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Fundamental and plane wave solution in non-local bio-thermoelasticity diffusion theory

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Abstract. This work is an attempt to design a dynamic model for a non local bio-thermoelastic medium with diffusion. The system of governing equations are formulated in terms of displacement vector field, chemical potential and the tissue temperature in the context of non local dual phase lag (NL DPL) theories of heat conduction and mass diffusion. Based on this considered model, we study the fundamental solution and propagation of plane harmonic waves in tissues. In order to analyze the behavior of the NL DPL model, we construct basic theorem in the terms of elementary function which determine the existence of three longitudinal and one transverse wave. The effects of various parameters on the characteristics of waves i.e., phase velocity and attenuation coefficients are elaborated by plotting various figures of physical quantities in the later part of the paper.

Keywords: non local; bio-thermoelastcity; diffusion; phase lag; fundamental solution; wave propagation

1. Introduction

The investigation of bioheat transfer is a complicated process because it entails a mixture of many mechanisms to take into account, such as thermal conduction in tissues, convection and blood perfusion, metabolic heat generation, vascular structure, changing of tissue properties depending on physiological condition and so on. This topic has a key role to predict accurately the temperature distribution in tissues, especially during biomedical applications. In biomedical applications laser-tissue interaction is of great interest. Thermal properties of the tissue and the thermal changes caused by the interaction of light and tissue are major aspects of the laser-tissue interaction. Lasers are widely used in biology and medicine and the majority of the hospitals utilize modern laser systems for diagnostic and therapeutic applications. Knowledge of laser-tissue interaction can help doctors or surgeons to select the optimal laser systems and to modify the type of their therapy.

The nonlocal effects arise in far from equilibrium processes, which involve extremely fast heat and mass transfer at very small time and length scales. Classical thermoelasticity may not be applicable to analyze at the micro or nanoscale as the characteristic length of the structure becomes comparable to the internal characteristic length, e.g., the mean free path, the wavelength.

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Thermoelastic analysis at micro and nanoscale is becoming important along with the miniaturization of the device and wide application of ultrafast lasers, although the novel laser burst technology, where size effect on heat conduction and elastic deformation increase and classical theory of thermoelastic coupling does not hold any more.

(Guyer and Krumhansl 1966) developed size dependent heat conduction model and solved the phonon Boltzmann equation by a linear assumption and formulated a transport model (GK model) containing the transient and non local terms. (Sobolew 1994) demonstrated that heat conduction at micro/nanoscale is essentially nonlocal, and classical heat conductive law should be modified by introducing material's characteristic length. (Dong *et al*. 2014) also obtain the similar expression as a GK model within thermomass theory and known as a microscopic interpretation of GK model.

(Eringen 2002) developed the non local elasticity in mechanical prospective by adopting a unified foundation for the fundamental field equations of nonlocal continuum field theories.

(Yu *et al*. 2013, Yu *et al*. 2014) extended non local theory into fractional order G-L theory and memory dependent based on L-S theory for the investigation of micro/nano scale sudden heating problem. (Yu *et al*. 2016) investigated the size effect of heat conduction where abnormal result within thermal wave model is eliminated by introducing spatially size effect. Non local thermoelasticity theory based on the non local heat conduction are very well formulated and various investigator studied various types of problems (Bachher and Sarkar 2018, Das *et al*. 2019). (Gupta and Mukhopadhyay 2019) investigated a one-dimensional elastic half space problem based on non local heat conduction introduced by (Tzou and Guo 2010). (Sarkar 2020) formulated new governing equations of thermoelasticity with nonlocal heat conduction.

Heat conduction in tissues is complicated process. Firstly (Pennes 1948) established the bioheat transfer equation and obtained the temperature profile in human forearm (Pennes' model). Shen *et al*. (2005) used Pennes' model to study the static thermo-mechanical responses of skin tissue at high temperature. (Xu *et al*. 2008, Xu *et al*. 2008) investigated the heat transfer, thermal damage and heat-induced stress of human skin. (Kim *et al*. 2016) analyzed the transient thermal-mechanical responses of innocuous tactile stimulation induced by laser. Nevertheless, it is found that the mechanical behavior has no effect on the distribution of temperature in these studies.

The concept of biothermomechanical behavior of tissue studied earlier is arose again and (Li. *et al* 2017) analyzed thermal distribution, thermal-induced mechanical deformation and thermal mechanical damage of soft tissues under thermal loads. (Li *et al*. 2018) developed the theory of a modified fractional order generalized bio-thermoelasticity with variable thermal material properties. Later, (Li *et al*. 2019) investigated the transient responses in the context of generalized bio-thermoelastic theories with temperature dependent blood perfusion rate in a triple layered skin tissue. (Kumar *et al*. 2019a) studied the non local heat conduction approach in bi-layer tissue during magnetic fluid hyperthermia (MFH). (Kumar *et al*. 2019a) studied the transient response due to three-phase-lag (TPL) model of heat conduction in skin tissues. Recently, (Li *et al*. 2020) established the dual phase lag thermo-viscoelastic model to capture the micro-scale responses of biological tissue

Fundamental solutions play a pivotal role in investigation of various problems of mathematical physics and continuum mechanics (Svandze 2018a). (Svanadze 2018b) constructed fundamental solutions in the theory of elasticity and thermoelasticity for solids with triple porosity. (Kansal 2019) found the fundamental solution of partial differential equations in the generalized theory of thermoelastic diffusion materials with double porosity. (Sharma *et al*. 2013) investigated the plane wave and fundamental solution in electro-microstretch elastic. (Sharma and Kumar 2014) investigated the temporal fluctuations in tissue by using Laplace and Hankel transforms. Recently, (Kumar *et al*. 2020) constructed the fundamental solution of the system of equations in the theory of bio-thermoelasticity and studied the waves in tissues.

The introduction of non local factor is significant because the size effects increases at microscopic level. The concept of non-local model along with bio-thermoelasticity theory has not been considered so far. In order to analyze the behavior of the NL DPL model, we investigate some basic theorem in the form of the fundamental solution and the effect of non local and phase lag parameters on phase velocity and attenuation coefficients.

2. Governing equations

According to the nonlocal elasticity theory of (Eringen 2002), the stress tensor at arbitrary points $\mathbf{x}$ of a nano-material body not only depends on strain tensor at $\mathbf{x}$ but also depends on all points of the body. The stress strain-temperature and chemical potential relations have the form (Xiong *et al*. 2017)

$$\tau(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|, \xi)\sigma(\mathbf{x})dV(\mathbf{x}'), \tag{1}$$

$$\sigma(\mathbf{x}) = \lambda_0(\nabla.\bar{\mathbf{u}})\mathbf{I} + 2\mu\varepsilon - \gamma_1\bar{T}\mathbf{I} - \gamma_2\bar{P}\mathbf{I}, \tag{2}$$

$$\varepsilon = \frac{1}{2}[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^{\mathbf{T}}], \tag{3}$$

where τ nonlocal stress tensor, σ local stress tensor, $|\mathbf{x}' - \mathbf{x}|$ Euclidean distance, $\alpha(|\mathbf{x}' - \mathbf{x}|)$ nonlocal kernal, ξ nonlocal parameter, $\bar{T}$ temperature increment, $\bar{T} = \Theta - T_0$, $\frac{\bar{T}}{T_0} \ll 1$, Θ tissue temperature, T_0 reference temperature; $\bar{P}$ chemical potential, $\bar{\mathbf{u}}$ displacement vector, (Eringen 2002) replaced the non local constitutive equations given by the integral formulation by the gradients. Thus, by applying the differential operator $(1 - \xi^2\nabla^2)$ to both sides of Eq. (1), we get the equivalent differential form of the nonlocal theory as

$$(1 - \xi^2\nabla^2)\tau = \lambda_0(\nabla.\bar{\mathbf{u}})\mathbf{I} + 2\mu\varepsilon - \gamma_1\bar{T}\mathbf{I} - \gamma_2\bar{P}\mathbf{I}, \tag{4}$$

which considers the size effect on the response of nano structures, where $\gamma_1 = \beta_1 + \frac{a}{b}\beta_2$, $\gamma_2 = \frac{\beta_2}{b}$, $\lambda_0 = \lambda - \frac{\beta_2^2}{b}$, and $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\lambda + 2\mu)\alpha_c$ λ, μ are Lame's constants; α_t is the linear diffusion expansion coefficient; α_c is the linear thermal expansion coefficient; a measure of the thermodiffusion effect; b measure of the diffusive effect; ∇^2 Laplacian operator.

The equations of motion can be written as in the following form

$$\nabla.\tau + \rho\bar{\mathbf{F}} = \rho\ddot{\bar{\mathbf{u}}}, \tag{5}$$

where $\bar{F}$ the body force per unit mass.

Using Eq. (4) in Eq. (5), the equations of motion in terms of the temperature, displacement and chemical potential fields is as follows

$$(\lambda_0 + \mu)\nabla(\nabla.\bar{\mathbf{u}}) + \mu\nabla^2\bar{\mathbf{u}} - \gamma_1\nabla\bar{T} - \gamma_2\nabla\bar{P} + \rho(1 - \xi^2\nabla^2)\bar{\mathbf{F}} = \rho(1 - \xi^2\nabla^2)\ddot{\bar{\mathbf{u}}} \tag{6}$$

Heat transfer in living biological tissues is complicated process. The Pennes' bioheat transfer model (Pennes 1948) is used most commonly for the prediction of thermal data. The conduction term in this model is based on the classical Fourier's law

$$\mathbf{q}(\mathbf{x}, t) = -k\nabla T(\mathbf{x}, t), \tag{7}$$

which implies unphysical infinite propagation speed of thermal disturbance, where $\mathbf{q}$ heat flux vector. Then In order to overcome this unphysical behavior, (Cattaneo 1958) and (Vernotte 1958) independently proposed a modified constitutive relation to overcome this phenomena by introducing a phase lag time (τ_q) in Fourier's law

$$\mathbf{q}(\mathbf{x}, t+\tau_q) = -k\nabla T(\mathbf{x}, t), \tag{8}$$

where τ_q captures the micro-scale responses in time. Due to the imperfect of thermal model for some situations, (Tzou 1996) established a dual-phase-lag (DPL) constitutive relation, i.e.

$$\mathbf{q}(\mathbf{x}, t+\tau_q) = -k\nabla T(\mathbf{x}, t+\tau_T), \tag{9}$$

The lagging time τ_T is interpreted the thermalization time caused by micro-structural interaction and the lagging time τ_q is relaxation time due to fast transient effect of thermal inertia which is called phase-lag of heat flux.

Later, (Tzou and Guo 2010), proposed non local model which assumes that

$$\mathbf{q}(\mathbf{x}+\zeta, t+\tau_q) = -k\nabla T(\mathbf{x}), \tag{10}$$

Kumar *et al*. (2019) reformulated NL DPL model following GK model as follows

$$\mathbf{q}(\mathbf{x}+\zeta, t+\tau_q) = -k\nabla T(\mathbf{x}+\tau_T), \tag{11}$$

where ζ nonlocal parameter.

A developed lagging response structure represented by Eq. (11) can be shown by expanding it in terms of time and space related expansions of Taylor's and holding the terms up to particular orders in the parameters

$$(1-\zeta^2\nabla^2+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2})\mathbf{q} = -k(1+\tau_T\frac{\partial}{\partial t})\nabla \mathrm{T}, \tag{12}$$

where

$$\nabla.\mathbf{q} = -\rho T_0\dot{S}+\rho_b c_b\omega_b(T_b-\Theta)+q_m+q_{ext}. \tag{13}$$

Constitutive relation is as follows

$$\rho S = \gamma_1 e_{kk}+l_1\bar{T}+d\bar{P}, \tag{14}$$

where $e_{kk}=\bar{u}_{i,i}$ i= 1,2,3, and $l_1=\frac{\rho c}{T_0}+\frac{a^2}{b}$.

From Eqs. (12)-(14), the non local bio-thermoelastic diffusive equation can be described as

$$k\left(1+\tau_T\frac{\partial}{\partial t}\right)\nabla^2\bar{T}-\left(1-\zeta^2\nabla^2+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\gamma_1 T_0\dot{e}_{kk}+l_1T_0\dot{\bar{T}}+dT_0\dot{\bar{P}}+\omega_b\rho_b c_b\bar{T}\right)= -\left(1-\zeta^2\nabla^2+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)(q_m+q_{ext}), \tag{15}$$

where T_b blood temperature, ρ_b blood mass density, ω_b blood perfusion rate, c_b specific heat of blood; ρ tissue mass density; c tissue specific heat. In Eq. (15), it is assumed that $T_b=T_0$.

Also the non local mass diffusion law in the context of dual phase lag model is expressed as follows

$$(1-\varsigma\nabla^2+\tau_v\frac{\partial}{\partial t}+\frac{\tau_v^2}{2}\frac{\partial^2}{\partial t^2})\eta = -D(1+\tau_\rho\frac{\partial}{\partial t})\nabla\mathbf{P}, \tag{16}$$

where

$$-\nabla.\eta = \dot{C} - M. \tag{17}$$

Constitutive relation is

$$C = \gamma_2 e_{kk} + d\bar{T} + n\bar{P}. \tag{18}$$

From Eqs. (16)-(18), the mass diffusion equation can be described as

$$D\left(1 + \tau_\rho \frac{\partial}{\partial t}\right)\nabla^2 \bar{P} - \left(1 - \varsigma^2\nabla^2 + \tau_\nu \frac{\partial}{\partial t} + \frac{\tau_\nu^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\gamma_2 \dot{e}_{kk} + d\dot{\bar{T}} + \eta\dot{\bar{P}}\right) = \\ -(1 - \varsigma^2\nabla^2 + \tau_\nu \frac{\partial}{\partial t} + \frac{\tau_\nu^2}{2}\frac{\partial^2}{\partial t^2})M, \tag{19}$$

where D thermoelastic diffusion constant, η mass diffusing vector, C concentration of diffusive materials, M mass diffusion source, $d = \frac{a}{b}$, $n = \frac{1}{b}$, ς non local parameter and τ_ν, τ_P are phase lag parameters.

For simplicity, we invoke the following dimensionless variables

$$\begin{aligned} x_{i'} &= \frac{\omega^*}{c_1}x_i,\ \bar{u}_i{}' = \frac{\omega^*}{c_1}\bar{u}_i, \xi' = \frac{\omega^*}{c_1}\xi,\ \zeta' = \frac{\omega^*}{c_1}\zeta,\ \varsigma' = \frac{\omega^*}{c_1}\varsigma, \\ t' &= \omega^* t, \tau_{q'} = \omega^*\tau_q,\ \tau_\rho{}' = \omega^*\tau_\rho,\ \tau_T{}' = \omega^*\tau_T,\ \tau_\nu{}' = \omega^*\tau_\nu, \\ \bar{T}' &= \frac{\gamma_1\bar{T}}{\rho c_1^2},\ \bar{P}' = \frac{\bar{P}}{b\gamma_2}, M' = \frac{\omega^*}{\gamma_2}M,\ \bar{\mathbf{F}}' = \frac{\rho c_1}{\mu\omega^*}\mathbf{F},\ q_{m'} = \frac{1}{\gamma_1 T_0\omega^*}q_{m'}, \\ q_{ext}{}' &= \frac{1}{\gamma_1 T_0 \omega^*}q_{ext'}, \end{aligned} \tag{20}$$

where $c_1^2 = \frac{\lambda+2\mu}{\rho}$, $\omega^* = \frac{\rho c c_1^2}{k}$, ω^* and c_1 are characteristics frequency and longitudinal wave velocity in the medium, respectively.

Using dimensionless variables given by Eq. (20) in Eqs. (6), (15) and (19), after suppressing the primes, we have

$$\begin{aligned} &a_1\nabla(\nabla.\bar{\mathbf{u}}) + a_2\nabla^2\bar{\mathbf{u}} - \nabla\bar{T} - a_3\nabla\bar{P} - (1 - \xi^2\nabla^2)\ddot{\bar{\mathbf{u}}} = -a_2(1 - \nabla^2\xi^2)\bar{\mathbf{F}}, \\ &\tau_{10}\nabla^2\bar{T} - \tau_{20}(a_4\dot{\nabla}.\bar{\mathbf{u}} + a_5\dot{\bar{T}} + a_6\dot{\bar{P}} - a_7\bar{T}) = \bar{F}_4, \\ &\tau_{30}\nabla^2\bar{P} - \tau_{40}(a_9\dot{\nabla}.\bar{\mathbf{u}} + a_{10}\dot{\bar{T}} + a_{11}\dot{\bar{P}}) = \bar{F}_5, \end{aligned} \tag{21}$$

where a_i for $i = 1\ldots12$, τ_{i0} for $i = 1,2,3,4$ and $\bar{\mathbf{F}}$, $\bar{F}_4$, $\bar{F}_5$ are given as follows

$$a_1 = \frac{\lambda_0 + \mu}{\rho c_1^2}, \quad a_2 = \frac{\mu}{\rho c_1^2}, \quad a_3 = \frac{b\gamma_2^2}{\rho c_1^2}, \quad a_4 = \frac{\gamma_1^2 T_0}{k\rho\omega^*},$$

$$a_5 = \frac{l_1 T_0 c_1^2}{k\omega^*}, \quad a_6 = \frac{dbT_0\gamma_1\gamma_2}{k\rho\omega^*}, a_7 = \frac{\omega_b\rho_b c_b c_1^2}{k\omega *^2}, \quad a_8 = \frac{\gamma_1^2 T_0}{k\rho\omega^*},$$

$$a_9 = \frac{c_1^2}{Db\omega^*}, \quad a_{10} = \frac{d\rho c_1^2\omega^*}{Db\gamma_1\gamma_2}, \quad a_{11} = \frac{nc_1^2}{D\omega^*}, \quad a_{12} = \frac{c_1^2}{Db\omega^*},$$

$$\tau_{10} = 1 + \tau_T\frac{\partial}{\partial t}, \quad \tau_{20} = 1 - \xi^2\nabla^2 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2},$$

$$\tau_{30} = 1 + \tau_P\frac{\partial}{\partial t}, \quad \tau_{40} = 1 - \varsigma^2\nabla^2 + \tau_\nu\frac{\partial}{\partial t} + \frac{\tau_\nu^2}{2}\frac{\partial^2}{\partial t^2},$$

$$\mathbf{F}^{(1)} = -a_2(1-\nabla^2\xi^2)\bar{\mathbf{F}}, \qquad \bar{F}_4 = -\tau_{21}a_8(q_m + q_{ext}), \qquad \bar{F}_5 = -\tau_{41}a_{12}M,$$

2.1 Steady oscillation

Taking harmonic variation of $\bar{\mathbf{u}}$, $\bar{T}$, $\bar{P}$ as

$$(\bar{u}_i, \bar{T}, \bar{P}, \bar{F}_j)(\mathbf{x}, t) = Re[(u_i, T, P, F_j)(\mathbf{x})e^{-i\omega t}], \tag{22}$$

where $i = 1,2,3$ and $j = 1,2,3,4,5$.

Let us take the second order matrix differential operator with constant coefficients as

$$A_{mn}(D_x) = (a_2\nabla^2 + \omega^2(1-\xi^2\nabla^2))\delta_{mn} + a_1\frac{\partial^2}{\partial x_m \partial x_n},\ A_{m4}(D_x) = -\frac{\partial}{\partial x_m}, A_{m5}(D_x) = -a_3\frac{\partial}{\partial x_m},$$
$$A_{4n}(D_x) = \tau_{21}a_4(i\omega), \qquad A_{44}(D_x) = \tau_{11}\nabla^2 + \tau_{21}(i\omega)(a_6 + a_7), \qquad A_{45}(D_x) = \tau_{21}a_6(i\omega),$$
$$A_{5n}(D_x) = \tau_{41}a_9(i\omega)\frac{\partial}{\partial x_n},\ A_{54} = \tau_{41}a_{10}(i\omega),\ A_{55}(D_x) = \tau_{31}\nabla^2 + \tau_{41}a_{11}(i\omega),$$
$$D_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}), m, n = 1,2,3,$$

where δ_{mn} is the kronecker delta function and τ_{1i} for $i = 1,2,3,4$ are given as follows

$$\tau_{11} = 1 + \tau_T(-i\omega), \qquad \tau_{21} = 1 - \zeta^2\nabla^2 + \tau_q(-i\omega) + \frac{\tau_q^2}{2}(-i\omega)^2,$$
$$\tau_{31} = 1 + \tau_p(-i\omega), \qquad \tau_{41} = 1 - \varsigma^2\nabla^2 + \tau_\nu(-i\omega) + \frac{\tau_\nu^2}{2}(-i\omega)^2.$$

With these considerations, Eq. (21) can be written as

$$\mathbf{A}(\mathbf{D_x}, \omega)\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{23}$$

i.e.

$$\mathbf{A}(\mathbf{D_x}) = \| G_{gh}(D_x) \|_{5\times5}, \tag{24}$$

where, $\mathbf{U} = (\mathbf{u}, T, P)$ and $\mathbf{F} = (F_1, \ldots, F_5)$ is a five component vector function, $\mathbf{x} \in R^3$.

3. Fundamental solutions

In this section fundamental solutions of the system of Eq. (21) is constructed as follows:

Definition The fundamental solution of the system of equation (the fundamental matrix of operator A) is the matrix $G(\mathbf{x}) = \| G_{gh}(\mathbf{x}) \|_{5\times5}$ satisfying the condition

$$\mathbf{A}(D_x)G(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}), \tag{25}$$

where $\delta(x)$ is the Dirac delta, $I = \| \delta_{gh} \|_{5\times5}$ is the unit matrix and $\mathbf{x} \in R^3$.

Now, we construct $G(\mathbf{x})$ in terms of the elementary functions.

Consider the system of non-homogeneous equations:

$$\begin{aligned}
&[a_2\nabla^2 + \omega^2(1-\xi^2\nabla^2)]u + a_1\nabla(\nabla.\mathbf{u}) + (\tau_{21}a_4(i\omega))\nabla T + \tau_{41}a_9(i\omega)\nabla\ \bar{P} = \mathbf{F}^{(1)},\\
&-\nabla.\ \mathbf{u} + (\tau_{11}\nabla^2 + \tau_{21}i\omega(a_5 + a_7))T + \tau_{41}a_{10}(i\omega)P = F_4,\\
&-a_3\nabla.u + (\tau_{21}a_6 i\omega)T + (\tau_{31}\nabla^2 + \tau_{41}a_{11}i\omega)P = F_5.
\end{aligned} \tag{26}$$

Introducing the matrix differential operator as

It can easily verify that the system of Eq. (26) can be written as

$$\mathbf{A}^T(D_x)\mathbf{U}(x) = \mathbf{F}(\mathbf{x}), \tag{27}$$

where A^T is the transpose of matrix A, $U = (\mathbf{u}, T, P)$ and $\mathbf{F} = (\mathbf{F}^{(1)}, F_4, F_5)$ is a five component vector function and $\mathbf{x} \in R^3$.

Applying divergence operator on the first equation of system (26), we obtain

$$\begin{aligned}&[(a_1 + a_2)\nabla^2 + \omega^2(1 - \xi^2\nabla^2)]\nabla.\mathbf{u} + (\tau_{21}a_4(i\omega))\nabla^2 T + \tau_{41}a_9(i\omega)\nabla^2 P = \nabla.\mathbf{F}^{(1)},\\ &-\nabla.\ \mathbf{u} + (\tau_{11}\nabla^2 + \tau_{21}i\omega(a_5 + a_7))T + \tau_{41}a_{10}(i\omega)P = F_4,\\ &-a_3\nabla.u + (\tau_{21}a_6 i\omega)T + (\tau_{31}\nabla^2 + \tau_{41}a_{11}i\omega)P = F_5.\end{aligned} \tag{28}$$

From Eq. (28), we have

$$\mathbf{B}(\nabla^2, \omega)\mathbf{V}(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x}), \tag{29}$$

where $\mathbf{V} = (\nabla.\mathbf{u}, T, P)$ and $\mathbf{\Phi} = (\phi_1, \phi_2, \phi_3) = (\nabla.\mathbf{F}^{(1)}, F_4, F_5)$ are there-component vector function,

$$B(\Delta, \omega) = (B_{ij}(\Delta, \omega))_{3\times 3},$$

$$B_{11}(\Delta, \omega) = (a_1 + a_2)\nabla^2 + \omega^2(1 - \xi^2\nabla^2), B_{12}(\Delta, \omega) = \tau_{21}a_4 i\omega, \qquad B_{13}(\Delta, \omega) = \tau_{41}a_9 i\omega$$

$$B_{21}(\Delta, \omega) = -1, B_{22}(\Delta, \omega) = \tau_{11}\nabla^2 + \tau_{21}, i\omega(a_5 + a_7), B_{23}(\Delta, \omega) = \tau_{41}a_{10}i\omega,$$

$$B_{31}(\Delta, \omega) = -a_{3,32}\,(\Delta, \omega) = \tau_{21}a_6 i\omega, \qquad B_{33}(\Delta, \omega) = \tau_{31}\nabla^2 + \tau_{41}a_{11}i\omega,$$

We introduce the notation

$$\Lambda_1(\nabla, \omega) = \frac{1}{[(a_1 + a_2 - \omega^2\xi^2)\tau_{11}\tau_{31}]} det\mathbf{B}(\nabla, \omega).$$

It is easily seen that $\Lambda_1(-\alpha^*, \omega) = 0$ is a third degree algebraic equation and there exist three roots $\lambda_1^2, \lambda_2^2, \lambda_3^2$ (w.r.t. α^*).

Then we have

$$\Lambda_1(\nabla, \omega) = \prod_{j=1}^{3} (\nabla + \lambda_j^2).$$

Eq. (29) imply that

$$\Lambda_1(\nabla, \omega)\mathbf{V} = \mathbf{\Phi}, \tag{30}$$

where $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)$,

$$\Phi_j = \frac{1}{(a_1 + a_2 - \omega^2\xi^2)\tau_{11}\tau_{31}} \sum_{l=1}^{3} B_{lj}^*\phi_l, \tag{31}$$

and B_{lj}^* is the cofactor of elements B_{lj} of the matrix B. Now applying the operator $\Lambda_1(\nabla, \omega)$ to the first equation of system (26) and taking into account Eq. (30), we obtain

$$\Lambda_2(\nabla, \omega)u = \tilde{\mathbf{F}}, \tag{32}$$

where $\Lambda_2(\nabla, \omega) = \Lambda_1(\nabla, \omega)(\nabla + \lambda_4)$, $\lambda_4 = \frac{\omega^2}{a_2 - \omega^2\xi^2}$ and

$$\tilde{\mathbf{F}} = \frac{1}{a_2 - \omega^2\xi^2}[\Lambda_1(\nabla, \omega)\mathbf{F}^{(1)} - a_1\nabla\Phi_1] - \frac{i\omega}{a_2 - \omega^2\xi^2}[\tau_{21}a_4\nabla\Phi_2 + \tau_{41}a_9\nabla\Phi_3]. \tag{33}$$

On the basis of Eqs. (30) and (32); we get

$$\Lambda(\nabla, \omega)U(x) = \tilde{\Phi}(x), \tag{34}$$

where $\widetilde{\Phi} = (\tilde{F}, \Phi_2, \Phi_3)$ is a five component vector function and

$$\Lambda(\nabla,\omega) = (\Lambda_{ij}(\nabla,\omega))_{5\times5},\ \Lambda_{11} = \Lambda_{22} = \Lambda_{33} = \Lambda_2, \tag{35}$$

$$\Lambda_{44} = \Lambda_{55} = \Lambda_1,\ \Lambda_{ij} = 0,\ i \neq j. \tag{36}$$

We introduce the notations

$$\begin{aligned} n_{11}(\nabla,\omega) &= -\frac{a_1}{a_2-\omega^2\xi^2} B_{l1}^* - \frac{i\omega}{a_2-\omega^2\xi^2}[\tau_{21}a_4B_{l2}^* + \tau_{41}a_9B_{l3}^*], \\ n_{lm}(\nabla,\omega) &= \frac{1}{k_0} B_{lm}^*(\nabla,\omega), l = 1,2,3, m = 2,3. \end{aligned} \tag{37}$$

In view of Eq. (37), from Eqs. (31) and (33) we have

$$\tilde{\mathbf{F}} = (\frac{1}{a_2-\omega^2\xi^2}\Lambda_1(\Delta,\omega)I + n_{11}(\Delta,\omega)\nabla div)F^{(1)} + \sum_{l=2}^{3} n_{l1}(\Delta,\omega)\nabla F_{l+2}, \tag{38}$$

$$\Phi_m = n_{1m}(\Delta,\omega) div F^{(1)} + \sum_{l=2}^{3} n_{lm}(\Delta,\omega) F_{l+2}, \tag{39}$$

where $I = (\delta_{ij})_{3\times3}$ is the unit matrix.

Thus, from Eq. (39) we have

$$\widetilde{\Phi}(x) = L^T(D_x,\omega)F(x), \tag{40}$$

where

$$\begin{aligned} L(D_x,\omega) &= (L_{ij}(D_x,\omega))_{5\times5}, \\ L_{ij}(D_x,\omega) &= \frac{1}{\mu}\Gamma_1(\nabla,\omega)\delta_{ij} + n_{11}(\nabla,\omega)\frac{\partial^2}{\partial x_i \partial x_j}, \\ L_{ir}(D_x,\omega) &= n_{1;r-2}(\nabla,\omega)\frac{\partial}{\partial x_i},\ L_{ri}(D_x,\omega) = n_{r-2;1}(\nabla,\omega)\frac{\partial}{\partial x_i}, \\ L_{rm}(D_x,\omega) &= n_{r-2;m-2}(\nabla,\omega),\ i,j = 1,2,3,\ r,m = 4,5. \end{aligned} \tag{41}$$

By virtue of Eqs. (27) and (40) from (34), it follows that $\Lambda U = L^T A^T U$. It follows that $L^T A^T = \Lambda$ and hence

$$A(D_x,\omega)L(D_x,\omega) = \Lambda(\nabla,\omega). \tag{42}$$

We assume that $\lambda_l^2 \neq \lambda_j^2$, where $l,j = 1,2,3,4$ and $l \neq j$. Let

$$\begin{aligned} Y(\mathbf{x},\omega) &= (Y_{lj}(\mathbf{x},\omega))_{5\times5}, \\ Y_{11}(\mathbf{x},\omega) &= Y_{22}(\mathbf{x},\omega) = Y_{33}(\mathbf{x},\omega) = \sum_{j=1}^{4} \eta_{2j}\gamma^{(j)}(\mathbf{x},\omega), \\ Y_{44}(\mathbf{x},\omega) &= Y_{55}(\mathbf{x},\omega) = \sum_{j=1}^{3} \eta_{1j}\gamma^{(j)}(\mathbf{x},\omega), \\ Y_{lj}(\mathbf{x},\omega) &= 0,\ l,j = 1,2,\ldots,5, \end{aligned} \tag{43}$$

where

$$\gamma^{(j)}(\mathbf{x},\omega) = -\frac{e^{i\lambda_j|x|}}{4\pi|\mathbf{x}|} \tag{44}$$

is the fundamental solution of Helmoholtz's equation, i.e. $(\nabla + \lambda_j^2)\gamma^{(j)}(\mathbf{x},\omega) = \delta(\mathbf{x})$ and

$$\begin{aligned} &\eta_{lm} = \prod_{l=1,l\neq m}^{3} (\lambda_i^2 - \lambda_m^2)^{-1},\ \eta_{2j} = \prod_{l=1,l\neq m}^{4} (\lambda_i^2 - \lambda_m^2)^{-1}, \\ &m = 1,2,3,\ j = 1,2,3,4. \end{aligned} \tag{45}$$

Lamma: The matrix $Y(x,\omega)$ is the fundamental solution of the operator $\Lambda(\Delta,\omega)$, that is

$$\Lambda(\Delta,\omega)Y(x,\omega) = \delta(x)J, \tag{46}$$

where $x \in R^3$.

Proof. It suffices to show that Y_{11} and Y_{44} are the fundamental solutions of operators $\Lambda_2(\Delta)$ and $\Lambda_1(\Delta)$, respectively, i.e.

$$\Lambda_2(\Delta)Y_{11}(x,\omega) = \delta(x), \tag{47}$$

$$\Lambda_1(\Delta)Y_{44}(x,\omega) = \delta(x). \tag{48}$$

Taking into account the equalities

$$\sum_{j=1}^{3} \eta_{1j} = 0,\ \sum_{j=2}^{3} \eta_{1j}(\lambda_1^2 - \lambda_j^2) = 0,\ \eta_{13}(\lambda_1^2 - \lambda_j^2)(\lambda_2^2 - \lambda_3^2) = 0,$$
$$(\nabla^2 + \lambda_l^2)\gamma^{(j)}(x,\omega) = \delta(x) + (\lambda_l^2 - \lambda_j^2)\gamma^{(j)}(x,\omega),$$
$$l,j = 1,2,3, \qquad x \in R^3$$

we have

$$\begin{aligned}\Gamma_1(\nabla,\omega)Y_{44}(x,\omega) &= \prod_{l=2}^{3} (\nabla + \lambda_l^2) \sum_{j=1}^{3} \eta_{1j}(\delta(x) + (\lambda_1^2 - \lambda_j^2)\gamma^{(j)}(x,\omega)) \\ &= \prod_{l=2}^{3} (\nabla + \lambda_l^2) \sum_{j=2}^{3} \eta_{1j}(\lambda_1^2 - \lambda_j^2)\gamma^{(j)}(x,\omega) \\ &= (\nabla + \lambda_3^2)\gamma^{(3)}(x,\omega) = \delta(x).\end{aligned} \tag{49}$$

Similarly we can prove Eq. (47). Introducing the following matrix

$$G(x,\omega) = L(D_x,\omega)Y(x,\omega). \tag{50}$$

Using Eqs. (42) and (46) from (50), we get the required results.

Theorem The matrix $G(\mathbf{x},\omega)$ defined by Eq. (50) is the fundamental solution of Eq. (23), where the matrix $L(D_x,\omega)$ and $Y(\mathbf{x},\omega)$ are given by formula (41) and (43), respectively.

Each element $\Gamma_{ij}(\mathbf{x},\omega)$ of the matrix $\Gamma(\mathbf{x},\omega)$ is represented in the following form

$$\begin{aligned}&\Gamma_{ij}(\mathbf{x},\omega) = L_{ij}(D_x,\omega)Y_{11}(\mathbf{x},\omega),\\ &\Gamma_{lm}(\mathbf{x},\omega) = L_{lm}(D_x,\omega)Y_{44}(\mathbf{x},\omega),\\ &l = 1,2,\ldots,5,\ j = 1,2,3,\ m = 4,5.\end{aligned} \tag{51}$$

3.1 Special case: Influence of heat source

Here we take the following special type of external heat source (Cheng and Kar 1997)

$$q_{ext} = \frac{2AP}{\pi r_0^2}\exp[-2(\frac{x_1^2+x_2^2+x_3^2}{r_0^2})]e^{-i\omega t}, \tag{52}$$

where P is the total power of the incident laser beam, A is the absorptivity of the workpiece, r_0 is the spot radius of the laser beam at $\frac{1}{e^2}$ point, and x_1, x_2 and x_3 are distances measured in Cartesian coordinates from the center of the laser beam.

Incorporating the considered heat source in the above basic theorem on fundamental solution, we will obtain the corresponding result due to external laser heat source.

4. Plane waves

In this section, we examine the behavior of plane waves in a homogeneous, isotropic, non local bio-themoelasic diffusive medium with phase lag. For this a two dimensional problem is considered for which the displacements, temperature and chemical potential are taken as

$$\boldsymbol{u} = \left(u_1(x_1,x_3,t),0,u_3(x_1,x_3,t)\right),\mathrm{T} = T(x_1,x_3,t),\mathrm{P} = P(x_1,x_3,t), \tag{53}$$

The relation between the displacement components and the potential functions is taken as

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3},\ u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1}. \tag{54}$$

Using Eq. (54) to the system of Eq. (21) in the absence of body force, metabolic source, external heat source and mass diffusion source, we have

$$((a_1+a_2)\nabla^2 - (1-\xi^2\nabla^2)\frac{\partial^2}{\partial t^2})\phi - T - a_3P = 0, \tag{55}$$

$$(a_2 - (1-\xi^2\nabla^2)\frac{\partial^2}{\partial t^2})\psi = 0, \tag{56}$$

$$(-a_4\tau_{21}\frac{\partial}{\partial t}\nabla^2)\phi + (\tau_{11}\nabla^2 - \tau_{21}a_5\frac{\partial}{\partial t} + \tau_{21}a_7)T + \tau_{21}a_6\frac{\partial}{\partial t}P = 0, \tag{57}$$

$$(-\tau_{41}a_9\frac{\partial}{\partial t}\nabla^2)\phi - \tau_{41}a_{10}\frac{\partial}{\partial t}T + (\tau_{31}\nabla^2 - a_{11}\tau_{41}\frac{\partial}{\partial t})P = 0. \tag{58}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}$.

We assume the solution of the form

$$(\phi,T,P,\psi) = (\tilde{\phi},\tilde{T},\tilde{P},\tilde{\psi})e^{i\xi_1(l_1x_1+l_3x_3-\omega t)} \tag{59}$$

where $\omega(=\xi_1 c)$ is the frequency; ξ_1 is the value number and c is the phase velocity; $\tilde{\phi},\tilde{T},\tilde{P},\tilde{\psi}$ are undetermined amplitudes depending on time t and coordinate $x_m(m=1,3)$; l_1 and l_3 are the direction cosines of the wave normal to the x_1-x_3 plane with the property $l_1^2+l_3^2=1$.

Using Eqs. (59) in (55)-(59), we obtain

$$(-(a_1+a_2)\xi_1^2 + (1+\xi^2\xi_1^2)\omega^2)\tilde{\phi} - \tilde{T} - a_3\tilde{P} = 0, \tag{60}$$

$$(-ia_4\tau_{21}^0\omega\xi_1^2)\tilde{\phi} + (\tau_{11}\xi_1^2 - \tau_{21}^0a_5i\omega + \tau_{21}^0a_7)\tilde{T} + (-i\omega\tau_{21}^0a_6)\tilde{P} = 0, \tag{61}$$

$$(-i\omega\xi_1^2\tau_{41}^0a_7)\tilde{\phi} + i\omega a_{10}\tau_{41}^0\tilde{T} + (-\tau_{31}\xi_1^2 + i\omega a_{11}\tau_{41}^0)\tilde{P} = 0 \tag{62}$$

$$(a_2 + (1+\xi^2\xi_1^2)\omega^2)\tilde{\psi} = 0 \tag{63}$$

For the non trivial solution of the system of Eqs. (60)-(62) can be obtained by equating the determinant of following matrix B. This yields a polynomial characteristics equation in ξ_1^2.

$$B = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

where matrix entries are as follows

$$c_{11} = -(a_1+a_2)\xi_1^2 + (1+\xi^2\xi_1^2)\omega^2, \qquad c_{12} = -1, \qquad c_{13} = -a_3,$$
$$c_{21} = -ia_4\omega\xi_1^2\tau_{21}^0, \qquad c_{22} = \tau_{11}\xi_1^2 + \tau_{21}^0(-a_5\omega + a_7), \qquad c_{23} = -i\omega\tau_{21}^0a_6,$$
$$c_{31} = -i\omega a_9\xi_1^2\tau_{41}^0,_{32} = i\omega a_{10}\tau_{41}^0, \qquad c_{33} = (-\tau_{31}\xi_1^2 + a_{11}i\omega\tau_{41}^0),$$
$$\tau_{21}^0 = 1 + \zeta^2\xi_1^2 + \tau_q(-i\omega) + \frac{\tau_q^2}{2}(-i\omega)^2, \tau_{41}^0 = 1 + \varsigma^2\xi_1^2 + \tau_v(-i\omega) + \frac{\tau_v^2}{2}(-i\omega)^2.$$

Solving the polynomial, we obtain six roots of ξ_1, in which three roots ξ_{11}, ξ_{12} and ξ_{13} correspond to positive x_3 direction and other three roots $-\xi_{11}$, $-\xi_{12}$ and $-\xi_{13}$ correspond to negative x_3 direction. Corresponding to roots ξ_{11}, ξ_{12} and ξ_{13} there exist three longitudinal waves in descending order of their velocities, namely longitudinal wave (P-wave), thermal wave

Table 1 Thermophysical parameters (Li *et al*. 2018)

Parameters	Units	Values
ρ	kgm^{-3}	1190
c	$Jkg^{-1}K^{-1}$	4196
ρ_b	kgm^{-3}	1060
c_b	$Jkg^{-1}K^{-1}$	3600
k	$Wm^{-1}K^{-1}$	0.613
ω_b	s^{-1}	1.87×10^{-3}
τ_q	s	8
τ_T	s	16

Table 2 Diffusion parameters (Xiong and Guo 2017)

Parameter	Units	Values
D	$kgsm^{-3}$	0.85×10^{-8}
a	$m^2s^{-2}K^{-1}$	1.2×10^4
b	$kg^{-1}m^5$	9×10^5
τ_P	s	8
τ_η	s	16

(T-wave) and mass diffusive wave (MD-wave). From Eq. (63), we obtain roots of ξ_1 as $\pm\xi_{14}$ and corresponding to this root there exists a transverse wave (SV) which is unaffected by the thermal and diffusive properties of tissues.

(i) **Phase velocity**

The Phase velocities is given by

$$PV_i = \frac{\omega}{real(\xi_{1i})}, i = 1,2,3.$$

where PV_1, PV_2, PV_3 are the phase velocities of P, T and MD waves respectively.

(ii) **Attenuation coefficients**

$$AQ_i = im(\xi_{1i}), i = 1,2,3,$$

where AQ_1, AQ_2, AQ_3 are attenuation coefficients of P, T and MD waves respectively.

5. Results and discussion

The plane waves are useful idealization and practical reality. The considered model gives four conceptually distinct waves. Out of which three are longitudinal and fast waves and one is transverse and much slower than the longitudinal waves. The wave distorts the tissue in two ways. Elements of the medium change shape (transverse, shear strain) and they are rotated. Furthermore, the shear strains are orders of magnitude greater than the bulk strain for a given applied stress. The longitudinal shear strains near bubbles in tissue are perhaps the primary concern for safety in the use of diagnostic ultrasound.

In this work, the wave characteristics (phase velocity and attenuation coefficients) in living

Table 3 Material constants (Li *et al*. 2018)

Parameter	Units	Values
λ	$kgm^{-1}s^{-2}$	8.27×10^{8}
μ	$kgm^{-1}s^{-2}$	3.446×10^{7}
α_t	K^{-1}	1×10^{-4}
α_c	K^{-1}	1.98×10^{-4}

Table 4 Non local parameters (Kumar *et al*., 2019a)

Parameter	Units	Values
ξ	m	0,0.02,0.04
ζ	m	0,0.02,0.04
ϑ	m	0,0.02,0.04

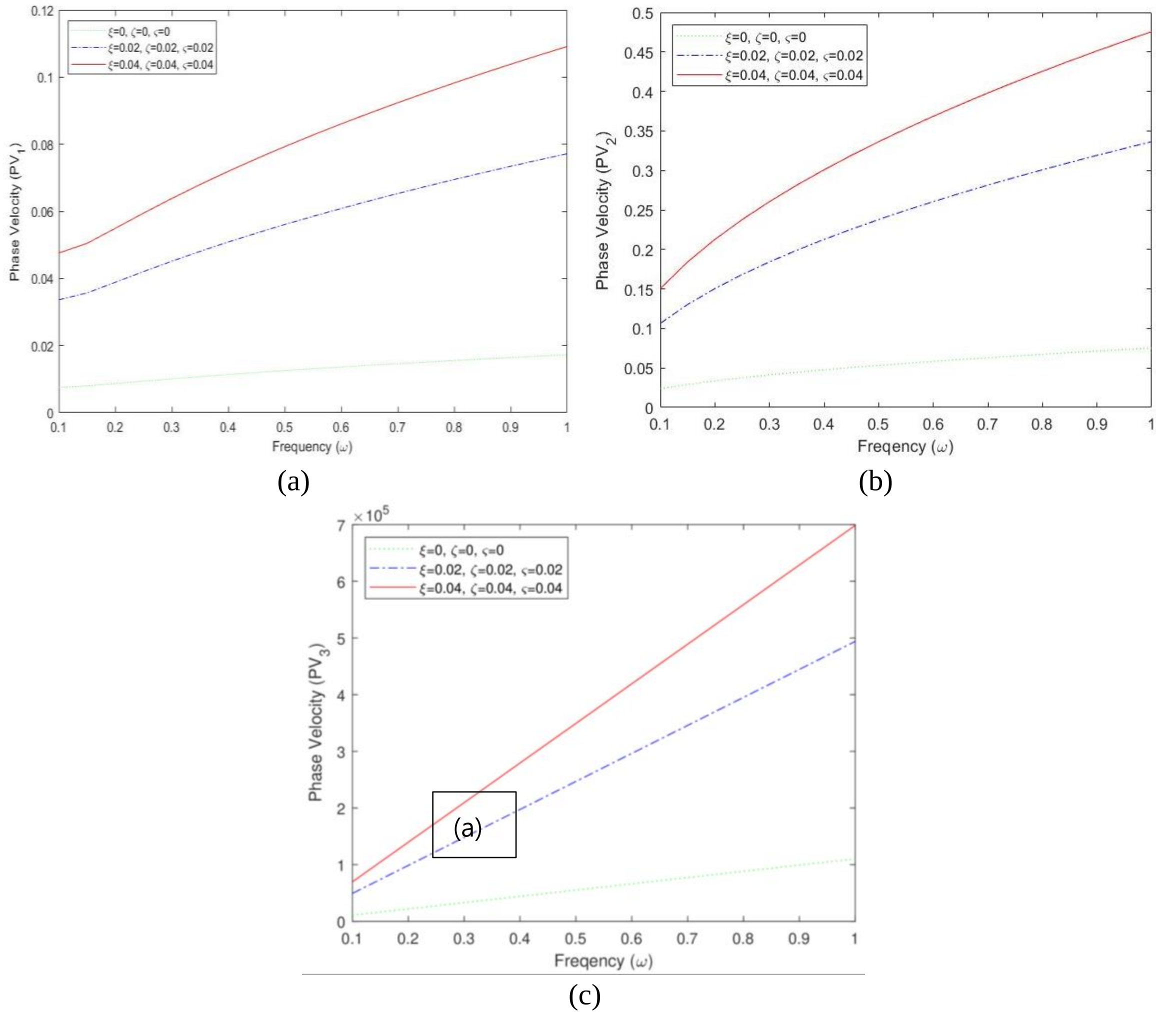
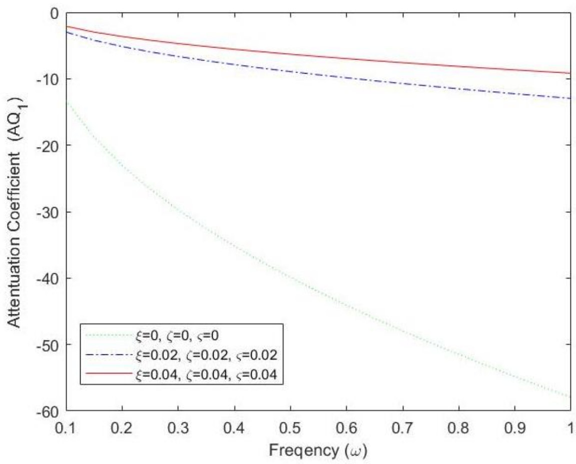


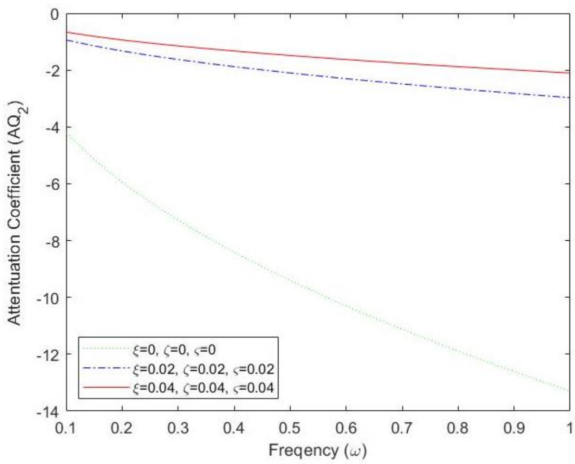
Fig. 1 Effect of non local parameters (ξ, ζ and ς) on phase velocities (PV_1, PV_2, and PV_3) of longitudinal waves

biological tissues obtained from NL DPL bio-thermoelastic diffusion model of bio-heat transfer is studied. The computation has been made by using the MATLAB-2019 software and results are presented in the figures. These graphical representations show the effects of non local and phase lag parameters on wave characteristics with frequency. The selected reference value of material constants and thermophysical, diffusion, non local, phase lag parameters to compute the profile of wave characteristics in living biological tissue in infinite domain are given in the Table 1, Table 2, Table 3 and Table 4.

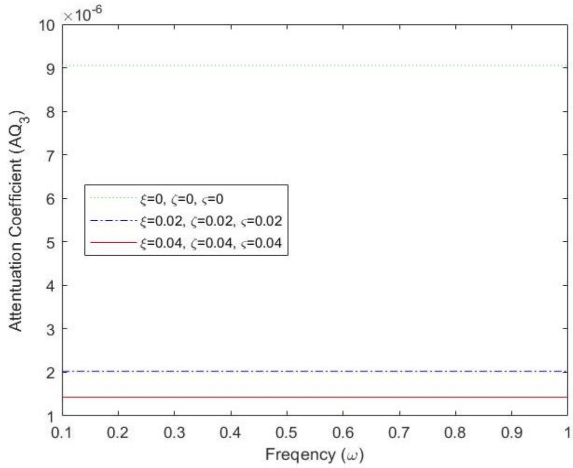
Red solid line corresponds to ($\xi = 0.04,\ \zeta = 0.04,\ \varsigma = 0.04$), blue dashed line corresponds to ($\xi = 0.02,\ \zeta = 0.02,\ \varsigma = 0.02$) and green dotted line corresponds to ($\xi = 0,\ \zeta = 0,\ \varsigma = 0$) in Fig. 1 and Fig. 2. Red solid line corresponds to ($\tau_T = 0.25,\ \tau_q = 0.15,\ \tau_P = 0.25, \tau_v = 0.15$), blue dashed line corresponds to ($\tau_T = 0,\ \tau_q = 0.15,\ \tau_P = 0, \tau_v = 0.15$) and green dotted line corresponds to ($\tau_T = 0,\ \tau_q = 0,\ \tau_P = 0, \tau_v = 0$) in Fig. 3 and Fig. 4.



(a)

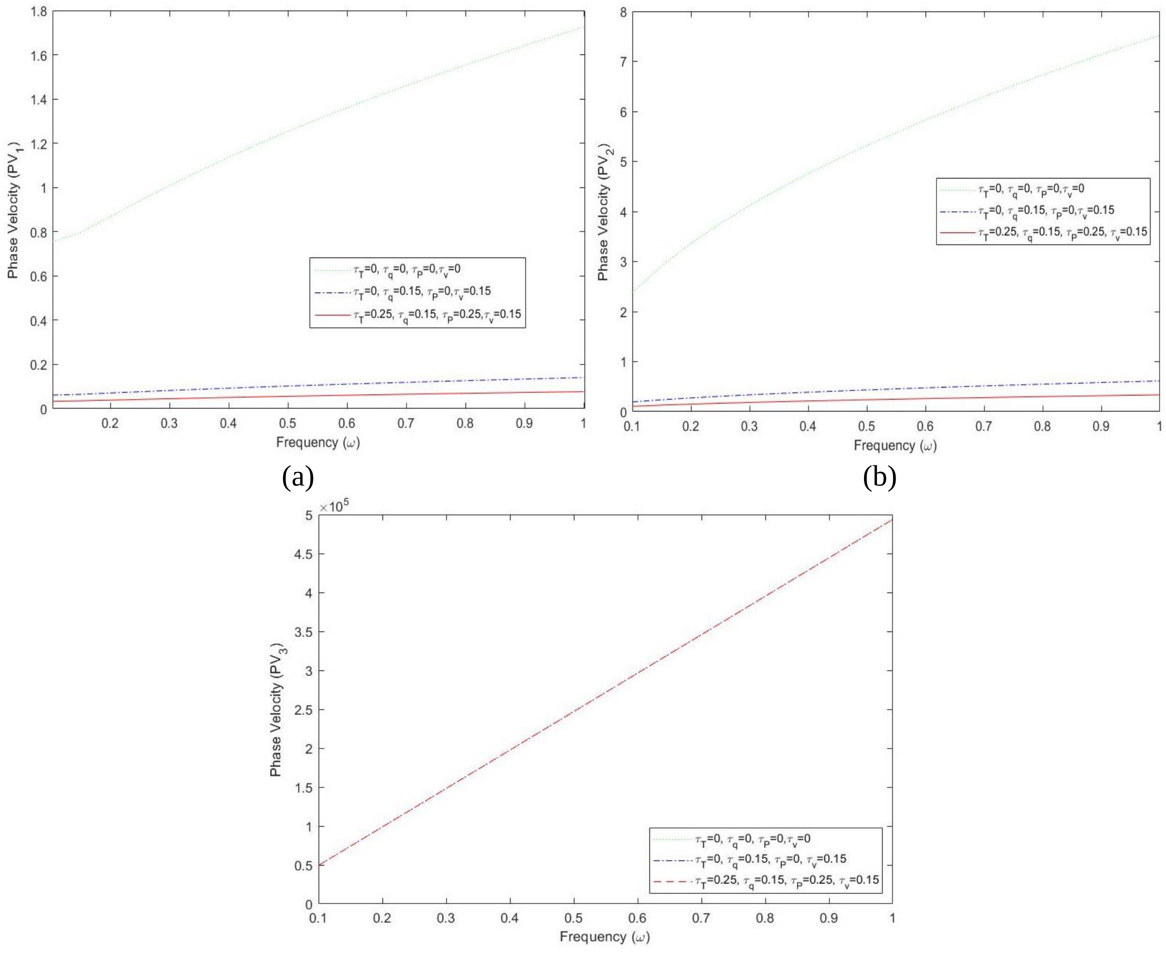


(b)



(c)

Fig. 2 Effect of non local parameters (ξ, ζ and ς) on attenuation coefficients (AQ_1, AQ_2, and AQ_3) of longitudinal waves



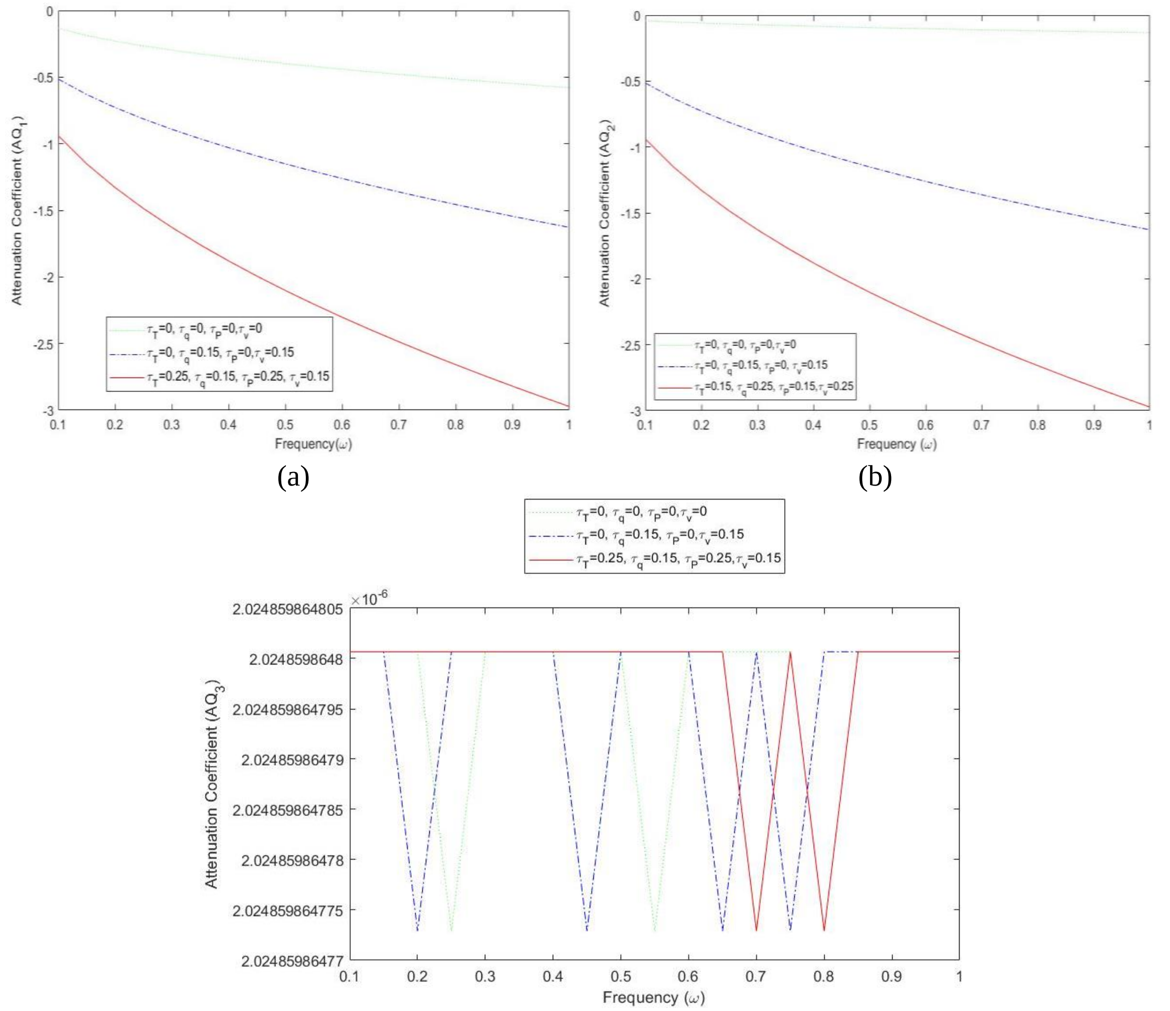
(a) (b)

(c)

Fig. 3 Effect of phase lag parameters (τ_q, τ_T, τ_P and τ_v) on phase velovities (PV_1, PV_2, and PV_3) of longitudinal waves

Fig. 1 shows the effect of non local parameters on phase velocities PV_1, PV_2 and PV_3 of longitudinal waves. The variation and behavior of PV_1 wave with non local parameters (ξ, ζ and ς) increases monotonically with frequency. As the value of non-local parameter increases the value of PV_1 also increases. Also the value of PV_1 in the absence of non local parameters denoted by green dotted line (....) gets decreased in comparison with three non local parameters (in the absence). The velocity PV_2 display the similar behavior and variation in absence and presence of non local parameters as shown in Fig. 1(a). However, the magnitude of PV_1 and PV_2 are distinct. These variations are depicted in Fig. 1(b). Fig. 1(c) displays the variation in phase velocity PV_3 along the frequency (ω) and it is seen that its value gets increased exponentially as frequency increases also the magnitude of PV_3 increases as the value of non local parameters increases. From Fig. 1(b) and 1(c) it is commonly seen that although the behavior appears to be similar but the magnitude value are distinct with the increase in the frequency (ω).

Fig. 2 shows the effect of non local parameters on attenuation coefficients AQ_1, AQ_2 and AQ_3



(a) (b) (c)

Fig. 4 Effect of phase lag parameters ($\tau_q,\ \tau_T, \tau_P$ and τ_v) on attenuation coefficients ($AQ_1,\ AQ_2,$ and AQ_3) of longitudinal waves

of longitudinal waves. Fig. 2(a) shows that the attenuation coefficient AQ_1. It is seen that in the absence of non local parameters the value gets sharply decreased. Although with the increase of three non local parameters the value of coefficient decrease with a lesser magnitude. Fig. 2(b) displays the trend of AQ_2 with frequency (ω). It appears that the trend is similar to AQ_1 but the magnitude value are quite distinct. Fig. 2(c) display a contrast behavior of oscillation to AQ_1. The value of AQ_3 gets decreased as the value of non local parameters increases and there is contrast difference in the magnitude value of AQ_3 in absence and presence of non local parameters.

Fig. 3 shows the effect of phase lag parameters ($\tau_q,\ \tau_T,\ \tau_v$, and τ_P) on phase velocities PV_1, PV_2 and PV_3 of longitudinal waves. Due to all phase lag parameters the magnitude of PV_1 and PV_2 get decreased in contrast to single phase lag and without phase lag parameters. Not much effect is noticed on PV_3.

Fig. 4 shows the effect of phase lag parameters on attenuation coefficients AQ_1, AQ_2 and AQ_3. It is seen that in Fig. 4(a) and 4(b) the magnitude of AQ_1 and AQ_2 get decreased for all phase parameters in contrast to the single and with phase lag parameters, whereas AQ_3 shows oscillatory variation.

6. Conclusions

In order to analyze the behavior of the NL DPL model, we investigated basic theorem in the form of the fundamental solution. Also the effects are examined on the basic characteristics of the wave, i.e., on phase velocity and attenuation coefficients. The problem has significant and practical meaning for structure or device with micro-scale subjected to transient response. On the basis of the above study it is possible:

1. to construct the surface and volume potentials in the considered theory and to establish their basic properties;
2. to investigate 3D boundary value problems (BVPs) of the linear theory of bio-thermoelasticity by means of the potential method (boundary integral equation method) and the theory of 2D singular integral equations;
3. to obtain the numerical solutions of the BVPs by using the boundary element method and the method of fundamental solutions; and
4. to construct the explicit solutions of the BVPs for the special cases of 3D domains (sphere, halfspace, etc.)
5. Phase velocity and attenuation quality factor with local phase remain smaller in comparison to the increase of non local parameters depicting the effect of non local parameter on the physical characteristics of wave.
6. Phase velocity PV_1, PV_2 and attenuation quality factor AQ_1, AQ_2 in case of single phase lag remains between the range of without and dual phase lag which shows the impact of dual phase lag.
7. Magnitude values of $PV1$, $PV2$ remains more in case of without phase lag in comparison with single phase and dual phase lag parameters whereas for PV_3 negligible effect is noticed while depicting the response of phase lag parameters on the velocities.
8. The values of AQ_1, AQ_2 and AQ_3 for single phase lag are in the intermediate range of without phase lag. It is also apparent for AQ_1 and AQ_2 the values are similar in case of without phase lag whereas for AQ_3 the value is more.

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