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Analytical and numerical algorithm for exploring dynamic response of non-classically damped hybrid structures

Shehata E. Abdel Raheem*[1,2]

[1]*Faculty of Engineering, Taibah University, Madinah 41411, P.O. 344, Saudia Arabia*
[2]*Civil Engineering Department, Faculty of Engineering, Assiut University, Assiut 71516, Egypt*

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Abstract. The dynamic characterization is important in making accurate predictions of the seismic response of the hybrid structures dominated by different damping mechanisms. Different damping characteristics arise from the construction of hybrid structure with different materials: steel for the upper part; reinforced concrete for the lower main part and interaction with supporting soil. The process of modeling damping matrices and experimental verification is challenging because damping cannot be determined via static tests as can mass and stiffness. The assumption of classical damping is not appropriate if the system to be analyzed consists of two or more parts with significantly different levels of damping. The dynamic response of structures is critically determined by the damping mechanisms, and its value is very important for the design and analysis of vibrating structures. A numerical algorithm capable of evaluating the equivalent modal damping ratio from structural components is desirable for improving seismic design. Two approaches are considered to explore the dynamic response of hybrid tower of cable-stayed bridges: The first approach makes use of a simplified model of 2 coupled lumped masses to investigate the effects of subsystems different damping, mass ratio, frequency ratio on dynamic characteristics and equivalent modal damping; the second approach employs a detailed numerical step-by step integration procedure.

Keywords: numerical algorithm; non-classical damping; hybrid structure; modal damping; coupling index

1. Introduction

As demonstrated by many field forced-excitation tests, the damping characteristics of hybrid cable-stayed bridges vary from bridge to bridge. This is due to the fact that the energy mechanisms predominant in the bridges are different (Kawashima *et al.* 1993). Therefore an analytical approach capable of evaluating the equivalent modal damping ratio of cable-stayed bridge from structural components is desirable for improving seismic design; by dividing a cable-stayed bridge into several substructures in which the energy dissipation mechanism can be regarded as the same (Johnson and Kienhholz 1982, Huang *et al.* 1995, Abdel Raheem and Hayashikawa 2007, 2008, 2009). In code-based seismic design of such hybrid structures several practical difficulties are encountered, due to inherent differences in the nature of damping of different parts. Such structures are irregularly damped and have complex modes of vibration, so that their analysis cannot be handled with readily available commercial software. Dynamic characterization is

*Corresponding author, Associate Professor, E-mail: shehataraheem@yahoo.com

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extremely important in formulating predictive models for seismic response of hybrid cable-stayed bridge structures subjected to earthquake loadings. Characterization of damping forces in a vibrating structure has long been an active area of research in structural dynamics (Prater and Singh 1990, Prells and Friswell 2000, Angeles and Ostrovskaya 2002, Du *et al.* 2002, Adhikari 2004a, b, Xu *et al.* 2004, Khanlari and Ghafory-Ashtiany 2005, Abdel Raheem and Hayashikawa 2007, 2008). There are many situations in which the un-damped and classically damped assumptions are invalid, since classical damping means that energy dissipation is almost uniformly distributed throughout the system. In general, this condition is not satisfied and thus damped hybrid systems cannot be decoupled by modal analysis (Ma and Morzfeld 2011); such as the structures made up of materials with different damping characteristics in different parts, structures equipped with passive and active control systems, soil structure interaction system and structures with layers of damping materials (Qu *et al.* 2003).

The process of modeling damping matrices and their experimental verification are challenging because damping cannot be determined via static tests as can mass and stiffness (Abdel Raheem 2014, Jehel *et al*. 2014). Furthermore, damping is more difficult to determine from dynamic measurements than natural frequency. There have been detailed studies on the material damping (Bert 1973) and on energy dissipation mechanisms in the joints (Bread 1979). The performance of a classical damping matrix, constructed either from the use of initial structural properties or current structural properties, in the step-by-step solution of a nonlinear multiple degree of freedom system is analytically evaluated (Chang 2013). However, in most real systems the damping is non-classical, even when classical damping is assumed for each sub-system in the analysis of mixed steel/concrete structure - soil interaction systems. In such problems, a more realistic model for the damping force should be used to capture the correct response, which leads to complex Eigen properties (Chopra 1995). Moreover, the damping matrix is required for most of standard analysis methods for a complete system. The seismic response of non-classical damping system can be substituted approximately by the seismic response calculated according to uniform damping ratio of concrete tower and steel stiffening girder respectively, which can simplify the calculation during preliminary analysis (Ding *et al.* 2011). The equivalent damping could be approximately estimated with different methods, such as the complex modal analysis, neglecting off-diagonal-elements in modal damping matrix and composite damping rule. In composite damping rule method; the equivalent damping ratio is computed as the sum of the damping ratio of each component weighted by the modal strain energy ratio of each component to that of total bridge system (Ragget 1975, Johnson and Kienhholz 1982, Lee *et al*. 2004). Papageorgiou and Gantes (2010, 2011) proposed equivalent modal/uniform damping ratios for structures with Rayleigh type damping and with simpler damping configurations; the basis of these works is a trial and error process of potential uniform damping ratios in substitution of the actual damping distribution of the structure. Villaverde (2008) proposed a method for using the complex modes of irregularly damped structures in combination with response spectra in order to compute the maxima of the structural response. Warburton and Soni (1977) proposed a parameter to assess the accuracy of the effective modal damping ratio that is computed by eliminating the off-diagonal elements of modal damping matrix.

The classical damping assumption is not appropriate if the system to be analyzed consists of two or more parts with significantly different levels of damping, although it may be reasonable for each region separately. The well separated different materials cause damping to be unevenly distributed for the complete bridge, known as non-classical damping (Qin and Lou 2000). In conventional analysis of hybrid structures, it is generally assumed that damping may be defined in terms of modal damping ratios for different types of sub-structures. One such example is mixed steel/concrete structure soil system; where the equivalent damping ratio for the hybrid system would typically be much different (Japan Road Association 1996, 2002) (15~20% for the soil region, 5~10% for footing compared to 2~5% for the steel super-structure). It is shown that the effect of non-classical damping is significant in systems that have nearly tuned modes and sufficiently small values of modal mass ratios. This study presents state-of-the-art knowledge on dynamic and modal response of 2 coupled SDOF systems as simplified simulation of coupled subsystems of different mass ratio; frequency ratio and damping ratio, hence an equivalent damping ratio, homogeneous over the composite structure could be used to construct the modal damping matrix for dynamic response by the direct integration method. An analytical approach capable of evaluating the equivalent modal damping ratio from structural components is desirable for improving seismic design. Numerical algorithm is formulated to efficiently evaluate the dynamic response of non-classical damped composite structures in earthquake engineering applications. Dynamic response of the systems with non-classical damping can be obtained by two approaches: direct integration method or modal analysis

The model analysis should be done through the calculation of the complex eigenvectors of the complete system, which are then used to transform the system to an uncoupled set of complex modal equations (Veletsos and Ventura 1986, Ibrahimbegovic and Wilson 1989, Ibrahimbegovic *et al*. 1990). However, the calculation of the complex mode shapes for a system with non-proportional damping requires approximately eight times the numerical effort as for the calculation of the un-damped free vibration mode shapes, as the eigen value problem is twice as large. Also, this method is the appropriate numerical method for the dynamic response of systems that can be accurately approximated by a small number of modes. This study introduces the dynamic characteristics of simplified 2 coupled SDOF systems, where equivalent modal damping and coupling index variations with structural parameter of mass ratio and frequency ratio are introduced. The equivalent modal damping could be used to defined the damping matrix for the direct integration method, the coupling index could be used to define the degree of the coupling of modes, hence could decide the range where the dynamic response could approximated using the coupled number of un-damped free vibration modes. Also check the effect of skipping the off diagonal terms in the modal damping matrix. The direct integration method, which is The most general and efficient approach for the solution of the dynamic response of structural systems in which a large number of high frequencies is excited, is the direct numerical integration of the dynamic equilibrium equations. However, the formation of damping matrix argument should be solved.

2. Theoretical approach

Classical modal analysis can be rigorously applied only to systems with proportional damping; when this is not the case, approximate solutions (Roesset *et al*. 1973, Kusainov and Clough 1988) based on the use of un-damped modal shapes and on the forced diagonalization of the transformed damping matrix, are adopted in most practical cases. It has been demonstrated, however, that these solutions can sometimes lead to unacceptable errors in response computations (Warburton and Soni 1977, Igusa *et al*. 1984). An alternative strategy consists of direct integration of a reduced set of coupled normal coordinates based on the use of classical modes (Clough and Mojtadhedi 1976);

however, with this approach, some of the advantages of modal analysis are lost. The decoupling approximation, as proposed by Rayleigh (1945), consists in neglecting the off-diagonal elements of the modal damping matrix. This approach is motivated by the smallness of the off-diagonal elements compared with the diagonal ones. Recently, some paradoxical results have been highlighted about this practiced assumption. Indeed, Morzfled *et al*. (2009) show that the relative smallness between off-diagonal and diagonal elements is not sufficient to ensure small decoupling errors. The complex modal analysis, originally proposed by Foss (1958), is an extension of the classical modal analysis to complex modes in a state space. This method allows decoupling the equations of motion in the state space of the system and therefore to perform deterministic or stochastic analyses in time or frequency domains without assumption about damping matrix. Wagner and Adhikari (2003) also proposed a state-space approach for the analysis of linear systems with exponential damping, adopting internal variables. Even if the state-space method can provide exact results, significant computations induced by the enlargement of the matrices size are required. Nevertheless, engineers and practitioners, especially in civil engineering, have never considered this approach as suitable, because of the difficult physical meaning of complex modes. The recent research activity in this field has been devoted to exact modal analysis that is to the use of the normal modes of the damped system; Adhikari (2002) and Corte´s and Elejabarrieta (2006) presented the modal analysis method and numerical method respectively to treat the eigenvalues and eigenvectors in the non-viscously damped linear dynamic systems. In an alternative approach, Ibrahimbegovic (1989) proposes a simple numerical algorithm considering off-diagonal damping forces as pseudo-forces applied to the uncoupled system. To avoid full transfer matrix inversion, Denoel and Degee (2009) propose an asymptotic expansion of the transfer matrix assuming the relative smallness of off-diagonal elements with regards to diagonal ones. Dynamic response of the systems with non-classical damping can be obtained by direct integration or by the calculation of the complex eigenvectors of the complete system, which are then used to transform the system to an uncoupled set of complex modal equations (Veletsos and Ventura 1986, Ibrahimbegovic and Wilson 1989, Ibrahimbegovic *et al*. 1990). For systems in which dynamic response can be accurately approximated by a small number of modes the mode superposition method is the appropriate numerical method. However, the calculation of the complex mode shapes for a system with non-proportional damping requires approximately eight times the numerical effort as for the calculation of the un-damped free vibration mode shapes, as the eigen value problem is twice as large. The most general and efficient approach for the solution of the dynamic response of structural systems in which a large number of high frequencies is excited, is the direct numerical integration of the dynamic equilibrium equations.

2.1 Dynamic equilibrium equation of motion

The dynamic equilibrium equation of motion for a structural system with *n*-degree of freedom (DOF) subject to ground motion excitation, whether classically or non-classically damped, can be expressed as follow

$$\mathrm{M}\ddot{\mathrm{u}} + \mathrm{C}\dot{\mathrm{u}} + \mathrm{K}\mathrm{u} = -\mathrm{M}r\ddot{u}_g \tag{1}$$

In which M and K are $n \times n$ mass and stiffness positive definite matrices, respectively. C is $n \times n$ damping semi-positive definite matrix, r is the dynamic load effect vector with all n elements being 1. The classical damping occur when the damping matrix is a linear combination of the mass and stiffness matrices. A commonly accepted for of the damping matrix C_p of classically damped system is Rayleigh's scheme, thus

$$C_p = \alpha M + \beta K \tag{2}$$

where α and β are real parameters, in formulating the Eigen value problem of the system, both sides of governing equation are pre-multiplied by M^{-1}, which is also done with Eq. (2), to yield

$$M^{-1}C_p = \alpha I + \beta\, M^{-1}K \tag{3}$$

Thereby making apparent the foregoing damping leads to a linear combination of what is known as identity matrix I and the dynamic matrix $M^{-1}K$. Therefore, matrices $M^{-1}\,C_p$ and $M^{-1}K$ share the same set of eigenvectors, which explain why under form (2); the mathematical model can be decoupled. Indeed, adding a linear combination of powers of the dynamic matrix at the right hand side of Eq. (3) yields a new matrix that still shares the same eigenvectors with the dynamic matrix. Such generalization of the C_p matrix according to generalized damping scheme. For both classically and non-classically damped structures; a coordinate transformation from physical coordinate's u to modal coordinate's q is adopted

$$u = \Phi\, q \tag{4}$$

where q is $m\times 1$ generalized coordinates vector (with $m\leq n$), and Φ is the $n\times m$ modal matrix, normalized with respect to mass matrix and given by the solution of the un-damped Eigen problem

$$K\,\Phi = M\,\Phi\,\Omega^2 \tag{5}$$

where Ω is a diagonal matrix listing the first m few natural circular frequencies. By using Eq. (4), the differential equations of motion (1) in the modal sub-space can be written as follows

$$\ddot{q} + \Xi\,\dot{q} + \Omega^2 q = p\,\ddot{u}_g \tag{6}$$

where $\mathbf{p}$ is the vector of participation coefficients and Ξ is the generalized damping matrix, given respectively by

$$p = -\,\Phi^T M\, r\,, \qquad \Xi = \Phi^T\, \mathbf{C}\, \Phi \tag{7}$$

For non-classically damped systems, the matrix Ξ is not diagonal. The relative maximum magnitude of the off-diagonal elements of Ξ with respect to the diagonal elements can be expressed by the following coupling index (Claret and Venancio-Filho 1991, Falsone and Muscolino 2004), which measures the degree of non proportionality of the damping.

$$\gamma = \max\left(\frac{\Xi_{ij}^2}{\Xi_{ii}\Xi_{jj}}\right) \quad i \neq j \tag{8}$$

2.2 State-space approach

The dynamic analysis of non-classically damped systems has been one of the most important topics in the field of structural dynamics (Perotti 1994). In the dynamic analysis of structures, the Eigen value problem of the system should be solved a priori in order to avoid resonance or to define the natural vibration characteristics. By introducing the state vector U (Veletsos and Ventura

1986), the equation of motion can be converted to a 2n-dimensional system of the first order differential equation given by

$$\mathrm{A}\,\dot{U} + \mathrm{B}U = -F\ddot{u}_g \tag{9}$$

$$\mathrm{A} = \begin{bmatrix} \mathrm{C} & \mathrm{M} \\ \mathrm{M} & 0 \end{bmatrix},\ \mathrm{B} = \begin{bmatrix} \mathrm{K} & 0 \\ 0 & -\mathrm{M} \end{bmatrix},\ \mathrm{F} = \begin{bmatrix} \mathrm{M}\,r \\ 0 \end{bmatrix},\ U = \{u^T \quad \dot{u}^T\}^T \tag{10}$$

The characteristic equation can be written as

$$(\lambda_j^2 M + \lambda_j C + K)\varphi_j = 0 \tag{11}$$

Eq. (6) can be rearranged into state space form as

$$\begin{bmatrix} 0 & \mathrm{I} \\ -\mathrm{M}^{-1}\mathrm{K} & -\mathrm{M}^{-1}\mathrm{C} \end{bmatrix} \begin{Bmatrix} \varphi_j \\ \lambda_j \varphi_j \end{Bmatrix} = \lambda_j \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{Bmatrix} \varphi_j \\ \lambda_j \varphi_j \end{Bmatrix} \tag{12}$$

where λ_j and φ_j are the *j* eigenvalue and the eigenvector of the structure system, respectively. The complex frequency has the form as

$$\lambda_j = -\alpha_j \pm i\beta_j \qquad (j = 1, 2, \ldots\ldots m) \quad \textit{m: number of vibration modes} \tag{13}$$

The equivalent modal frequency w_{mj} and equivalent modal damping ratio $\zeta_{mj}$of the complex frequencies are give as

$$w_{mj} = \sqrt{\alpha_j^2 + \beta_j^2}\ , \zeta_{mj} = \alpha_j / w_j \quad (j = 1, 2, \ldots\ldots m) \quad \textit{m: number of vibration modes} \tag{14}$$

The free vibration or natural modes of a non-classically damped system are to be distinguished from those of a corresponding system with classical damping by the fact that the components of the former modes differ in phase as well as amplitude. The effect of this variable phase on the motion of the system is that the motion is no longer characterized by the presence of fixed nodes as is the case for un-damped or classically damped systems. The nodes are no longer stationary but rather wander along the modal shape.

3. Numerical results and discussion

The structural behavior during an earthquake can be explained with the aid of modes of vibration of a structure. The major response in a system is primarily due to vibrations of its subsystems. The hybrid tower structure is composed of different substructures representing steel tower superstructure, reinforcement concrete footing/pier and supporting soil. Different damping characteristics arise from the construction of the tower with different materials; steel for the upper part; reinforced concrete for the lower main part and supporting soil. Two approaches are considered to define and investigate dynamic characteristics of hybrid tower of cable-stayed bridges: The first approach method makes use of a simplified approximation of 2 coupled lumped masses to investigate the structure irregularity effects including damping of different material, mass ratio, frequency ratio on dynamic characteristics and modal damping; the second approach employs a detailed numerical step-by step integration procedure in which the damping matrices of the upper and the lower substructures are modeled with the Rayleigh damping formulation.

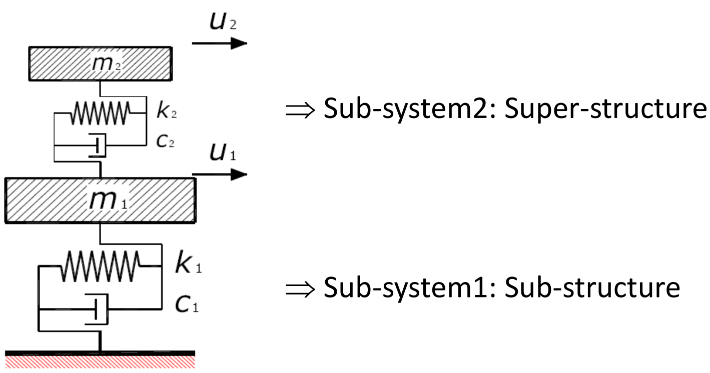


Fig. 1 2-DOF simplified tower model

3.1 Simplified 2-DOF model

An elementary analysis based on a simplified 2-DOF model "SDOF-SDOF coupled sub-systems", Fig. 1, as representative of non-classically damped tower foundation soil interaction system is used to demonstrate the dynamic characteristics. Tower structure system of substructures with different damping characteristics is represented by simple model to study the effect of system structural parameters on modal properties including natural frequency, modal damping characteristics, damping matrix, vibration mode shapes and modal participation factor. The governing equation of motion for damped Free System with 2 coupled DOFs could be given by

$$\mathrm{M\ddot{u}} + \mathrm{C\dot{u}} + \mathrm{Ku} = 0 \tag{15}$$

M, C and K are the mass, damping and stiffness matrices; respectively. $\mathrm{\ddot{u}}$, $\mathrm{\dot{u}}$ and u represent the acceleration, velocity, and displacement vectors, respectively. Details of the matrices and vectors elements can be defined as

$$\mathrm{M} = \begin{bmatrix} \mathrm{m_1} & 0 \\ 0 & \mathrm{m_2} \end{bmatrix}, \quad \mathrm{C} = \begin{bmatrix} \mathrm{c_1 + c_2} & -\mathrm{c_2} \\ -\mathrm{c_2} & +\mathrm{c_2} \end{bmatrix}, \quad \mathrm{K} = \begin{bmatrix} \mathrm{k_1 + k_2} & -\mathrm{k_2} \\ -\mathrm{k_2} & +\mathrm{k_2} \end{bmatrix}, \quad \mathrm{u} = \begin{Bmatrix} \mathrm{u_1} \\ \mathrm{u_2} \end{Bmatrix} \tag{16}$$

The sub-systems modal parameters including natural frequency and damping ratio for each sub-system are given as follows

$$w_j = \sqrt{k_j/m_j} \quad , \quad c_j = 2\zeta_j w_j m_j \quad (j = 1, 2) \tag{17}$$

The frequency domain approach is used to find dynamic response characteristics. For systems with classical damping distributions, in each mode the phase angles between DOF’s are always zero degrees (in-phase) or 180 degrees (completely out-of-phase). The result is that, as time progresses, the shape (not the amplitude) of the free vibration response in a given mode shape remains constant. For systems with non-classical damping distributions, the phase angles generally lie between zero and 180 degrees and thus the DOF’s are either in-phase or completely out-of-phase. The result is that the shape of the free vibration response in a given mode shape changes with time. From the frequency response function plot as shown in Fig. 2, mass ratio (m_2 $/m_1$) and frequency ratio (w_2 $/w_1$) equal to unity for different level of damping, it can be noticed that the amplitude of the peak decreases significantly, verifying the damping effects. The lower plot shows the phase relation between the base and the mass for different frequencies. At low frequencies, the phase is zero degrees means that the mass is in phase with base. The phase becomes 90 degree at resonance. At high frequencies, the phase is -180 degree; the mass is out of phase with the base and move in opposite directions. The damping ratios affect the slope of the phase. Damping ratios affect both the slope of the phase diagram and the sharpness of the magnitude peaks vs. frequency diagram.

The effect of mass ratio (m_2/m_1) and frequency ratio (w_2/w_1) on the acceleration frequency response function is investigated for equal damping ratio case ($\zeta_1 = \zeta_2 = 0.02$) and different damping ratio of sub-systems ($\zeta_1 = 0.20$, $\zeta_2 = 0.02$), as shown in Fig. 3. As the mass ratio decreases, the vibration modes get closer, the soft substructure dominant frequency increase, while the stiff substructure dominant frequency decreases. The lower mass ratio leads to significant modal coupling especially for equal dominant frequency of substructures. Also it is noted the horizontal shift in the position of the natural frequencies as the mass ratio increases. In a complex valued eigenvector, each element describes the relative magnitude and phase of the motion of the DOF associated with that element when the system is excited at that mode only. The relative position of each DOF can be out of phase by the amount indicated by the complex part of the mode shape element; all DOFs vibrate with the same phase angle if the mode shape is real-valued. For frequency domain analysis, high damping lowers the resonant peak of a frequency response function. A further complication of large damping arises when natural frequencies are close, which is a common situation for high frequency modes in complex systems. In such situations, the modal bandwidth of adjacent resonant peaks might exceed the natural frequency difference, leading to a merger of the resonant peaks into one broader peak, which is known as mode coupling. This can make it difficult to distinguish the individual modes.

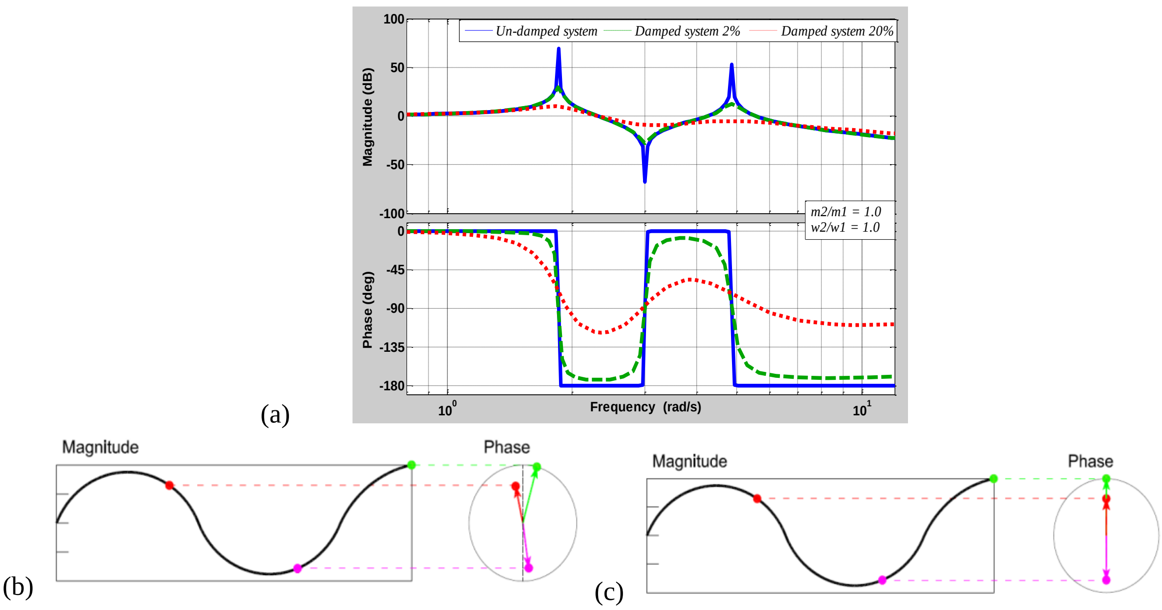
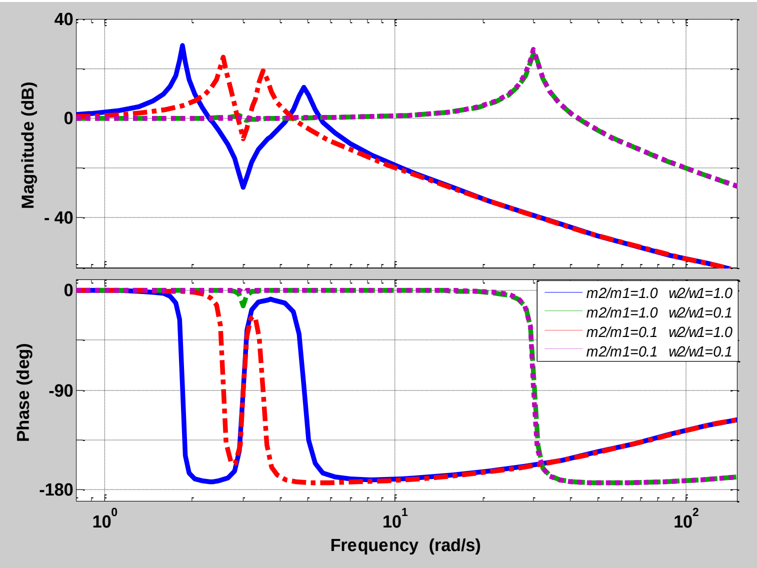
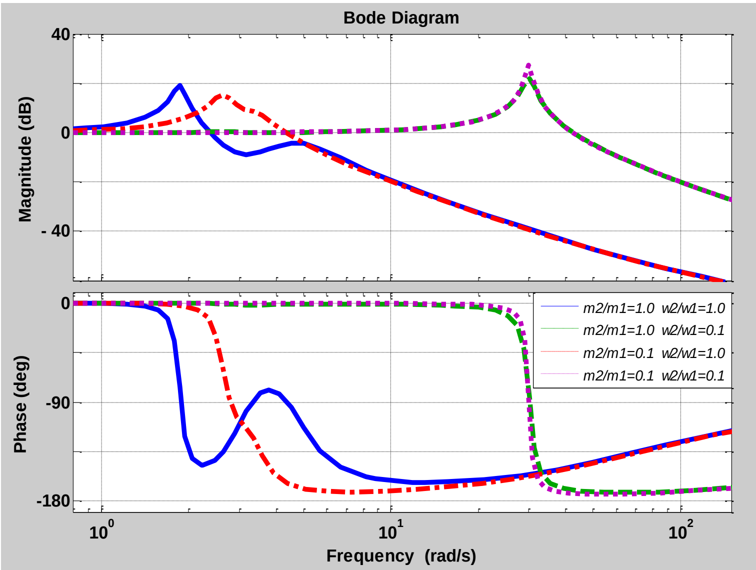


Fig. 2 Frequency response function: (a) damping effect on vibration modes; (b) Classical (real normal) mode schematic and (c) Non-classical (complex) mode schematic



(a) $\zeta_1 = 0.02$, $\zeta_2 = 0.02$



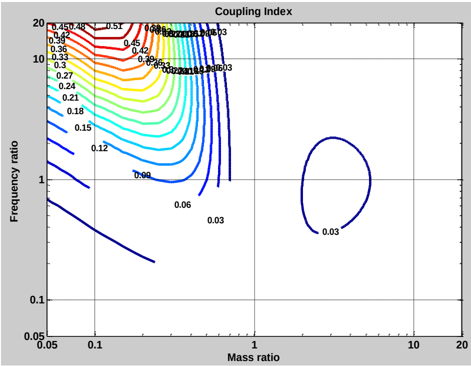
(b) $\zeta_1 = 0.20$, $\zeta_2 = 0.02$

Fig. 3 Frequency response function

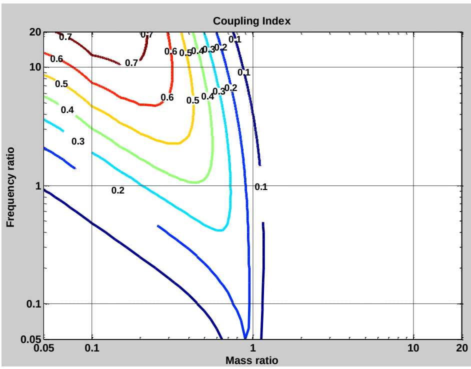
It is shown that the effect of non-classical damping is significant in systems that have nearly tuned modes and sufficiently small values of modal mass ratios. The type of modes of vibration would depend on relative stiffness and mass of different subsystems of tower structure, to cover a range of parameter variations including tower superstructure to footing structure frequency ratio; mass ratio, and different damping ratio for sub-systems. One of the major effects of non-classical damping on tower structures is to cause the damped modal vectors to be coupled with respect to the damping matrix, which is reflected mathematically by the non-zero off-diagonal elements in the transferred damping matrix and measured by coupling index given by Eq. (8). Fig. 4 shows the variation of coupling index with mass ratio (m_2 /m_1) and frequency ratio (w_2 /w_1) for uniform and different damping substructures. For sub-systems with uniform damping (almost classical damping scheme), the coupling index increases with structure irregularity as frequency ratio increases and the trend of increase become slight as the system approach regular mass distribution (mass ratio = 1.0), Fig. 4(a).

For a structure system with substructures of different material damping, the damping behavior has two distinct trend: the first trend as could be shown in Figs. 4(b) and 4(d), where the substructure 2 has higher damping relative to that of substructure 1, the contour area for the case of uniform damping is stretched to lower value of frequency ratio, and the coupling index is characterized with bell shape with high rate of change with mass ratio at low frequency ratio, the peak values increase as frequency ratio increases; the second as could be shown in Figs. 4(c) and 4(e), where the substructure 1 has higher damping relative to that of substructure 2. The effect of mass ratio on coupling index has bell shape with its peak locus shift to lower mass ration with the increase of frequency ratio increase, moreover the peak value get high value. For sub-systems with different damping (non-classical damping scheme, super structure with damping 2%, and sub-structure with higher damping 5% and 20%), new trend behavior of the coupling index behavior with the variation of stiffness and mass irregularity of structure appears. For sub-structure with 5% damping, the coupling index have two different regions, Fig. 4(c), the first trend is the same as that showed in Fig. 4(a), but this region get smaller. The second trend is that the coupling index increases with structure irregularities of mass and stiffness, the peak locus make slight angle to the right with the vertical axis. As frequency ratio increases, the coupling index decrease and expand over a wide range of mass ratio variation. The second region expands while the first region and trend disappear as shown in Fig. 4(e).

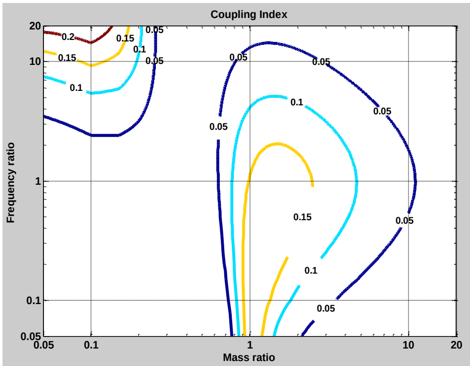
Figs. 5 and 6 show the variation of 1st and 2nd mode modal damping with mass/frequency ratio, for sub-systems with equal damping, the first mode modal damping decreases and the second mode modal damping increases with the increase of frequency ratio at constant level of mass ratio. At certain level of frequency ratio, the first model modal damping decreases and the second model modal damping increases, this behavior gets their peaks around uniform mass structure then keep constant horizontal trend "no effects". For sub-systems with different damping (non-classical damping scheme, super structure with damping 2%, and sub-structure with higher damping 5% and 20%), new trend behavior of the coupling index behavior with the variation of stiffness and mass irregularity of structure appears. For sub-structure with 5%damping, modal have two different regions, Figs. 5(c) and 6(c), the first trend is the same as that showed in Figs. 5(a) and 6(a), but this region get smaller. The second trend; the modal damping has bell shape with the variation of mass/frequency ratio. As frequency ratio increases, the modal damping expands over a wide range of mass ratio variation. The second region expands while the first region and trend disappear as seen in Figs. 5(e) and 6(e). Fig. 7 presents the natural frequency variation of 1st and 2nd vibration modes.



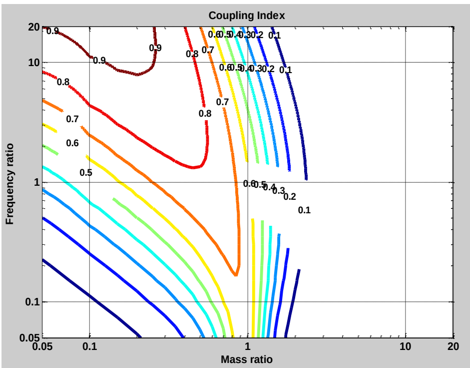
(a) $\zeta_1 = \zeta_2 = 0.02$



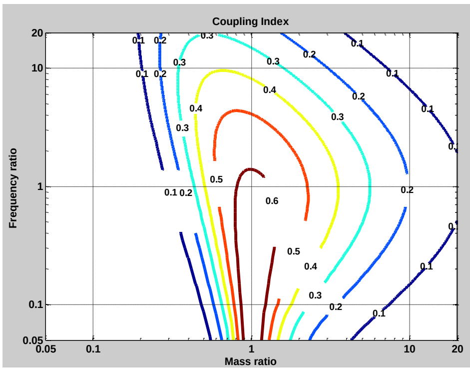
(b) $\zeta_1 == 0.02$, $\zeta_2 = 0.05$



(c) $\zeta_1 == 0.05$, $\zeta_2 = 0.02$

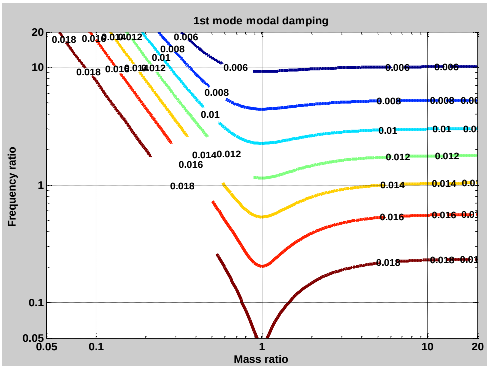


(d) $\zeta_1 == 0.02$, $\zeta_2 = 0.20$

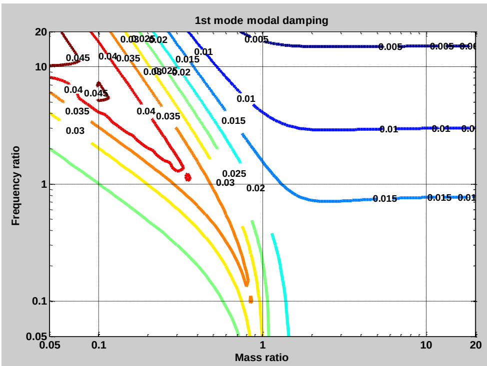


(e) $\zeta_1 == 0.20$, $\zeta_2 = 0.02$

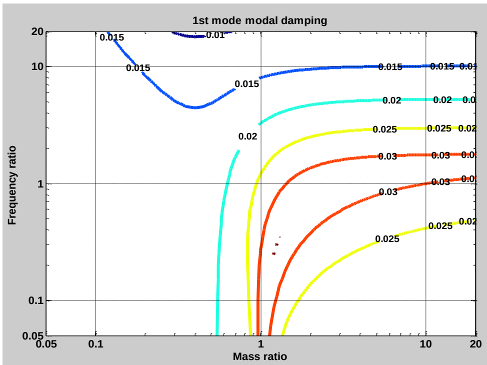
Fig. 4 Coupling index variation with mass/frequency ratio



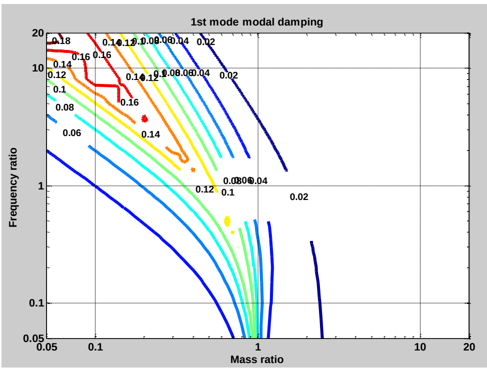
(a) $\zeta_1 = \zeta_2 = 0.02$



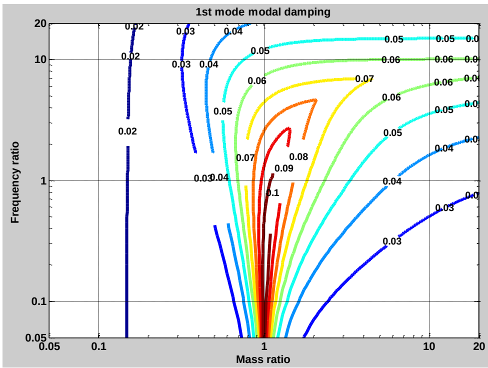
(b) $\zeta_1 == 0.02, \zeta_2 = 0.05$



(c) $\zeta_1 == 0.05, \zeta_2 = 0.02$

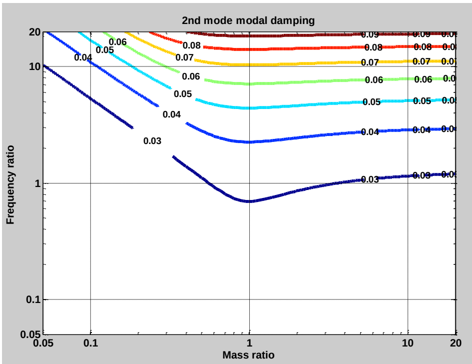


(d) $\zeta_1 == 0.02, \zeta_2 = 0.20$

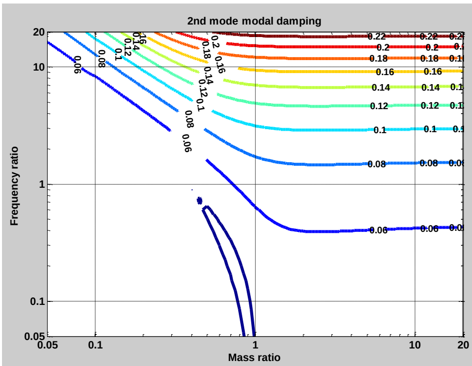


(e) $\zeta_1 == 0.20, \zeta_2 = 0.02$

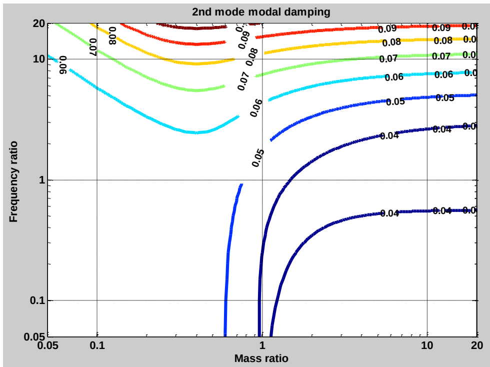
Fig. 5 1^{st} mode modal damping variation with mass/frequency ratio



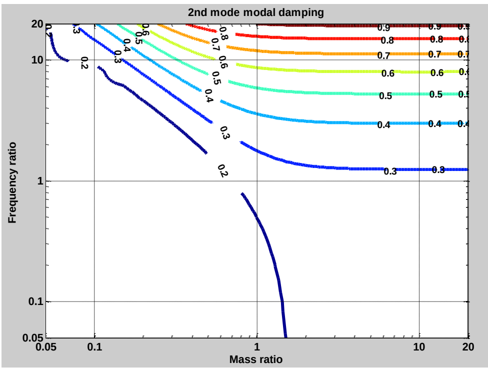
(a) $\zeta_1 = \zeta_2 = 0.02$



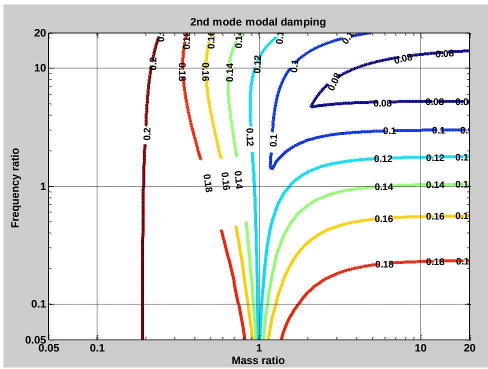
(b) $\zeta_1 == 0.02$, $\zeta_2 = 0.05$



(c) $\zeta_1 == 0.05$, $\zeta_2 = 0.02$

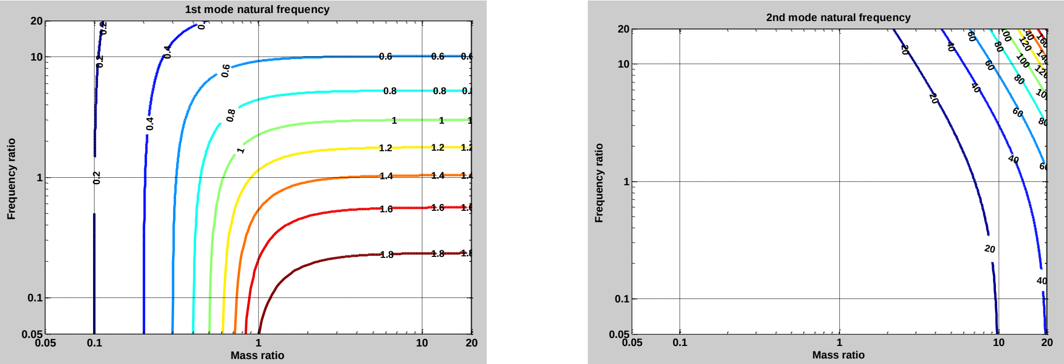


(d) $\zeta_1 == 0.02$, $\zeta_2 = 0.20$



(e) $\zeta_1 == 0.20$, $\zeta_2 = 0.02$

Fig. 6 2nd mode modal damping variation with mass/frequency ratio



(a) 1st mode of vibration (b) 2nd mode of vibration

Fig. 7 Natural frequency variation with mass/frequency ratio (ζ_1 = 0.02/0.20, ζ_2 = 0.02)

3.2 Finite element model of hybrid tower structure

A cable-stayed bridge located in Hokkaido, Japan is considered. Since the cable-stayed bridges are not structurally homogeneous, the tower, deck and cable stays affect the structural response in a wide range of vibration modes. The tower is taken out of the cable-stayed bridge and modeled as three-dimensional frame structure as shown in Figs. 8 and 9. The dimensions units are based on SI system, meter; m. The finite element of soil foundation superstructure interaction model is formulated based on the design drawings (Abdel Raheem *et al.* 2003, Park and Hashash 2004, Hayashikawa *et al.* 2004, Abdel Raheem and Hayashikawa 2013a, b). Damping, which dissipates energy, as the velocities of motion and strain are varied, is important to dynamic structural analyses. The damping matrix for the complete system is constructed by directly assembling the damping matrices for the individual subsystems, assumed to be classically damped. Soil damping is captured primarily through the hysteretic energy dissipating response. Viscous damping, using the Rayleigh damping formulation, is often added to represent damping at very small strains where many soil models are primarily linear (Park and Hashash 2004). For concrete structures, the elastic damping ratio is taken as 5% related to critical damping, a value that is supported by recent experiments (Petrini *et al.* 2008).

For cable stayed bridges without special dampers it can be assumed that the steel structural parts exhibit a uniformly distributed 2% viscous damping, and that the concrete parts exhibit a uniformly distributed 5% damping. Moreover, the soil damping is captured primarily through the hysteretic energy using nonlinear soil springs for large strain of soil plus viscous damping using dashpot for small strain range of soil. Consequently, a pair of Rayleigh damping coefficients α_{0s} and β_{1s} can be used to describe the element damping matrices of all steel structural components, and another pair of Rayleigh damping coefficients α_{0f} and β_{1f} can be used to describe the element damping matrices of all concrete structural components. The damping matrices could be constructed by Rayleigh's damping procedures, thus the damping matrices for the structure and the foundation soil; 5% for footing and 2% for the steel super-structure is used.

$$C_s = \alpha_{0s} M_s + \beta_{1s} K_s \quad , \quad C_f = \alpha_{0f} M_f + \beta_{1f} K_f \tag{18}$$

The natural frequency and the corresponding effective modal mass and modal damping ratio are plotted for different three level of modeling of Composite "steel/concrete/underneath soil" cable-stayed tower:

Model I: steel superstructure with fixed base (classical damping system);

Model II: steel superstructure & concrete substructure massive pier with fixed base (non-classical damping composite system);

Model III: steel superstructure & concrete massive substructure pier with soil interaction (non-classical damping composite system).

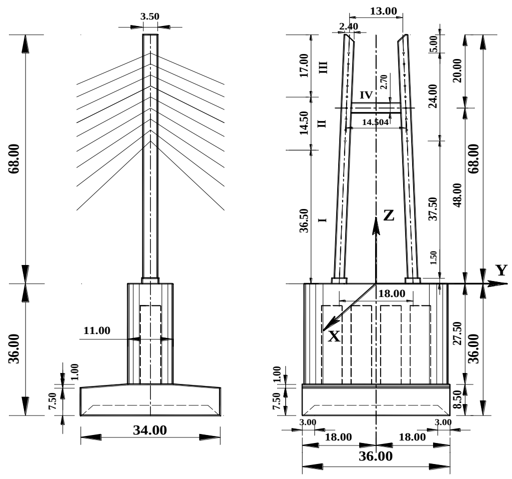


Fig. 8 Tower structure of cable-stayed bridge (*m*)

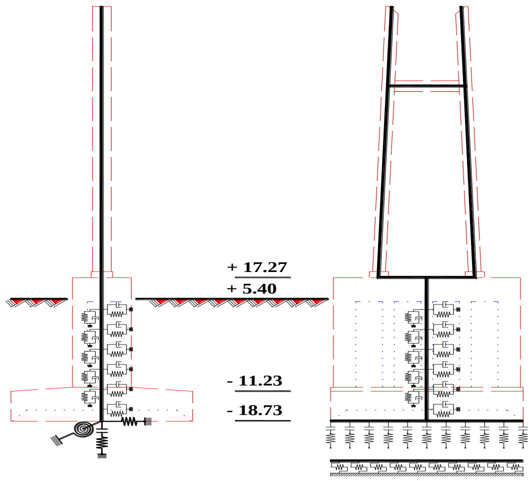


Fig. 9 Hybrid tower model

3.2.1 Vibration analysis hybrid tower model

Because response of cable-stayed bridges significantly depends on damping ratio, it is of great importance to correctly evaluate the damping ratio for seismic design (Kawashima *et al*. 1993, Atkins and Wilson 2000). Therefore an analytical approach capable of evaluating the system-level damping based upon the damping information of components is desirable, by dividing a cable-stayed bridge into several substructures. In this study, the damping characteristics soil foundation superstructure interaction model of cable-stayed bridges tower is studied. The formulation, analysis methods, and results are first compared for classically and non-classically damped structural systems. The effect of non-classical damping on the properties of natural frequency; vibration modes; effective modal mass and modal damping eigenvectors of soil foundation super-structure interaction model is presented and compared with that of fixed base model. From the un-damped natural vibration analysis, the dynamic characteristics including natural frequency and effective modal mass are investigated. Fig. 10 shows natural frequencies for the first 50 Eigen modes of the different tower models, the first seven modes up to 2.6 Hz almost coincide for the different models, while significant differences grow for higher modes due to footing and soil effects. The vibration modes of higher effective modal mass are significantly changed, which could be seen from Fig. 11. The type of modes of vibration would depend on relative stiffness and mass of different subsystems of tower structure. It is shown that tower superstructure fixed base model of classical damping, the modal damping increase linearly with natural frequency, while for tower footing soil interaction model give higher modal damping and increase with high rate, nonlinearly change as shown in Fig. 12.

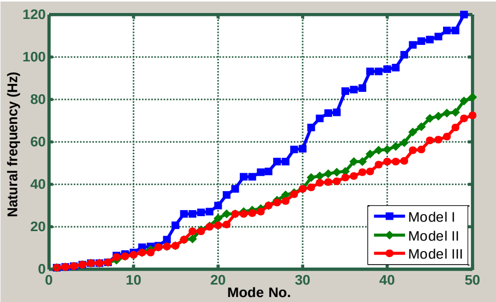


Fig. 10 Natural frequencies for the first 50 Eigen modes of the different tower models

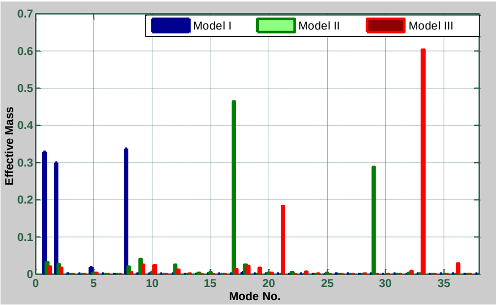


Fig. 11 Modal effective mass ratio for the first 50 Eigen modes of the different tower models.

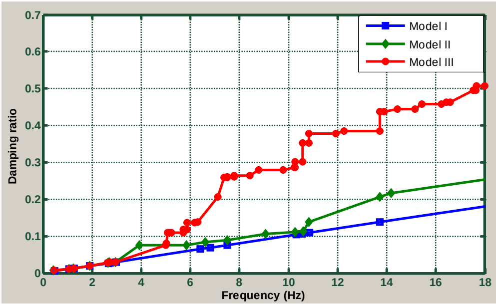


Fig. 12 Modal damping ratio for different tower models

It is shown that in classically damped structures increasing the damping decreases the natural frequencies of the system; with non-classical damping some of the natural frequencies of the damped system may be greater than the corresponding natural frequencies of the un-damped system. Also the coupling index is calculated of the tower structure system, it is equal to 0.174 for the tower footing fixed base model, and equal to 0.439 for the tower superstructure footing soil interaction model, the modal coupling could attribute different damping characteristics, dynamic (frequency ratio) and structural (mass and stiffness ratio) of the substructures of tower structure. So in the dynamic analysis of such structure, where the damping matrix is required for the complete system, more attention should be considered in the formulation of damping matrix. Neglecting the non-classical damping effect would result in un-conservative results. The Rayleigh's damping can cause significant error in the calculation of the damping matrix if the combined structures have significant different substructure damping ratios.

3.2.2 Nonlinear dynamic analysis hybrid tower model

A nonlinear dynamic analysis, including soil-structure interaction, is developed to estimate the seismic response characteristics and to predict the earthquake response of cable-stayed bridges towers with spread foundation. An incremental iterative finite element technique is adopted for a more realistic dynamic analysis of nonlinear soil-foundation-superstructure interaction system under great earthquake ground motion. In the dynamic response analysis, the seismic motion by an inland direct strike type earthquake that was recorded during the 1995 Hyogoken-Nanbu earthquake of high intensity but short duration is used as an input ground motion to assure the seismic safety of bridges. The horizontal and the vertical accelerations recorded at the station of JR-Takatori observatory are used for the dynamic response analysis of the cable-stayed bridge tower. The kinematic interaction can be modeled by a transfer function, which can be used to modify a free-field ground motion so that an effective input motion for a soil-structure system can be obtained. A simple model based on analytical modeling of rigid foundation is adopted, in which the effective seismic motion is obtained starting from the free-field motion by an approximate analytical solution (Harada *et al*. 1981, Ganev *et al*. 1995).

From the Fourier spectra study of tower acceleration response at different levels of tower for soil foundation superstructure nonlinear interaction model, it is shown that there is amplification of different modes over a wide frequency range as seen in Figs. 13 and 14. The in-plane superstructure base response spectrum is larger than that at footing base at spectral frequency less than 2.0 Hz because of amplification induced by flexible superstructure and massive rigid substructure interaction, while at high frequency above 2.0 Hz, the response spectra is slightly attenuated due to inertial interaction. The tower top response spectra is significantly amplified at low frequency range and is almost totally attenuated at high frequency range due to tower superstructure flexibility. The massive foundation has the effect of amplifying the response over a wide frequency band. The vertical acceleration response at the footing base level shows relative high frequency amplification as the response spectra within the frequency range 2 ~ 3 Hz is slightly amplified at superstructure base level, and it is dramatically amplified at tower top. On the other hand, the response spectrum at high frequency range is attenuated by superstructure flexibility filter of the tower response. The nonlinear seismic response of bridge piers is distinctly different from that of the linear response. There is a great difference whether it is in vibration amplitude or in frequency property. The nonlinear properties of foundations make the stiffness of the structure low, the response of rotational angle increase and the response of bending moment decrease.

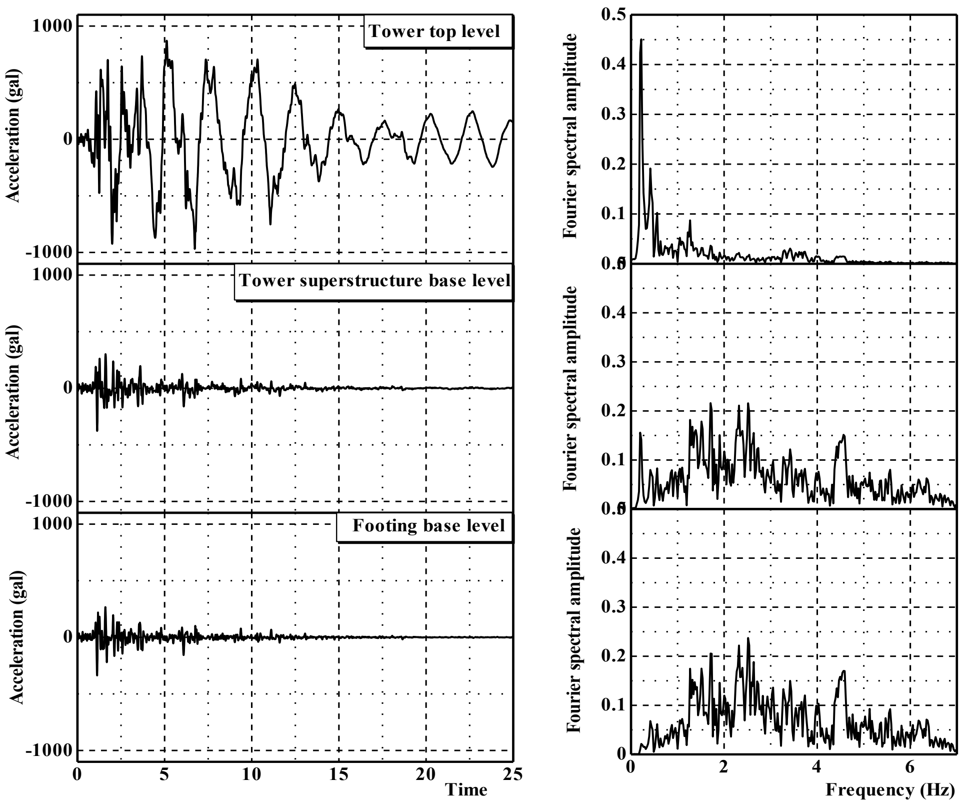


Fig. 13 In-plane acceleration time history and response spectra

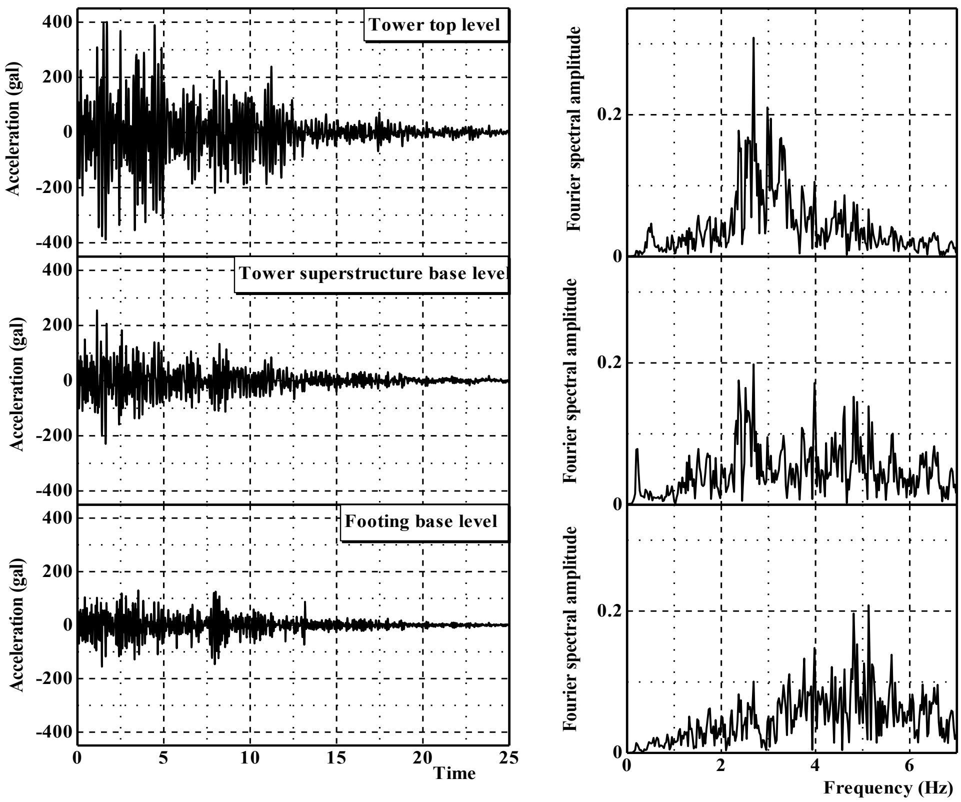


Fig. 14 Vertical acceleration time history and response spectra

4. Conclusions

In code-based seismic design of hybrid structures, several practical difficulties are encountered, due to inherent differences in the nature of dynamic response of each part. The specific issue addressed here is the analysis complications due to the different damping ratios of the different parts. Such structures are irregularly damped and have complex modes of vibration. In the dynamic analysis of a non-classically damped and coupled system, such steel/concrete hyrid with soil structure interaction systems, a major step is to define and compute the damping matrix for the combined system either in the time domain or the modal domain. The task of computing the damping matrix for the coupled system is a nontrivial process especially when the components within the system have large dissimilar damping characteristics, and are dominated by different energy dissipation mechanisms. This study presents state-of-the-art knowledge on dynamic and modal response of 2 coupled SDOF systems as simplified simulation of coupled subsystems of different mass ratio; frequency ratio and damping ratio, hence an equivalent damping ratio, homogeneous over the composite structure could be used to construct the modal damping matrix for dynamic response by the direct integration method. An analytical approach capable of evaluating the equivalent modal damping ratio from structural components is desirable for improving seismic design. Numerical algorithm is formulated to efficiently evaluate the dynamic response of non-classical damped composite structures in earthquake engineering applications. Dynamic response of the systems with non-classical damping can be obtained by two approaches: direct integration method or modal analysis. The formulation, analysis methods and results have been compared in this paper for classically and non-classically damped structural systems. The following conclusions can be drawn from this study:

One of the major effects of non-classical damping on MDOF structures is to cause the un-damped modal vectors to be coupled with respect to the damping matrix. However, the degree of modal coupling in tower footing soil structure model is much higher than that in fixed-base structure model. It is also illustrated that proportional modal damping can result in incorrect responses in non-classically damped systems. In classically damped systems, increasing the damping decreases the natural frequencies of the system: with non-classical damping some of the natural frequencies of the damped system may be greater than the corresponding natural frequencies of the un-damped system. The coupling of the various modes along with their specific damping characteristics should be taken into account in the model of the structure. Damping is extremely important in formulating predictive models of structures, especially combined structures such as tower superstructure footing soil system. The choice of a proper damping ratio is critical to the design/analysis of tower structure response. If the dynamic interaction of the tower superstructure and the supporting footing structure is deemed significant; then the damping of the combined system can exhibit non-classical damping. Non-classical damping gives rise to complex-valued mode shapes. If the tower superstructure is tuned with a dominant mode of the supporting structure and has damping values much lower than those of the supporting structure, neglecting the non-classical damping effect would result in un-conservative results. The Rayleigh's damping can cause significant error in the calculation of the damping matrix if the combined structures have significant different substructure damping ratios. It is shown that the effect of non-classical damping is significant in systems that have nearly tuned modes and sufficiently small values of modal mass ratios. The inclusion of massive foundation and nonlinearity of soil effects leads to amplification of higher modes of vibration and activates the high frequency translational motion of the input ground motion and generates foundation-rocking responses.

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