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# Static deflection and dynamic behavior of higher-order hyperbolic shear deformable compositionally graded beams

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**Abstract.** In this work we introduce a higher-order hyperbolic shear deformation model for bending and frees vibration analysis of functionally graded beams. In this theory and by making a further supposition, the axial displacement accounts for a refined hyperbolic distribution, and the transverse shear stress satisfies the traction-free boundary conditions on the beam boundary surfaces, so no need of any shear correction factors (SCFs). The material properties are continuously varied through the beam thickness by the power-law distribution of the volume fraction of the constituents. Based on the present refined hyperbolic shear deformation beam model, the governing equations of motion are obtained from the Hamilton's principle. Analytical solutions for simply-supported beams are developed to solve the problem. To verify the precision and validity of the present theory some numerical results are compared with the existing ones in the literature and a good agreement is showed.

**Keywords**: deflection; dynamic analysis; functionally graded material; hyperbolic shear deformation theory; refined theory

## 1. Introduction

Functionally graded materials (FGMs) are the novel class of composite materials that have continuous mutation in material properties from one surface to a further along the thickness direction. This genius concept of FGMs was primary initiated in 1984 by a group of material scientists while preparing a space-plane plan, in Japan (Koizumi 1997). The primary constituents for these materials are metal with ceramic or from a combination of materials. The FGM is thus appropriate for various applications, such as thermal coatings of barrier for ceramic engines, gas turbines, nuclear fusions, optical thin layers, biomaterial electronics, and in many other fields.

Consequently, studies and computational (numerical) techniques are devoted to analyze the static and dynamic behaviors of FGM beams and plates are also in huge demand in research sectors day-by-day. However, the behavior of FG beams can be predicted using either, the classical beam theory (CBT), first-order shear deformation beam theory (FSBT), third-order shear

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deformation beam theory (TSBT), higher-order shear deformation beam theory (HSBT) and three-dimensional (3D) elasticity theory.

However, it is well known that due to the impotency of classical beam theory or Euler-Bernoulli beam theory (EBT) to consider the effect of the transverse shear deformations as well as shear correction factor dependency of Timoshenko beam theory (TBT), a numeral of higher-order theories are provided and applied in analysis of FG structures. Sallai *et al*. (2009) analyzed the bending responses of a sigmoid FG thick beam by using different higher order beam theories. Sankar (2001) provided an exact solution for bending analysis of FG beams subjected to transverse loads based on Euler-Bernoulli beam theory. Zhong and Yu (2007) used a 3D elasticity theory to predict the bending responses of cantilever FG beams under concentrated and uniformly distributed loads. The third-order shear deformation theory (TSDT) in form of finite element method are employed by Kadoli *et al*. (2008) to study the bending of FG beams by considering different boundary conditions (BCs) at the edges. Li (2008) analyzed the static and dynamic behaviors of FG beams by using a new unified approach, the rotary inertia and shear deformation have been included. Li *et al*. (2010) proposed analytical solutions for static and dynamic analysis of FG beams using TSBT. Benatta *et al*. (2009) derived an analytical solution to the bending analysis of a symmetric FG beam by including warping of the cross-section and shear deformation effect. Şimşek (2010a) used different higher order beam theories to study the free vibration of an FG beam. Then in another one, Şimşek (2010b) conducted an investigation on the dynamic deflections and the stresses of an FG simply-supported beam subjected to a moving mass by using Euler–Bernoulli, Timoshenko and the parabolic shear deformation beam theory. Thai and Vo (2012) used various shear deformation beam theories for studying the bending and vibration responses of FG beams. Simsek and Yurtcu (2013) developed an analytical solution for static bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory. Bouremana *et al*. (2013) developed a new first shear deformation beam theory model based on neutral surface position for FG beams. Hadji (2014) investigated the static and free vibration of FGM beam by using a higher order shear deformation theory. Recently, Nguyen *et al*. (2013) proposed a new first order shear deformation beam theory for studying the static bending and free vibration of axially loaded FG beams in which, an improved transverse shear stiffness has been introduced without using shear correction factor. Ould *et al*. (2013) derived an efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams. The effects of shear and normal deformations may become more considerable for moderately thick beams and plates. Hence, various novel plate theories are recommended accounting for both transverse shear and normal deformations and satisfy the zero traction boundary conditions on the surfaces of the plate (Hebalii *et al*. 2014, Belabed *et al*. 2014, Bousahla *et al*. 2014, Hamidi *et al*. 2015, Bennoun *et al*. 2016). Meradjah *et al*. (2015) constructed a novel shear deformation beam model including the stretching effect for studying the flexural and free vibration responses of functionally graded beams. Vo *et al*. (2015) developed a finite element model to study the dynamic vibration and buckling of FG sandwich beams by the newly quasi-3D theory in which both shear deformation and thickness stretching effects are incorporated. Bourada *et al*. (2015) also presented a new simple shear and normal deformations theory for functionally graded beams without using shear correction factors. The wave propagation of an FG porous plate based on various simple higher-order shear deformation theories has been investigated by Ait Yahia *et al*. (2015). Free vibration and buckling analysis of functionally graded (FG) sandwich beams under various boundary conditions using higher-order shear and normal deformation theory was examined by Bennai *et al*. (2015). Mahi *et al*. (2015) established a novel hyperbolic shear deformation theory to study the bending and free vibration responses of isotropic, functionally graded, sandwich and laminated composite plates. More recently, a new three unknowns model as the case of the classical plate theory (CPT) was elaborated by Tounsi *et al*. (2016) and Houari *et al*. (2016) for static, buckling and vibration analysis of both functionally graded and sandwich plates. Some plate theories are used also to explore the behavior of nanostructures as is described in Refs (Bounouara *et al*. 2016, Belkorissat *et al*. 2015).

In this study, static bending and vibration behaviors of compositionally graded beams are analyzed based on a novel simple higher order refined beam model which captures the shear deformation effects using a shear strain function without the need of any shear correction factor. The governing equations of motion in the framework of the present refined hyperbolic beam model are derived through Hamilton's principle and resolved applying an analytical solution for simply-supported boundary conditions. The material properties are graded through the beam's depth by the power-law model. Numerical examples are supplied to demonstrate the impacts of power-law exponent, length of the FG beam, thickness to length ratios on the bending and free vibration of functionally graded beams.

## 2. Mathematical formulation

We consider a functionally graded beam having length $L$ and rectangular cross section $b\times h$, with $b$ represents the width and $h$ the thickness which its coordinates is depicted in Fig. 1. The beam is made of elastic and isotropic material with material properties varying smoothly in the $z$ thickness direction.

### *2.1 Material properties*

The effective material properties of the non-homogeneous beam such as Young's modulus $E_f$, shear modulus $G_f$ and mass density $\rho_f$ are supposed to vary continuously in the thickness direction according to a power function of the volume fractions of the constituents.

According to the Voigt rule of mixture, the effective material properties, $P_f$, can be expressed as (Simsek and Yurtcu 2013, Bouremana *et al*. 2013, Ould Larbi *et al*. 2013, Ebrahimi and barati 2016) like

$$P_f = P_c V_c + P_m V_m \tag{1}$$

Where $P_m$, $P_c$, $V_m$ and $V_c$ are the material properties and the volume fractions of the metal and the ceramic constituents related by

$$V_c + V_m = 1 \tag{2}$$

The volume fraction of the ceramic constituent of the FG beam is supposed to be given by

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{3}$$

$k$ is a variable parameter that dictates material variation profile through the thickness and $z$ is the distance from the mid-plane of the FG beam. The FG beam becomes a fully ceramic beam

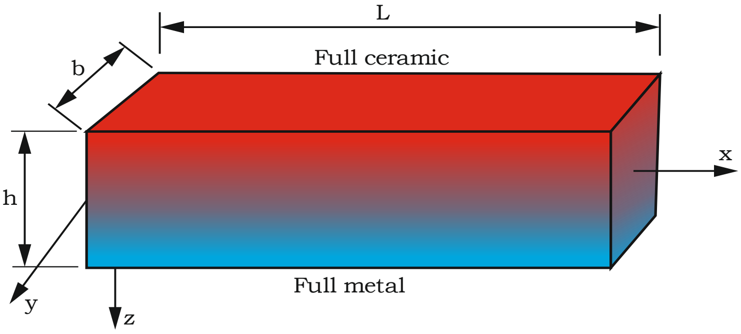


Fig. 1 Geometry and coordinate of a FG beam

When $p$ is set to be zero, whereas infinite $p$ indicates a fully metallic beam. Hence, from Eqs. (1) and (2), the effective material properties of the FG beam such as Young's modulus ($E$), mass density ($\rho$) are given as,

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m \tag{3}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_m \tag{4}$$

### 2.2 Kinematic relations and strains

On the basis on the proposed refined theory, the displacement field of the present beam model can be written in a simpler form as follow (Thai *et al*. 2014, Nguyen 2014)

$$u(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} + f(z)\frac{\partial \varphi}{\partial x} \tag{5a}$$

$$w(x,z,t) = w_0(x,t) \tag{5b}$$

It is noted that the displacement field of the recent refined existing FG beam theories (Bourada *et al* 2015, Meradja *et al*. 2015, Ould *et al*. 2013) are obtained by splitting the transverse displacement into bending and shear parts. Therefore, by making another supposition to existing ones, the displacement field and the governing equations of motion resulting in this study will be completely diverse with those cited above.

$f(z)$ is the shape function which controls the shearing stress distribution across the thickness of the FG beam and is chosen according to (Nareen and Shimpi 2014) as

$$f(z) = \alpha\frac{z}{h} + \beta\sinh\left(\frac{z}{h}\right) \tag{6}$$

The strains related with the displacements in Eq. (5) are written in following compact form

$$\varepsilon_x = \varepsilon_x^0 + z\kappa^b + f(z)\kappa^s \text{ and } \gamma_{xz} = g(z)\gamma^0 \tag{7}$$

Where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \ \kappa^b = -\frac{\partial^2 w}{\partial x^2}, \ \kappa^s = -\frac{\partial^2 \varphi}{\partial x^2}$$
$$\gamma^0 = \frac{\partial \varphi}{\partial x}, \ g(z) = 1 - f'(z), \text{ and } f'(z) = \frac{df(z)}{dz} \tag{8}$$

Supposing that the material of FG beam obeys Hooke's law, the constitutive relations can be given as

$$\sigma_x = Q_{11}(z)\varepsilon_x \text{ and } \tau_{xz} = Q_{55}(z)\gamma_{xz} \tag{9}$$

Where

$$Q_{11}(z) = E(z) \text{ and } Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{10}$$

### *2.3 Equations of motion*

The equations of motion are derived by using Hamilton's principle. The principle can be stated in an analytical form as (Reddy 2002)

$$\delta\int_0^T (U + V - K)dt = 0 \tag{11}$$

Where $\delta U$ is the variation of the strain energy; $\delta V$ represents the potential energy; and the variation of the kinetic energy is given by $\delta K$. The variation of the strain energy of the beam can expressed by the following form

$$\delta U = \int_0^L \int_A (\sigma_x \delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz}) dA dx$$
$$= \int_0^L \left( N\frac{d\delta u_0}{dx} - M\frac{d^2\delta w_0}{dx^2} - P\frac{d^2\delta\varphi}{dx^2} + Q\frac{d\delta\varphi}{dx} \right) dx \tag{12}$$

Where *N*, *M*, *P* and *Q* represent the stress resultants and they are expressed as

$$(N, M, P) = \int_A (1, z, f)\sigma_x dA \text{ and } Q = \int_A g\,\tau_{xz} dA \tag{13}$$

The variation of work done by externally transverse loads *q* can be given as

$$\delta V = -\int_0^L q\delta w_0 \, dx \tag{14}$$

The variation of the kinetic energy can be expressed as

$$\delta K = \int_0^L \int_A \rho \left[ \dot{u}\delta\dot{u} + \dot{w}_0 \delta \dot{w}_0 \right] dA dx$$

$$= \int_?^L \left\{ I_0 \left[ \dot{u}_0 \delta \dot{u}_0 + \dot{w}_0 \delta \dot{w}_0 \right] - I_1 \left( \dot{u}_0 \frac{d\delta \dot{w}_0}{dx} + \frac{d\dot{w}_0}{dx} \delta \dot{u}_0 \right) + I_2 \left( \frac{d\dot{w}_0}{dx} \frac{d\delta \dot{w}_0}{dx} \right) \right. \tag{15}$$

$$\left. - J_1 \left( \dot{u}_0 \frac{d\delta\dot{\varphi}}{dx} + \frac{d\dot{\varphi}}{dx} \delta \dot{u}_0 \right) + K_2 \left( \frac{d\dot{\varphi}}{dx} \frac{d\delta\dot{\varphi}}{dx} \right) + J_2 \left( \frac{d\dot{w}_0}{dx} \frac{d\delta\dot{\varphi}}{dx} + \frac{d\dot{\varphi}}{dx} \frac{d\delta \dot{w}_0}{dx} \right) dx$$

Where dot-superscript sign defines the differentiation with sense to the time variable $t$; $\rho$ is the mass density; and ($I_0$ , $I_2$, $J_2$, $K_2$) are the mass inertias expressed as

$$\left( I_0, I_1, J_1, I_2, J_2, K_2 \right) = \int_A \left( 1, z, f, z^2, zf, f^2 \right) \rho dA \tag{16}$$

Substituting the expressions for $\delta U$, $\delta V$, and $\delta K$ from Eqs. (12), (14), and (15) into Eq. (11) and integrating by parts, and collecting the coefficients of $\delta u_0$, $\delta w_0$ and $\delta\varphi$, the following equations of motion of the FG beam are obtained

$$\delta u_0 : \frac{dN}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_0}{dx} - J_1 \frac{d\ddot{\varphi}}{dx} \tag{17a}$$

$$\delta w_0 : \frac{d^2 M}{dx^2} + q = I_0 \ddot{w}_0 + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2 \ddot{w}_0}{dx^2} - J_2 \frac{d^2 \ddot{\varphi}}{dx^2} \tag{17b}$$

$$\delta\varphi : \frac{d^2 P}{dx^2} + \frac{dQ}{dx} = I_0 \ddot{w}_0 + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2 \ddot{w}_0{}^2}{dx^2} - K_2 \frac{d^2 \ddot{\varphi}}{dx^2} \tag{17c}$$

Eqs. (17a)-(17c) can be expressed in terms of displacements ($u_0$, $w_0$, and $\varphi$) by using Eqs. (5), (7), (9) and (13) as follows

$$A \frac{d^3 u_0}{dx^2} - B \frac{d^3 w_0}{dx^2} - B_s \frac{d^3 \varphi}{dx^3} = I_0 \ddot{u} - I_1 \frac{d\ddot{w}_0}{dx} - J_1 \frac{d\ddot{\varphi}}{dx}$$

$$B \frac{d^3 u_0}{dx^3} - D \frac{d^3 w_0}{dx^3} - D_s \frac{d^4 \varphi}{dx^4} + q = I_0 \ddot{w}_0 + I_1 \frac{d\ddot{u}}{dx} - I_2 \frac{d^2 \ddot{w}_0}{dx^2} - J_2 \frac{d^2 \ddot{\varphi}}{dx^2} \tag{18a}$$

$$B_s \frac{d^3 u_0}{dx^3} - D_s \frac{d^4 w_0}{dx^4} - H_s \frac{d^4 \varphi}{dx^4} + A_s \frac{d^2 \varphi}{dx^2} = I_0 \ddot{w}_0 + J_1 \frac{d\ddot{u}}{dx} - J_2 \frac{d^2 \ddot{w}_0}{dx^2} - K_2 \frac{d^2 \ddot{\varphi}}{dx^2}$$

Where $A$, $B$, etc., are the beam stiffness, defined by

$$\left( A, B, D, B_s, D_s, H_s \right) = \int_{-h/2}^{h/2} Q_{11} \left( 1, z, z^2, f, z\,f, f^2 \right) dz, \tag{19a}$$

$$A_s = \int_{-h/2}^{h/2} Q_{55} g^2 \, dz, \tag{19b}$$

## 3. Analytical solution

The equations of motion cited above are analytically resolved for bending and free vibration problems. The Navier solution procedure is employed to determine the analytical solutions for a simply supported FG beam. The variables $u_0$, $w_0$, $\varphi$ can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_0 \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_n \cos(\lambda x) e^{i\omega t} \\ W_n \sin(\lambda x) e^{i\omega t} \\ \phi_n \sin(\lambda x) e^{i\omega t} \end{Bmatrix}, \tag{20}$$

Where $U_n$, $W_n$, and $\Phi_n$ are arbitrary parameters to be determined, $\omega$ is the eigenfrequency associated with $m^{\text{th}}$ eigenmode, and $\lambda = m\pi/L$. The transverse load $q$ is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda x) \tag{21}$$

Where $Q_m$ is the load amplitude calculated from

$$Q_m = \frac{2}{L} \int_0^L q(x) \sin(\lambda x)\, dx, \tag{22}$$

The coefficients $Q_m$ are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m=1 \text{ and } Q_1 = q_0 \tag{23}$$

And for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \quad (m = 1, 3, 5....) \tag{24}$$

Substituting the expansions of $u_0$, $w$, $\varphi$, and $q$ from Eqs. (20) and (21) into the equations of motion Eq. (18), the analytical solutions can be obtained from the following equations

$$\left( \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \right) \begin{Bmatrix} U_n \\ W \\ \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ 0 \end{Bmatrix}, \tag{25}$$

Where

$$s_{11} = A\lambda^2 \,,\; s_{12} = -B\lambda^3 \,, s_{13} = -B_s\lambda^3 \,,\; s_{22} = D\lambda^4 \,,\; s_{23} = D_s\lambda^4 \,, s_{33} = H_s\lambda^4 + A_s\lambda^2$$
$$m_{11} = I_0 \,,\; m_{12} = -I_1\lambda \,,\; m_{13} = -J_1\lambda \,,\; m_{22} = I_0 + I_2\lambda^2 \,,\; m_{23} = I_0 + J_2\lambda^2 \,, \tag{26}$$
$$m_{33} = I_0 + K_2\lambda^2$$

Table 1 Comparison of nondimensional static deflections and stresses of FG beams under uniform load

| $k$ | Model | $L/h$=5 | | | | $L/h$=20 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\bar{w}$ | $\bar{u}$ | $\bar{\sigma}_x$ | $\bar{\tau}_{xz}$ | $\bar{w}$ | $\bar{u}$ | $\bar{\sigma}_x$ | $\bar{\tau}_{xz}$ |
| 0 | Li *et al*. (2010) | 3.1657 | 0.9402 | 3.8020 | 0.7500 | 2.8962 | 0.2306 | 15.0130 | 0.7500 |
| | Thai and Vo 2012 | 3.1654 | 0.9397 | 3.8017 | 0.7312 | 2.8962 | 0.2306 | 15.0129 | 0.7429 |
| | Ould *et al*. (2013) | 3.1651 | 0.9406 | 3.8043 | 0.7489 | 2.8962 | 0.2305 | 15.0136 | 0.7625 |
| | Present | 3.1323 | 0.9214 | 3.7511 | 0.7546 | 2.8934 | 0.2302 | 15.0000 | 0.7672 |
| 0.5 | Li *et al*. (2010) | 4.8292 | 1.6603 | 4.9925 | 0.7676 | 4.4645 | 0.4087 | 19.7005 | 0.7676 |
| | Thai and Vo 2012 | 4.8285 | 1.6595 | 4.9920 | 0.7484 | 4.4644 | 0.4087 | 19.7003 | 0.7599 |
| | Ould *et al*. (2013) | 4.8282 | 1.6608 | 4.9956 | 0.7676 | 4.4644 | 0.4087 | 19.7013 | 0.7795 |
| | Present | 4.7888 | 1.6267 | 4.9033 | 0.7717 | 4.4609 | 0.4081 | 19.6781 | 0.7840 |
| 1 | Li *et al*. (2010) | 6.2599 | 2.3045 | 5.8837 | 0.7500 | 5.8049 | 0.5686 | 23.2054 | 0.7500 |
| | Thai and Vo 2012 | 6.2594 | 2.3036 | 5.8831 | 0.7312 | 5.8049 | 0.5685 | 23.2052 | 0.7429 |
| | Ould *et al*. (2013) | 6.2590 | 2.3052 | 5.8875 | 0.7489 | 5.8049 | 0.5685 | 23.2063 | 0.7625 |
| | Present | 6.2036 | 2.2728 | 5.7977 | 0.7546 | 5.8001 | 0.5680 | 23.1834 | 0.7672 |
| 2 | Li *et al*. (2010) | 8.0602 | 3.1134 | 6.8812 | 0.6787 | 7.4415 | 0.7691 | 27.0989 | 0.6787 |
| | Thai and Vo 2012 | 8.0675 | 3.1127 | 6.8819 | 0.6685 | 7.4420 | 0.7691 | 27.0989 | 0.6802 |
| | Ould *et al*. (2013) | 8.0683 | 3.1146 | 6.8878 | 0.6870 | 7.4421 | 0.7691 | 27.1005 | 0.7005 |
| | Present | 7.9581 | 3.0747 | 6.7699 | 0.6930 | 7.4335 | 0.7684 | 27.0705 | 0.7057 |
| 5 | Li *et al*. (2010) | 9.7802 | 3.7089 | 8.1030 | 0.5790 | 8.8151 | 0.9133 | 31.8112 | 0.5790 |
| | Thai and Vo 2012 | 9.8271 | 3.7097 | 8.1095 | 0.5883 | 8.8181 | 0.9134 | 31.8127 | 0.5998 |
| | Ould *et al*. (2013) | 9.8345 | 3.7128 | 8.1187 | 0.6084 | 8.8186 | 0.9134 | 31.8151 | 0.6218 |
| | Present | 9.5482 | 3.6507 | 7.9457 | 0.6153 | 8.7981 | 0.9124 | 31.7712 | 0.6281 |
| 10 | Li *et al*. (2010) | 10.8979 | 3.8860 | 9.7063 | 0.6436 | 9.6879 | 0.9536 | 38.1372 | 0.6436 |
| | Thai and Vo 2012 | 10.9375 | 3.8859 | 9.7111 | 0.6445 | 9.6905 | 0.9536 | 38.1383 | 0.6572 |
| | Ould *et al*. (2013) | 10.9413 | 3.8898 | 9.7203 | 0.6640 | 9.6907 | 0.9537 | 38.1408 | 0.6788 |
| | Present | 10.5487 | 3.8338 | 9.5781 | 0.6706 | 9.6673 | 0.9525 | 38.0824 | 0.6847 |

• Ceramic ($P_c$: Alumina, $Al_2O_3$): $E_c$=380 GPa; $v$=0.3; $\rho_c$=3960 kg/m$^3$
• Metal ($P_m$: Aluminium, Al): $E_m$=70 GPa; $v$=0.3; $\rho_m$=2707 kg/m$^3$

## 4. Results and discussion

Here, some numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the bending and free vibration responses of simply supported FG beams. The FG beam is taken to be made of aluminum and alumina $Al/Al_2O_3$ with the following material properties:

### *4.1 Results for static analysis*

For the validation of our model in the case of bending analysis, we consider an FG beam

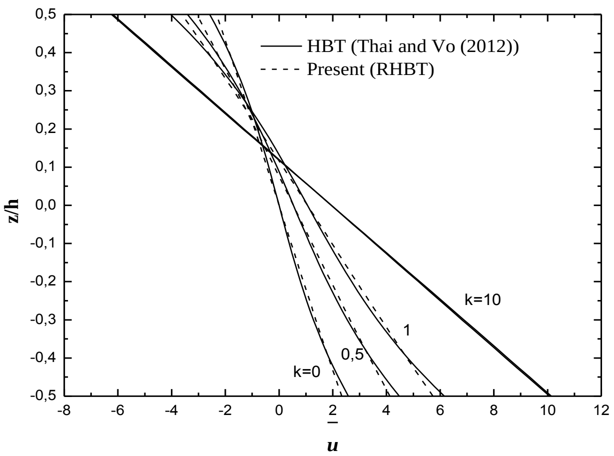


Fig. 2 Variation of the longitudinal displacement $\overline{u}$ through-the-thickness of a FG beam

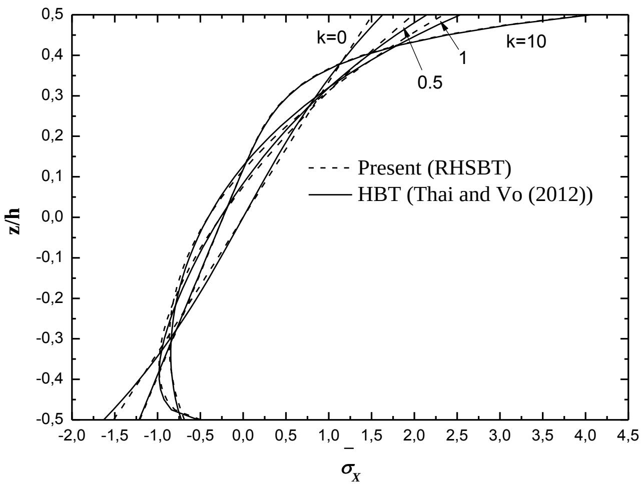


Fig. 3 The variation of the axial stress $\overline{\sigma}_x$ across-the-thickness of a FG beam ($L$=2 h)

subjected to a uniform load. For convenience, the following dimensionless forms are used

$$\overline{w}=100\frac{E_m h^3}{q_0 L^4}w\left(\frac{L}{2}\right),\ \overline{u}=100\frac{E_m h^3}{q_0 L^4}u\left(0,-\frac{h}{2}\right),\ \overline{\sigma}_x=\frac{h}{q_0 L}\sigma_x\left(\frac{L}{2},\frac{h}{2}\right),\ \overline{\tau}_{xz}=\frac{h}{q_0 L}\tau_{xz}\left(0,0\right),$$

$$\overline{\omega}=\frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}\ ,$$

The results obtained of our model are displayed in Table 1, and for various nondimensional displacements and stresses of FG beams under uniform load $q_0$ for different values gradient index $k$ and slenderness ratio $L/h$. One can observe that our results are in good correlations with the provided results by the existing efficient shear deformation beam theories (Li *et al*. 2010, Thai and Vo 2012, Ould *et al*. 2013).

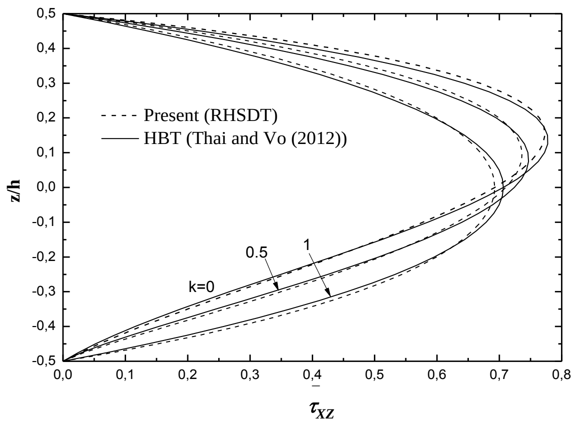


Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ across-the-thickness of a FG beam ($L$=2 h)

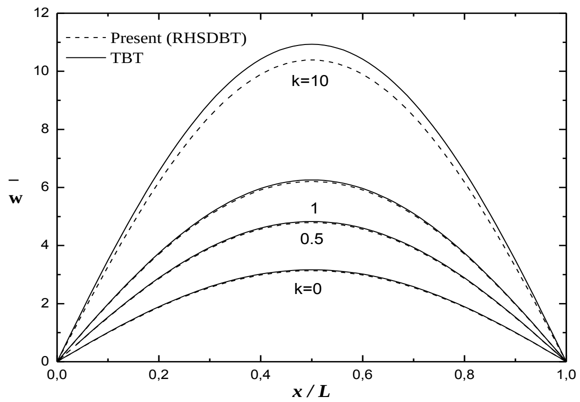


Fig. 5 Variation of the transverse displacement $\bar{w}$ versus non-dimensional length of a FG beam ($L$=5 h) bending stiffness of the FG beam

Figs. 2-4 demonstrate the variation of the longitudinal displacement $\bar{u}$, longitudinal stresses $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$ through the depth of the FG thick beam ($L$=2$h$), for the case of uniform load. A comparison with the analytical solutions given by Thai and Vo (2012) is also depicted in these figures using different values of the gradient index $k$. It is observed that there is a good concordance between the actual higher-order hyperbolic beam model and those of Thai and Vo (2012). It can also observed from these Figs. 2-4 that the rise of the power law exponent $k$ leads to an increase of the axial displacement $\bar{u}$, longitudinal stresses $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$. This is due to the fact that an increase in the volume fraction of metal will decrease the Fig. 5 presents the evolution of the dimensionless transversal displacement $\bar{w}$ versus nondimensional length for various values of the volume fraction exponent $k$. It is shown also that the present refined beam model offers very closes results to Reddy (TBT). Furthermore, the results show that the rise of the power law exponent $k$ leads to an increase of transversal displacement $\bar{w}$.

### *4.2 Free vibration analysis*

Table 2 Variation of dimensionless frequency $\bar{\omega}$ with various gradient indices $k$ for FG beam

| $L/h$ | Model | $k$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 0 | 0.5 | 1 | 2 | 5 | 10 |
| 5 | Simsek (2010) | 5.1527 | 4.4111 | 3.9904 | 3.6264 | 3.4012 | 3.2816 |
| | Thai and Vo (2012) | 5.1527 | 4.4107 | 3.9904 | 3.6265 | 3.4014 | 3.2817 |
| | Present model | 5.1527 | 4.4109 | 3.9904 | 3.6264 | 3.4013 | 3.2856 |
| 10 | Simsek (2010) | 5.4603 | 4.6516 | 4.2050 | 3.8361 | 3.6485 | 3.5389 |
| | Thai and Vo (2012) | 5.4603 | 4.6516 | 4.2050 | 3.8361 | 3.6485 | 3.5390 |
| | Present model | 5.4603 | 4.6616 | 4.2050 | 3.8361 | 3.6484 | 3.5391 |

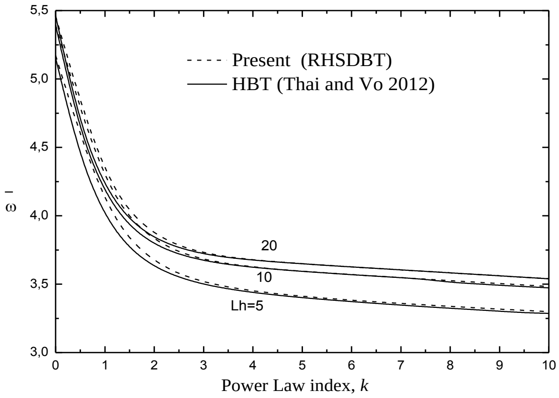


Fig. 6 Influence of the power law index $k$ on the dimensionless frequency $\bar{\omega}$ of FG beam with various span-to-depth ratio $L/h$

For the evaluate of the present refined hyperbolic shear deformation beam model in the case of free vibration, dimensionless fundamental frequencies $w$ obtained by the present theory are compared with those obtained by Thai and Vo (2012) and Simsek (2010) of FG beams for different values of power law index $k$ and slenderness ratio $L/h$ and the results are tabulated in Table 2. It can be observed that the results are in good correlations with the results obtained by Thai and Vo (2012), Simsek (2010).

The non-dimensional frequency of FG beam as a function of gradient index $k$ and for different values of slendness ratio $L/h$ using both the present theory and HBT (Thai and Vo 2012) is plotted in Fig. 6 and a close agreement between the present theory and HBT is shown. It is also seen that the frequency values decrease with increasing the power law index $k$. The full ceramic beams ($k$=0) lead to a highest frequency. However, the lowest frequency values are obtained for full metal beams ($k \rightarrow \infty$). The reason is an increasing in the value of the power indexes lead to grow the percentage of metal phase which make the FG beam more flexible, and thus a reduction in the fundamental frequency values.

## 5. Conclusions

In the present research work, static deflection and vibration analysis of functionally graded

(FG) beams are proposed according to a higher-order hyperbolic shear deformation beam model; in which the transverse shear stress vary hyperbolically through the thickness satisfying shear stress free surface conditions on the top and bottom surfaces of the beam without the need any shear correction factors. The governing differential equations of motion and the boundary conditions of FG beam are formulated through Hamilton's principle, and resolved analytically by Navier-type model for simply-simply boundary condition. According to the obtained results, it is found that the proposed model can provide very accurate results compared o the other solution results. The impacts of the power-law index and span-to-depth ratio on the deflection, stresses, and natural frequencies as well as load-frequency are explored.

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