Biomaterials and Biomechanics in Bioengineering, *Vol. 2, No. 4 (2015) 207-223*
DOI: http://dx.doi.org/10.12989/bme.2015.2.4.207

Efficient non-linear analysis and optimal design of biomechanical systems

I. Shojaei[1a], A. Kaveh[*2], H. Rahami[3b] and B. Bazrgari[1b]

[1]*Department of Biomedical Engineering, University of Kentucky, Lexington, KY, USA*
[2]*Centre of Excellence for Fundamental Studies in Civil Engineering, Iran University of Science and Technology, Tehran, Iran*
[3]*School of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran*

(Received October 31, 2014, Revised March 26, 2015, Accepted April 24, 2015)

Abstract. In this paper a method for simultaneous swift non-linear analysis and optimal design/posture of mechanical/biomechanical systems is presented. The method is developed to get advantages of iterations in non-linear analysis and/or generations in genetic algorithm (GA) for the purpose of efficient analysis within the optimal design/posture. The method is applicable for both size and geometry optimizations wherein material and geometry non-linearity are present. In addition to established mechanical systems, the method can solve biomechanical models of human musculoskeletal system. Optimization-based procedures are popular methods for resolving the redundancy at joints wherein the number of unknown muscle forces is far more than the number of equilibrium equations. These procedures involve optimization of a cost function(s) which is assumed to be consistent with the central nervous system's strategy when activating muscles to assure equilibrium. However, because of the complexity of biomechanical problems (i.e., due to non-linear biomaterial, large deformation, redundancy of the problem and so on) efficient analysis are required within optimization procedures as suggested in this paper.

Keywords: optimal design; biomechanical systems; non-linear analysis; genetic algorithm

1. Introduction

Regarding economic considerations, single-objective optimization has been traditionally used for the design of structural/mechanical systems (Wang *et al*. 2003, Lee *et al*. 2004, Rahami *et al*. 2008). However, such single-objective algorithms fail to improve the performance and efficiency of the design (i.e., as secondary objectives). To account for these multiple objectives (or other customized objectives) in the design, multi-objective optimizations were well developed in literature (e.g., Konak *et al*. 2006, Marler *et al*. 2004, Kaveh *et al*. 2013). Classical methods of optimization are quick and accurate mathematical tools but solve only a limited number of simple problems. In engineering problems, one usually deals with complicated objective functions with

∗Corresponding author, Professor, E-mail: alikaveh@iust.ac.ir
[a]MSc.
[b]Ph.D.

http://www.techno-press.org/journals/bme&subpage=7 ISSN: 2288-3738 (Print), 2288-3746 (Online)

various linear and non-linear constraints such that the classical methods fail to solve the problem. Alternatively, heuristic algorithms have successfully been developed for complicated optimization problems (Dorigo *et al.* 1996). Though solving a wide range of optimization problems (i.e., the benefit over classical methods), heuristic algorithms find the optimal solutions using too much time-consuming iterations (the weakness of heuristic algorithms); especially in non-linear analyses wherein an iterative procedure is required in each iteration of optimization algorithm. Despite these difficulties in analysis, less works have been done in literature for efficient solutions. Swift solution of regular and near-regular structural/mechanical systems using graph products, group theoretical methods, sub-structuring methods and manipulating stiffness matrix were developed to decrease the computational complexity of the problems (e.g., Shojaei *et al.* 2015, Kaveh *et al.* 2015, Shojaei *et al.* 2013, Kaveh 2014, Zingoni 2002, Zingoni 2009, Zingoni 2014). Simple analysis of structures using modified neural networks for the prediction of behavior of structures under seismic loads was performed by Kaveh *et al.* (2013).

While optimization procedures were traditionally used in structural/mechanical systems for finding more cost-efficient solutions (i.e., as an important but optional part of the design), these procedures are the necessary part of the solution in many biomechanical problems. Neuromuscular optimization assumptions are employed for prediction of active behavior of muscle forces to resolve redundancy of the problem (Goel *et al.* 1993, Ezquerro *et al.* 2004, Stokes *et al.* 2001). Though such redundancy exists in many joints of human body, the spine analysis is a very relevant problem for the application of optimization procedure in a redundant system. A large number of muscles control the equilibrium, stability and motion of the spine. The developed active muscle forces of the spine are adjusted by complicated neuromuscular strategies which are not well known yet. However, it has been accepted that due to energy-efficiency requirements the estimated muscle forces should optimize a cost function(s). Heuristic algorithms are required to solve such a problem because the objective function and constraints are complicated. Furthermore, non-linear analyses should be performed due to non-linear biomaterial and large deformations of spine components. These result in very laborious and time-consuming procedures unless efficient approaches are employed within the solution procedure to reduce computational complexity of the problem.

In this paper, efficient nonlinear analysis of mechanical/biomechanical systems involved in optimal design/posture by GA is presented. A short review of the method for linear analysis is presented and then the main non-linear approach is developed in detail. Both material and geometry non-linearity are involved in the formulation of analysis and both size and geometry optimization are considered in the design. Finally, A biomechanical model of human spine is solved using the current method and time efficiency and optimal postures are presented.

2. Efficient linear analysis in optimal design/posture with GA

One critical issue in structural/mechanical design using GA is extreme and time-wasting computations. The most laborious phase in a GA is usually the solution of the governed equation. In structural/mechanical designs, solving the relationship $F = K\Delta$ and finding the results as $\Delta = K^{-1}F = DF$ is the main part of calculations. To design real-life systems such as huge buildings not only the stiffness matrix K is very large to be inverted but also the procedure should be repeated in each generation of a heuristic method like GA. In this section, efficient formulations are developed to decrease the computational complexity of mechanical/biomechanical problems in optimal design/postures by GA. Effective methods of solving the equation $F = K\Delta$ were previously presented using finite element formulation (Bathe 1996) and group theoretical methods (Zingoni 2002, Zingoni 2009, Zingoni 2014). Formulations for the swift analysis of arbitrary structural/mechanical systems with some changed members, supports and nodes were developed by our research group (Rahami *et al.* 2015). In the mentioned study, we assumed that an analyzed system is available (Eq. (1) and Eq. (2)) (i.e., the inverse of its stiffness matrix exists) and changes are applied to some members, supports or nodes (Eq. (3) and Eq. (4)) (i.e., rehabilitation). It was aimed to obtain a swift solution for the modified system using its available initial solution before the applied changes. An excellent application of the method is in the GAs wherein for each generation the solution of the previous generation is available (i.e., existing solution) and changes occur in the current generation (i.e., in some members (size) or in some nodes (geometry)) for obtaining the optimal design/posture. Although GAs are powerful tools for optimal design, numerous time-consuming analyses are required within the procedure. Alternatively, finding efficient solutions in GAs leads to the valuable simultaneous efficient analysis and optimal design. Efficient solution of mechanical/biomechanical models includes inverting the stiffness matrix in a swift way. Four inverted matrices $D_{modified}$ were previously formed for member, support and nodal modifications.

Suppose the solution of the original structure has the following pattern. The blocks corresponding to the members which should be modified, are put in a separated partition

$$\begin{bmatrix} \Delta_m \\ \Delta_n \\ \vdots \\ \dots \\ \Delta_k \\ \Delta_s \\ \vdots \end{bmatrix} = \begin{bmatrix} D_{mm} & D_{mn} & & \vdots & D_{mk} & D_{ms} & \\ & D_{nn} & & \vdots & D_{nk} & D_{ns} & \\ & & \ddots & \vdots & & & \ddots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & & \vdots & D_{kk} & D_{ks} & \\ & & & \vdots & & D_{ss} & \\ Sym & & & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} F_m \\ F_n \\ \vdots \\ \dots \\ F_k \\ F_s \\ \vdots \end{bmatrix} \tag{1}$$

In a more compact form, one can write

$$\begin{bmatrix} \Delta_I \\ \cdots \\ \Delta_{II} \end{bmatrix} = \begin{bmatrix} D_{I,I} & \vdots & D_{I,II} \\ \cdots & \vdots & \cdots \\ D_{II,I} & \vdots & D_{II,II} \end{bmatrix} \begin{bmatrix} F_I \\ \dots \\ F_{II} \end{bmatrix} \tag{2}$$

The stiffness matrix of the modified structure will have the following pattern

$$\begin{bmatrix} F_m \\ F_n \\ \vdots \\ \dots \\ F_k \\ F_s \\ \vdots \end{bmatrix} = \begin{bmatrix} K_{mm} + K'_{mm} & K_{mn} + K'_{mn} & & \vdots & K_{mk} & K_{ms} & \\ & K_{nn} + K'_{nn} & & \vdots & K_{nk} & K_{ms} & \\ & & \ddots & \vdots & & & \ddots \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & & \vdots & K_{kk} & K_{ks} & \\ & & & \vdots & & K_{ss} & \\ Sym & & & \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} \Delta'_m \\ \Delta'_n \\ \vdots \\ \dots \\ \Delta'_k \\ \Delta'_s \\ \vdots \end{bmatrix} \tag{3}$$

where the blocks K belong to the original structure and the blocks K' are associated with the modified ones due to applying the changes. In a more compact form one can write

$$\begin{bmatrix} F_I \\ \cdots \\ F_{II} \end{bmatrix} = \begin{bmatrix} K_{I,I} + K'_{I,I} & \vdots & K_{I,II} \\ \cdots & \vdots & \cdots \\ K_{II,I} & \vdots & K_{II,II} \end{bmatrix} \begin{bmatrix} \Delta'_I \\ \cdots \\ \Delta'_{II} \end{bmatrix} \tag{4}$$

In a structural/mechanical system with some altered members (Rahami *et al.* 2015), the inverted stiffness matrix will have the following form

$$D_{modified} = \begin{bmatrix} \bar{D}_{I,I} D_{I,I} & \vdots & \bar{D}_{I,I} D_{I,II} \\ \cdots & \vdots & \cdots \\ -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,I} + D_{II,I} & \vdots & -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,II} + D_{II,II} \end{bmatrix} \tag{5}$$

where $\bar{D}_{I,I} = [\, I + D_{I,I} K'_{I,I} \,]^{-1}$.

For systems with some support modifications (Rahami *et al.* 2015), the inverted stiffness matrices are obtained via the following equations

$$D_{modified} = \begin{bmatrix} D_{I,I} + D_{I,I} D_{I,I} K_{I,II} [K_{II,II} - K_{II,I} D_{I,I} K_{I,II}]^{-1} K_{II,I} & \vdots & -D_{I,I} K_{I,II} [K_{II,II} - K_{II,I} D_{I,I} K_{I,II}]^{-1} \\ \cdots & \vdots & \cdots \\ [K_{II,II} - K_{II,I} D_{I,I} K_{I,II}]^{-1} K_{II,I} D_{I,I} & \vdots & [K_{II,II} - K_{II,I} D_{I,I} K_{I,II}]^{-1} \end{bmatrix} \tag{6}$$

and

$$D_{modified} = D_{I,I} - D_{I,II} D_{II,II}^{-1} D_{II,I} \tag{7}$$

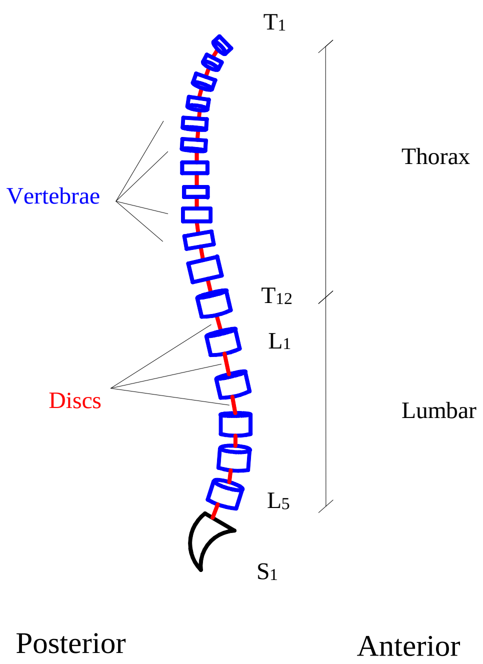

Fig. 1 The biomechanical model of spine

Lastly, for systems with some nodal changes (Rahami *et al.* 2015), one can write

$$D_{modified} = \begin{bmatrix} \bar{D}_{I,I} D_{I,I} & \vdots & \bar{D}_{I,I} D_{I,II} \\ \cdots & \vdots & \cdots \\ -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,I} + D_{II,I} & \vdots & -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,II} + D_{II,II} \end{bmatrix} \tag{8}$$

Now, consider a GA with a population of 40 individuals and a generation number equal to 30. In this optimization procedure 1200 stiffness matrices should entirely be inverted that is very arduous. Instead, the equations above are inserted into the GA to solve the problem. Normally, main operations in a GA consist of reproduction, crossover and mutation. There are different ways for performing any of these operations. In the reproduction operation better fit individuals are picked and duplicated. In the crossover operation, individuals are positioned in a matting pool to mate. In this operation, usually, two strings of the mating pool are selected and some parts of them are swapped to produce two new offsprings. In a single-point crossover it is done by selecting a random point of the parents' strings and exchanging the bits in the right side of the point. This is where the solution based on the modified formulations above is applied. Consider the biomechanical model of spine with 17 flexible discs shown in Fig. 1. Rotations of discs are considered as variables in the GA.

Suppose a string of 85-bits length is utilized as a chromosome for a set of rotational variables in the posture optimization (i.e., a gene of 5-bits length for each disc). Consider the following strings with a random point in the 12th bit

Parent 1: 01001 10110 10⋮010 11101 10010..... 01001 11101 01000 11100 10010

Parent 2: 10001 11000 00⋮111 10110 00110….. 11010 10011 11010 00110 10100

Therefore, the offsprings will be

Offspring 1: 10001 11000 00010 11101 10010….. 01001 11101 01000 11100 10010

Offspring 2: 01001 10110 10111 10110 00110….. 11010 10011 11010 00110 10100

It is observed the variable numbers 4 to 17 (black ones) are equal for parent 1 and offspring 1 and for parent 2 and offspring 2. Applying this concept to the corresponding postures leads to Fig. 2 and Fig. 3.

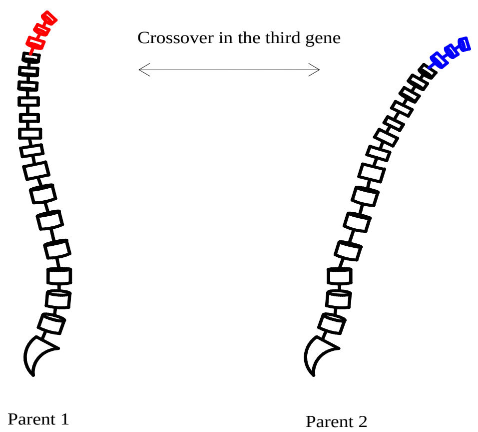

Fig. 2 The spinal postures corresponding to parents

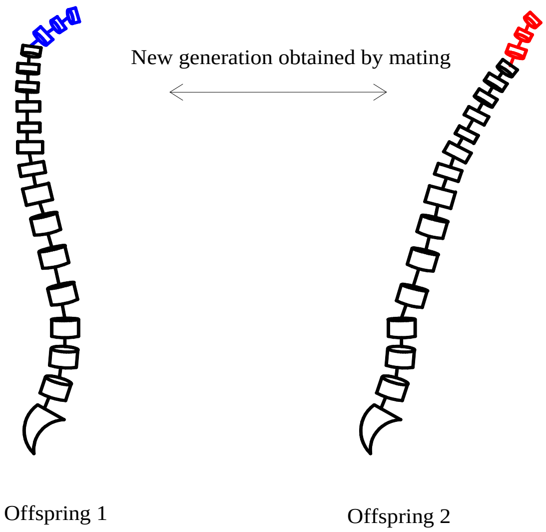

Fig. 3 The spinal postures corresponding to offsprings

The postures in Fig. 2 are original postures (previous generation) for the postures in Fig. 3 (current generation) as the modified new ones. Consequently, regarding Eq. (5), obtained for the solution of structures with some changed members, the solution of offsprings (postures with some changed rotations) is quickly found

$$\begin{bmatrix} \Delta'_I \\ \cdots \\ \Delta'_{II} \end{bmatrix} = \begin{bmatrix} \bar{D}_{I,I} D_{I,I} & \vdots & \bar{D}_{I,I} D_{I,I} \\ \cdots & \vdots & \cdots \\ -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,I} + D_{II,I} & \vdots & -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,II} + D_{II,II} \end{bmatrix} \begin{bmatrix} F_I \\ \cdots \\ F_{II} \end{bmatrix} \tag{9}$$

and

$$D_{modified} = \begin{bmatrix} \bar{D}_{I,I} D_{I,I} & \vdots & \bar{D}_{I,I} D_{I,II} \\ \cdots & \vdots & \cdots \\ -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,I} + D_{II,I} & \vdots & -D_{II,I} K'_{I,I} \bar{D}_{I,I} D_{I,II} + D_{II,II} \end{bmatrix} \tag{10}$$

The matrix $D_{modified}$ will itself be used as the available solution for the next generation where the offsprins in this step will play the role of parents in the subsequent generation. This process will be continued up to the final generation. More details about this section and efficiency of the method are discussed in Rahami *et al.* (2015). Same but much simpler procedure can be completed during the mutation operation wherein an arbitrary bit in a chromosome is changed from its original state. Mutation operation is associated with the change in only one rotational variable above (i.e., only one gene). Therefore, the changed gene will be corresponding to the block K' in Eq. (3) and Eq. (4). Due to the small change of the modified offspring relative to its corresponding parent during the mutation operations, the dimension of the block K' will be small such that the matrix $D_{modified}$ is calculated using less computations.

Optimal design of a mechanical system is associated with finding the optimal size and geometry of a structure such to minimize the material cost. Optimal posture of a biomechanical system is associated with obtaining a set of optimal kinematics which results in minimizing a cost function (i.e., sum of muscle forces, sum of muscles stress, squared sum of muscles stress and so on []). However, the procedure which is completed to define this optimal design/posture in the mechanical/biomechanical system is another aspect of the problem. During optimization procedure numerous iterations are completed to find the final design/posture. Each iteration of optimization algorithm itself involves an iterative non-linear analysis of the system that is very laborious. Therefore, while it is aimed to find the optimal design/posture, reducing the computational complexity of the solution (i.e., optimal analysis) is of interest. In the following sections, efficient non-linear analyses (i.e., optimal analyses) are presented within the optimal design/posture of mechanical/biomechanical systems.

3. Efficient non-linear (Geometric and material) solution of structural/mechanical systems used in GAs for optimal design

3.1 Nonlinear analysis of structural/mechanical systems

Newton-Raphson method is the usual method for non-linear analysis of a structural/mechanical system. The method starts with a linear solution

$$K_0 \Delta_0 = F_0 \tag{11}$$

where K_0, Δ_0 and F_0 are the initial stiffness matrix, initial displacement vector and external force vector.

The final goal of a non-linear structural/mechanical analysis is finding a displacement vector Δ such that its corresponding internal nodal forces F_I vector is in equilibrium with the external loads vector F_0

$$g(\Delta) = F_0 - F_I = 0 \tag{12}$$

Using the first two terms of the Taylor series about the vector of the initial displacements Δ_0 results in

$$g(\Delta) = g(\Delta_0) + \left.\frac{\partial g}{\partial \Delta}\right|_{\Delta_0} (\Delta - \Delta_0) \tag{13}$$

Substituting Eq. (12) into Eq. (13) leads to

$$\left.\frac{\partial F_I}{\partial \Delta}\right|_{\Delta_0} (\Delta - \Delta_0) = F_0 - F_I \tag{14}$$

The tangent stiffness matrix can be obtained as

$$K_t = \left.\frac{\partial F_I}{\partial \Delta}\right|_{\Delta_0} \tag{15}$$

The terms of the tangent stiffness matrix K_t include

$$K_t = K_0 + K_g + K_m \tag{16}$$

Where $K_g=K_g(\Delta)$ and $K_m=K_m(\Delta)$ are the geometric stiffness matrix and material stiffness matrix, respectively. The increment nodal displacement is obtained as

$$\delta\Delta = K_t^{-1}\delta F \tag{17}$$

Now, the start point is updated

$$\Delta = \Delta_0 + \delta\Delta \tag{18}$$

Where Δ_0 is obtained using the initial stiffness matrix K_0

The calculations are repeated until the stop conditions $\|\delta F\| \leq \varepsilon_F$ and $\|\delta\Delta\| \leq \varepsilon_\Delta$ are satisfied.

Now, this iterative analysis should be inserted into a GA for the optimal design. In each generation of the GA iterative non-linear analyses should be performed that is laborious and time-consuming. The difficulty with the non-linear analysis comes from: 1) calculation inverse of the matrix K_t in each iteration 2) the number of iteration to reach to the stop criteria. In the following sections efficient analyses for reducing the computational complexity of the problem is presented. The optimal solutions are presented for both optimal size design and optimal geometry design.

3.2 Efficient analysis in the optimal size and geometry design (non-linear material)

Consider the optimal size and geometry design of a structure using non-linear analysis. Suppose the non-linear behavior is only due to the non-linear material. Now, one is in the first generation of the GA and wants to start the non-linear analysis for the initial set of members and node locations (size and geometry). In each iteration the matrix K_t should be inverted. The matrix can be written as

$$K_{t_{i+1}} = K_{t_i} + K_m \tag{19}$$

Where K_{t_i} and $K_{t_{i+1}}$ are the tangent stiffness matrices in the ith and i+1th iteration respectively and K_m is the material stiffness matrix. From one iteration to another one the stiffness of some members change due to material nonlinearity, therefore

$$K_{t_{i+1}} = \begin{bmatrix} (K_{t_i})_{I,I} + K_m & \vdots & (K_{t_i})_{I,II} \\ \cdots & \vdots & \cdots \\ (K_{t_i})_{II,I} & \vdots & (K_{t_i})_{II,II} \end{bmatrix} \tag{20}$$

Where the block I, I is corresponding to the nodes which are affected by K_m.

The inverse of the stiffness matrix $K_{t_{i+1}}$ is simply calculated as

$$D_{i+1} = \begin{bmatrix} (\overline{D}_i)_{I,I}(D_i)_{I,I} & \vdots & (\overline{D}_i)_{I,I}(D_i)_{I,II} \\ \cdots & \vdots & \cdots \\ -(D_i)_{II,I}K_m(\overline{D}_i)_{I,I}(D_i)_{I,I} + (D_i)_{II,I} & \vdots & -(D_i)_{II,I}K_m(\overline{D}_i)_{I,I}(D_i)_{I,II} + (D_i)_{II,II} \end{bmatrix} \tag{21}$$

Where D_i is the inverse of the matrix K_{t_i} that is available from ith iteration and the matrix $(\overline{D}_i)_{I,I}$ is defined as

$$[\, I + (D_i)_{I,I}K_m \,]^{-1} = (\overline{D}_i)_{I,I} \tag{22}$$

The computational complexity of inverting a matrix of the dimension n is $O(n^{2.373})$. Here, the whole process comprises inverting the small matrix $[\, I + (D_i)_{I,I}K_m \,]$, of the dimension equal to that of small matrix K_m (K_m is small since material nonlinearity extends gradually from one iteration to another one). Using the present equations the inverse of the matrix K_t is simply obtained in each iteration up to the point the stop criteria is achieved (the end of first generation).

Analyses of the first generation of the GA got done. Now, offsprings are produced by parents similar to what is shown in Fig. 2 and Fig. 3. In a traditional method of analysis, the new generation (offspring) is considered as an independent structure and is analyzed from the first step. However, since only the size of some members change from parents to offsprings, one wants to get advantages of the available solution of parents (previous generation) in the solution of offsprings (new generation).

Suppose in the *j*th iteration of the first generation the stop criteria got satisfied and the corresponding displacement vector is Δ_j. Now, the second generation can be solved for this displacement vector (start point) and find the internal nodal forces vector. Using Δ_j as the start point of the second generation not only make a swift solution for the tangent stiffness matrix of the second generation (see below) but also decrease the number of iterations significantly (see Fig. 4).

When using the vector Δ_j as the start point, the members of the structure in the *j*th iteration of the first generation and the corresponding members in the first iteration of the second generation, except the members with changed size and members connected to changed-location nodes (size and geometry modifications applied by GA), will have the same position on the σ-ε curve and therefore the same stiffness matrices. This way the stiffness matrices of the two structures are similar except in the changed-location nodes and the nodes affected by the changed-size members

$$(K_{t_1})_2 = (K_{t_j})_1 + K_{s,g} \tag{23}$$

Where $(K_{t_j})_1$ is the tangent stiffness matrix of the *j*th (i.e., last) iteration of the first generation, $(K_{t_1})_2$ is the tangent stiffness matrix of the first iteration of the second generation and $K_{s,g}$is the stiffness matrix because of changed size and geometry applied by GA.

$$(K_{t_1})_2 = \begin{bmatrix} ((K_{t_j})_1)_{I,I} + K_{s,g} & \vdots & ((K_{t_j})_1)_{I,II} \\ \cdots & \vdots & \cdots \\ ((K_{t_j})_1)_{II,I} & \vdots & ((K_{t_j})_1)_{II,II} \end{bmatrix} \tag{24}$$

Where the block *I*,*I* is corresponding to the nodes which are affected by $K_{s,g}$.

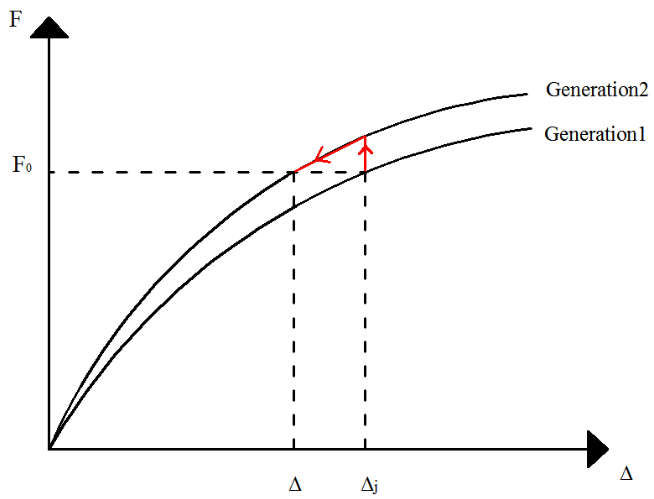

Fig. 4 The illustrative interpretation of jumping from one generation to another one

The inverse of the stiffness matrix $(K_{t_1})_2$ is calculated as

$$(D_1)_2 = \begin{bmatrix} ((\overline{D}_j)_1)_{I,I}((D_j)_1)_{I,I} & \vdots & ((\overline{D}_j)_1)_{I,I}((D_j)_1)_{I,II} \\ \cdots & \vdots & \cdots \\ -((D_j)_1)_{II,I}K_s((\overline{D}_j)_1)_{I,I}((D_j)_1)_{I,I} + ((D_j)_1)_{II,I} & \vdots & -((D_j)_1)_{II,I}K_s((\overline{D}_j)_1)_{I,I}((D_j)_1)_{I,II} + ((D_j)_1)_{II,II} \end{bmatrix} \quad (25)$$

Where $(D_j)_1$ is the inverse of the matrix$(K_{t_j})_1$which is available from *j*th iteration of the first generation and the matrix $((\overline{D}_j)_1)_{I,I}$ is defined as

$$[\, I + \, ((D_j)_1)_{I,I}K_{s,g} \,]^{-1} = ((\overline{D}_j)_1)_{I,I} \quad (26)$$

Therefore, the inverse of the stiffness matrix $(K_{t_1})_2$ is obtained by inverting the small matrix $[\, I + \, \left((D_j)_1\right)_{I,I}K_{s,g}]$, of the dimension equal to that of the small matrix $K_{s,g}$(see below).

Briefly, the first iteration of the second generation is solved for Δ_j and the corresponding internal nodal forces vector is found. This is like jumping from the F-Δ curve of the first generation to that of the second generation vertically at the point Δ_j. Then, the tangent stiffness matrix of the first iteration of the second generation is obtained and inverted using Eq. (24) and Eq. (25), respectively. Now, moving toward the point of solution is started iteratively (i.e., the point Δ corresponding to the external load F_0). Similar to the previous generation, from one iteration to another one, the tangent stiffness matrix is swiftly inverted using Eq. (21). In comparison with a traditional method, one is now much closer to the point of solution Δ since utilizing Δ_j as the start point.

The illustrative interpretation of the descriptions above is shown in Fig. 4.

The efficiency of the optimal solution depends on three factors (Shojaei *et al.* 2013; Rahami *et al.* 2015): 1) the size of matrix K_m in Eq. (22) 2) the size of matrix $K_{s,g}$ in Eq. (26) and 3) the similarity of the curves of two following generations (more similar the curves are, less iterations are needed). Previously, it was shown that in the GA from one generation to another one about 75% of a structure remains unchanged, averagely (Rahami *et al.* 2015). This assures that the second and third factors work efficiently. Moreover, since non-linearity extends gradually in the elements of a structure, the first factor works efficiently as well. The method can always be used with no concern because even in the worst condition its efficiently is similar to that of a traditional method.

3.3 Efficient analysis in the optimal size and geometry design (non-linear geometry)

The optimal size and geometry design of a structure using an optimal non-linear analysis is presented. Suppose the non-linear behavior is only due to the non-linear geometry. Suppose one is in the first generation of the GA and want to start the non-linear analysis for the initial set of members and node locations (size and geometry). The tangent matrix K_t is inverted in each iteration. The matrix can be written as

$$K_{t_{i+1}} = K_{t_i} + K_g \quad (27)$$

Where K_{t_i} and $K_{t_{i+1}}$ are the tangent stiffness matrices in the *i*th and *i*+1th iteration respectively and K_g is the geometric stiffness matrix. However, from one iteration to another one, the size of matrices K_{t_i} and K_g are the same because unlike the material non-linearity which happens gradually, geometric non-linearity happens entirely in each iteration. Therefore, in the first generation up to the point Δ_j a usual method is utilized.

Now, solving the second generation for Δ_j and finding the internal nodal forces vector are aimed. When using the vector Δ_j as the start point, the members of the structure in the jth iteration of the first generation and the corresponding members in the first iteration of the second generation, except the members with changed size and members connected to the changed-location nodes, will have the same stiffness matrices. This way the stiffness matrices of the two structures are similar except in the changed-location nodes and the nodes affected by the changed members

$$(K_{t_1})_2 = (K_{t_j})_1 + K_{s,g} \tag{28}$$

Where $(K_{t_j})_1$ is the tangent stiffness matrix of the *j*th iteration of the first generation, $(K_{t_1})_2$ is the tangent stiffness matrix of the first iteration of the second generation and $K_{s,g}$ is the stiffness matrix because of changed size and geometry.

$$(K_{t_1})_2 = \begin{bmatrix} ((K_{t_j})_1)_{I,I} + K_{s,g} & \vdots & ((K_{t_j})_1)_{I,II} \\ \cdots & \vdots & \cdots \\ ((K_{t_j})_1)_{II,I} & \vdots & ((K_{t_j})_1)_{II,II} \end{bmatrix} \tag{29}$$

Where the block *I*, *I* is corresponding to the nodes which are affected by $K_{s,g}$.

Similar to the calculations shown in Eq. (25) and Eq. (26), the stiffness matrix $(K_{t_1})_2$ is inverted. Now, moving toward the point of solution is started iteratively. Again from one iteration to iteration the tangent stiffness matrix is inverted using a usual method because geometric non-linearity happens entirely in each iteration. In comparison with a traditional method, one is now much closer to the point of solution Δ since utilizing Δ_j as the start point (See Fig. 4).

The efficiency of the optimal solution depends on two factors: 1) the size of matrix $K_{s,g}$ in Eq. (29) and 2) the similarity of the curves of two following generations (more similar the curves are, less iteration is needed). As mentioned, in the GA from one generation to another one about 75% of a structure remains unchanged, averagely. This assures that the two factors work efficiently (Rahami *et al.* 2015).

4. Efficient analysis of a biomechanical model of the lumbar spine for predicting the muscle forces (non-linear material and geometry)

Unlike the structural/mechanical problems where the external loads are usually available and displacements are found, in biomechanical problems of human musculoskeletal system, measuring the kinematics data is much simpler than finding the applied loads. In such problems, the applied loads consist of both the external loads and the internal tissue responses. However, direct measurement of these internal tissue responses (i.e., muscle forces, ligament forces and so on) is almost impossible such that alternative inverse solutions (i.e., measuring the kinematics data and estimating the internal tissue responses using those kinematics measures and external loads) are required. For instance, in problems related to the biomechanics of the lumbar spine (i.e., the lower portion of vertebral column starting from the T12 vertebra and ending at the S1), kinematics of the spine can be obtained using motion capture system and by tracking markers attached to different bony land marks on the lumbar spine. Though knowledge of mass distribution along the spine facilitates estimation of external loads (e.g., gravity and inertia) that act on the spine, measurement of internal muscle responses is not possible. Existing mechanical models of the lumbar spine predict such muscle responses by solving an inverse dynamic problem and obtaining moments at one or multiple spinal levels to be balanced by muscles attached to those levels. The generated equilibrium problems for estimation of muscle forces are however redundant; there are more unknown muscle responses as compared to equilibrium equations. Different methods have been adopted to resolve such redundancy problem including optimization method wherein from all possible solution a set of muscle forces that optimize a given cost function is selected. One of the cost functions that has been shown to predict muscle forces consistent with electromyography measurement of muscle activity is the minimization of sum of squared muscle stress.

Here, a previously developed and validated non-linear model of the lumbar spine (Bazrgari *et al.* 2007) is used to show the time-efficiency of the algorithm in this paper. This model, shown in Fig. 5 and Fig. 6, will be used to estimate muscle forces and spinal loads under different conditions and using three methods: 1) non-linear inverse- dynamic analysis with assumption of constant segmental rotations (Method-1) (Arjmand *et al.* 2005, Bazrgari *et al.* 2007), 2) non-linear inverse dynamic analysis with segmental rotations obtained using the optimal design (i.e., posture) (Method-2) and 3) modified non-linear analysis, as presented in this paper, with segmental rotations obtained using the optimal design (i.e., posture) (Method-3). Available Kinematics data for these simulations included rotation of the T12 and the S1 spinal vertebrae in the sagittal plane (i.e., the plane that divide the body to right and left parts) and were obtained from an earlier study (Arjmand *et al.* 2005). Vertebral rotations in the lumbar region in the Method-1 are estimated as constant percentages of total lumbar rotation (i.e., T12 minus S1 rotations). These percentages

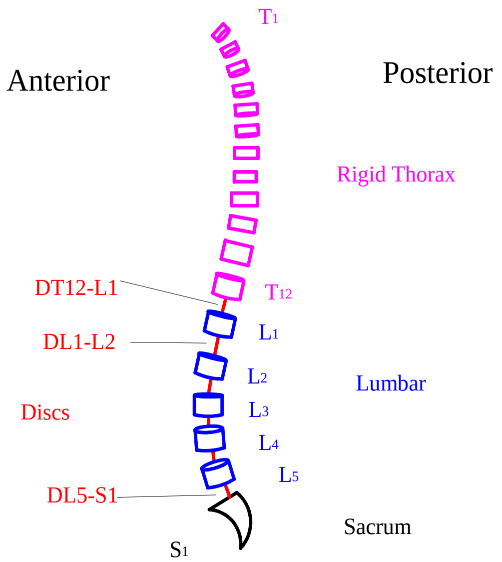

Fig. 5 A schematic model of the spine

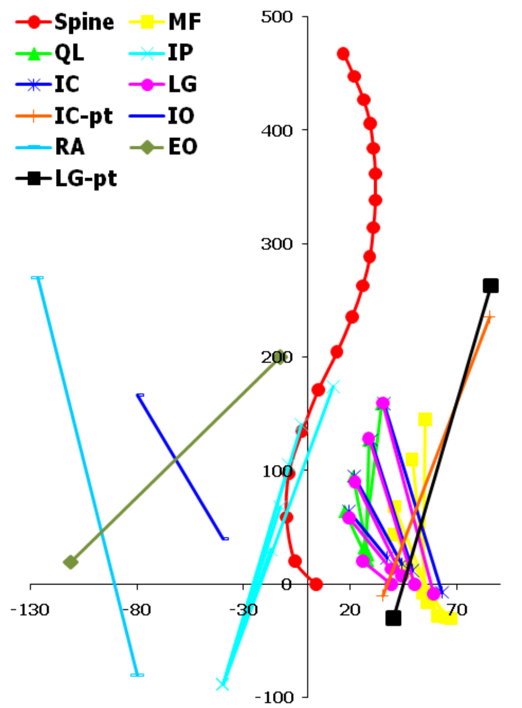

Fig. 6 The lateral view of the spine and the attached muscles

have been obtained from earlier studies of segmental range of motion and include 8% for the T12-L1, 13% L1-L2, 16% for L2-L3, 23% for L3-L4, 26% for L4-L5 and 14% for L5-S1 (Arjmand *et al.* 2005). In the Method-2 and the Method-3 however vertebral rotations in the lumbar region were considered as unknown (dependent on input muscle forces) in the optimization problem (see Eq. (30)) but were bounded within their reported physiological range of motion. The Method-2 and the Method-3 are expected to predict similar results but different execution times because the analyses in Method-2 are performed using a usual algorithm while the analyses in Method-3 are done using the present optimal analysis in this paper. However, vertebral rotation and spinal loads will be different between the two last methods and those of the Method-1 and will be hence compared here. Description of using the nonlinear model of spine in the Method-1 can be found elsewhere (Bazrgari *et al.* 2007), but its application within Method-2 and Method-3 are described here.

The models account for 56 muscles that are symmetrically distributed with respect to sagittal plane and will be used as input to the optimization procedure. These include 10 muscles in each level from the T12 to the L4 and 6 muscles in the level L5 that is 56 muscles totally. Because of the symmetry of the muscles and symmetry of the motion task (i.e., flexion in sagittal plane) 28 muscles are considered as optimization inputs. The optimization problem is formulated as

$$\left\{\begin{array}{c} \boldsymbol{F} = [F_1, F_2, F_3, \ldots, F_{28}] \\ 0 \le F_i \le (\sigma_{max})_m \times PCSA_i \qquad i = 1{:}\,28 \\ \text{Minmize } \left(\sum_{i=1}^{n=28} \left(\frac{F_i}{PCSA_i}\right)^2 \left(1 + \alpha \sum_{i=1}^{nc=14} max[0, g_m]\right)\right) \\ \text{Subject to} \\ \theta_{T_{12}} = \theta_1 \\ \theta_{S_1} = \theta_2 \\ -9.6^\circ \le \theta_{T_{12}} - \theta_{L_1} \le 6^\circ \\ -9.6^\circ \le \theta_{L_1} - \theta_{L_2} \le 6^\circ \\ -12^\circ \le \theta_{L_2} - \theta_{L_3} \le 3.6^\circ \\ -14.4^\circ \le \theta_{L_3} - \theta_{L_4} \le 1.2^\circ \\ -15.6^\circ \le \theta_{L_4} - \theta_{L_5} \le 2.4^\circ \\ -10.8^\circ \le \theta_{L_5} - \theta_{S_1} \le 6^\circ \\ |\sigma_{D_j}| \le (\sigma_{\max})_d \qquad j = 1{:}\,6 \end{array}\right. \tag{30}$$

Where F_i and PCSA_i respectively denote the force and the physiological cross section area of ith lower back muscle, $(\sigma_{\max})_m$ is the maximum allowable stress in the muscle (i.e., 1.0 MPa), *nc* is the number of constraints (i.e., 14), *gm* is the violations of the optimization constraints, α is a large number (Kaveh *et al.* 2013), θ_{T12} and θ_{S1} are respectively the rotation of T12 and S1 vertebrae and are inserted in the constraints by the user for a given flexion angle, θ_1 and θ_2 are calculated rotations of T12 and S1 vertebrae in any generation for the corresponding set of muscle forces, θ_{L1} to θ_{L5} are respectively vertebral rotations of L1 to L5 in any generation for the corresponding set of muscle forces and $(\sigma_{\max})_d$ is the maximum allowable stress in the disc. The rotational inequality constraints denote sagittal plane range of motion of lumbar motion segments with negative sign denoting flexion.

To solve the defined optimization problem in Eq. (30), a heuristic method is employed wherein a GA with 100 generations and 30 individuals in each generation is utilized. Therefore, 3000 non-linear analysis should be performed totally that is very laborious and time-consuming. In this optimization procedure from one generation to another one, external loads (i.e., muscle forces) as the variables of the problem will change. However, as mentioned before, about 75% of the loads remain unchanged in two subsequent generations. Here, the non-linear behavior is due to both non-linear material and geometry. Suppose one is in the first generation of the GA and want to start the non-linear analysis for the initial set of muscle forces. The tangent matrix K_T is inverted in each iteration. The matrix can be written as

$$K_{t_{i+1}} = K_{t_i} + K_m + K_g \tag{31}$$

Where K_{t_i} and $K_{t_{i+1}}$ are the tangent stiffness matrices in the ith and i+1th iteration respectively, K_m is the material stiffness matrix and K_g is the geometric stiffness matrix. Although from one iteration to another one the size of matrix K_m is smaller than that of matrices K_{t_i} and K_g, since matrices K_{t_i} and K_g are of the same dimension, a usual method is utilized in the first generation up to the point Δ_j.

Since about 0.75 of the loads remain unchanged in the second generation, instead of solving the new generation from the first step, one starts to solve it for Δ_j which is much closer the target displacement Δ of the second generation. Solving the second generation for Δ_j and finding the corresponding internal nodal forces vector means jumping from the curve of generation 1 to the curve of generation 2 vertically. When using the vector Δ_j as the start point, the members of the structure in the jth iteration of the first generation and the corresponding members in the first iteration of the second generation will have the same position on the σ-ε curve as well as same geometry. Consequently, the stiffness matrices of the two structures are similar

$$(K_{t_1})_2 = (K_{t_j})_1 \tag{32}$$

Where $(K_{t_j})_1$ is the tangent stiffness matrix of the jth iteration of the first generation, $(K_{t_1})_2$ is

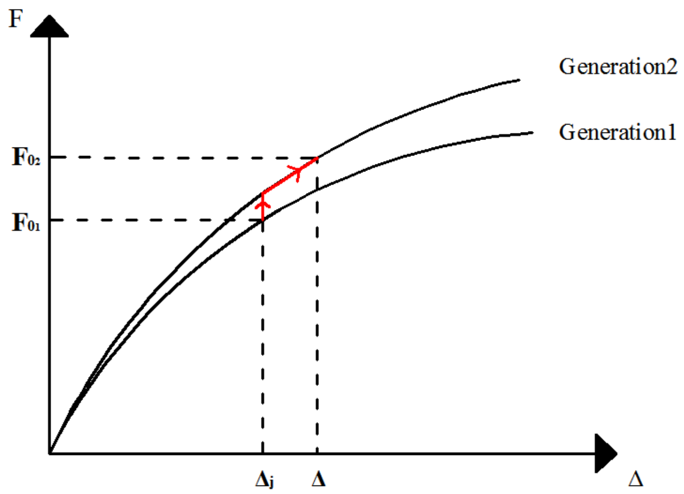

Fig. 7 The illustrative interpretation of jumping from one generation to another one

the tangent stiffness matrix of the first iteration of the second generation.

The inverse of the stiffness matrix $(K_{t_j})_1$is available from *j*th iteration of the first generation that means the inverse of matrix $(K_{t_1})_2$ is available as well. Now, moving toward the point of solution is started iteratively. From one iteration to iteration the tangent stiffness matrix is inverted using a usual method. As shown in Fig. 7, one is now much closer to the target point Δ in comparison with a usual method

Unlike Fig. 4, where the external loads are the same but the size of members and nodes location change in two generations, here the external loads (i.e., muscle forces) change from one generation to another one (i.e., the different between structural (mechanical) and biomechanical approaches)

Using the three methods explained above, muscle forces and spinal loads were calculated for different lumbar rotations and external loading (with and without holding a 4.55 kg weight in hand) conditions. Tables 1 and 2 contain compression and shear forces at the lower most level of lumbar spine as calculated by the Method-1 and the Method-2 (or Method-3) for all condition studied here. For each condition, the predicted muscle forces were used to calculate the value of cost function considered in the Method-2 (or Method-3) and is also reported in Tables 1 and 2. As mentioned above, the only difference between Method-2 and Method-3 is related to the execution time. The time-efficiency of the Method-3 (the present algorithm) versus the Method-2 (the usual algorithm) was reflected in the total time needed to complete the optimization procedure for each condition. In average, simulation durations for completing the optimization procedure for one condition were respectively 12.5 and 7.25 hours. This suggests an average of 42% decrease in execution time of the Method-3. The average value (different lumbar rotations) for the execution time, number of iterations, and cost function during different generations of GA is presented in Table 3.

Table 1 The value of cost function and compression and shear forces of the L5-S1 level (no external load)

Lumbar Rotation (degree)		25	30	35	40	45	50	55
Cost Function*e-12	**Method-1**	0.93	1.29	1.66	2.49	2.73	3.25	3.52
	Method-2	0.16	0.319	0.50	0.638	0.75	1.04	1.68
Compression (N)	**Method-1**	1127	1400	1579	1848	1978	2282	2473
	Method-2	935	1177	1343	1565	1716	2028	2400
Shear (N)	**Method-1**	378	424	440	447	437	400	362
	Method-2	402	449	468	480	477	450	385

Table 2 The value of cost function and compression and shear forces of the L5-S1 level (4.55 kg external load)

Lumbar Rotation (degree)		25	30	35	40	45	50	55
Cost Function*e-12	**Method-1**	1.20	1.66	2.18	2.93	3.59	4.17	4.49
	Method-2	0.26	0.48	0.67	0.94	1.46	1.74	2.22
Compression (N)	**Method-1**	1287	1618	1840	2136	2452	2778	2952
	Method-2	1059	1371	1570	1849	2260	2581	2882
Shear (N)	**Method-1**	430	491	513	520	504	458	420
	Method-2	454	517	546	564	549	511	445

Table 3 The average value for the execution time, number of iterations, and cost function during different generations of GA

Total Analyses (Generations*Individuals)		500	1000	1500	2000	2500	3000
Execution Time(h)	**Method-2**	2.14	4.19	6.45	8.62	10.65	12.50
	Method-3	1.26	2.42	3.67	4.97	6.21	7.25
Number of Iterations*e-2	**Method-2**	5000	10000	15000	20000	25000	30000
	Method-3	2831	5527	8257	12016	14283	17531
Cost Function*e-12		4.9	3.1	2.03	1.35	1.13	1.11

5. Conclusions

In this paper simultaneous efficient non-linear analysis and optimal design for mechanical/biomechanical systems are presented. Considering the difficulties with numerous iterations in non-linear analyses and numerous generations in the heuristic optimization procedures like GAs, swift solutions for material and geometry non-linear analyses within the optimization design procedures are presented. The quickness of the method comes from decreasing both the size of the inverted matrix and number of iterations. The method is developed in biomechanical modeling as well where the optimization (due to neuromuscular assumption) and non-linear analysis (due to large deformations and non-linear biomaterial) are necessary parts of the solution. In average, Method -3, performed by the simultaneous optimal analyses (i.e., reduced execution time and number of iterations) and optimal design (i.e., the posture associated with minimum cost function), could reduce the cost function and execution time by about 66% and 42%, respectively, compared to those values obtained by Method-1 and Method-2.

References

Arjmand, N. and Shirazi-Adl, A. (2005), "Biomechanics of changes in lumbar posture in static lifting", *Spine*, **30**(23), 2637-2648.

Bathe, K.J. (1996), *Finite Element Procedures*, Englewood Cliffs, Prentice-Hall, NJ, USA.

Bazrgari, B., Shirazi-Adl, A. and Arjmand, N. (2007), "Analysis of squat and stoop dynamic liftings: muscle forces and internal spinal loads", *European Spine J.*, **16**(5), 687-699.

Dorigo, M., Maniezzo, V. and Colorni, A. (1996), "Ant system: optimization by a colony of cooperating agents. Systems, Man, and Cybernetics, Part B: Cybernetics", *IEEE Trans.*, **26**(1), 29-41.

Ezquerro, F., Simón, A., Prado, M. and Pérez, A. (2004), "Combination of finite element modeling and optimization for the study of lumbar spine biomechanics considering the 3D thorax-pelvis orientation", *Med. Eng. Phys.*, **26**(1), 11-22.

Goel, V.K., Kong, W., Han, J.S., Weinstein, J.N. and Gilbertson, L.G. (1993), "A combined finite element and optimization investigation of lumbar spine mechanics with and without muscles", *Spine*, **18**(11), 1531-1541.

Kaveh, A., Kalateh-Ahani, M. and Fahimi-Farzam, M. (2013), "Constructability optimal design of reinforced concrete retaining walls using a multi-objective genetic algorithm", *Struct. Eng. Mech.*, **47**(2), 227-245.

Kaveh, A., Shojaei, I., Gholipour, Y. and Rahami, H. (2013), "Seismic design of steel frames using multi-objective optimization", *Struct. Eng. Mech*., **45**(2), 211-232.

Kaveh, A., Shojaei, I. and Rahami, H. (2015), "New developments in the optimal analysis of regular and near-regular structures: decomposition, graph products, force method", *Acta Mech*., **226**(3), 665-681.

Kaveh, A. (2014), *Computational Structural Analysis and Finite Element Methods*, Springer verlag, Wien.

Konak, A., Coit, D.W. and Smith, A.E. (2006), "Multi-objective optimization using genetic algorithms: A tutorial", *Reliab. Eng. Syst. Saf.*, **91**(9), 992-1007.

Lee, K.S. and Geem, Z.W. (2004), "A new structural optimization method based on the harmony search algorithm", *Comput. Struct.*, **82**(9), 781-798.

Marler, R.T. and Arora, J.S. (2004), "Survey of multi-objective optimization methods for engineering", *Struct. Multidiscip. Optim.*, **26**(6), 369-395.

Rahami, H., Kaveh, A. and Gholipour, Y. (2008), "Sizing, geometry and topology optimization of trusses via force method and genetic algorithm", *Eng. Struct*., **30**(9), 2360-2369.

Rahami, H., Kaveh, A. and Shojaei, I. (2015), "Swift analysis for size and geometry optimization of structures", *Adv. Struct. Eng.*, **18**(3), 365-380.

Shojaei, I., Kaveh, A. and Rahami, H. (2013), "Analysis of structures convertible to repeated structures using graph products", *Comput. Struct.*, **125**, 153-163

Shojaei, I., Rahami, H. and Kaveh, A. (2015), "Efficient finite element solution of regular and near-regular systems using graph products", *Acta Mech*., DOI 10.1007/s00707-015-1337-z.

Stokes, I.A. and Gardner-Morse, M. (2001), "Lumbar spinal muscle activation synergies predicted by multi-criteria cost function", *J. Biomech.*, **34**(6), 733-740.

Wang, M.Y., Wang, X. and Guo, D. (2003), "A level set method for structural topology optimization", *Comput. Methods Appl. Mech. Eng.*, **192**(1), 227-246.

Zingoni, A. (2002), *Group-theoretic applications in solid and structural mechanics: a review in Computational structures technology*, Civil-Comp Press.

Zingoni, A. (2009), "Group-theoretic exploitations of symmetry in computational solid and structural mechanics", *Int. J. Numer. Meth. Eng.*, **79**(3), 253-289.

Zingoni, A. (2014), "Group-theoretic insights on the vibration of symmetric structures in engineering", *Phil. Trans. R. Soc. A*, **372**(2008), 20120037**.**

MC