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Static bending and free vibration of FGM beam using an exponential shear deformation theory

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Abstract. In this paper, a refined exponential shear deformation beam theory is developed for bending analysis of functionally graded beams. The theory account for parabolic variation of transverse shear strain through the depth of the beam and satisfies the zero traction boundary conditions on the surfaces of the beam without using shear correction factors. Contrary to the others refined theories elaborated, where the stretching effect is neglected, in the current investigation this so-called "stretching effect" is taken into consideration. The material properties of the functionally graded beam are assumed to vary according to power law distribution of the volume fraction of the constituents. Based on the present shear deformation beam theory, the equations of motion are derived from Hamilton's principle. Analytical solutions for static are obtained. Numerical examples are presented to verify the accuracy of the present theory.

Keywords: bending; dynamic analysis; functionally graded; stretching effect; vibration

1. Introduction

Functionally graded materials (FGMs) are a class of composites that have continuous variation of material properties from one surface to another, thus eliminating the stress concentration found in laminated composites.

A typical FGM is made from a mixture of two material phases, for example, a ceramic and a metal. The FGMs are widely used in mechanical, aerospace, nuclear, and civil engineering. Consequently, studies devoted to understand the static and dynamic behaviors of FGM beams, plates have being paid more and more attentions in recent years.

Sankar (2001) investigated an elasticity solution for bending of functionally graded beams (FG beams) based on Euler-Bernoulli beam theory. Zhong and Yu (2007) provided an analytical solution for cantilever beams subjected to various types of mechanical loadings using the Airy stress function. Li (2008) investigated static bending and transverse vibration of FGM Timoshenko beams, in which by introducing a new function, the governing equations for bending and vibration of FGM beams were decoupled and the deflection, rotational angle and the resultant force and moment were expressed only in the terms of this new function. Benatta *et al.* (2009) proposed an analytical solution to the bending problem of a symmetric FG beam by including warping of the

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cross-section and shear deformation effect. Sallai *et al*. (2009) investigated the static responses of a sigmoid FG thick beam by using different beam theories. Şimşek (2010a) studied the free vibration analysis of an FG beam using different higher order beam theories. In a recent study, Şimşek (2010b) has studied the dynamic deflections and the stresses of an FG simply-supported beam subjected to a moving mass by using Euler–Bernoulli, Timoshenko and the parabolic shear deformation beam theory. El Meiche *et al*. (2011) proposed a novel hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate. Benachour *et al*. (2011) employed a four-variable refined plate theory to study the free vibrations response of FG plates with arbitrary gradient. Bachir Bouiadjra *et al*. (2012) used a four-variable refined plate theory for the buckling response of FG plates under thermal loads. Bourada *et al*. (2012) developed a new four-variable refined plate theory for thermal buckling of FG sandwich plates. Fekrar *et al*. (2012) analyzed the buckling response of FG hybrid composite plates using a new four variable refined plate theory. Bouremana *et al*. (2013) proposed a novel first shear deformation beam theory based on neutral surface position for FG beams. Bachir Bouiadjra *et al*. (2013) investigated the nonlinear thermal buckling response of FG plates using an efficient sinusoidal shear deformation theory. Bessaim *et al*. (2013) examined the bending and free vibration behaviours of sandwich plates with FG isotropic face sheets by using a new higher-order shear and normal deformation theory. Tounsi *et al*. (2013a) presented an analytical investigation on the thermoelastic bending of FG sandwich plates using a refined trigonometric shear deformation theory. Bouderba *et al*. (2013) studied the thermomechanical bending behaviour of FG plates supported by Winkler-Pasternak elastic foundations. Kettaf *et al.* (2013) proposed a novel hyperbolic shear displacement model to study the thermal buckling behaviour of FG sandwich plates. Ould larbi latifa *et al*. (2013) developed an efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams. Zidi *et al*. (2014) investigated the bending response of FG plates subjected to a hygro-thermo-mechanical loading by using a four variable refined plate theory. Ait Amar Meziane *et al*. (2014) developed an efficient and simple refined theory for buckling and free vibration response of exponentially graded sandwich plates under various boundary conditions. Draiche *et al*. (2014) studied the free vibration of rectangular composite plates with patch mass using a trigonometric four variable plate theory. Nedri *et al*. (2014) studied the free vibration behavior of laminated composite plates resting on elastic foundations by using a refined hyperbolic shear deformation theory. Khalfi *et al*. (2014) employed a refined and simple shear deformation theory for thermal buckling behavior of solar FG plates resting on elastic foundation. Klouche Djedid *et al*. (2014) developed an *n*-order four variable refined theory for bending and free vibration of FG plates. Recently, Hadji (2014) studied the static and free vibration of FGM beam using a higher order shear deformation theory. Belabed *et al*. (2014) developed an efficient and simple higher order shear and normal deformation theory for FG plates. Hebali *et al*. (2014) analyzed the bending and free vibration behaviour of FG plates using a novel quasi-3D hyperbolic shear deformation theory. The stretching effect was included also in the analysis of the mechanical responses of thick FG plates (Houari *et al*. 2013, Bousahla *et al*. 2014, Fekrar *et al*. 2014). Some beam theories are applied also to different type of structures as is described in Refs (Heireche *et al*. 2008, Tounsi *et al*. 2008, Benzair *et al*. 2008, Tounsi *et al*. 2009, Amara *et al*. 2010, Tounsi *et al*. 2013b, c, Berrabah *et al*. 2013, Benguediab *et al*. 2014). A general revue for FG structures such as beams plates and shells is presented by Tounsi *et al*. (2013d).

In the present study, bending and free vibration of simply supported FG beams was investigated by using a refined exponential shear deformation beam theory with $(\varepsilon_z \neq 0)$. The most interesting feature of this theory is that it accounts for a parabolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. Then, the present theory together with Hamilton's principle, are employed to extract the motion equations of the functionally graded beams. Analytical solutions for static and free vibration are obtained. Numerical examples are presented to verify the accuracy of the present theory.

2. Problem formulation

Consider a functionally graded beam with length L and rectangular cross section $b \times h$, with b being the width and h being the height as shown in Fig. 1. The beam is made of isotropic material with material properties varying smoothly in the thickness direction.

2.1 Material properties

The properties of FGM vary continuously due to the gradually changing volume fraction of the constituent materials (ceramic and metal), usually in the thickness direction only. The power-law function is commonly used to describe these variations of materials properties. The expression given below represents the profile for the volume fraction.

$$V_C = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{1a}$$

k is a parameter that dictates material variation profile through the thickness. The value of k equal to zero represents a fully ceramic beam, whereas infinite k indicates a fully metallic beam, and for different values of k one can obtain different volume fractions of metal.

The material properties of FG beams are assumed to vary continuously through the depth of the beam by the rule of mixture (Marur 1999) as

$$P(z) = (P_t - P_b)\mathrm{V_C} + P_b \tag{1b}$$

where P denotes a generic material property like modulus, $\mathrm{P_t}$ and $\mathrm{P_b}$ denotes the property of the top and bottom faces of the beam respectively, Here, it is assumed that modules E , G and ν vary according to the Eq. (1).

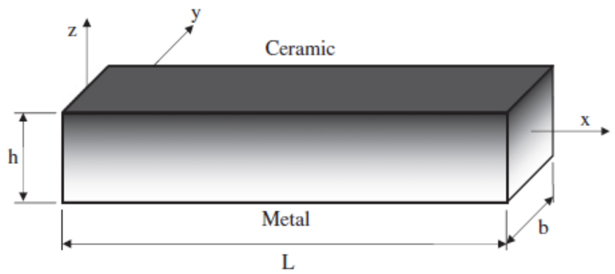

Fig. 1 Geometry and coordinate of a FG beam

2.2 Kinematics and constitutive equations

The displacement field of the proposed theory takes the simpler form as follows

$$u(x,z)=u_0(x,t)-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial w_s}{\partial x}$$
$$u(x,z)=w_b(x)+w_s+g(z)\varphi_z(x) \tag{2}$$

Clearly, the displacement field in Eq. (2) contains only four unknowns $\left(u,w_b,w_s,\varphi_z\right)$. The strains associated with the displacements in Eq. (2) are

$$\varepsilon_x=\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w_b}{\partial x^2}-f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{3a}$$

$$\varepsilon_z=g^{'}(z)\,\varphi_z \tag{3b}$$

$$\gamma_{xz}=g(z)\left(\frac{\partial w_s}{\partial x}+\frac{\partial \varphi_z}{\partial x}\right) \tag{3c}$$

Where $f(z)=z-ze^{-2(z/h)^2}$ and $g(z)=1-f'(z)$. It can be seen from Eqs. (3(c)) that the transverse shears strain γ_{xz} is equal to zero at the top $(z=h/2)$ and bottom $(z=-h/2)$ surfaces of the beam, thus satisfying the zero transverse shear stress conditions.

The state of stress in the beam is given by the generalized Hooke’s law as follows

$$\sigma_x=Q_{11}(z)\varepsilon_x+Q_{13}(z)\varepsilon_z \tag{4a}$$

$$\tau_{xz}=Q_{55}(z)\gamma_{xz} \tag{4b}$$

$$\sigma_z=Q_{13}(z)\varepsilon_x+Q_{33}(z)\varepsilon_z \tag{4c}$$

The Q_{ij} expressions in terms of engineering constants are

$$Q_{11}(z)=Q_{33}(z)=\frac{E(z)}{1-\upsilon^2},\ Q_{13}(z)=\nu\,Q_{11}(z),\quad Q_{55}(z)=\frac{E(z)}{2(1+\upsilon)} \tag{4d}$$

2.3 Equations of motion

Hamilton’s principle is used herein to derive the equations of motion. The principle can be stated in analytical form as (Thai and Vo 2012)

$$\delta\int_{t_1}^{t_2}(U+V-K)dt=0 \tag{5}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; δV is the virtual variation of the potential energy; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\delta U = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_x \delta\, \varepsilon_x + \sigma_z \delta\varepsilon_z + \tau_{xz} \delta\, \gamma_{xz}\right) dz dx$$
$$= \int_0^L \left(N_x \delta \frac{\partial u_0}{\partial x} - M_x \delta \frac{\partial^2 w_b}{\partial x^2} - P_x \delta \frac{\partial^2 w_s}{\partial x^2} + R_z \delta\varphi_z + Q_{xz} \delta\left(\frac{\partial w_s}{\partial x} + \frac{\partial \varphi_z}{\partial x} \right)\right) dx \tag{6}$$

where N, M, P and Q are the stress resultants defined by

$$(N, M_x, P_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f)\, \sigma_x dz\,, \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} g\, \tau_{xz} dz \quad \text{and} \quad R_z = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_z g'(z) dz \tag{7}$$

The variation of the potential energy by the applied transverse load q can be written as

$$\delta V = -\int_0^L q\delta\left(w_b + w_s + g\varphi_z \right) dx \tag{8}$$

The variation of the kinetic energy can be expressed as

$$\delta K = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z) \left[\ddot{u}\, \delta\, u + \ddot{w} \delta w \right] dz dx$$
$$= \int_0^L \left\{ I_0 \left[\ddot{u}_0\, \delta u_0 + \left(\ddot{w}_b + \ddot{w}_s \right)\left(\delta w_b + \delta w_s \right) \right] + I_1 \left(\frac{\partial^3 u_0}{\partial x \partial t^2} \delta w_b - \frac{\partial^3 w_b}{\partial x \partial t^2} \delta u_0 \right) \right.$$
$$+ J_1 \left(\frac{\partial^3 u_0}{\partial x \partial t^2} \delta w_s - \frac{\partial^3 w_s}{\partial x \partial t^2} \delta u_0 \right) - I_2 \left(\frac{\partial^4 w_b}{\partial x^2 \partial t^2} \delta w_b \right) - J_2 \left(\frac{\partial^4 w_b}{\partial x^2 \partial t^2} \delta w_s + \frac{\partial^4 w_s}{\partial x^2 \partial t^2} \delta w_b \right)$$
$$\left. - K_2 \left(\frac{\partial^4 w_s}{\partial x^2 \partial t^2} \delta w_s \right) + L_1 \left(\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) \delta\varphi_z + \left((\delta w_b + \delta w_s) \frac{\partial^2 \varphi_z}{\partial t^2} \right) \right) + L_2 \left(\frac{\partial^2 \varphi_z}{\partial t^2} \delta\varphi_z \right) \right\} dx \tag{9}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2, L_1, L_2) are the mass inertias defined as

$$\left(I_0, I_1, J_1, I_2, J_2, K_2, L_1, L_2 \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(1, z, f, z^2, zf, f^2, g, g^2 \right) \rho(z) dz \tag{10}$$

Substituting the expressions for δU, δV and δT from Eqs. (6), (8) and (9) into Eq. (5) and integrating by parts versus both space and time variables, and collecting the coefficients of $\delta\, u_0$, $\delta\, w_b$, $\delta\, w_s$ and $\delta\, \varphi_z$, the following equations of motion of the functionally graded beam are obtained

$$\delta u_0 : \frac{\partial N_x}{\partial x} = I_0 \frac{d^2 u_0}{dt^2} - I_1 \frac{d^3 w_b}{dxdt^2} - J_1 \frac{d^3 w_s}{dxdt^2} \tag{11a}$$

$$\delta w_b : \frac{\partial^2 M_x^b}{\partial x^2} + q = I_1 \frac{\partial^3 u_0}{\partial x dt^2} - I_2 \frac{\partial^4 w_b}{\partial x^2 dt^2} + I_0 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) - J_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2} + L_1 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{11b}$$

$$\delta w_s : \frac{\partial^2 P_x}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + q = J_1 \frac{\partial^3 u_0}{\partial t^2} - J_2 \frac{\partial^4 w_b}{\partial x^2 \partial t^2} + I_0 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) - K_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2} + L_1 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{11c}$$

$$\delta \varphi_z : -R_z + \frac{\partial Q_{xz}}{\partial x} + gq = L_1 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) + L_2 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{11d}$$

Eqs. (11) can be expressed in terms of displacements (u_0, w_b, w_s, φ_z) by using Eqs. (2), (3), (4) and (7) as follows:

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} + X_{13} \frac{\partial \varphi_z}{\partial x} = I_0 \frac{d^2 u_0}{dt^2} - I_1 \frac{d^3 w_b}{dxdt^2} - J_1 \frac{d^3 w_s}{dxdt^2} \tag{12a}$$

$$B_{11} \frac{\partial^3 u_0}{\partial x^3} - D_{11} \frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + Y_{13} \frac{\partial^2 \varphi_z}{\partial x^2} + q = I_1 \frac{\partial^3 u_0}{\partial x dt^2} - I_2 \frac{\partial^4 w_b}{\partial x^2 dt^2} + I_0 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) - J_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2} + L_1 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{12b}$$

$$B_{11}^s \frac{\partial^3 u_0}{\partial x^3} - D_{11}^s \frac{\partial^4 w_b}{\partial x^4} - H_{11} \frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + \left(Y_{13}^s + A_{55}^s \right) \frac{\partial^2 \varphi_z}{\partial x^2} + q = J_1 \frac{\partial^3 u_0}{\partial t^2} - J_2 \frac{\partial^4 w_b}{\partial x^2 \partial t^2}$$

$$+ I_0 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) - K_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2} + L_1 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{12c}$$

$$-X_{13} \frac{\partial u_0}{\partial x} + Y_{13} \frac{\partial^2 w_b}{\partial x^2} + \left(Y_{13}^s + A_{55}^s \right) \frac{\partial^2 w_s}{\partial x^2} + A_{55}^s \frac{\partial^2 \varphi_z}{\partial x^2} - Z_{33} \varphi_z + gq = L_1 \left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2} \right) + L_2 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{12d}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} dz,\ B_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z dz,\ B_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} f dz,\ X_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} g' dz \tag{13a}$$

$$D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z^2 dz,\ D_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z.f dz,\ Y_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} z.g' dz,\ H_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} f^2 dz \tag{13b}$$

$$Y_{13}^{s} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}.f.g^{'}.dz,\ Z_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}\left[g^{'}\right]^{2} dz,\ A_{55}^{s} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55} g^{2} dz \tag{13c}$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s, φ_z, can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi_z \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ w_{bm} \sin(\lambda x) e^{i\omega t} \\ w_{sm} \sin(\lambda x) e^{i\omega t} \\ \phi_{zm} \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \tag{14}$$

where U_m, W_{bm}, W_{sm} and ϕ_{zm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $\lambda = m\pi / L$. The transverse load q is also expanded in Fourier series as

$$\mathrm{q(x)} = \sum_{\mathrm{m=1}}^{\infty} \mathrm{Q_m} \sin\lambda(\mathrm{x}) \tag{15}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda\, x)dx \tag{16}$$

The coefficients Q_m are given below for some typical loads. For the case of uniform distributed load, we have

$$\mathrm{Q_m} = \frac{4\mathrm{Q_0}}{\mathrm{m}\pi},\quad (\mathrm{m} = 1,3,5\ldots) \tag{17}$$

Substituting the expressions of u_0, w_b, w_s, φ_z from Eqs. (14) and (15) into the equations of motion Eq. (12), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ \phi_{zm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ Q_m \\ gQ_m \end{Bmatrix} \tag{18}$$

where

$$a_{11} = A_{11}\lambda^2, a_{12} = -B_{11}\lambda^3, a_{13} = -B^s_{11}\lambda^3, a_{14} = -X_{13}\lambda, a_{22} = D_{11}\lambda^4, a_{23} = D^s_{11}\lambda^4,$$
$$a_{24} = Y_{13}\lambda^2, a_{33} = H_{11}\lambda^4 + A^s_{55}\lambda^2, a_{34} = Y^s_{13}\lambda^2 + A^s_{55}\lambda^2, a_{44} = A^s_{55}\lambda^2 + Z_{33} \tag{19a}$$

$$m_{11} = I_0, m_{12} = -I_1\lambda, m_{13} = J_1\lambda, m_{14} = 0, m_{22} = I_0 + I_2\lambda^2, m_{23} = I_0 + J_2\lambda,$$
$$m_{24} = L_1, m_{33} = I_0 + K_2\lambda^2, m_{34} = L_1, m_{44} = L_2 \tag{19b}$$

4. Results and discussion

In this section, various numerical examples are presented and discussed to verify the accuracy of present theories in predicting the bending and free vibration responses of simply supported FG beams. The FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (P_C : Alumina, Al2O3): $E_c = 380$ GPa; $\nu = 0.3$; $\rho_c = 3960$ kg/m3.
Metal (P_M : Aluminium, Al): $E_m = 70$ GPa; $\nu = 0.3$; $\rho_m = 2702$ kg/m3.

And their properties change through the thickness of the beam according to power-law. The bottom surfaces of the FG beams are aluminum rich, whereas the top surfaces of the FG beams are alumina rich.

For convenience, the following dimensionless form is used

$$\bar{w} = 100\frac{E_m h^3}{q_0 L^4} w\left(\frac{L}{2}\right), \quad \bar{u} = 100\frac{E_m h^3}{q_0 L^4} u\left(0, -\frac{h}{2}\right), \quad \bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(\frac{L}{2}, \frac{h}{2}\right), \quad \bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}(0,0),$$
$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

4.1 Results for bending analysis

Table 1 contains nondimensional deflection and stresses of FG beams under uniform load q_0 for different values of power law index k and span-to-depth ratio L/h. The obtained results are compared with various shear deformation beam theories (i.e., SSDBT, PSDBT).

It can be observed that our results with $(\varepsilon_z \neq 0)$ are in an excellent agreement to those predicted using the various shear deformation beam theories (i.e., SSDBT, PSDBT) with $(\varepsilon_z = 0)$ for all values of power law index p and span-to-depth ratio L/h.

Figs. 2-4 show the variations of axial displacement $\bar{u}$, axial stress $\overline{\sigma_x}$, and transverse shear stress $\overline{\tau_{xz}}$, respectively, through the depth of a very deep beam $(L = 2h)$ under uniform load. In general, the present theory and the shear deformation beam model of Reddy (PSDBT) give almost identical results.

Table 1 Nondimensional deflections and stresses of FG beams under uniform load

k	Method	$L/h=5$				$L/h=20$			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Li *et al.* (2010)	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	SSDBT	**3.1649**	**0.9409**	**3.8052**	**0.7546**	**2.8962**	**0.2306**	**15.0137**	**0.7672**
	PSDBT	**3.1654**	**0.9397**	**3.8019**	**0.7330**	**2.8962**	**0.2306**	**15.0129**	**0.7437**
	Present ($\varepsilon_z \neq 0$)	3.1673	0.9233	3.9129	0.7883	2.8807	0.2290	15.4891	0.7890
0.5	Li *et al.* (2010)	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	SSDBT*	**4.8278**	**1.6613**	**4.9969**	**0.7717**	**4.4644**	**0.4087**	**19.7014**	**0.7840**
	PSDBT*	**4.8285**	**1.6595**	**4.9923**	**0.7501**	**4.4644**	**0.4087**	**19.7003**	**0.7606**
	Present ($\varepsilon_z \neq 0$)	4.8045	1.6091	5.1538	0.8053	4.4160	0.3998	20.3969	0.8057
1	Li *et al.* (2010)	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	SSDBT*	**6.2586**	**2.3058**	**5.8891**	**0.7546**	**5.8049**	**0.5686**	**23.2065**	**0.7672**
	PSDBT*	**6.2594**	**2.3036**	**5.8835**	**0.7330**	**5.8049**	**0.5685**	**23.2051**	**0.7437**
	Present ($\varepsilon_z \neq 0$)	6.1805	2.2115	6.0709	0.7883	5.6965	0.5498	24.0095	0.7890
2	Li *et al.* (2010)	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	SSDBT*	**8.0683**	**3.1153**	**6.8899**	**0.6931**	**7.4421**	**0.7692**	**27.1008**	**0.7058**
	PSDBT*	**8.0677**	**3.1127**	**6.8824**	**0.6704**	**7.4421**	**0.7691**	**27.0989**	**0.6812**
	Present ($\varepsilon_z \neq 0$)	7.9106	2.9629	7.0925	0.7274	7.2458	0.7366	27.9844	0.7287
5	Li *et al.* (2010)	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	SSDBT*	**9.8367**	**3.7140**	**8.1219**	**0.6153**	**8.8188**	**0.9134**	**31.8156**	**0.6282**
	PSDBT*	**9.8281**	**3.7097**	**8.1104**	**0.5904**	**8.8182**	**0.9134**	**31.8127**	**0.6013**
	Present ($\varepsilon_z \neq 0$)	9.6933	3.5429	8.3581	0.6513	8.6182	0.8775	32.8183	0.6540
10	Li *et al.* (2010)	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	SSDBT*	**10.9419**	**3.8913**	**9.7236**	**0.6706**	**9.6908**	**0.9537**	**38.1411**	**0.6847**
	PSDBT*	**10.9381**	**3.8859**	**9.7119**	**0.6465**	**9.6905**	**0.9536**	**38.1382**	**0.6586**
	Present ($\varepsilon_z \neq 0$)	10.8680	3.7462	9.9878	0.7064	9.5513	0.9262	39.2717	0.7091

* Results form Ref (Huu-Tai Thai 2012)

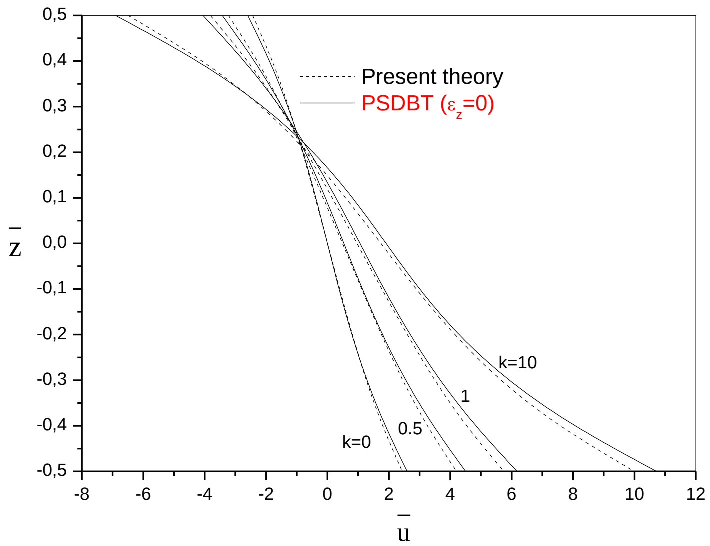

Fig. 2 The variation of the axial displacement $\bar{u}$ through-the-thickness of a FG beam ($L = 2h$)

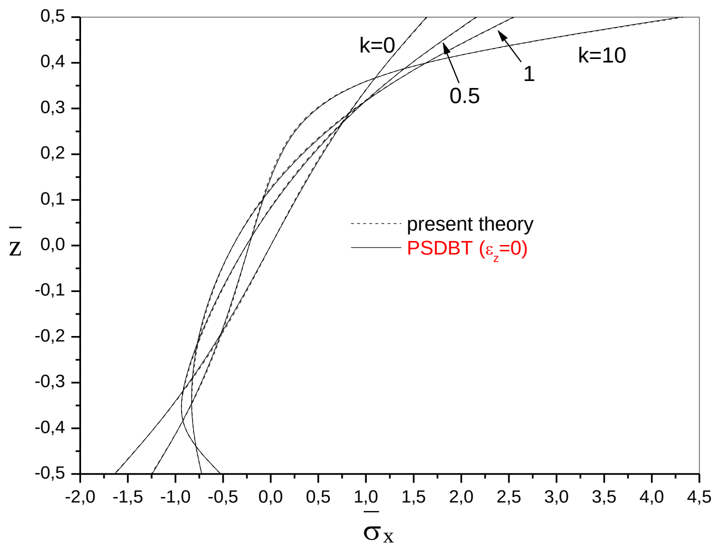

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ through-the-thickness of a FG beam ($L = 2h$)

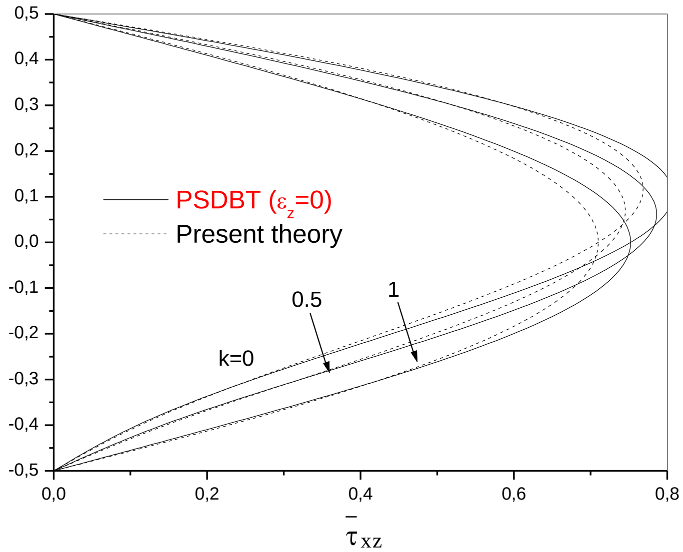

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a FG beam ($L = 2h$)

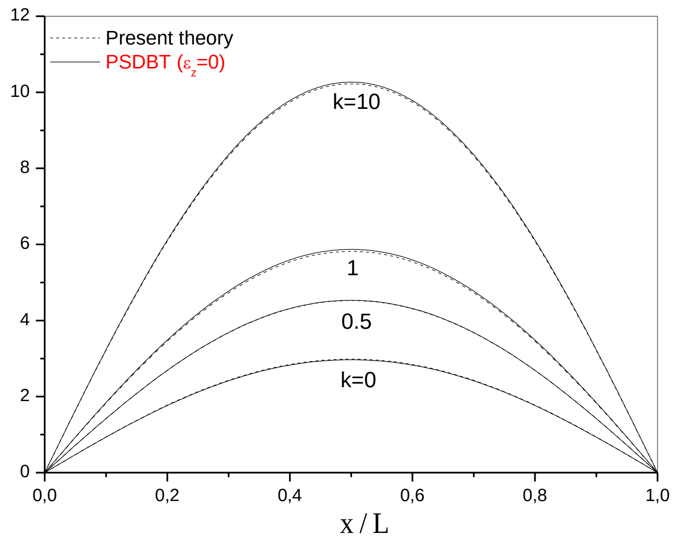

Fig. 5 Variation of the transverse displacement $\bar{w}$ versus non-dimensional length of a FG beam ($L = 5h$)

Fig. 5 illustrates the variation of the non-dimensional transversal displacement $\bar{w}$ versus non-dimensional length for different power law index k. It can be seen also that the present beam theory gives almost identical results to Reddy (PSDBT). In addition, the results show that the increase of the power law index k leads to an increase of transversal displacement $\bar{w}$.

4.2 Results for free vibration analysis

Table 2 show the nondimensional fundamental frequencies $\bar{\omega}$ of FG beams for different values of power law index k and span-to-depth ratio L/h. The calculated frequencies are compared with those given by Simsek (2010a) with $(\varepsilon_z = 0)$. An excellent agreement between the present solutions and results of Simsek (2010a) are found.

Table 3 shows the variations of first three nondimensional frequencies $\bar{\omega}$ of FG beams using the present theory and the results given by PSDBT (Simsek 2010a) and CBT for different values of power law index k and span-to-depth ratio L/h. The present frequencies are in good agreement with the results of Simsek (2010a). It should be remembered that the frequencies predicted by the present theory are smaller than those predicted by the classical beam theory. and the difference between the frequencies of CBT and the shear deformable beam theories decreases as the value of L/h increases.

Table 2 Variation of fundamental frequency $\bar{\omega}$ with the power-law index for FG beam

L / h	Theory	k					
		0	0.5	1	2	5	10
5	**PSDBT#**	**5.1527**	**4.4111**	**3.9904**	**3.6264**	**3.4012**	**3.2816**
	Present ($\varepsilon_z \neq 0$)	5.1788	4.4441	4.0354	3.6798	3.4425	3.3103
20	**PSDBT#**	**5.4603**	**4.6516**	**4.2050**	**3.8361**	**3.6485**	**3.5389**
	Present ($\varepsilon_z \neq 0$)	5.4770	4.6781	4.2463	3.8890	3.6918	3.5660

Results form Ref (M Simsek,,201a)

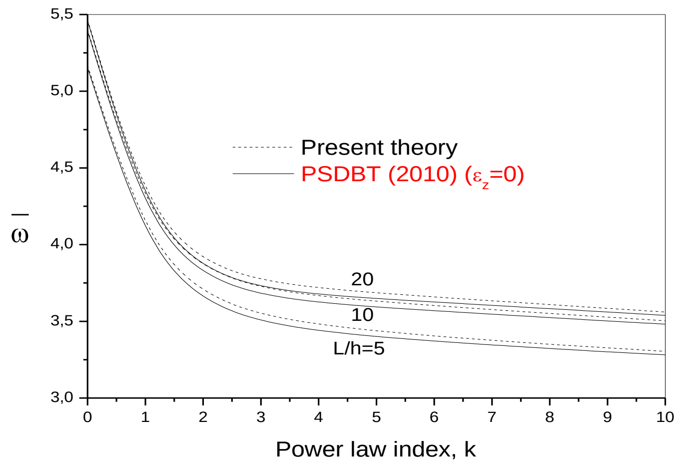

Fig. 6 Variation of the nondimensional fundamental frequency $\bar{\omega}$ of FG beam with power law index k and span-to-depth ratio L/h

Table 3 First three nondimensional frequencies $\bar{\omega}$ of FG beams

L / h	Mode	Theory	k = 0	0.5	1	2	5	10
5	1	CBT[#]	5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
		PSDBT[#]	5.1527	4.4107	3.9904	3.6264	3.4012	3.2816
		Present ($\varepsilon_z \neq 0$)	5.1788	4.4441	4.0354	3.6798	3.4425	3.3103
	2	CBT[#]	20.6187	17.5415	15.7982	14.3260	13.5876	13.2376
		PSDBT[#]	17.8812	15.4588	14.0100	12.6405	11.5431	11.0240
		Present ($\varepsilon_z \neq 0$)	18.0493	15.6362	14.2133	12.8536	11.6921	11.1470
	3	CBT[#]	43.3483	36.8308	33.0278	29.7458	28.0850	27.4752
		PSDBT[#]	34.2097	29.8382	27.0979	24.3152	21.7158	20.5561
		Present ($\varepsilon_z \neq 0$)	34.6743	30.2943	27.5781	24.7783	22.0198	20.8356
20	1	CBT[#]	5.4777	4.6641	4.2163	3.8472	3.6628	3.5547
		PSDBT[#]	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
		Present ($\varepsilon_z \neq 0$)	5.4770	4.6781	4.2463	3.8890	3.6918	3.5660
	2	CBT[#]	21.8438	18.5987	16.8100	15.3334	14.5959	14.1676
		PSDBT[#]	21.5732	18.3962	16.6344	15.1619	14.3746	13.9263
		Present ($\varepsilon_z \neq 0$)	21.6488	18.5102	16.8029	15.3739	14.5462	14.0362
	3	CBT[#]	48.8999	41.6328	37.6173	34.2954	32.6357	31.6883
		PSDBT[#]	47.5930	40.6526	36.7679	33.4689	31.5780	30.5369
		Present ($\varepsilon_z \neq 0$)	47.7924	40.9292	37.1587	33.9478	31.9577	30.7896

Fig. 6 shows the non-dimensional fundamental natural frequency $\bar{\omega}$ versus the power law index k for different values of span-to-depth ratio L/h using both the present theory and theory PSDBT (Simsek, 2010a). An excellent agreement between the present theory and the PSDBT is showed from Fig. 6. It is observed that an increase in the value of the power law index leads to a reduction of frequency. The highest frequency values are obtained for full ceramic beams $(k=0)$ while the lowest frequency values are obtained for full metal beams $(k \to \infty)$. This is due to the fact that an increase in the value of the power law index results in a decrease in the value of elasticity modulus. In other words, the beam becomes flexible as the power law index increases, thus decreasing the frequency values.

5. Conclusions

A refined exponential shear deformation theory is proposed for bending analysis of functionally graded beams. The theory accounts for the stretching and shear deformation effects without requiring a shear correction factor. It is based on the assumption that the transverse displacements consist of bending, shear and thickness stretching parts. Based on the present refined exponential beam theory, the equations of motion are derived from Hamilton's principle. Numerical examples show that the proposed theory gives solutions which are almost identical with those obtained using other shear deformation theories.

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