35215683	0.35096603
-0.6	0.03262933	0.03217274	0.30260966	0.30117737
-0.5	0.03208611	0.03154511	0.25295810	0.25138653
-0.4	0.02901579	0.02840695	0.20321740	0.20159090
-0.3	0.02392317	0.02326343	0.15339713	0.15177707
-0.2	0.01731154	0.01661778	0.10350286	0.10192749
-0.1	0.00967934	0.00896816	0.05353738	0.05202578
0.0	0.00152009	0.00080780	0.00350177	0.00206049
0.1	-0.00667536	-0.00737269	-0.04660377	-0.04797267
0.2	-0.01441546	-0.01508225	-0.09677903	-0.09807006
0.3	-0.02120459	-0.02182606	-0.14702295	-0.14822071
0.4	-0.02654082	-0.02710348	-0.19733286	-0.19840851
0.5	-0.02991595	-0.03040789	-0.24770374	-0.24861580
0.6	-0.03081889	-0.03122988	-0.29812783	-0.29882852
0.7	-0.02874368	-0.02906544	-0.34859460	-0.34904314
0.8	-0.02320304	-0.02342875	-0.39909138	-0.39927474
0.9	-0.01374877	-0.01387092	-0.44960488	-0.44956649
1.0	0.00000000	0.0000000	-0.50012401	-0.50000000

represents the plot for TFN $\tilde{\phi} = [0, 0.1, 0.2]$.

By following the concept of *r*-cut, the interval form of $\tilde{\phi} = [0, 0.1, 0.2]$ may be obtained as

$$\tilde{\phi} = [0.1r, 0.2 - 0.1r] \text{ where } r \in [0,1].$$

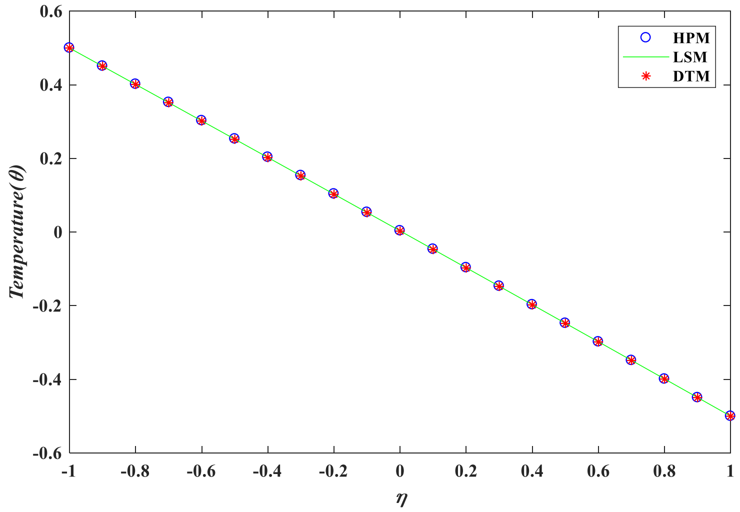


Fig. 3(a) Temperature profile for Cu-SA nanofluid when r=0, β=0.05, δ=Ec=Pr=1

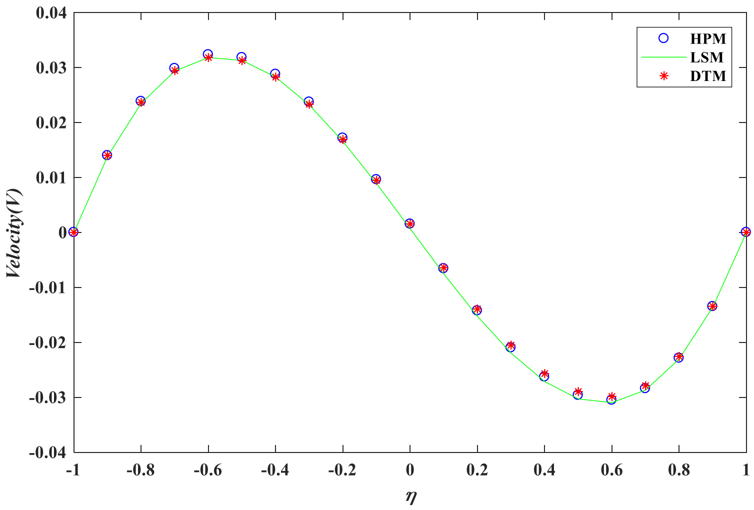
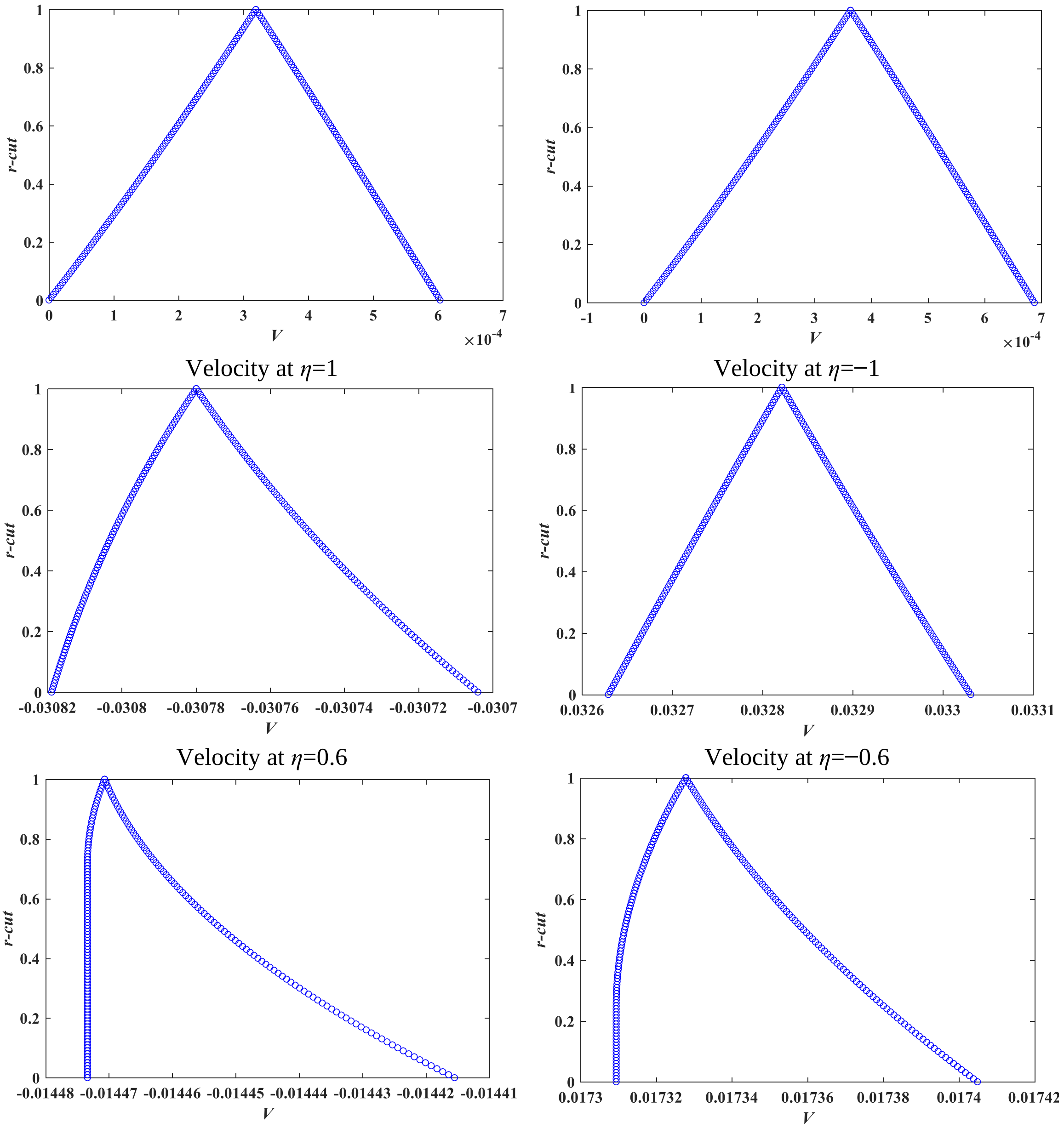


Fig. 3(b) Velocity profile for Cu-SA nanofluid when r=0, β=0.05, δ=Ec=Pr=1

Again by parametric approach, it may be transferred to parametric form as

$$\phi(r,\beta) = 0.1r + 0.2\beta(1-r) \text{ where } r,\beta \in [0,1].$$

It is worth mentioning here that parameters r and β control the fuzzy term. One may observe that by putting r=0 in the interval form of volume fraction, it will cover the whole interval that is $\tilde{\phi} = [0, 0.2]$. Further for β=0 in crisp form of volume fraction we may get ϕ=0, which convert the fuzzy coupled differential Eqs. (5.7) and (5.8) into the modeled problem for real fluid. As such, for validation of the present result Table 2 shows the comparison between existing results for real fluid (Ziabakhsh and Domairry 2009) with the present result when r=β=0. Also by substituting r=0 and any non-zero value of β in Eqs. (5.7) and (5.8) we may get model coupled differential equations for the considered natural convection problem of nanofluid in crisp case. Figs. 3(a) and 3(b) depict the



Velocity at η=1 Velocity at η=−1

Velocity at η=0.6 Velocity at η=−0.6

Velocity at η=0.2 Velocity at η=−0.2

Fig. 4 Fuzzy plots of velocity for Cu-SA nanofluid when δ=0.5, *Ec*=*Pr*=1

comparison of present results for Cu-SA nanofluid when r=0 and β=0.05 with the corresponding existing crisp result for ϕ=0.01 (Hatami and Ganji 2014). Figs. 4 and 5 represent the fuzzy plot of velocity and temperature respectively for different values of η. From Fig. 4, it may be concluded that fuzzy plots of velocity at η=a and η=−a are almost similar for $a \in [-1, 1]$. The same thing for temperature may be observed from Fig. 5.

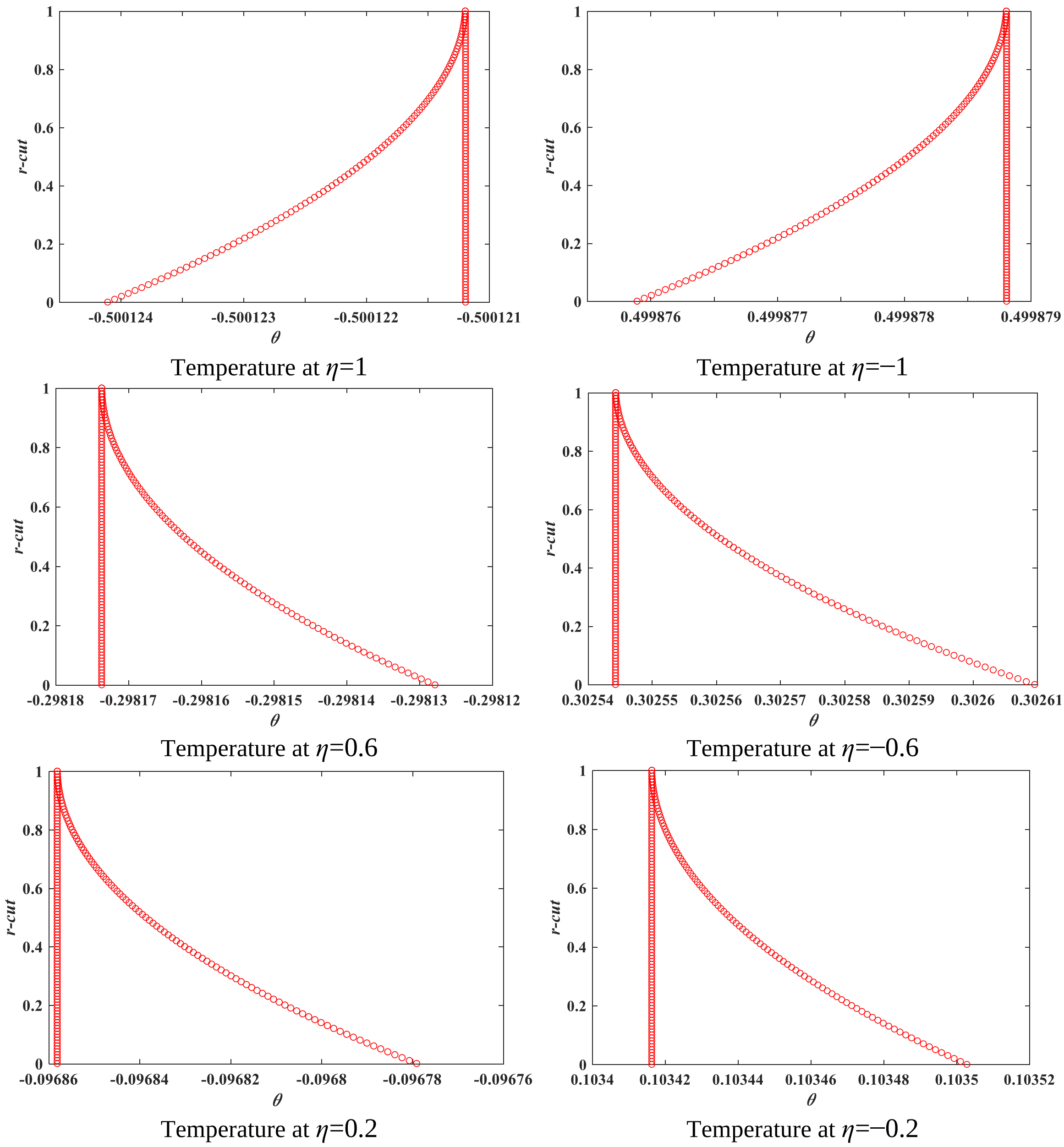


Fig. 5 Fuzzy plots of temperature for Cu-SA nanofluid when δ=0.5, Ec=Pr=1

7. Conclusions

HPM has been successfully implemented to find the fuzzy form of velocity and temperature of Cu-SA nanofluid flow between vertical parallel plates. Uncertain form of nanoparticle volume fraction has been considered as triangular fuzzy number. The parameters r and β control fuzziness in the present article. By substituting r=0 and varying β between 0 and 1 in final solution, we may get velocity and temperature profiles for different values of volume fraction. Obtained results are compared with existing results viz. crisp case and they are found to be in good agreement. Fuzzy plots of velocity and temperature for different values of η have also been illustrated.

Acknowledgment

The 1st author is thankful to the Council of Scientific and Industrial Research (CSIR), New Delhi, India for the support and funding to carry out the present research work.

References

Bakar, N.A.A., Bachok, N. and Arifin, N.M. (2016), "Boundary layer flow and heat transfer in nanofluid over a stretching sheet using buongiorno model and thermophysical properties of nanoliquids", *Ind. J. Sci. Technol.*, **9**, 31.

Biswal, U., Chakraverty, S. and Ojha, B. (2019), "Natural convection of non-Newtonian nanofluid flow between two vertical parallel plates", *Int. J. Numer. Meth. Heat Fluid Flow*, **29**(6), 1984-2008. https://doi.org/10.1108/HFF-06-2018-0302.

Brinkman, H.C. (1952), "The viscosity of concentrated suspensions and solutions", *J. Chem. Phys.*, **20**(4), 571-571. https://doi.org/10.1063/1.1700493.

Chakraverty, S. and Nayak, S. (2012), "Fuzzy finite element method for solving uncertain heat conduction problems", *Coupl. Syst. Mech.*, **1**(4), 345-360. http://dx.doi.org/10.12989/csm.2012.1.4.345.

Chakraverty, S. and Nayak, S. (2017), *Neutron Diffusion: Concepts and Uncertainty Analysis for Engineers and Scientists*, CRC Press, Taylor and Francis, Boca Raton, Florida.

Chakraverty, S. and Perera, S. (2018), *Recent Advances in Applications of Computational and Fuzzy Mathematics*, Springer, Singapore.

Chakraverty, S., Tapaswini, S. and Behera, D. (2016), *Fuzzy Arbitrary Order System: Fuzzy Fractional Differential Equations and Applications*, John Wiley & Sons.

Chakraverty, S., Tapaswini, S. and Behera, D. (2016), *Fuzzy Differential Equations and Applications for Engineers and Scientists*, CRC Press.

Ellahi, R., Hassan, M. and Zeeshan, A. (2015), "Study of natural convection MHD nanofluid by means of single and multi-walled carbon nanotubes suspended in a salt-water solution", *IEEE Tran. Nanotechnol.*, **14**(4), 726-734. https://doi.org/10.1109/TNANO.2015.2435899.

Garoosi, F., Jahanshaloo, L., Rashidi, M.M., Badakhsh, A. and Ali, M.E. (2015), "Numerical simulation of natural convection of the nanofluid in heat exchangers using a Buongiorno model", *Appl. Math. Comput.*, **254**, 183-203. https://doi.org/10.1016/j.amc.2014.12.116.

Hanss, M. (2005), *Applied Fuzzy Arithmetic: an Introduction with Engineering Applications*.

Hatami, M. and Ganji, D.D. (2014), "Natural convection of sodium alginate (SA) non-Newtonian nanofluid flow between two vertical flat plates by analytical and numerical methods", *Case Stud. Therm. Eng.*, **2**, 14-22. https://doi.org/10.1016/j.csite.2013.11.001.

He, J.H. (1999), "Homotopy perturbation technique", *Comput. Meth. Appl. Mech. Eng.*, **178**(3-4), 257-262. https://doi.org/10.1016/S0045-7825(99)00018-3.

He, J.H. (2005), "Application of homotopy perturbation method to nonlinear wave equations", *Chaos Solit. Fract.*, **26**(3), 695-700. https://doi.org/10.1016/j.chaos.2005.03.006.

He, J.H. (2006), "Homotopy perturbation method for solving boundary value problems", *Phys. Lett. A*, **350**(1-2), 87-88. https://doi.org/10.1016/j.physleta.2005.10.005.

Karunakar, P. and Chakraverty, S. (2017), "Comparison of solutions of linear and non-linear shallow water wave equations using homotopy perturbation method", *Int. J. Numer. Meth. Heat Fluid Flow*, **27**(9), 2015-

2029. https://doi.org/10.1108/HFF-09-2016-0329.
Karunakar, P. and Chakraverty, S. (2018), "2-D shallow water wave equations with fuzzy parameters", *Recent Advances in Applications of Computational and Fuzzy Mathematics*, Springer, Singapore.
Mandal, A. and Maity, D. (2019), "Seismic analysis of dam-foundation-reservoir coupled system using direct coupling method", *Coupl. Syst. Mech.*, **8**(5), 393-414. https://doi.org/10.12989/csm.2019.8.5.393.
Maxwell, J.C. (1881), *A Treatise on Electricity and Magnetism*, Vol. 1, Clarendon Press.
Nayak, S. and Chakraverty, S. (2018), *Interval finite element method with MATLAB*, Academic Press.
Rajagopal, K.R. and Na, T.Y. (1985), "Natural convection flow of a non-Newtonian fluid between two vertical flat plates", *Acta Mechanica*, **54**(3-4), 239-246. https://doi.org/10.1007/BF01184849.
Rao, T.D. and Chakraverty, S. (2017), "Modeling radon diffusion equation in soil pore matrix by using uncertainty based orthogonal polynomials in Galerkin's method", *Coupl. Syst. Mech.*, **6**(4), 487-499. https://doi.org/10.12989/csm.2017.6.4.487.
Sheikholeslami, M. and Seyednezhad, M. (2018), "Simulation of nanofluid flow and natural convection in a porous media under the influence of electric field using CVFEM", *Int. J. Heat Mass Transf.*, **120**, 772-781. https://doi.org/10.1016/j.ijheatmasstransfer.2017.12.087.
Sus, C. and Eastman, J.A. (1995), "Enhancing thermal conductivity of fluids with nanoparticles", *International Mechanical Engineering Congress and Exhibition (IMECE)*, San Francisco.
Ziabakhsh, Z. and Domairry, G. (2009), "Analytic solution of natural convection flow of a non-Newtonian fluid between two vertical flat plates using homotopy analysis method", *Commun. Nonlin. Sci. Numer. Simul.*, **14**(5), 1868-1880. https://doi.org/10.1016/j.cnsns.2008.09.022.
Ziaolhagh, S.H., Goudarzi, M. and Sani, A.A. (2016), "Free vibration analysis of gravity dam-reservoir system utilizing 21 node-33 Gauss point triangular elements", *Coupl. Syst. Mech.*, **5**(1), 59-86. https://doi.org/10.12989/csm.2016.5.1.059.

CC

Nomenclature

b	Constant number
C_p	Specific heat
Ec	Eckert number
k	Thermal conductivity
Pr	Prandtl number
T_i, i=1,2	Temperature
V	Dimensionless velocity

Greek symbols

δ	Dimensionless non-Newtonian viscosity
ϕ	Nanoparticle volume fraction
θ	Dimensionless temperature
ρ	Effective density
μ	Effective dynamic viscosity
ρC_p	Heat capacitance

Subscripts

s	solid particle
f	fluid
nf	nanofluid