Advances in Computational Design, *Vol. 5, No. 4 (2020) 363-380*
DOI: https://doi.org/10.12989/acd.2020.5.4.363

On mixing the Rayleigh-Ritz formulation with Hankel's function for vibration of fluid-filled functionally graded cylindrical shell

Muzamal Hussain*[1], Muhammad Nawaz Naeem[1], Aamir Shahzad[2], Muhammad Taj[3], Sehar Asghar[1], Alireza Fatahi-Vajari[4], Rahul Singh[5] and Abdelouahed Tounsi[6,7]

[1]Department of Mathematics, Govt. College University Faisalabad, 38000, Faisalabad, Pakistan
[2]Department of Physics, Govt. College University Faisalabad, Faisalabad 38040, Pakistan
[3]Department of Mathematics, University of Azad Jammu and Kashmir, Muzaffarabad, Azad Kashmir, Pakistan
[4]Department of Mechanical Engineering, Shahryar Branch, Islamic Azad University, Shahryar, Iran
[5]University Department of Mechanical Engineering, Rajasthan Technical University, Kota 324010, India
[6]Materials and Hydrology Laboratory University of Sidi Bel Abbes, Algeria Faculty of Technology Civil Engineering Department
[7]Department of Civil and Environmental Engineering, King Fahd University of Petroleum & Minerals, 31261 Dhahran, Eastern Province, Saudi Arabia

(Received December 31, 2019, Revised February 9, 2020, Accepted February 13, 2020)

Abstract. In this paper, a cylindrical shell is immersed in a non-viscous fluid using first order shell theory of Sander. These equations are partial differential equations which are solved by approximate technique. Robust and efficient techniques are favored to get precise results. Employment of the Rayleigh-Ritz procedure gives birth to the shell frequency equation. Use of acoustic wave equation is done to incorporate the sound pressure produced in a fluid. Hankel's functions of second kind designate the fluid influence. Mathematically the integral form of the Lagrange energy functional is converted into a set of three partial differential equations. Throughout the computation, simply supported edge condition is used. Expressions for modal displacement functions, the three unknown functions are supposed in such way that the axial, circumferential and time variables are separated by the product method. Comparison is made for empty and fluid-filled cylindrical shell with circumferential wave number, length- and height-radius ratios, it is found that the fluid-filled frequencies are lower than that of without fluid. To generate the fundamental natural frequencies and for better accuracy and effectiveness, the computer software MATLAB is used.

Keywords: Langrange functional; fluid-filled; MATLAB; Hankel's functions; Rayleigh-Ritz method

1. Introduction

Vibration of fluid-filled cylindrical shell problems occurs in industrial and engineering fields. The vibration analysis predicts to approximate their experimental results. The nature of a shell material plays an important role in specifying their vibration frequencies. Stability of a cylindrical

*Corresponding author, Ph.D., E-mail: muzamal45@gmail.com; muzamalhussain@gcuf.edu.pk

http://www.techno-press.org/?journal=acd&subpage=7 ISSN: 2383-8477 (Print), 2466-0523 (Online)

shell depends highly on these aspects of material.

More the shell material sustains a load due to physical situations, the more the shell is stable. Any predicted fatigue due to burden of vibrations is evaded by estimating their dynamical aspects. Study of vibration characteristics of fluid-filled cylindrical shells is a widely area of research in applied mathematics and theoretical mechanics. Analytical investigation of vibrations of these shells are performed to estimate the probable dynamical response. Addition of more physical parameters may give rise more instability in a system of a submerged cylindrical shell. During the recent years, study of submerged cylindrical shell has gained the attention of researchers doing work on their vibration characteristics. Advanced composite materials keep extreme particular stiffness, strength and are resistant to corrosion. The acoustic wave equation is applied to extract influence of a fluid on shell vibrations. Firstly, Love (1888) presented the Kirchhoff's hypotheses for plates. This theory became a foundation stage for building new ones by changing physical terms expressions. More than one type of materials is used to structure the functionally graded materials and their physical properties vary from one surface to the other surface. In these surfaces, one has highly heat resistance property while other may preserve great dynamical perseverance and differs mechanically and physically in regular manner from one surface to other surface, making them of dual physical appearance. All these materials have changeable outer and inner sides and their physical properties greatly differ from each other (Suresh and Mortensen 1997, Koizumi 1997). These materials are organized by various techniques and their applications are seen in dynamical elements such as plates, beams and shells. Moreover, they are also observed in space crafts, nuclear reactors and missiles technology, etc. Loy and Lam (1997) investigated shell vibrations with ring supports that restricted the motion of cylindrical shells in the transverse direction. This influence was inducted by the polynomial functions. Xiang *et al*. (2002) formed some closed form solution functions for studying vibrations of cylindrical shells. The mid-way ring supports were clamped around the shells. Sewall and Naumann (1968) considered the vibration analysis of cylindrical shell based on analytical and experimental methods. The shells were strengthened with longitudinal stiffeners. Sharma and Johns (1971) analyzed vibration frequencies circular cylinder with using the Rayleigh-Ritz formulation and made comparisons of his results with some experimental ones. Chung *et al*. (1981) investigated the vibrations of fluid-filled cylindrical shell and presented an analysis of experimental and analytical investigation. Goncalves and Batista (1988) gave an analytical investigation of submerged cylindrical shell with fluid.

Jiang and Olson (1994) recommended the characteristics of analysis of stiffened shell using finite element method to diminish large computational efforts which are required in the conventional finite element analysis. Wang *et al*. (1997) scrutinized the vibrations of ring-stiffened cylindrical shells using Ritz polynomial functions. Materials of both shells and rings were of isotropic nature. These shells were stiffened with isotropic rings having three types of locations on the shell outer surface. To increase the stiffness of cylindrical shells was stabilized by ring-stiffeners. Isotopic materials are the constituents of these rings. A large use of shell structures in practical applications makes their theoretical analysis an important field of structural dynamics. Since a shell problem is a physical one, so their vibrational behaviors are distorted by variations of physical and material parameters. To elude any complications which may risk a physical system their analytical investigation was done. Sofiyev *et al*. (2012) truncated the conical shells subjected to combined loads and resting on elastic foundations for two boundary condition. The functionally graded material properties are assumed to vary continuously through the thickness of the conical shell. Sharma *et al*. (1998) determined frequencies of composite cylindrical shells containing fluid.

They estimated the axial modal deformations by trigonometric functions. Amabili *et al*. (1998) assessed the free and forced non-linear vibrations of circular cylindrical shell with the quiescent, dense, inviscid and incompressible fluid. The analyses are made for the large moderately vibrations using Donnell's shallow-shell model. Also, the dense fluid is studied for the influence of both the internal and external side of the shell. In the external side of the shell, the fluid was considered as an unbounded domain in the radial direction, while internally, the shell was considered as filled completely. Mercan *et al*. (2016) investigated the free vibration response of circular cylindrical shells with functionally graded material. The constitutive relations are based on the Love's first approximation shell theory. The material properties are graded in the thickness direction according to a volume fraction power law indices. Wang and Lai (2000) examined a novel approach for the evaluation of eigen - frequencies of cylindrical shells. The numerical process adopted by them was alike the wave propagation approach. Ergin and Temarel (2002) did a vibration study of cylindrical shells. The shells lied in a horizontal direction and contained fluid and submerged in it. Zhang (2002) studied vibrations of cylindrical shells submerged in a fluid. It was seen that the fluid factor impressed vibration shell frequencies to a significant limit. Najafizadeh and Isvandzibaei (2007) applied ring supports to cylindrical shells for vibration analysis of along the tangential direction and founded their research on angular deformation theory of higher order. The angular deformation was used for shell equations and determined the effects of constituent volume fractions and shell configurations on the shell vibrations. Functionally graded material parameters were changed step by step.

Shah *et al*. (2009) and Sofiyev and Avcar (2010) studied stability of cylindrical shells based on Rayleigh-Ritz and Galerkin technique using elastic foundations. The structures of cylindrical shell are tackled under the exponential law and axial load. Ersoy *et al*. (2018) investigated numerically the free vibration analysis of curved structural components such as truncated conical shells. The method of discrete singular convolution and the method Differential Quadrature (DQ) are used for numerical simulations, respectively. Naeem *et al*. (2013) conducted the vibrational behavior of submerged functionally graded cylindrical shells. The problem of submerged cylindrical shells was frequently met where fluid envelopes a structure. The present problem consists of a cylindrical shell submerged in a fluid and surrounded by ring supports. There is no evidence found where this problem has not been studied earlier. Farahani and Barati (2015) focused the vibration analysis of functionally graded cylindrical shell submerged in an incompressible fluid. The equation is established considering axial and lateral hydrostatic pressure based on first order shear deformation theory of shell motion using the wave propagation approach and classic Flügge shell equations. Ansari *et al*. (2015) performed nonlocal model for the frequencies of multi-walled carbon nanotubes with small effects subject to various boundary conditions using Rayleigh-Ritz technique. The governing equation was formulated based on Flügge's and nonlocal shell theory. Some new resonant frequencies were identified with the association of vibrational modes and circumferential modes into shell model. Khayat *et al*. (2018) examined the free vibration of cylindrical shells made up of functionally graded material. The properties of functionally graded shells are assumed to be temperature-dependent and vary continuously in the thickness direction according to a simple power law distribution in terms of the volume fraction of ceramic and metal. Hussain *et al*. (2017) demonstrated an overview of Donnell theory for the frequency characteristics of two types of SWCNTs. Fundamental frequencies with different parameters have been investigated with wave propagation approach. Hussain and Naeem (2017) examined the frequencies of armchair tubes using Flügge's shell model. The effect of length and thickness-to-radius ratios against fundamental natural frequency with different indices of armchair tube was investigated. On continuing their work, Hussain and Naeem (2018a, b) used Donnell's shell model to calculate the dimensionless frequencies for two types of single-walled carbon nanotubes. The frequency influence was observed with different parameters. Li *et al*. (2019) analyzed the free vibration characteristics of uniform and stepped functionally graded circular cylindrical shells under complex boundary conditions. The analytical model is established based on multi-segment partitioning strategy and first-order shear deformation theory.

Sharma *et al*. (2019) studied the functionally graded material using sigmoid law distribution under hygrothermal effect. The Eigen frequencies are investigated in detail. Frequency spectra for aspect ratios have been depicted according to various edge conditions. Avcar (2019) presented the free vibration of beams made of imperfect functionally graded materials including porosities is investigated. Because of faults during process of manufacture, micro voids or porosities may arise in the functionally graded material, and this situation causes imperfection in the structure. Recently, Hussain and Naeem (2019a, 2019b) performed the vibration of SWCNTs based on wave propagation approach and Galerkin's method.

According to our knowledge, up to now little is known about the vibration analyses of fluid-filled functionally graded cylindrical shells based on Rayleigh-Ritz method. A large use of shell structures in practical applications makes their theoretical analysis an important field of structural dynamics. Since a shell problem is a physical one, so their vibrational behaviors are distorted by variations of physical and material parameters. It is also exhibited that the effect of frequencies by varying the different layers with constituent material. The coupled frequencies changes with these layers according to the material formation of fluid-filled functionally graded cylindrical shells. Also, the Sander's theory based on the Rayleigh-Ritz method for estimating fundamental natural frequency has been developed to converge more quickly than other methods and models. The presented vibration modeling and analysis of cylindrical shell may be helpful especially in applications such as oscillators and in non-destructive testing. To elude any complications which may risk a physical system their analytical investigation is done.

2. Functionally graded material

The modeling of functionally graded cylindrical shell is due to mixing two or more than two materials like ceramic and metal and the distribution of various functions and properties (physical and material), is termed as rule of mixture. Power law function has been utilized for with particular index using material properties in the thickness direction. The temperature and properties variations have been obtained by using the property of temperature and volume fraction. The distributions of volume fraction for all types of cylindrical shell are assumed as (Chi and Chung 2006)

$$V_f = \left[\frac{z}{h} + \frac{1}{2}\right]^N \tag{1}$$

where *N*, *h* and z, respectively, denoted for power law index, thickness and the coordinate, where *z* which varies from zero to infinity.

A functionally graded cylindrical shells consisting of two constituent materials. In these types, nickel and stainless steel are used as the interior surfaces and the exterior surface respectively, but their arrangement has profound influence on the formation of functionally graded cylindrical shells. If E_1 and E_2 as Young's moduli, ν_1 and ν_2 as Poisson's ratios, ρ_1 and ρ_2 mass densities respectively. Then effective material quantities for functionally graded material are

$$E_{FGM} = [E_1 - E_2]\left[\frac{2z+h}{2h}\right]^N + E_2, \qquad v_{FGM} = [v_1 - v_2]\left[\frac{2z+h}{2h}\right]^N + v_2 \tag{2}$$

$$\rho_{FGM} = [\rho_1 - \rho_2]\left[\frac{2z+h}{2h}\right]^N + \rho_2 \tag{3}$$

Toulokian (1967) stated the material properties C at high temperature environ, with temperature-dependents which is a function of temperature. In Eq. (3), the constants $(C_0,\ C_{-1},\ C_1,\ C_2,\ C_3)$ are different for different material.

$$C = C_0(C_{-1}T^{-1} + C_1T + C_2T^2 + C_3T^3) \tag{4}$$

3. Theoretical formation

Geometrical structure of a cylinder is sketched in Fig. 1. The tube is assumed to have length L, thickness h and the radius R for cylindrical shell with its coordinate system (x, θ, z) as shown in Fig. 1. The x, θ co-ordinates are assumed to be along longitudinal and circumferential direction, respectively and z-co-ordinates are taken in its radial directions.

When the material and geometrical parameters are considered, the formula for a strain energy, S of a vibrating cylindrical shell is expressed as

$$S = \frac{R}{2}\int_0^L\int_0^{2\pi} [A_{11}e_1^2 + A_{22}e_2^2 + 2A_{12}e_1e_2 + A_{66}e_{12}^2 + 2(B_{11}e_1k_1 + B_{11}e_1k_1 + B_{11}e_1k_1 + B_{11}e_1k_1 + 2B_{66}e_{12}k_{12}) + D_{11}k_1^2 + D_{22}k_2^2 + 2D_{12}k_1k_2 + D_{66}^2k_{12}^2]d\theta dx \tag{5}$$

where e_1, e_2 and e_3 designate the reference surface strains and k_1, k_2 and k_3 denote the reference surface curvatures respectively. The extensional stiffness, A_{ij}, coupling stiffness, B_{ij} and bending stiffness, D_{ij} are written as

$$\{A_{ij}, B_{ij}, D_{ij}\} = \int_{h/2}^{h/2} Q_{ij}\{1, z, z^2\}dz(i,j = 1,2,6) \tag{6}$$

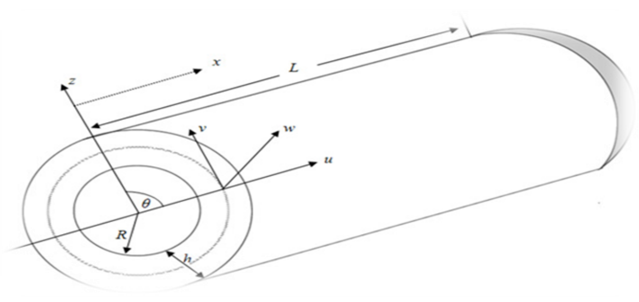

Fig. 1 Geometry of cylindrical shell

Here the reduced stiffness, Q_{ij}'s for isotropic material are written as

$$Q_{11}=Q_{22}=\frac{E}{1-v^2},\qquad Q_{12}=\frac{vE}{1-v^2},\qquad Q_{66}=\frac{E}{2(1+v)} \tag{7}$$

4. Application of Budiansky and Sander's shell theory

The strain-displacement relations from Budiansky and Sanders (1963) theory are furnished as

$$e_{12}=\frac{\partial y}{\partial x},\qquad e_{22}=\frac{1}{R}\left(\frac{\partial v}{\partial \theta}-w\right),\qquad e_{12}=\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial \theta} \tag{8}$$

and the expressions for the curvature - displacement relations are represented as

$$k_{11}=\frac{\partial^2 w}{\partial x^2},\qquad k_{22}=\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial v}{\partial \theta}\right),\qquad k_{12}=\frac{1}{R}\left(\frac{\partial^2 w}{\partial x\partial \theta}+\frac{3}{4}\frac{\partial v}{\partial x}-\frac{1}{4R}\frac{\partial u}{\partial \theta}\right) \tag{9}$$

Making substitutions of these relations from the Eqs. (8) and (9) into the Eq. (5), the shell strain energy, *S* takes the following forms

$$\begin{aligned}S=\frac{1}{2}\iint_{0\,0}^{2\pi L}&\Big[A_{11}\left(\frac{\partial u}{\partial x}\right)^2+A_{22}\frac{1}{R^2}\left(\frac{\partial v}{\partial \theta}+w\right)^2+2A_{12}\frac{1}{R}\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial \theta}+w\right)+A_{66}\left(\frac{\partial v}{\partial \theta}+\frac{1}{R}\frac{\partial u}{\partial x}\right)^2\\&-2B_{11}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x^2}\right)-2B_{12}\frac{1}{R^2}\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial \theta^2}-\frac{\partial v}{\partial \theta}\right)-2B_{12}\frac{1}{R}\left(\frac{\partial v}{\partial \theta}+w\right)\left(\frac{\partial^2 w}{\partial x^2}\right)\\&-2B_{22}\frac{1}{R^3}\left(\frac{\partial v}{\partial \theta}+w\right)\left(\frac{\partial^2 w}{\partial \theta^2}-\frac{\partial v}{\partial \theta}\right)-4B_{66}\frac{1}{R}\left(\frac{\partial v}{\partial \theta}+\frac{1}{R}\frac{\partial u}{\partial x}\right)\left(\frac{\partial^2 w}{\partial x\partial \theta}-\frac{3}{4}\frac{\partial v}{\partial x}+\frac{1}{4R}\frac{\partial u}{\partial \theta}\right)\\&+D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{D_{22}}{R^4}\left(\frac{\partial^2 w}{\partial \theta^2}-\frac{\partial v}{\partial \theta}\right)^2+2D_{12}\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial \theta^2}-\frac{\partial v}{\partial \theta}\right)+4D_{66}\frac{1}{R^2}\left(\frac{\partial^2 w}{\partial x\partial \theta}\right.\\&\left.-\frac{3}{4}\frac{\partial v}{\partial x}+\frac{1}{4R}\frac{\partial u}{\partial \theta}\right)^2\Big]R\,dx\,d\theta\end{aligned} \tag{10}$$

The shell kinetic energy, *K* of the cylindrical shell is written as

$$K=\frac{1}{2}\iint_{0\,0}^{2\pi L}\rho_t\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]R\,dx\,d\theta \tag{11}$$

ρt designates the mass density per unit length and is written as

$$\rho_t=\int_{h/2}^{h/2}\rho\,dz \tag{12}$$

where ρ stands for the mass density. The Lagrangian energy functional is obtained by combing the above said energies.

$$\prod=K-S \tag{13}$$

5. Solution methodology

Here Rayleigh's method is used to solve the cylindrical shell problem of differential equations in an efficient and comprehensive way. This method needs the axial modal approximates dependence on the characteristic function. The governing equation was formulated based on Sander's thin shell theory with energy functional. Over the past several years vibration of tube/shell and plate structures of various configurations and boundary conditions have been extensively studied (Hussain and Naeem 2018a, 2020, Hussain *et al*. 2018a, 2018b, 2019a, 2019b, 2019c, Sehar *et al*. 2020). The Rayleigh-Ritz method is very powerful technique for the prediction of vibration of shells/tubes. Here the Rayleigh-Ritz method is employed to obtain the shell frequency equation. For separating the space and time variables, the product method for PDEs has been utilized and modal deformation displacement functions are supposed in the form of product of functions of the axial, tangential and time variables. A cylindrical shell problem is described in which the three unknown functions represent the deformation displacement. The vibration investigation of submerged cylindrical shells is studied here. This problem crops up while examining response of waves in water, noise and vibration, etc.

$$\begin{aligned} u(x,\theta,t) &= U(x)\sin(n\theta)\sin(\omega t) \\ v(x,\theta,t) &= V(x)\cos(n\theta)\sin(\omega t) \\ w(x,\theta,t) &= W(x)\sin(n\theta)\sin(\omega t) \end{aligned} \tag{14}$$

The number of circumferential wave mode *n* and axial wave mode *m* demonstrate for modes of vibrations of a fluid-filled functionally graded cylindrical shell. $U(x)$, $V(x)$ and $W(x)$ are the axial modal deformation, respectively, in the direction of longitudinal, tangential and transverse.

For generality, the following non-dimension parameters are written as

$$\bar{U} = \frac{U}{h}\ , \quad \bar{V} = \frac{V}{h}\ , \quad \bar{W} = \frac{W}{R}\ , \quad a = \frac{L}{R}\ , \quad b = \frac{h}{R}\ , \quad X = \frac{x}{L} \tag{15}$$

Using these quantities, the expressions designate in the equation (18) is re-framed as

$$\begin{gathered} u(x,\theta,t) = h\bar{U}\sin(n\theta)\sin(\omega t), \qquad v(x,\theta,t) = h\bar{V}\cos(n\theta)\sin(\omega t) \\ w(x,\theta,t) = R\bar{W}\sin(n\theta)\sin(\omega t) \end{gathered} \tag{16}$$

The displacement function was first invoked by Flügge (1962) to clarify the problem of cylindrical shells and then used by Forsberg (1964) and Warburton (1965). The following expressions for Ritz polynomial functions are taken for measuring the axial modal deformations: $\bar{U}$, $\bar{V}$ and $\bar{W}$ that fulfill the edge conditions

$$\begin{aligned} \bar{U} &= \sum_{i=1}^{N} a_i\,\bar{U}_i = \sum_{i=1}^{N} a_i\,X^{i-1}X^{n_u^0}\ (1-X)^{n_u^1}, \\ \bar{V} &= \sum_{i=1}^{N} b_i\,\bar{V}_i = \sum_{i=1}^{N} b_i\,X^{i-1}X^{n_v^0}\ (1-X)^{n_v^1} \\ \bar{W} &= \sum_{i=1}^{N} c_i\,\bar{W}_i = \sum_{i=1}^{N} c_i\,X^{i-1}X^{n_w^0}\ (1-X)^{n_w^1} \end{aligned} \tag{17}$$

where the exponents $n_u^0,\ n_u^1,\quad n_v^0,\ n_v^1,\quad n_w^1\ and\quad n_w^0$ are used for boundary conditions.

The Lagrange function is achieved in the dimensionless quantities given in Eq. (13) as follows

$$\prod = \int_0^1 \Big\{\frac{\pi h L R^3}{2}[(b^2\sum_{i=1}^{N} a_1\,\bar{U}_i)^2 + (b^2\sum_{i=1}^{N} b_i\,\bar{V}_i)^2 + (b^2\sum_{i=1}^{N} c_i\,\bar{W}_i)^2\ ]\bar{\rho}_t\omega^2 cos^2\omega t - \frac{\pi h L R}{2}$$
$$[a^2b^2\bar{A}_{11}(\sum_{i=1}^{N} a_i\frac{d\bar{U}_i}{dX})^2 + \bar{A}_{22}(-nb\sum_{i=1}^{N} b_i\bar{V}_i + \sum_{i=1}^{N} c_i\,\bar{W}_i)^2 + 2ab\bar{A}_{12}(\sum_{i=1}^{N} a_i\frac{d\bar{U}_i}{dX})\left(-nb\sum_{i=1}^{N} b_i\,\bar{V}_i\right.$$
$$\left.+\sum_{i=1}^{N} c_i\,\bar{W}_i + \bar{A}_{66}(ab\sum_{i=1}^{N} b_i\frac{d\bar{V}_i}{dX}s + nb\sum_{i=1}^{N} a_i\,\bar{U}_i)^2 - 2a^3b^2\bar{B}_{11}(\sum_{i=1}^{N} a_i\frac{d\bar{U}_i}{dX}\right)\left(\sum_{i=1}^{N} c_i\frac{d^2\bar{W}_i}{dX^2}\right)$$
$$-2ab^2\bar{B}_{12}\left(\sum_{i=1}^{N} a_i\frac{d\bar{U}_i}{dX}\right)\left(-n^2\sum_{i=1}^{N} c_i\,\bar{W}_i + nb\sum_{i=1}^{N} b\bar{V}_i\right) - 2a^2b\bar{B}_{12}\left(-nb\ \sum_{i=1}^{N} b_i\bar{V}_i + \sum_{i=1}^{N} c_i\,\bar{W}_i\right)$$
$$\left(\sum_{i=1}^{N} c_i\frac{d^2\bar{W}_i}{dX^2}\right) - 2b\bar{B}_{22}\left(-nb\sum_{i=1}^{N} b_i\bar{V}_i + \sum_{i=1}^{N} c_i\,\bar{W}_i\right)\left(-n^2\sum_{i=1}^{N} c_i\,\bar{W}_i + nb\ \sum_{i=1}^{N} b_i\bar{V}_i\ \right) - 4b\bar{B}_{66} \tag{18}$$
$$(ab\sum_{i=1}^{N} b_i\frac{d\bar{V}_i}{dX} + nb\sum_{i=1}^{N} a_i\,\bar{U}_i)(na\sum_{i=1}^{N} c_i\frac{d\bar{W}_i}{dX} - \frac{3ab}{4}\sum_{i=1}^{N} b_i\,\bar{V}_i + \frac{nb}{4}\sum_{i=1}^{N} a_i\,\bar{U}_i\ )\ +\ a^4b^2\bar{D}_{11}$$
$$(\sum_{i=1}^{N} c_i\frac{d^2\bar{W}_i}{dX^2})^2 +\ b^2\bar{D}_{22}(-n^2\sum_{i=1}^{N} c_i\,\bar{W}_i + nb\sum_{i=1}^{N} b_i\,\bar{V}_i)^2 + 2a^2b^2\bar{D}_{12}(\sum_{i=1}^{N} c_i\frac{d^2\bar{W}_i}{dX^2})\left(-n^2\sum_{i=1}^{N} c_i\,\bar{W}_i\right.$$
$$\left.+nb\sum_{i=1}^{N} b_i\,\bar{V}_i\right) + 4b^2\bar{D}_{66}\left(\ na\sum_{i=1}^{N} c_i\frac{d\bar{W}_i}{dX} -\ \ \frac{3ab}{4}\sum_{i=1}^{N} b_i\,\bar{V}_i + \frac{nb}{4}\sum_{i=1}^{N} a_i\,\bar{U}_i)^2\right]\ sin^2wt\}dX$$

6. Use of the Rayleigh-Ritz procedure

To obtain necessary extreme conditions the Lagrangian functional $\prod$ is differentiated with regard to the generalized Fourier coefficients $a_i,\ b_i,\ c_i$. The following conditions are obtained

$$\frac{\partial\Pi}{\partial a_i} = \frac{\partial\Pi}{\partial b_i} = \frac{\partial\Pi}{\partial c_i} = 0 \qquad \text{where } i = 1,2,\ldots N \tag{19}$$

These equations are written in the following complete forms after concealing a huge amount of algebraic process

$$\sum_{j=1}^{N} a_j\,[a^2b^2\bar{A}_{11}\int_0^1 \frac{d\bar{U}_i}{dx}\frac{d\bar{U}_j}{dx}dX + n^2b^2(\bar{A}_{66} - b\bar{B}_{66} + \frac{b^2\bar{D}_{66}}{4})\int_0^1 \bar{U}_i\bar{U}_j\,dX]\ \ + \sum_{j=1}^{N} b_j\,[-nab^2$$
$$(\bar{A}_{12} + b\bar{B}_{12})\int_0^1 \frac{d\bar{U}_i}{dX}\bar{V}_j dX + nab^2(\bar{A}_{66} + b\bar{B}_{66} - \frac{3b^2\bar{D}_{66}}{4})\int_0^1 \bar{U}_i\frac{d\bar{V}_j}{dX}dX]\ \ + \sum_{j=1}^{N} c_j\,[ab(\bar{A}_{12} \tag{20}$$
$$+n^2b\bar{B}_{12})\int_0^1 \frac{d\bar{U}_i}{dX}\bar{W}_j dX + n^2ab^2(-2\bar{B}_{66} + b\bar{D}_{66})\int_0^1 \bar{U}_i\frac{d\bar{W}_j}{dX}dX - a^3b^2\bar{B}_{11}\int_0^1 \frac{d\bar{U}_j}{dX}\frac{d^2\bar{W}_j}{dX^2}dX]$$
$$= R^2\bar{\rho}_t\omega^2\textstyle\sum_{j=1}^{N} a_j\,b^2\int_0^1 \bar{U}_i\bar{U}_j\,dX$$

$$\sum_{j=1}^{N} a_j\,[-nab^2(\bar{A}_{12}+b\bar{B}_{12})\int_0^1 \frac{d\bar{U}_j}{dX}\bar{V}_i dX + nab^2(\bar{A}_{66}+b\bar{B}_{66}-\frac{3b^2\bar{D}_{66}}{4}\int_0^1 \bar{U}_j\frac{d\bar{V}_i}{dX}dX$$
$$+\sum_{j=1}^{N} b_j\,[a^2b^2(\bar{A}_{66}+3b\bar{B}_{66}+\frac{9b^2\bar{D}_{66}}{4})\int_0^1 \frac{d\bar{V}_i}{dX}\frac{d\bar{V}_j}{dX}dX + (\bar{A}_{22}+2b\bar{B}_{22}+b^2\bar{D}_{22})$$
$$+\sum_{j=1}^{N} c_j\,[-nb(\bar{A}_{22}+(n^2+1)b\bar{B}_{22}+n^2b^2\bar{D}_{22})\int_0^1 \bar{V}_i\,\bar{W}_j d - n^2b^2(2\bar{B}_{66}+3b\bar{D}_{66}) \tag{21}$$
$$\int_0^1 \frac{d\bar{V}_j}{dX}\frac{d\bar{W}_j}{dX}dX + nab^2(\bar{B}_{12}+b\bar{D}_{66})\int_0^1 \bar{V}_i\frac{d^2\bar{W}_i}{dX^2}dX] = R^2\bar{\rho}_t\omega^2\sum_{j=1}^{N} b_j\,b^2\int_0^1 \bar{V}_i\,\bar{V}_j dX$$

$$\sum_{j=1}^{N} a_j\,[ab(\bar{A}_{12}+n^2b\bar{B}_{12})\int_0^1 \frac{d\bar{U}_j}{dX}\bar{W}_j dX + n^2ab^2(-2\bar{B}_{12}+b\bar{D}_{66})\int_0^1 \bar{U}_j\frac{d\bar{W}_i}{dX}dX - n^2b^2\bar{B}_{11}$$
$$\int_0^1 \frac{d\bar{U}_j}{dX}\frac{d^2\bar{W}_i}{dX}dX] + \sum_{j=1}^{N} b_j\,[-nb(\bar{A}_{12}+(n^2+1b\bar{B}_{22}+n^2b^2\bar{D}_{22})\int_0^1 \bar{V}_i\,\bar{W}_i dX - na^2b^2(2\bar{B}_{66}$$
$$+3b\bar{D}_{66})\int_0^1 \frac{d\bar{V}_j}{dX}\frac{d\bar{W}_j}{dX}dX + na^2b^2(2\bar{B}_{12}+3b\bar{D}_{66})\int_0^1 \frac{d\bar{V}_j}{dX}\frac{d\bar{W}_i}{dX}dX + n\int_0^1 \frac{d\bar{V}_j}{dX}\frac{d\bar{W}_j}{dX}dXb^2(\bar{B}_{12}$$
$$+b\bar{D}_{12}) \;+ n\int_0^1 \frac{d\bar{V}_j}{dX}\frac{d\bar{W}_j}{dX}dXb^2(\bar{B}_{12}+b\bar{D}_{12})\int_0^1 \bar{V}_j\frac{d^2\bar{W}_i}{dX^2}dX] + \sum_{j=1}^{N} c_j\,[(\bar{A}_{22}+2n^2b\bar{B}_{22} \tag{22}$$
$$+n^4b^2\bar{D}_{22})\int_0^1 \bar{W}_i\,\bar{W}_j dX + 4n^2a^2b^2\bar{D}_{66}\int_0^1 \frac{d\bar{W}_i}{dX}\frac{d\bar{W}_j}{dX}dX - a^2b(\bar{B}_{12}+n^2b\bar{D}_{12})\int_0^1 (\bar{W}_i\frac{d^2\bar{W}_j}{dX^2}$$
$$+\frac{d^2\bar{W}_i}{dX^2}\bar{W}_i)dX + a^4b^2\bar{D}_{11}\int_0^1 \frac{d^2\bar{W}_i}{dX^2}\frac{d^2\bar{W}_j}{dX^2}dX] = R^2\bar{\rho}_t\omega^2\sum_{j=1}^{N} c_j\,b^2\int_0^1 \bar{W}_i\,\bar{W}_j dX$$

After the ordering of above equations, the shell vibration frequency is expressed by the generalized eigenvalue relation as

$$\{[K]-\Delta[M]\}[\underline{x}]^T = 0 \tag{23}$$

where $[K]$ and $[M]$ represent the stiffness and mass matrices for the cylindrical shells and

$$[\underline{x}]^T = (a_1, a_2, \dots, a_N, b_2, \dots b_N, c_1, c_2, \dots, c_N) \tag{24}$$

and

$$\Delta_1 = R^2\overline{\rho}_t\omega^2 \tag{25}$$

The eigenvalues represent with the shell frequencies and the corresponding eigenvectors designate the mode shapes.

7. Annexation of fluid term

The acoustic wave equation represents pressure of sound in fluid and this equation of motion describing fluid is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \tag{26}$$

where (x, θ, r) are the cylindrical coordinates and ϕ, t, c, r stands respectively, for acoustic pressure, time variable, fluid sound speeds and the axial coordinate adopted from the shell axis. The acoustic pressure expression for an immersed cylinder in a fluid that satisfied the acoustic wave Eq. (26), written in following form

$$\phi=\phi_m\, sin(n\theta)H_n^{(2)}(k_r r)\psi(x)\,cos\,\omega\,t \tag{27}$$

where ϕ_m symbolizes the pressure amplitude, $H_n^{(2)}(k_r r)$ denotes the second kind of Hankel functions with order n. The radial and axial wave numbers k_r and k_x respectively linked by the vector equation, $k_r=(k_0-k_x)^{1/2}$, where $k_0=\frac{w}{c}$ is written for the acoustic wave number of the fluid. Here k_r has many values and depends on the variable k_x. In order to ensure that the sound field fulfills the suitable conditions of radiation and decay as $r\to\infty$, the branch that meets the condition $k_r=\sqrt{k_0-k_x}$ for $k_0\geq k_r$ and $k_r=-i\sqrt{k_x-k_0}$ for $k_0<k_x$, is chosen. For the assurance of keeping fluid connection with shell wall, the radial displacement of the fluid must be equal to that at the boundary of the outer wall of shell and the fluid.

The coupling condition is applied and is written as

$$-\{1/(iw\phi_\rho)\}(\partial\phi/\partial r)|_{r=R}=(\partial w/\partial t)|_{r=R} \tag{28}$$

Subsequently the above condition has new form

$$\phi_m=\left[\omega^2\rho_f/k_r H_n^{\prime(2)}(k_r r)\right]C_m \tag{29}$$

where ρ_f signifies the fluid density and the dot written upon the $H_n^{(2)}(k_r r)$ represents the differentiation w, r, t the argument $k_r R$. With the application of the coupling condition Eq. (29) along with the relation Eq. (25), the shell frequency of the submerged cylindrical shell is given by i as

$$\{[K+FL]-\Delta[M]\}[\underline{x}]^T=0 \tag{30}$$

FL defines the fluid loading term and is written as

$$FL=\Omega^2(\rho_f/\rho_s)(R/h)(k_r R)^{-1}\left[H_n^{(2)}(k_r R)/H_n^{\prime(2)}(k_r R)\right] \tag{31}$$

When the fluid term reduces to zero, the frequency equation fluid-filled cylindrical shells converts into that for the empty shell case.

Table 1 Comparison of the frequency parameter $\omega^* = \omega R\sqrt{(1-\nu^2)\rho/E}$ for a long rotating isotropic cylindrical shell with m=1, h/R=0.002, L/R = 200, ν = 0

Finite element method (Chen *et al.* 1993)	RRM
0.00167	0.00166
0.00447	0.00445
0.00847	0.00846
0.01364	0.01362

Table 2 Convergence of Rayleigh-Ritz method frequencies

Method		Modal order (m, n)			
		(1,3)	(2,3)	(3,3)	(3,4)
Coupled frequency	Zhang *et al.* (2001)	8.94	10.64	14.66	19.96
	Present	8.90	10.62	14.59	19.85
Uncoupled frequency	Zhang *et al.* (2001)	19.61	23.28	31.98	39.78
	Present	19.6	23.31	32.01	39.81

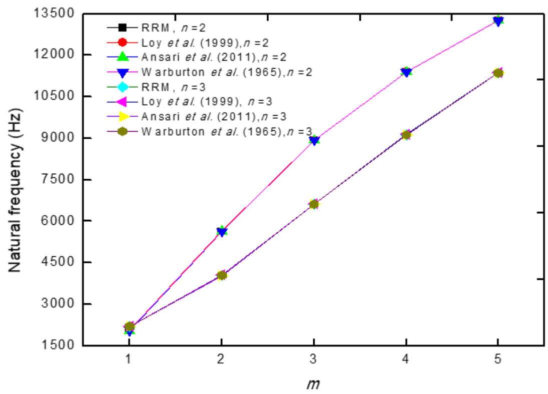

Fig. 2 Convergence of Rayleigh-Ritz method frequencies (L=8 in, h=0.1 in, R=2 in, E=30×10^6 lbf in^{-2}, ν=0.3, ρ=7.35×10^{-4} lbf s^2 in^{-4})

8. Simulation results and discussion

In this section, the versatile numerical technique Rayleigh-Ritz method has been used in current study to study the frequency analysis of fluid-filled functionally graded cylindrical shells. For the convergence rate of cylindrical shell, the non-dimensional frequency enumerated in the current work, i.e., using Rayleigh-Ritz method. The proposed model based on Rayleigh-Ritz method can incorporate in order to accurately predict the acquired results of material data point. Table 1 shows the comparison of present results with finite element method (Chen *et al*. 1993). In Table 2, the coupled and uncouple frequency results are well matched those evaluated by Zhang *et al*. (2001) for different modal numbers for C-C shells. The proposed model based on Rayleigh-Ritz method can incorporate in order to accurately predict the acquired results of material data point. In this section, using Rayleigh-Ritz method happened to be in a good consistency along with the so-called

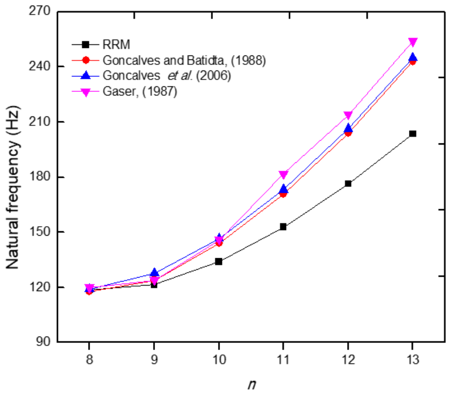

Fig. 3 Convergence of Rayleigh-Ritz method frequencies (m=1, L=0.41 m, h=0.001 m, R=0.3015 m, E=2.1 ×10^{11} N/m^2, ν=0.3, ρ=7850 Kg/m^3)

Table 3 Comparison of fluid-filled and without fluid versus circumferential wave number (m=1, L=8 m, h=0.004 m, R=1 m, E=2.1x10^{11} N/m^2, ν=0.3, ρ=7850 Kg/m^3)

n	Frequencies (Hz)	
	Without fluid	With fluid
1	75.439	33.693
2	27.333	12.238
3	15.220	5.5650
4	16.499	7.1548
5	24.103	10.452
6	34.718	14.923
7	47.563	20.336
8	62.469	26.695
9	79.3907	33.760
10	98.314	41.744

exact results furnished by Loy *et al*. (1999), Rahimi *et al*. (2011) and Warburton (1965), those were established by working out with the deformation theory provided in Fig. 2. There is once again comparison of empty and fluid-filled cylindrical shell with Gonclaves *et al*. (2006) as shown in Fig 3. The proposed model based on Rayleigh-Ritz method can incorporate in order to accurately predict the acquired results of material data point.

This section influence of inclusion of fluid in an isotropic cylindrical shell is analyzed for some shell parameters. These parameters are associated with the physical quantities which describes a cylindrical shell problem. Table 3 indicates that the frequency values versus circumferential wave number. It is observed that the frequencies are highly visible for without fluid are higher than those for ones. They have been considerably reduced by the impact of terms. In Table 4, variations of natural frequencies (Hz) for an empty cylindrical shell and a cylindrical shell with fluid are exhibited versus the axial half wave modes (m). The rest shell factors are labeled in the Table. As m grows, frequencies for the empty and fluid-filled cylindrical shells boost indefinitely. For a cylindrical shell with fluid, frequencies are diminished considerably when their comparison is

Table 4 Comparison of fluid-filled and without fluid versus half axial wave mode (m) (n=1, L=8 m, h=0.004 m, R=1 m, E=2.1x10^{11} N/m^2, ν=0.3, ρ=7850 Kg/m^3, ρ_f=1000 Kg\m^3)

n	Frequencies (Hz)	
	Without fluid	With fluid
1	75.4385	33.6914
2	221.672	94.1282
3	368.978	156.0425
4	494.248	208.8342
5	589.228	248.8925
6	655.284	276.7597
7	699.488	295.4044
8	729.187	307.9337
9	749.662	316.5669
10	764.239	322.7088

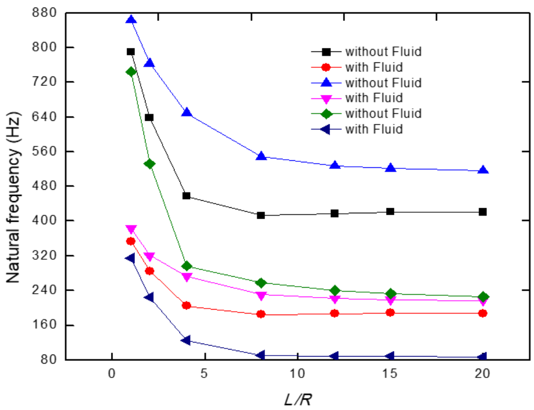

Fig. 4 Comparison of fluid-filled and without fluid versus length-to-radius ratio (m=1, L=0.41 m, h=0.001 m, R=0.3015 m, E=2.1 ×10^{11} N/m^2, ν=0.3, ρ=7850 Kg/m^3

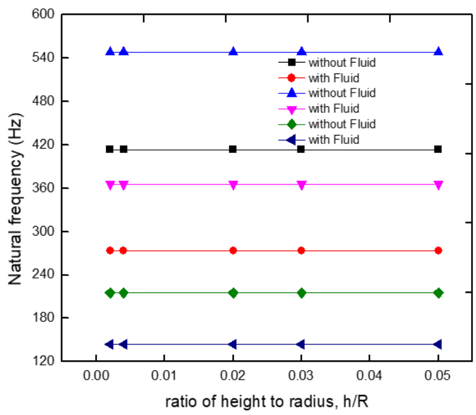

Fig. 5 Comparison of fluid-filled and without fluid versus height-to-radius ratio (L=8 in, h=0.1 in, R=2 in, E=30×10^6 lbf in^{-2},ν=0.3, ρ=7.35×10^{-4} lbf s^2 in^{-4})

made with ones for the empty cylindrical shells. Fig. 4 represent frequencies with length-to-radius ratio using simply supported edge conditions. There is a pronounced decrease in frequencies when the cylindrical shell has been filled with fluid. Moreover, it is observed that with increments in *L*/*R*, the fundamental frequencies decrease. Fig. 5 shows the frequencies against *h*/*R*. As *h*/*R* is increased, the fundamental frequency increases for a fixed value of *L*/*R*. In these figures, the frequencies are considerably reduced when the shell are filled with fluid as compared to without fluid. They are approximately half of those for a cylindrical shell not filled with fluid. This reduction is approximately fifty percent.

9. Conclusions

The Rayleigh-Ritz method has been employed in this paper to analyze the vibration characteristics of fluid-filled functionally graded cylindrical shells using Sander's shell theory. The fundamental natural frequency of functionally graded cylindrical shells of parameter versus ratios of length-to-diameter and height-to-diameter for a wide range has been reported and investigated through the study. Hankel's functions of second kind are utilized to represent this phenomenon. The frequencies are higher for higher values of circumferential wave number. The frequency first increases and gain maximum value with the increase of circumferential wave mode. It has been investigated that the frequencies lower down on implicating the fluid term. The uncoupled frequencies are higher than that of coupled frequencies. The shells are submerged in a fluid and terms describing fluid effects are added with the shell motion equations. The problem is formulated by applying the Rayleigh-Ritz method and the shell fluid condition is annexed with the third equation of shell motion equations. The longitudinal modal displacement functions are assessed by characteristic beam ones meet end conditions applied at the shell edges. Stability of a cylindrical shell depends highly on these aspects of material. More the shell material sustains a load due to physical situations, the more the shell is stable. Any predicted fatigue due to burden of vibrations is evaded by estimating their dynamical aspects. An extension of present study can be done for investigating the rotating functionally graded cylindrical shells with ring supports.

Declaration of conflcting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

References

Amabili, M. (1999), "Vibration of circular tubes and shells filled and partially immersed in dense fluids", *J. Sound Vib*., **221**(4), 567-585. https://doi.org/10.1006/jsvi.1998.2050.
Amabili, M., Pellicano, F. and Paidoussis, M.P. (1998), "Nonlinear vibrations of simply supported, circular

cylindrical shells, coupled to quiescent fluid", *J. Fluids Struct.*, **12**(7), 883-918. https://doi.org/10.1006/jfls.1999.0225.

Ansari, R. and Rouhi, H. (2015), "Nonlocal Flügge shell model for the axial buckling of single-walled carbon nanotubes: An analytical approach", *Int. J. Nano Dimens.*, **6**(5), 453-462. https://doi.org/10.7508/IJND.2015.05.002.

Arshad, S.H., Naeem, M.N. and Sultana, N. (2007), "Frequency analysis of functionally graded cylindrical shells with various volume fraction laws", *J. Mech. Eng. Sci.*, **221**, 1483-1495. https://doi.org/10.1243/09544062JMES738.

Asghar, S., Naeem, M.N. and Hussain, M. (2019), "Non-local effect on the vibration analysis of double walled carbon nanotubes based on Donnell shell theory", *Physica E Low Dimens. Syst. Nanostruct.*, **116**, 113726. https://doi.org/10.1016/j.physe.2019.113726.

Avcar, M. (2019), "Free vibration of imperfect sigmoid and power law functionally graded beams", *Steel Composite Struct.*, **30**(6), 603-606. https://doi.org/10.12989/scs.2019.30.6.603.

Chen, Y., Zhao, H.B. and Shea, Z.P. (1993), "Vibrations of high-speed rotating shells with calculations for cylindrical shells", *J. Sound Vib.*, **160**, 137-160. https://doi.org/10.1006/jsvi.1993.1010.

Chi, S.H. and Chung, Y.L. (2006), "Mechanical behavior of functionally graded material plates under transverse load-part II: Numerical results", *Int. J. Solids Struct.*, **43**, 3657-3691. https://doi.org/10.1016/j.ijsolstr.2005.04.010.

Chung, H., Turula, P., Mulcahy, T.M. and Jendrzejczyk, J.A. (1981), "Analysis of cylindrical shell vibrating in a cylindrical fluid region", *Nucl. Eng. Des.*, **63**(1), 109-120. https://doi.org/10.1016/0029-5493(81)90020-0.

Dong, S.B. (1977), "A block-stodola eigen solution technique for large algebraic systems with non-symmetrical matrices", *Int. J. Number Methods Eng.*, **11**, 247-267. https://doi.org/10.1002/nme.1620110204.

Ergin, A. and Temarel, P. (2002), "Free vibration of a partially liquid-filled and submerged, horizontal cylindrical shell", *J. Sound Vib.*, **254**(5), 951-965. https://doi.org/10.1006/jsvi.2001.4139.

Ersoy, H., Mercan, K. and Civalek, Ö. (2018), "Frequencies of FGM shells and annular plates by the methods of discrete singular convolution and differential quadrature methods", *Composite Struct.*, **183**, 7-20. https://doi.org/10.1016/j.compstruct.2016.11.051.

Farahani, H. and Barati, F. (2015), "Vibration of submerged functionally graded cylindrical shell based on first order shear deformation theory using wave propagation method", *Struct. Eng. Mech.*, **53**(3), 575-587. https://doi.org/10.12989/sem.2015.53.3.575.

Fatahi-Vajari, A., Azimzadeh, Z. and Hussain, M., (2019), "Nonlinear coupled axial-torsional vibration of single-walled carbon nanotubes using Galerkin and homotopy perturbation method", *Micro Nano Lett.*, **14**(14), 1366-1371. https://doi.org/10.1049/mnl.2019.0203.

Flügge, W. (1962), *Stresses in shells*, Springer-Verlag, Berlin.

Gasser, L.F.F. (1987), "Free vibrations on thin cylindrical shells containing liquid", M.Sc. Dissertation, Federal University of Rio de Janerio, Rio de Janerio, Portugal.

Goncalves, P.B. and Batista, R.C. (1988), "Non-linear vibration analysis of fluid-filled cylindrical shells", *J. Sound Vib.*, **127**(1), 133-143. https://doi.org/10.1016/0022-460X(88)90354-9.

Goncalves, P.B., Da Silva, F.M.A. and Prado, Z.J.G.N. (2006), "Transient stability of empty and fluid-filled cylindrical shells", *J. Braz. Soc. Mech. Sci. Eng.*, **28**(3), 331-333. http://dx.doi.org/10.1590/S1678-58782006000300011.

Gonçalves, P.B., Silva, F. and del Prado, Z.J. (2006), "Transient stability of empty and fluid-filled cylindrical shells international symposium on dynamic problems of mechanics, *J. Brazilian Society Mech. Sci. Eng.*, **28**(3), 331-338. https://doi.org/10.1590/S1678-58782006000300011.

Hussain, M, Naeem, M.N. and Taj. M. (2019b), "Vibration characteristics of zigzag and chiral FGM rotating carbon nanotubes sandwich with ring supports", *Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci.*, **233**(16), 5763-5780. https://doi.org/10.1177/0954406219855095.

Hussain, M. and Naeem, M.N. (2017), "Vibration analysis of single-walled carbon nanotubes using wave propagation approach", *Mech. Sci.*, **8**(1),155-164. https://doi.org/10.5194/ms-8-155-2017.

Hussain, M. and Naeem, M.N. (2018a), *Advance Testing and Engineering*, Intechopen, London, U.K.
Hussain, M. and Naeem, M.N. (2018b), *Novel Nanomaterials: Synthesis and Applications*, Intechopen, London, U.K. https://doi.org/10.5772/intechopen.73503.
Hussain, M. and Naeem, M.N. (2019a), "Effects of ring supports on vibration of armchair and zigzag FGM rotating carbon nanotubes using Galerkin's method", *Compos. Part B Eng.*, **163**, 548-561. https://doi.org/10.1016/j.compositesb.2018.12.144.
Hussain, M. and Naeem, M.N. (2019b), "Rotating response on the vibrations of functionally graded zigzag and chiral single walled carbon nanotubes", *Appl. Math. Model.*, **75**, 506-520. https://doi.org/10.1016/j.apm.2019.05.039.
Hussain, M. and Naeem, M.N. (2020), "Mass density effect on vibration of zigzag and chiral SWCNTs", *J. Sandw. Struct. Mater.*, 1099636220906257. https://doi.org/10.1177/1099636220906257.
Hussain, M., Naeem, M., Shahzad, A. and He, M. (2017), "Vibrational behavior of single-walled carbon nanotubes based on cylindrical shell model using wave propagation approach", *AIP Adv.*, **7**(4), 045114. https://doi.org/10.1063/1.4979112.
Hussain, M., Naeem, M.N and Tounsi, A. (2020), "Simulating vibration of single-walled carbon nanotube based on Relagh-Ritz Method", *Adv. Nano Res.*, **8**(3), 215-228. https://doi.org/10.12989/anr.2020.8.3.215.
Hussain, M., Naeem, M.N. and Isvandzibaei, M. (2018a), "Effect of Winkler and Pasternak elastic foundation on the vibration of rotating functionally graded material cylindrical shell", *Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci.*, **232**(24), 4564-4577. https://doi.org/10.1177/0954406217753459.
Hussain, M., Naeem, M.N. and Taj, M. (2019c), "Effect of length and thickness variations on the vibration of SWCNTs based on Flügge's shell model", *Micro Nano Lett.*, **15**(1), 1-6. https://doi.org/10.1049/mnl.2019.0309.
Hussain, M., Naeem, M.N., Shahzad, A., He, M. and Habib, S. (2018b), "Vibrations of rotating cylindrical shells with functionally graded material using wave propagation approach", *Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci.*, **232**(23), 4342-4356. https://doi.org/10.1177/0954406218802320.
Hussain, M., Naeem, M.N., Tounsi, A. and Taj, M. (2019a), "Nonlocal effect on the vibration of armchair and zigzag SWCNTs with bending rigidity", *Adv. Nano Res.*, **7**(6), 431-442. https://doi.org/10.12989/anr.2019.7.6.431.
Jiang, J. and Olson, M.D. (1994), "Vibrational analysis of orthogonally stiffened cylindrical shells using super elements", *J. Sound Vib.*, **173**, 73-83. https://doi.org/10.1006/jsvi.1994.1218.
Khayat, M., Dehghan, S.M., Najafgholipour, M.A. and Baghlani, A. (2018), "Free vibration analysis of functionally graded cylindrical shells with different shell theories using semi-analytical method", *Steel Compos. Struct.*, **28**(6), 735-748. https://doi.org/10.12989/scs.2018.28.6.735.
Koizumi, M. (1997), "FGM activities in Japan", *Compos. Part B*, **28**(1-2), 1-4. https://doi.org/10.1016/S1359-8368(96)00016-9.
Lam, K.Y. and Loy, C.T. (1998), "Influence of boundary conditions for a thin laminated rotating cylindrical shell", *Compos. Struct.*, **41**, 215-228. https://doi.org/10.1016/S0263-8223(98)00012-9.
Li, H., Pang, F., Du, Y. and Gao, C. (2019), "Free vibration analysis of uniform and stepped functionally graded circular cylindrical shells", *Steel Compos. Struct.*, **33**(2), 163-180. https://doi.org/10.12989/scs.2019.33.2.163.
Love, A.E.H. (1888), "The small free vibrations and deformation of a thin elastic shell", *Philos. Trans. R. Soc. Lond. B Biol. Sci.*, **179**, 491-546. https://doi.org/10.1098/rsta.1888.0016.
Loy, C.T. and Lam, K.Y. (1997), "Vibration of cylindrical shells with ring supports", *J. Mech. Eng.*, **39**, 455-471. https://doi.org/10.1016/S0020-7403(96)00035-5.
Loy, C.T., Lam, K.L. and Shu, C. (1997), "Analysis of cylindrical shells using generalized differential quadrature", *Shock Vib.*, **4**(3), 193-198. https://doi.org/10.3233/SAV-1997-4305.
Loy, C.T., Lam, K.Y. and Reddy, J.N. (1999), "Vibration of functionally graded cylindrical shells", *Int. J. Mech. Sci.*, **41**, 309-324. https://doi.org/10.1016/S0020-7403(98)00054-X.
Mercan, K., Demir, Ç. and Civalek, Ö. (2016), "Vibration analysis of FG cylindrical shells with power-law index using discrete singular convolution technique", *Curved Layer. Struct.*, **3**(1), 0007. https://doi.org/10.1515/cls-2016-0007.

Naeem, M.N., Ghamkhar, M., Arshad, S.H., and Shah, A.G. (2013), "Vibration analysis of submerged thin FGM cylindrical shells", *J Mech. Sci. Technol.*, **27**(3), 649-656. https://doi.org/10.1007/s12206-013-0119-6.

Najafizadeh, M.M. and Isvandzibaei, M.R. (2007), "Vibration of (FGM) cylindrical shells based on higher order shear deformation plate theory with ring support", *Acta Mech.*, **191**, 75-91. https://doi.org/10.1007/s00707-006-0438-0.

Orsberg, K. (1964), "Influence of boundary conditions on modal characteristics of cylindrical shells", *J. Amer. Institute Aeronautics Astronautics,* **2**, 182-189.

Rahimi, G.H., Ansari, R. and Hemmatnezhad, M. (2011), "Vibration of functionally graded cylindrical shells with ring support", *Sci. Iran.*, **18**(6), 1313-1320. https://doi.org/10.1016/j.scient.2011.11.026.

Sehar, A., Hussain, M., Naeem M.N. and Tounsi, A. (2020), "Prediction and assessment of nonlocal natural frequencies DWCNTs: Vibration analysis", *Comput. Concrete*, **25**(2), 133-144. https://doi.org/10.12989/cac.2020.25.2.133.

Sewall, J.L. and Naumann, E.C. (1968), *An Experimental and Analytical Vibration Study of Thin Cylindrical Shells with and without Longitudinal Stiffeners*, NASA, Washington D.C., U.S.A.

Shah, A.G., Mahmood, T. and Naeem, M.N. (2009), "Vibrations of FGM thin cylindrical shells with exponential volume fraction law", *Appl. Math. Mech.*, **30**(5), 607-615. https://doi.org/10.1007/s10483-009-0507-x.

Sharma, C.B. and Johns, D.J. (1971), "Vibration characteristics of a clamped-free and clamped-ring-stiffened circular cylindrical shell", *J. Sound Vib.*, **14**(4), 459-474. https://doi.org/10.1016/0022-460X(71)90575-X.

Sharma, C.B., Darvizeh, M. and Darvizeh, A. (1998), "Natural frequency response of vertical cantilever composite shells containing fluid", *Eng. Struct.*, **20**(8), 732-737. https://doi.org/10.1016/S0141-0296(97)00102-8.

Sharma, P., Singh, R. and Hussain, M. (2019), "On modal analysis of axially functionally graded material beam under hygrothermal effect", *Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci.*, **234**(5), 1085-1101. https://doi.org/10.1177/0954406219888234.

Sodel, W. (1981), *Vibration of Shell and Plates*, Mechanical Engineering Series, New York, U.S.A.

Sofiyev, A.H. and Avcar, M. (2010), "The stability of cylindrical shells containing an FGM layer subjected to axial load on the Pasternak foundation", *Eng.*, **2**, 228-236. https://doi.org/10.4236/eng.2010.2403.

Sofiyev, A.H., Alizada, A.N., Akin, Ö., Valiyev, A., Avcar, M. and Adiguzel, S. (2012), "On the stability of FGM shells subjected to combined loads with different edge conditions and resting on elastic foundations", *Acta Mech.*, **223**(1), 189-204. https://doi.org/10.1007/s00707-011-0548-1.

Suresh, S. and Mortensen, A. (1997), "Functionally gradient metals and metal ceramic composites part 2: Thermo mechanical behavior", *Int. Mater. Rev.*, **42**(3), 85-116. https://doi.org/10.1179/imr.1997.42.3.85.

Toulokian, Y.S. (1967), *Thermophysical Properties of High Temperature Solid Materials*, Macmillan, New York, U.S.A.

Wang, C. and Lai, J.C.S. (2000), "Prediction of natural frequencies of finite length circular cylindrical shells", *Appl. Acoust.*, **59**(4), 385-400. https://doi.org/10.1016/S0003-682X(99)00039-0.

Wang, C.M., Swaddiwudhipong, S. and Tian, J. (1997), "Ritz method for vibration analysis of cylindrical shells with ring-stiffeners", *J. Eng. Mech.*, **123**, 134-143. https://doi.org/10.1061/(ASCE)0733-9399(1997)123:2(134).

Warburton, G.B. (1965), "Vibration of thin cylindrical shells", *J. Mech. Eng. Sci.*, **7**, 399-407. https://doi.org/10.1243/JMES-JOUR-1965-007-062-02.

Wuite, J. and Adali, S. (2005), "Deflection and stress behavior of nanocomposite reinforced beams using a multiscale analysis", *Compos. Struct.*, **71**(3-4), 388-96. https://doi.org/10.1016/j.compstruct.2005.09.011.

Xiang, Y., Ma, Y.F., Kitipornchai, S. and Lau, C.W.H. (2002), "Exact solutions for vibration of cylindrical shells with intermediate ring supports", *Int. J. Mech. Sci.*, **44**(9), 1907-1924. https://doi.org/10.1016/S0020-7403(02)00071-1.

Xuebin, L. (2008), "Study on free vibration analysis of circular cylindrical shells using wave propagation", *J. Sound Vib.*, **311**, 667-682. https://doi.org/10.1016/j.jsv.2007.09.023.

Zhang, X.M. (2002), “Parametric analysis of frequency of rotating laminated composite cylindrical shells with the wave propagation approach”, *Comput*. *Methods Appl*. *Mech*. *Eng*., **191**, 2057-2071. https://doi.org/10.1016/S0045-7825(01)00368-1.

Zhang, X.M., Liu, G.R. and Lam, K.Y. (2001), “Coupled vibration of fluid-filled cylindrical shells using the wave propagation approach”, *Appl*. *Acoust*., **62**, 229-243. https://doi.org/10.1016/S0003-682X(00)00045-1.

WT