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Effect of boundary conditions on harmonic response of laminated plates

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Abstract. This work presents a modified Fourier-Ritz approach for first time is used to study dynamic transverse response of laminated plates with different boundary conditions based on classical plate's theory. The transverse displacement component of the plate is represented by Fourier series which is modified by adding auxiliary functions to cosine series so as to accelerate the convergence of the series and the solution, proposed by (W.L. Li, Journal of Sound and Vibration, 273, 619–635, 2004) is *corrected in present work*. Different boundary conditions, types of lamination cross and angle ply, material types, range of force frequency and thickness schemes, are investigated flexibly and the results are in good agreement with those obtained by other solution techniques.

Keywords: forced vibration; Ritz method; laminated plates; general boundary conditions

1. Introduction

Composite laminated plates are important structural element and are widely used in many engineering applications, as a result they may work under severe conditions such as dynamic loading and or with different support conditions; therefore, it is of great significance for design, to investigate the response of laminated plates with general support conditions in practical structure designs.

Analytical, semi-analytical, numerical methods and experimental techniques were developed to investigate dynamic behavior of plates such as Naviers solution, Rayleigh-Ritz method and finite element technique (FEM). Superposition method used for free vibration problem has been modified by Gorman and Singhal (2009) to handle dynamic response of cantilever plates of two different geometries to a harmonically excited base with range of frequencies, while Henry Khov *et al.* (2009), extended Li *et al.* investigation on isotropic beams and plates, to study static and vibration free analyses of orthotropic plates with general elastic boundary supports. Rahbar Ranji and Rostami Hoseyn Abadi (2010) presented Kantorovich method to obtain the response of orthotropic plates subjected to harmonic load with different combinations of boundary conditions. Trigonometric Ritz method (TRM) is used by Dozio (2011) to investigate vibration problem of rectangular orthotropic Kirchhoff plates with different load and edges conditions, natural frequencies of the plate are

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obtained using functions, originally developed by Beslin and Nicolas (Journal of Sound and Vibration 1997, 202, 633-655). Hu *et al.* (2011) analyzed free vibration and harmonic forced vibration of orthotropic plate with different boundary conditions using a unified method by which only the fundamental equations are used, therefore the trial function is not selected anymore and gave a more flexible method compared with traditional semi-inverse method. Ritz method is used by Hashemi and Fazeli (2012) to obtain natural frequencies of a fiber reinforced Mindlin plate. Seyyed *et al.* (2013) employed method of eigenfunction expansion in elliptic coordinates based on CLP to obtain an exact response in time-domain of a thin elastically supported elliptical plate. Gupta *et al.* (2014) obtained response of non-homogeneous rectangular plate of variable thickness which varied linearly, to uniform distributed harmonic lateral load using CLP and for boundary condition SFSF, Mishra (2015), involved extensive experimental works to investigate the free vibration of industry driven woven fiber glass/epoxy composite plates with different boundary conditions including free–free cases. Maithry and Chanra Mohan Rao (2016), carried out dynamic response of the laminated composite plates with holes subjected to excitations, varying arbitrarily with time. Static, Modal and Transient dynamic analysis of laminated composite plates, simply supported at all the edges has been obtained.

Landge and Prajapati (2017), investigated mode shapes, their frequencies and harmonically excited displacement of simply supported laminated plates using F.E.M program ANSYS15. Zhang (2017) performed the mechanical behavior of laminated CNT-reinforced quadrilateral composite due to a sudden transverse dynamic load is carried out using the IMLS-Ritz method based on first-order shear deformation theory. Adhikari1 and Singh (2017), proposed finite element technique based on a new simple Quasi 3-D displacement field to obtain response of laminated plate under deferent time dependent loads.

Gliszczynski *et al*. (2018), investigated effect of impact load with low velocity in thin-walled plates experimentally and compared with the theoretical model (one degree of freedom mass-spring system), for Glass Fiber Reinforced Polymer (GFRP) laminate with a quasi-isotropic, quasi-orthotropic and angle ply plate with different size and shape as a function of boundary conditions, layer arrangements and impact energy. Prasad and Sahu (2018), analyzed free vibration of fiber-metal-laminated (FML) plates, a new aircraft material using finite element method and experimental program, effect of many parameters such as side-to-thickness ratio, ply orientation, and boundary conditions on the dynamic behavior investigated. Vescovini (2018) analyzed free vibration and buckling analysis of composite plates using Ritz method. Focus is given on the choice of the trial functions in relation to the degree and the kind of anisotropy exhibited by the plates. Qin *et al.* (2019) obtained free vibration behavior of laminated rectangular plate using Jacobi-Ritz method with arbitrary boundary conditions, based on first-order shear deformation theory (FSDT)

Different plates theories are developed and used by researchers, Meksi *et al*. (2019) introduced, a new shear deformation plate theory is to illustrate the bending, buckling and free vibration responses of functionally graded material sandwich plates. A new displacement field containing integrals is proposed which involves only four variables also Hellal *et al*. (2019) proposed a new simple “four-variable shear deformation” plate model to demonstrate the hygro-thermal environment effects on dynamic and buckling of functionally graded material “sandwich plates” supported by “Winkler–Pasternak” elastic foundations. The proposed model uses only “four variables” and considers trigonometric variation of “transverse shear stress, and Sahla *et al*. (2019) developed, a simple four-variable trigonometric shear deformation model with undetermined integral terms to obtain the dynamic response of anti-symmetric laminated composite and soft core sandwich plates. Unlike the existing higher order theories, the current one contains only four unknowns while Tounsi *et al*. (2020) proposed, a simple four-variable trigonometric integral shear deformation model for the static behavior of advanced functionally graded (AFG) ceramic-metal plates supported by a two-parameter elastic foundation and subjected to a nonlinear hygro-thermo-mechanical load but Chaabane *et al*. (2019), used a hyperbolic shear deformation theory (HySDT) to study the static and dynamic behaviors of simply supported resting on the elastic foundation (Winkler-Pasternak types)functionally graded beams (FGB).

Bourada *et al*. (2019) studied free vibration of simply supported perfect and imperfect (porous) FG beams using a high order trigonometric deformation theory, unlike other theories, the number of unknown is only three. Khiloun *et al*. (2020) presented an efficient and original high-order shear and normal deformation theory for the static and free vibration analysis of functionally graded plates. Unlike any other theory, the number of unknown functions involved in displacement field is only four, as against five or more in the case of other shear and normal deformation theories. Kaddari *et al*. (2020) studied free vibration of functionally graded porous plates resting on elastic foundations based on a new type of quasi-3D hyperbolic shear deformation theory and Boutaleb *et al*. (2019) obtained the dynamic analysis of the functionally graded rectangular nanoplates using theory of nonlocal elasticity based on the quasi 3D high shear deformation theory (quasi 3D HSDT), in HSDT a cubic function is employed in terms of thickness coordinate to introduce the influence of transverse shear deformation and stretching thickness also Addou *et al*. (2019) investigated the effect of Winkler/Pasternak/Kerr foundation and porosity on dynamic behavior of FG plates using a simple quasi-3D hyperbolic theory which considers only four-unknown variables to determine the four coupled vibration responses (axial-shear-flexion-stretching) while Boulefrakh *et al*. (2019) employed, a simple quasi 3D hyperbolic shear deformation model for bending and dynamic behavior of functionally graded (FG) plates resting on visco-Pasternak. Boukhlif *et al*. (2019) presented a dynamic investigation of functionally graded (FG) plates resting on elastic foundation using a simple quasi-3D higher shear deformation theory (quasi-3D HSDT) in which the stretching effect is considered , also the kinematic is defined with only 4 unknowns, which is even lower than the first order shear deformation theory (FSDT). The elastic foundation is included in the formulation using the Pasternak mathematical model. Zaoui *et al*. (2019), established a two dimensional (2D) and quasi three dimensional (quasi-3D) shear deformation theories, which can model the free vibration of FG plates resting on elastic foundations (Pasternak (two-parameters)) using a new shear strain shape function. The proposed theories have a novel displacement field which includes undetermined integral terms and contains fewer unknowns with taking into account the effects of both transverse shear and thickness stretching.

Belbachir *et al*. (2019) addressed a refined plate theory in order to obtain the response of anti-symmetric cross-ply laminated plates subjected to a uniformly distributed nonlinear thermo-mechanical loading. In this theory, the undetermined integral terms are used and the variables number is reduced to four instead of five or more in other higher-order theories also Balubaid *et al*. (2019) used nonlocal two variables integral refined plate theory to study free vibrational behavior of the simply supported FG nano-plate while Karami *et al*. (2019), analyzed the size-dependent wave propagation analysis of functionally graded (FG) anisotropic nanoplates based on a nonlocal strain gradient refined plate model.

Medani *et al*. (2019) investigated static and dynamic behavior of Functionally Graded Carbon Nanotubes (FG-CNT)-reinforced porous sandwich (PMPV) polymer plate, based on the first order shear deformation theory (FSDT) and Bousahla *et al*. (2020) developed a novel integral first order shear deformation theory the to investigate buckling and vibrational behavior of the composite beam armed with single-walled carbon nanotubes (SW-CNT) resting on Winkler-Pasternak elastic foundation are. The current theory contains three variables and uses the shear correction factors.

Admissible functions play a critical role in the Rayleigh–Ritz method, from above literature many researches are investigated free and forced vibration of laminated plates using different displacement functions, in present work a modified Fourier-Ritz approach for first time is used to study dynamic transverse response of laminated plates with different boundary conditions proposed by W.L. Li (2004) and *corrected in present work.*

2. Theoretical analysis

Based on **(CLPT)** assumptions (neglecting extension-bending and twisting-bending stiffness coupling terms), equation of motion is written as follows, Reddy (2004)

$$D_{11}\frac{\partial^4 w}{\partial x^4}+(2D_{12}+4D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial x^4}=-I_o\frac{\partial^2 w}{\partial t^2}+I_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \qquad (1)$$

Where D_{ij} are bending stiffness elements and $I_{0,2}$ are mass moment of inertia.

3. Boundary conditions

The twisting moments and bending, transversal shear forces can be written in terms of the displacement function as, Li (2004)

$$M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}\,,\ M_y=-D_{22}\frac{\partial^2 w}{\partial y^2}-D_{12}\frac{\partial^2 w}{\partial x^2} \qquad (1,2)$$

$$M_{xy}=-2D_{66}\frac{\partial^2 w}{\partial x\partial y}\ ,\ Q_x=-D_{11}\frac{\partial^3 w}{\partial x^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial y^2\partial x} \qquad (3,4)$$

$$Q_x=-D_{22}\frac{\partial^3 w}{\partial y^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial x^2\partial y} \qquad (5)$$

For a flexibly restricted rectangular plate shown in Fig. 1, the boundary conditions are

$$k_{x0}w=Q_x\ ,\ K_{x0}\frac{\partial w}{\partial x}=-M_x\ \text{ at x=0} \qquad (7\text{-}8)$$

$$k_{x1}w=-Q_x\ ,\ K_{x1}\frac{\partial w}{\partial x}=-M_x\ \text{ at x=a} \qquad (9\text{-}10)$$

$$k_{y0}w=Q_y\ ,\ K_{y0}\frac{\partial w}{\partial y}=-M_y\ \text{ at y=0} \qquad (11\text{-}12)$$

$$k_{y1}w=-Q_y, K_{y1}\frac{\partial w}{\partial y}=-M_y\ \text{ at y=b} \qquad (13\text{-}14)$$

Where k_{y0}, k_{y1} and k_{x0}, k_{x1} are the transitional stiffness of spring,$K_{y0}, K_{y1}$and K_{x0}, K_{x1} are the rotations stiffness of spring at plate adages. Eqs. (7)-(14) define different supporting conditions by simply putting the stiffness of spring equalize to a very small or large number. From Eqs. (7-14), can be finally written as

$$k_{x0}w=-D_{11}\frac{\partial^3 w}{\partial x^3}-(D_{12}+4D_{66})\frac{\partial^3 w}{\partial y^2\partial x}\ ,\ k_{x1}w=D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+4D_{66})\frac{\partial^3 w}{\partial y^2\partial x} \qquad (15,16)$$

$$K_{x0}\frac{\partial w}{\partial x}=D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2}\ ,\ K_{x1}\frac{\partial w}{\partial x}=D_{11}\frac{\partial^2 w}{\partial x^2}+D_{12}\frac{\partial^2 w}{\partial y^2} \qquad (17,18)$$

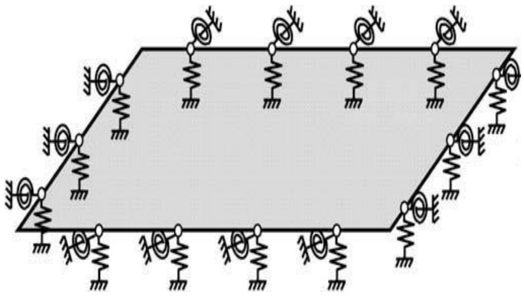


Fig. 1 Elastic support of rectangular plate along edges Li (2004)

And similarly found other four equations in the *y* direction.

4. Total mechanical energy

Response of laminated plate to harmonic sinusoidal distributed load is investigated in present work using Ritz method, so energy expression based on CLP theory is, Reddy (2004)

$$E = U + K + W \tag{19}$$

Where: U=Total potential energy of a plate, E=Total mechanical energy of a plate, K=Total kinetic energy of a plate, W=Total work done by harmonic force, which are defined as

$$U = \frac{1}{2}\int_0^b\int_0^a \left[D_{11}\left(\frac{\partial^2 w}{\partial^2 x}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial^2 y}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] dx\,dy + \frac{1}{2}\int_0^b\left[k_{x0}w^2 + K_{x0}\left(\frac{\partial w}{\partial x}\right)^2\right]_{x=0} dy + \frac{1}{2}\int_0^b\left[k_{x1}w^2 + K_{x1}\left(\frac{\partial w}{\partial x}\right)^2\right]_{x=a} dy + \frac{1}{2}\int_0^a\left[k_{y0}w^2 + K_{y0}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=0} dx + \frac{1}{2}\int_0^a\left[k_{y1}w^2 + K_{y1}\left(\frac{\partial w}{\partial y}\right)^2\right]_{y=b} dx \tag{20}$$

$$W = \frac{1}{2}\int_0^b\int_0^a q_o w\,dx\,dy \tag{21}$$

And

$$K = \frac{1}{2}\Omega^2 \iint I_o w^2\,dx\,dy \tag{22}$$

Where: q_o the amplitude of the load, for harmonic excitation, we assume a harmonic response

$$w(x, y, t) = w(x, y)e^{i\Omega t} \tag{23}$$

Where: $w(x, y, t)$ is the dynamic displacement, Ω is the frequency of applied force.

5. Admissible functions

In Ritz method the allowable functions play an important part. If the beam function is expanded in y-direction also, then plate function can be expressed as the product of beam functions in x and y direction and the result is Li (2004)

$$w(x,y)=\sum_{m,n=1} A_{mn}X_m(x)\,Y_n(y) \tag{24}$$

Where $X_m(x)$ and $Y_n(y)$ are the displacement functions for beams. A developed Fourier series function for beam with general support, is written as Li (2000)

$$w(x)=\sum_{m=0}^{\infty} a_m \cos\lambda_{am}x + p(x) \quad ,\left(\lambda_{am}=\frac{m\pi}{a}\right), \qquad 0\le x\le a. \tag{25}$$

Where *p* (*x*) in Eq. (25), is constantly selected to satisfy the boundary equations as following

$$P'''(0)=W'''(0)=\alpha_{0,} \qquad , \quad P'''(a)=W'''(a)=\alpha_{1,} \tag{26-27}$$

$$P'(0)=W'(0)=\beta_{0,} \quad and \qquad P'(a)=W'(a)=\beta_{1,} \tag{28-29}$$

To compensate the discontinuities of the displacement function and its derivative in adage points, *p* (*x*) is added. Thus, it can be chosen in several required forms such as a polynomial function Li (2004)

$$P(x)=\sum_{n=0}^{4} C_n\, P_n\left(\frac{x}{a}\right), \tag{30}$$

It is clear that the function *p* (*x*) must be at least a 4th polynomial to satisfy Eqs. (26)-(29), it is written in matrix form as

$$\text{P(x)} = \zeta_a(x)^T\bar{\boldsymbol{\alpha}} \quad \text{, where:} \qquad \bar{\boldsymbol{\alpha}}=\{\alpha_0,\alpha_1,\beta_0,\beta_1\}^T \tag{31,32}$$

And

$$\zeta_a(x)^T=\begin{Bmatrix} -(15x^4-60ax^3+60a^2x^2-8a^4)/360a \\ (15x^4-30a^2x^2+7a^4)/360a \\ (6ax-2a^2-3x^2)/6a \\ (3x^2-a^2)/6a \end{Bmatrix} \tag{33}$$

The results in Eqs. (31)-(33) are already derived from a simpler approach Li (2000). So as to obtain the support constants, $\alpha_0,\alpha_1,\beta_0$, and β_1, substitution of Eqs. (25) and (31) into the supporting conditions Eqs. (15)-(18) results in:

$$\bar{\boldsymbol{\alpha}}=\sum_{m=0}^{\infty} H_a^{-1}Q_{am}a_m \tag{34}$$

Where:

$$H_a=\begin{bmatrix} 1+\frac{8k_{x0}a^3}{360D_{11}} & \frac{7k_{x0}a^3}{360D_{11}} & \frac{-k_{x0}a}{3D_{11}} & \frac{-k_{x0}a}{6} \\ \frac{7k_{x1}a^3}{360D_{11}} & 1+\frac{8k_{x1}a^3}{360D_{11}} & \frac{-k_{x1}a}{3D_{11}} & \frac{-k_{x1}a}{6} \\ \frac{a}{3} & \frac{a}{6} & \frac{K_{x0}}{D_{11}}+\frac{1}{a} & \frac{-1}{a} \\ \frac{a}{6} & \frac{a}{3} & \frac{-1}{a} & \frac{K_{x1}}{D_{11}}+\frac{1}{a} \end{bmatrix} \tag{35}$$

And

$$Q_{am}=\left\{(-1)\frac{k_{x0}}{D_{11}} \quad (-1)^m\frac{k_{x1}}{D_{11}} \quad -\lambda_{am}^2 \quad (-1)^m\lambda_{am}^2\right\}^T \tag{36}$$

It must be reminded that for a totally free beam, matrix Ha will become singular, this can be solved by artificially setting stiffness of springs with smallest value at boundary of a beam Li (2000), however, the characteristic functions are very suitable and can be easily used as the allowable functions in Ritz technique. Finally Eq. (25) became as

$$w(x) = \sum_{m=0}^{\infty} a_m \varphi_m^a(x) \text{ , where: } \varphi_m^a(x) = \cos \lambda_{am} x + \zeta_a(x) H_a^{-1} Q_{am} \tag{37,38}$$

As mentioned above, Eq. (24) can be consequently rewritten as

$$w(x,y) = \sum_{m,n=0}^{\infty} A_m \varphi_m^a(x) \varphi_n^b(y) \text{ , where: } \varphi_n^b(y) = \cos \lambda_{bn} y + \zeta_b(y) H_b^{-1} Q_{bn} \tag{39,40}$$

Eq. (39) is derived in present work and present the corrected form of modified Fourier function that proposed by Li (2004), the terms for $\zeta_b(y), H_b and\ Q_{bn}$ can be, correspondingly, obtained from Eq. (38) by just changing the x-concerning parameters by the y- concerning.

6. Response of plate to harmonic force

To calculate the response of plate under harmonic excitation force, the derived Fourier function in above article is substituted in Eqs. (20), (21) and Eq. (22), differentiations and integrations are required and then by using Ritz method (minimizing energy), the following equations are obtained

$$\frac{\partial E}{\partial A_{mn}} = 0 \tag{41}$$

Eq. (41) will give a set of linear homogenous equations, from which stiffness and mass matrices are obtained, and then get the response from following equation.

$$\left(K - \Omega^2 M\right) A_{mn} = Q \tag{42}$$

Where $M, K\ and\ Q$ Are the generalized mass matrix, stiffness matrix and forcing matrix, A_{mn} the amplitude to be determined, by solving Eq. (41), the amplitudes A_{mn} can be determined. Then from Eq. (39), the response of the rectangular plate can be finally obtained. While equating Eq. (42) to zero will lead to an Eigen-value problem as flowing

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,(m*n)} \\ \vdots & \ddots & \vdots \\ a_{(m*n),1} & \cdots & a_{(m*n),(m*n)} \end{bmatrix} \begin{Bmatrix} A_{11} \\ \vdots \\ A_{mn} \end{Bmatrix} = 0 \tag{43}$$

Where a_{ij} are the coefficients of the nonzero unknowns A_{mn}. Finding the determinant of Eq. (43) will lead to get the natural frequency ω.

7. Results and discussion

Cross and angle-ply (symmetric and anti-symmetric) rectangular plate under various edge conditions are analyzed and their mode shapes, natural frequencies and response to harmonic excitation are evaluated using Ritz method. In present study MATLAB R2015a is used to solve the response of harmonic force. Validity of the derived equations is examined for free and forced vibrations response of laminated plates by comparing present results with those obtained using program ANSYS (R15), they show good agreement.

Table 1 Dimensionless natural frequency, for square plates

[30 -30 30] plates of different boundary conditions, ($E_1/E_2 = 2.45, G_{12} = 0.48E_2, \nu_{12} = 0.23$.			
References	SSSS	CCCC	SCSC
Present work	7.311	13.02	9.949
Ansys	7.237	13.04	9.913
$[45\ -45]_4$ plates ($E_1/E_2 = 10, G_{12} = 0.5E_2, \nu_{12} = 0.25$.			
References	SSSS	CCCC	SCSC
Present work	13.409	21.632	17.914
Ansys	13.111	21.165	17.533
[0 90 90 0] plates ($E_1/E_2 = 40, G_{12} = 0.5E_2, \nu_{12} = 0.25$).			
References	SSSS	CCCC	SCSC
Present work	18.817	41.216	38.668
Ansys	18.703	40.662	38.099

Table 2 Dimensionless natural frequency $\left(\overline{\omega} = \omega_o a^2\sqrt{\rho_c/E_2}/h\right)$, for (SSSS) [0 90]2s plates with effect of aspect and modulus ratios, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$)

References	*a/b*	E_1/E_2=10	25	40
Present work	0.5	8.875	13.652	17.147
Ansys		8.875	13.62	17.064
Present work	1	10.502	15.237	18.817
Ansys		10.48	15.176	18.702
Present work	1.5	14.364	19.748	23.954
Ansys		14.367	19.747	23.948

Table 3 Dimensionless natural frequency $\left(\overline{\omega} = \omega_o a^2\sqrt{\rho_c/E_2}/h\right)$, for [45 -45 45 -45] square plate with effect of modulus ratios, ($G_{12} = G_{13} = 0.6E_2, G_{23} = 0.5E_2, \nu_{12} = 0.25$)

Refs.	E_1/E_2	SSSS	SSSC	SSCC
Reddy 2004	2	7.02	8.39	10.24
Present work		7.09	8.47	9.67
Reddy 2004	10	12.54	14.43	16.90
Present work		13.40	15.41	17.22
Reddy 2004	20	17.02	19.43	22.53
Present work		18.49	21.09	23.47
Reddy 2004	30	20.53	23.37	27.00
Present work		22.46	25.54	28.37
Reddy 2004	40	23.53	26.73	30.83
Present work		25.82	29.33	32.55

Dimensionless natural fundamental frequency $\left(\overline{\omega} = \omega_o a^2\sqrt{\rho_c/E_2}/h\right)$, for different laminated plates schemes (cross and angle ply) with different material and boundary conditions are presented in Tables 1, 2 and 3, they show good agreement with those obtained by ANSYS and Reddy (2004) while verification for central displacement of plate under harmonic excitation are shown in Table 4, also fist fourth mode shapes for (SSSS) [30 -30 30] laminated square plate are shown in Fig. 2, which show that fundamental natural frequency for CCCC plates is larger than other boundary conditions and when orthotropy ratio increases (E_1/E_2) the frequency increases since they have larger stiffness.

Table 4 Comparison of displacement amplitudes for square isotropic plate under different harmonic excitation force $W(x_1,y_1)/p_o\, a^4 /D$

ω/ω_1	Present work	LAURA (1975) Galerkin	LAURA (1975) Exact
Simply supported plate			
0.3	0.0045332	0.0045	0.004473
0.5	0.0055003	0.0055	0.005448
0.8	0.0114591	0.0115	0.01145
Clamped plate			
0.3	0.00132919	0.00139	-
0.5	0.00177003	0.00170	-
0.8	0.003681969	0.00362	-

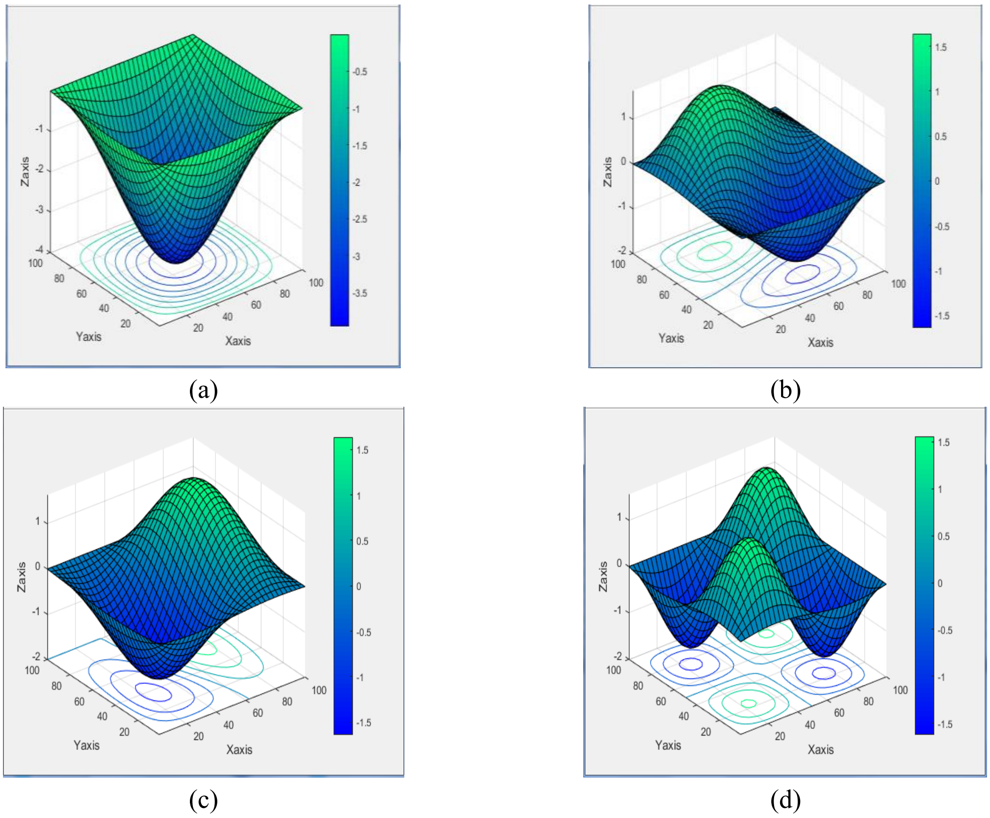
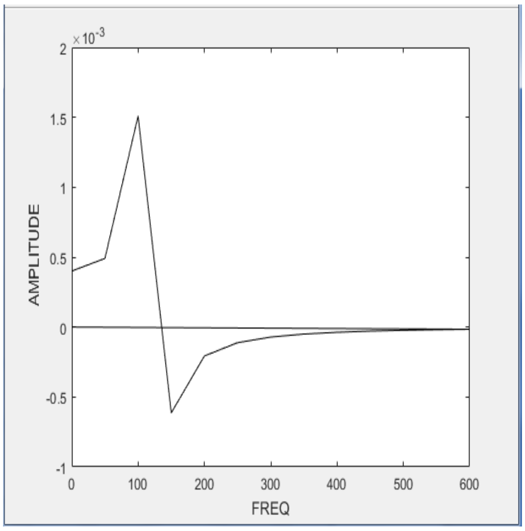
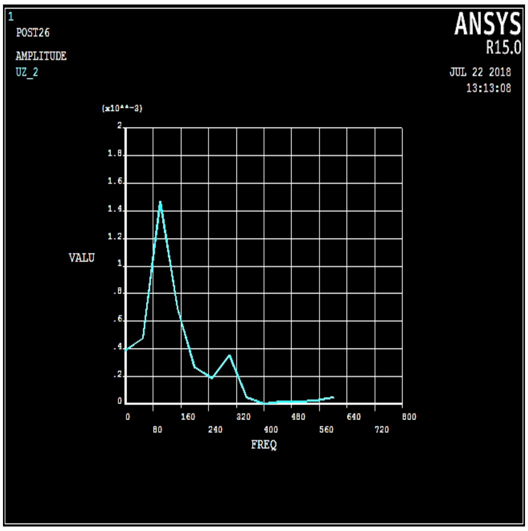


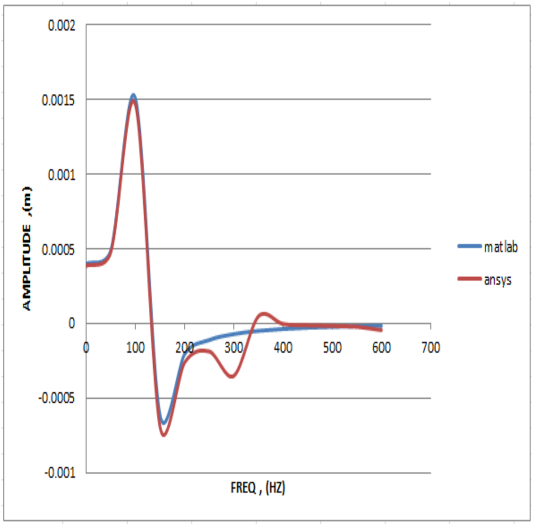
Fig. 2 Mode shape for free vibration of (SSSS) for [30 -30 30] laminated square plate (a) first (b) second (c) third (d) fourth modes



(a)



(b)



(c)

Fig. 3 Frequency - central deflection variation, for [0 90 90 0] plate with CCCC

The force frequency range is selected depending on natural frequencies of each plate. Maximum central displacements (*w*) for symmetric cross-ply [0 90 90 0] with different boundary condition as shown in Fig. 3 to Fig. 5, from which it's obvious that maximum displacement of this plate is when it is under CFCF and minimum displacement under CCCC since stiffness is larger for plate with these boundary. While, for anti-symmetric cross-ply$[0\ 90]_4$ the maximum central displacement is under SSSS and minimum central displacement is under CCCC as obtained in Fig. 6 to Fig. 8. It can be noted that the amplitudes are smaller for anti-symmetric cross ply than that for symmetric cross ply laminates for example all edges are clamped, (0.0001895 m, 0.0015 m) respectively. For harmonically excited plates, the material properties for this study are, E_1=180 Gpa, E_2=1 Gpa, v_{12}=0.28, v_{13}=0.28, v_{23}=0.35, G_{12}=7.3 Gpa, G_{13}=6.1 Gpa, G_{12}=6.1 Gpa, q_o=2000 Mpa, ρ=1600 kg.m^{-3}.

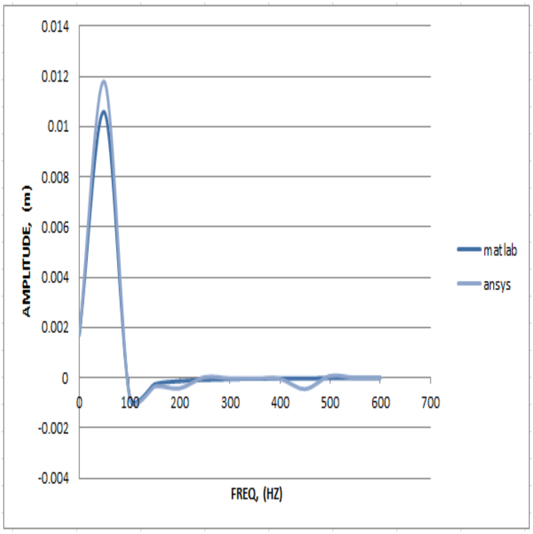


Fig. 4 Frequency - central deflection variation, for [0 90 90 0] plate with SSSS

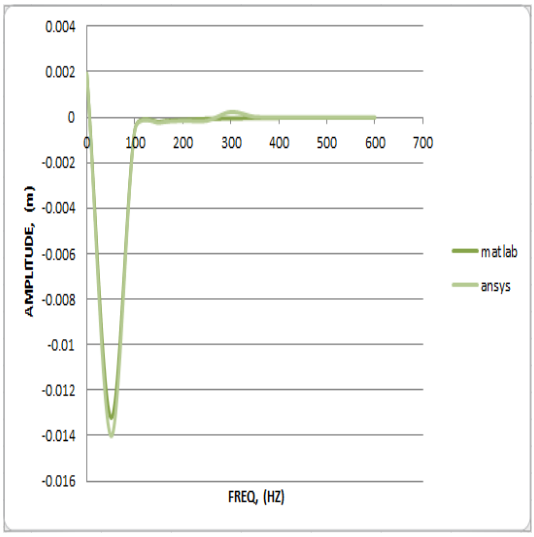


Fig. 5 Frequency - central deflection variation, for [0 90 90 0] plate with CFCF

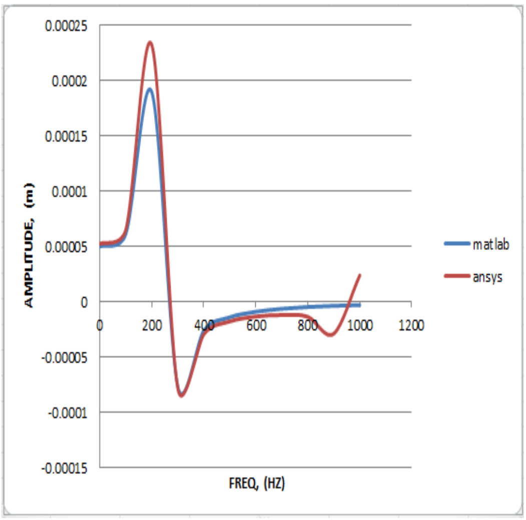


Fig. 6 Frequency -central deflection variation, for ($[0\ 90]_2$, h=0.02) plate with CCCC

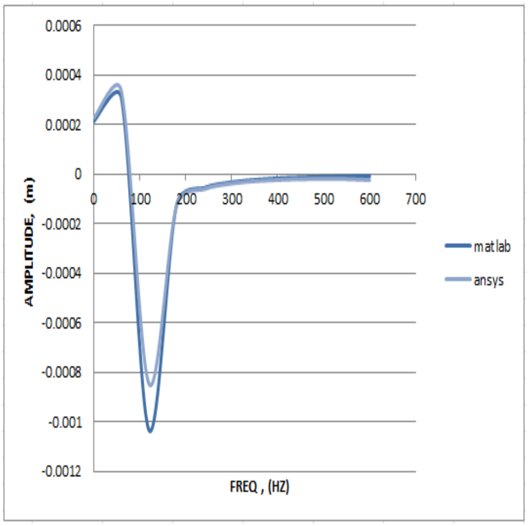


Fig. 7 Frequency-central deflection variation, for $[0\ 90]_4$ plate with SSSS

The discrepancy of results belongs to that, for ANSYS analysis based on plate first order shear deformation theory while for present work classical plate theory is used, also different presentation for boundary in present work, spring with stiffness that given appreciate value for each boundary type i.e., for clamped boundary all spring stiffness=10^{12} while for free it is not zero but very small value 10^{-12}, maximum discrepancy for cross ply plate is for anti symmetric one with 18.37% and also for $[30\ -30]_4$ angle ply is 27.4%.

Maximum central displacements (w) for anti-symmetric angle-ply $[30-30]_4$ and $[-45\ 45]_4$ with different boundary conditions as shown in Fig. 9 to Fig. 11 and Fig. 12 to Fig. 14 respectively,

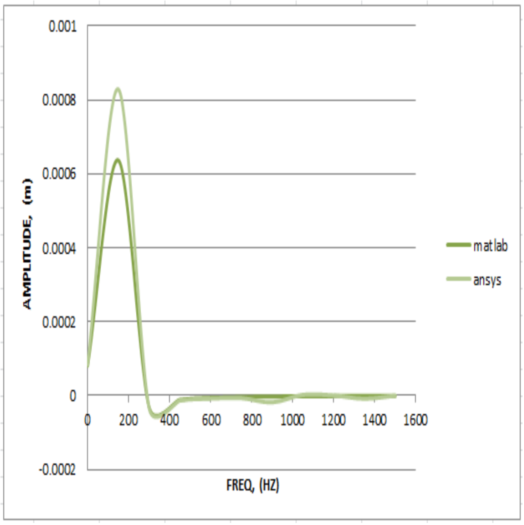


Fig. 8 Frequency-central deflection variation, for $[0\ 90]_4$ plate with CFCF

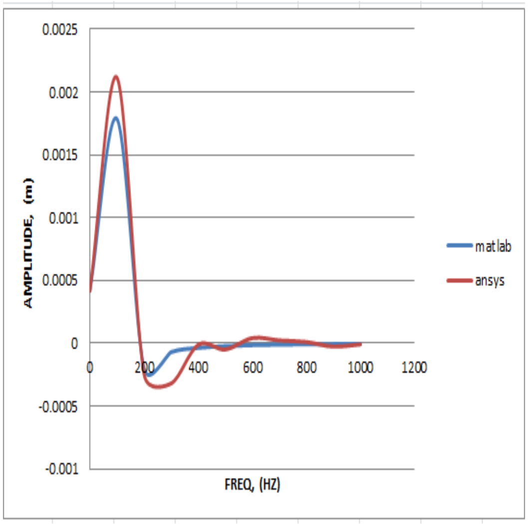


Fig. 9 Frequency-central deflection variation, for $([30-30]_4$ h=0.01) plate with CCCC

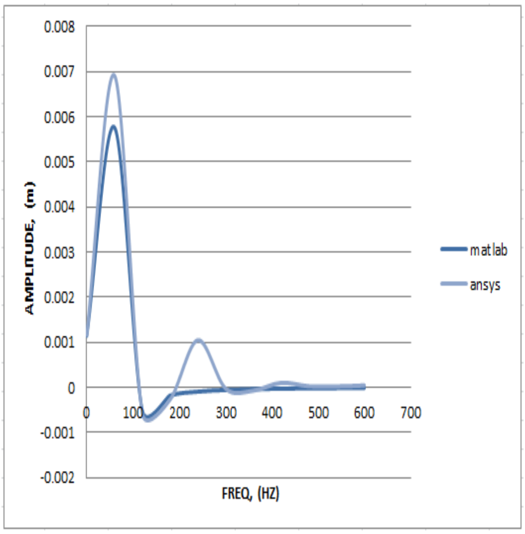


Fig. 10 Frequency-central deflection variation, for $([30-30]_4$ h=0.01) plate with SSSS

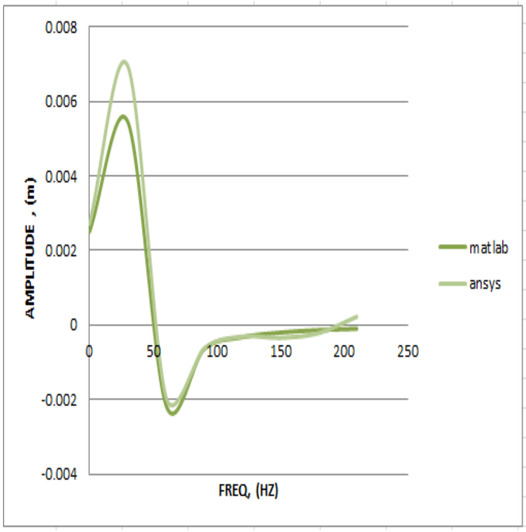


Fig. 11 Frequency-central deflection variation, for $([30-30]_4$ h=0.01) plate with CFCF

from which it's obvious that maximum displacement of these plates is under for [$\pm\theta$=30^o] simply supported condition and minimum displacement is under the clamped edge condition for [$\pm\theta$=45^o] since they have larger stiffness.

8. Conclusions

Natural frequency and response of laminated plate under harmonic load with different boundary

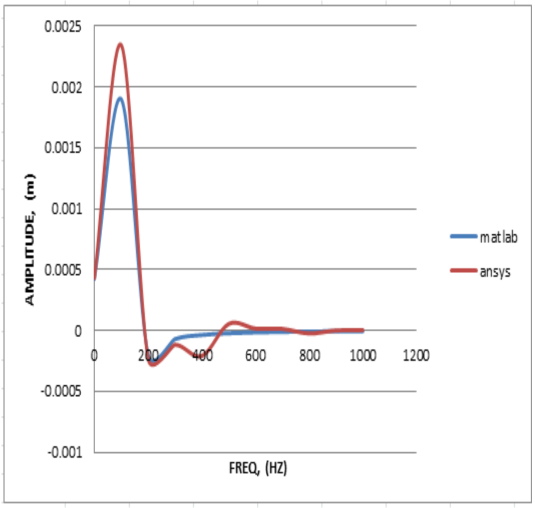


Fig. 12 Frequency-central deflection variation, for ($[-45\ 45]_4$, h=0.01) plate with CCCC

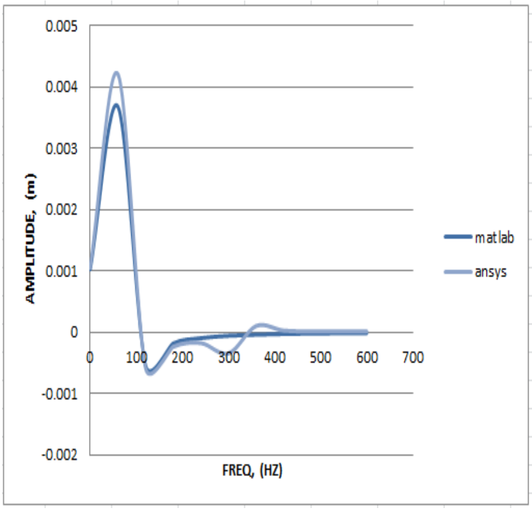


Fig. 13 Frequency-central deflection variation, for ($[-45\ 45]_4$, h=0.01) plate with SSSS

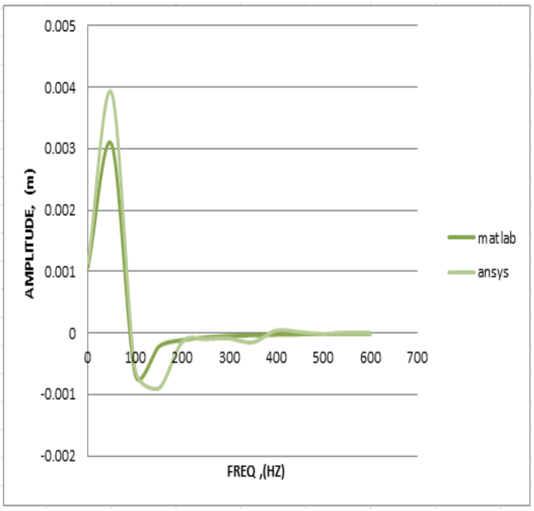


Fig. 14 Frequency-central deflection variation, for ($[-45\ 45]_4$, h=0.01) plate with CFCF

conditions, number of layers and symmetric and anti symmetric cross and angle ply response is obtained using Ritz approach. One displacement modified Fourier function is used to study dynamic transverse response of laminated plates for different boundary conditions.

The results show that the present method enables rapid convergence and good accuracy. Present work can be used for different types of supporting conditions, such as all the classical cases and their combinations without changing solution procedure and function of transverse displacement with changing the support type.

If compared with most existing methods, developed solution is more flexible and less complicated from other solution methods such as Levy or F.E.M to get response of laminated plate with different boundary conditions.

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