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A low computational cost method for vibration analysis of rectangular plates subjected to moving sprung masses

Ali Nikkhoo[1], Soheil Asili[1], Shabnam Sadigh[1], Iman Hajirasouliha*[2] and Hossein Karegar[1]

[1]*Department of Civil Engineering, University of Science and Culture, Tehran, Iran*
[2]*Department of Civil and Structural Engineering, The University of Sheffield, Sheffield, UK*

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Abstract. A low computational cost semi-analytical method is developed, based on eigenfunction expansion, to study the vibration of rectangular plates subjected to a series of moving sprung masses, representing a bridge deck under multiple vehicle or train moving loads. The dynamic effects of the suspension system are taken into account by using flexible connections between the moving masses and the base structure. The accuracy of the proposed method in predicting the dynamic response of a rectangular plate subjected to a series of moving sprung masses is demonstrated compared to the conventional rigid moving mass models. It is shown that the proposed method can considerably improve the computational efficiency of the conventional methods by eliminating a large number of time-varying components in the coupled Ordinary Differential Equations (ODEs) matrices. The dynamic behaviour of the system is then investigated by performing a comprehensive parametric study on the Dynamic Amplification Factor (DAF) of the moving loads using different design parameters. The results indicate that ignoring the flexibility of the suspension system in both moving force and moving mass models may lead to substantially underestimated DAF predictions and therefore unsafe design solutions. This highlights the significance of taking into account the stiffness of the suspension system for accurate estimation of the plate maximum dynamic response in practical applications.

Keywords: vibration analysis; moving mass; maximum dynamic response; multiple vehicular load; sprung mass

1. Introduction

Calculating the dynamic characteristics of solids and structures subjected to moving loads has been a significant engineering problem to gain insights into the dynamic response of railways, bridge type structures, pavements and mechanical devices. Fryba (2013) conducted a comprehensive study on the dynamic behaviour of structures subjected to moving force and provided corresponding mathematical formulations. In another study, Ouyang (2011) inspected different aspects of moving load problem in a review article and classified the problem into several distinct categories.

Dynamic behaviour of beams subjected to moving loads has been studied by several

*Corresponding author, Professor, E-mail: i.hajirasouliha@sheffield.ac.uk

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researchers employing different moving force and mass models (e.g. Piccardo *et al*. 2012; Takabatake 2013; Lotfollahi-Yaghin *et al*. 2015). Michaltsos and Kounadis (2001) showed that the inertial loads caused by the vertical motion of the moving mass can affect the dynamic characteristics of the beam. The findings of their research indicated that by increasing the mass of the moving load, the increase in the beam response due to this additional inertial load becomes more considerable. Tahmasebi Yamchelou and Nouri (2016) investigated the dynamic behaviour of beam-type bridges under a varying speed moving mass using two distinct approaches. Their findings indicate that an accelerating moving mass causes smaller response amplitude in forced vibration phase in comparison with a decelerating motion. This trend is quite reverse in free vibration phase and an accelerating moving mass creates larger responses after exiting the beam compared to a decelerating one. By substituting the moving mass with a moving force and a concentrated mass fixed at the mid-span of the beam, Jiang (2011) proposed a simplified method to calculate the dynamic response of a simply-supported beam under a moving mass. More recently, Eftekhari (2016) used a modified differential quadrature procedure to simulate the dynamic response of Euler–Bernoulli and Timoshenko beams traversed by a point moving load. Luo *et al*. (2018) presented a closed-form solution in the frequency-wavenumber domain and a moving coordinate system for calculating the dynamic responses of a Timoshenko beam on elastic foundation under a moving harmonic line load. The closed-form solution was obtained by substituting the poles of the characteristic equation into Cauchy's residue theorem.

The dynamic response of an infinite beam resting on a Winkler foundation subjected to a single moving load was studied by Raftoyiannis *et al*. (2012). It was shown that, compared to the conventional free single-span beams, the possible speeds of moving loads that can be reached in the beams resting on elastic foundations are very high even for cohesionless soils. Celebi (2006) carried out a comprehensive numerical investigation to study the dynamic response of train-tracks and sub-soil profiles due to the surface vibrations under moving loads. The numerical results were used to assess the influence of different design parameters on the amplitude reduction factor due to the presence of an open trench. In another relevant research, vibration of a bi-directional functionally graded Timoshenko beam under a moving load was assessed by Şimşek (2015) considering different boundary conditions. Similarly, Roshandel *et al*. (2015) presented benchmark solutions for the dynamic response of a Timoshenko beam with varying thickness under a moving mass using modal expansion method. More recently, Dimitrovová (2018) investigated the dynamic response of a beam on a visco-elastic foundation subjected to a moving mass employing a semi-analytical solution, in which the steady state dynamic response was accurately obtained by using a closed-form method.

A plate traversed by moving loads can reasonably represent a slab-type bridge subjected to moving vehicles. Gbadeyan and Dada (2011) investigated the dynamic response of an elastic plate under different types of moving masses. The cases they considered consisted of concentrate, linearly distributed, and uniformly distributed moving masses. Yamchelou *et al*. (2017) studied dynamic response of a plate under a single moving mass and concluded that maximum response amplitudes do not necessary take place at the center point of the plate. Their findings revealed that as the length to width ratio of the plate increases, the absolute maximum response converges to that obtained at the center point. Shadnam *et al*. (2001) calculated the dynamic response of a simply-supported thin rectangular plate carrying a moving mass using the eigenfunction expansion method. The same problem was studied by Nikkhoo and Rofooei (2012) presenting a comprehensive parametric study to explore the effect of plate's aspect ratio, moving load magnitude and speed as well as the load's trajectory on the dynamic response of the structure. In

another relevant study, an analytical solution was proposed by Ghazvini *et al*. (2016) for vibration analyses of a thin plate with non-uniform thickness subjected to a moving mass using orthogonal polynomials. Hassanabadi *et al*. (2016) utilized a semi-analytical method to capture the dynamic response of a rectangular plate under a series of moving masses and proposed an approximate method to determine the resonance velocities based on the fundamental frequency of the system. Using Galerkin's method, Mamandi *et al*. (2015) developed a computational model to evaluate the dynamic behaviour of a geometrically nonlinear rectangular plate subjected to a moving mass. In one of the subsequent studies, Torkan *et al*. (2018) investigated the dynamic instability of simply supported thin rectangular plates on an elastic foundation under successive moving masses. In their adopted method, the effects of all components of moving mass inertia were considered, and the governing partial differential equation of motion was solved via the Galerkin method. Malekzadeh and Monajjemzadeh (2015) also investigated the nonlinear response of functionally graded plates under moving loads, while Şimşek et el (2015) used the modified couple stress theory to study the vibration of a microplate under the action of a moving load. Rofooei *et al*. (2017) presented a finite element (FE) code to obtain the dynamic response of a geometrically nonlinear rectangular Elastic plates subjected to a moving mass. In their code, unlike the existing FE programs, all inertial components associated with the moving mass are taken into account. In another study, the dynamic behaviour of rectangular plates subjected to a distributed moving mass was investigated by Sorrentino and Catania (2018). For this purpose, they modelled a damped rectangular Kirchhoff plate simply supported on two opposite edges and free on the other two edges. The dynamic responses of the plate were then obtained via the Rayleigh–Ritz method.

Developing simple and accurate moving oscillator models has been of great interest to engineering practitioners concerning with the effects of vehicle flexibility on the dynamic response of bridges. Pesterev *et al*. (2003) proposed an analytical solution to the asymptotic states of the moving oscillator/beam interaction considering soft and rigid springs for the general case of non-zero beam initial conditions. They concluded that in case of high spring stiffness, the moving oscillator problem is not equivalent to the moving mass problem. The results of their study also highlighted that while these two problems can be equivalent in terms of the beam deflection, they are not equivalent in terms of stress distributions.

The dynamic behaviour of a truss structure excited by two moving mass oscillators was investigated by Baeza and Ouyang (2009) by utilizing Timoshenko beam theory in a finite element (FE) program. They concluded that the dynamic response of the system and the maximum contact force can be considerably higher than the static response at high speeds. In another relevant study, Lee and Chung (2014) simulated the dynamic interaction of a tensioned beam with a moving oscillator. They considered four different dynamic models and reported that in general the discrepancy between the contact forces and the deflections predicted by the models increase by decreasing the beam tension force and the moving velocity. Lu and Liu (2013) proposed a method for identification of physical parameters of a coupled bridge-vehicle system under moving oscillator. However, it was found that their method is insensitive to artificial measurement noises. Ghafoori *et al*. (2011) proposed a semi-analytical method using an adaptive FE model to predict the dynamic response of a simply supported rectangular plate under a moving sprung mass having arbitrary trajectory. Their work has been followed by Hassanabadi *et al*. (2014) accounting for various plate boundary conditions. However, in their work only a single moving oscillator was considered.

A finite element method with Newmark's time integration procedure was used by Mohebpour *et al*. (2011) to calculate the dynamic response of a shear deformable laminated composite plate

traversed by a moving oscillator. Similarly, Thai-Hoang *et al*. (2011) proposed an alternative alpha finite element method with discrete shear gap technique to analyse laminated composite plates. Using a different approach, Chang (2014) performed a stochastic dynamic FE analysis of a bridge-vehicle system subjected to random material properties and loadings. The bridge was modelled as a laminated composite beam with Gaussian random elastic modulus and mass density distribution subjected to random moving forces.

As discussed above, most of the previous research studies have investigated the dynamic behaviour of beam-type bridges under consecutive loads by using either a simplified model of the moving force simulation neglecting the load-bridge interaction, or a moving mass simulation which is an asymptotic state of the moving oscillator for an infinitely large spring constant. On the other hand, the superposition principle for multiple moving loads used in many of these methods is not valid if the vehicle-bridge interaction is considered (Klasztorny and Langer 1990; Pesterev *et al*. 2001; Nikkhoo *et al*. 2014; Khoraskani *et al*. 2016). Therefore, to obtain the dynamic response of such systems, coupled equations of motions (representing multiple moving inertial objects traversing a continuum) should be solved simultaneously, which will increase the complexity of the mathematical model.

To address the above mentioned issues, this study aims to develop a more computationally efficient method for vibration analysis of rectangular plates by taking into account the effects of suspension system flexibility on the dynamic amplification factor. The dynamic response of a Kirchhoff plate under simultaneous action of multiple moving sprung masses is examined and compared to the response under a stream of moving forces and masses. Subsequently, an efficient analytical-numerical solution based on modal expansion method is developed, which can considerably reduce the computational costs compared to the conventional modal expansion technique. The results are then used to investigate the effects of the key parameters of the moving oscillator, such as stiffness and the amount of inertial force, on the maximum dynamic amplification factor of the plate through several numerical examples.

2. Mathematical framework

2.1 Governing equations and solution

To develop vibration constitutive model of a typical slab-type bridge under train loads, a rectangular plate hinged at the two parallel edges and free at the other two longitudinal sides (SFSF boundary condition) is considered as shown in Fig. 1. The plate is subjected to a series of traversing sprung masses to simulate the travelling vehicle. Unlike the moving mass and moving force approaches, the general solution of this model is obtained by taking into account the suspension system of the travelling vehicle. The vibration of the slab is then assessed under a series of equidistant moving loads at the middle line of the plate (parallel to x axis), which can represent a moving train crossing the bridge. For simplicity, similar speed and mechanical characteristics are assumed for the traveling oscillators in this study.

By assuming a Kirchhoff plate theory for an isotropic, linear media with infinitesimal strains, the following equations can be easily obtained to determine the plate vibration under multiple pulsating loads (Hassanabadi *et al*. 2015):

$$D\nabla^4 w(x,y,t)+\mu\frac{\partial^2 w(x,y,t)}{\partial t^2}=B(x,y)\sum_{m=1}^{N}P_m\delta\left(x-X_m(t)\right)\delta\left(y-Y_m(t)\right) \tag{1}$$

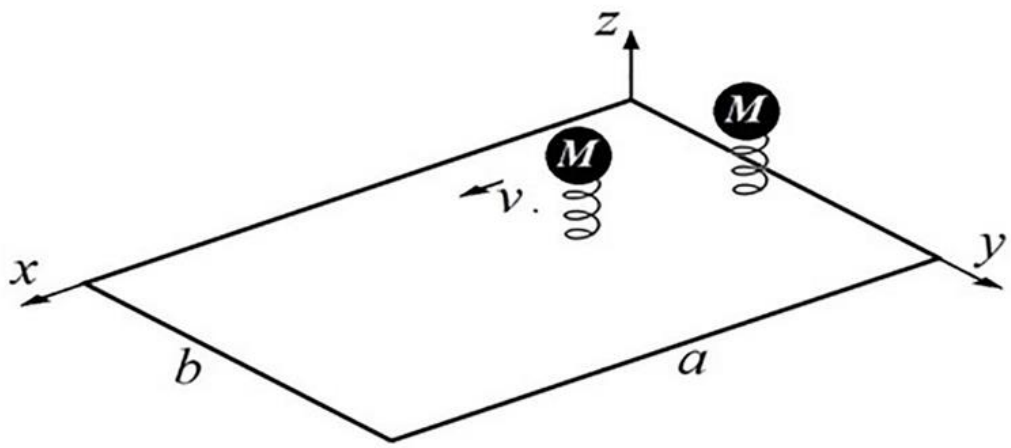

Fig. 1 A rectangular plate subjected to moving sprung masses

$$B_m\left(x,y\right)=[H(x)-H(x-a)][H(y)-H(y-b)] \tag{2}$$

$$M_m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(v_m(t)+w(X_m(t),Y_m(t),t))+P_m+M_m g=0 \tag{3}$$

$$P_m=k_m(v_m(t)-v_{0m}) \tag{4}$$

Eq. (1) denotes the general form of the governing equation, for a thin plate influenced by arbitrary multiple dynamic loads. Eq. (2) describes a window function, which allows the load to excite the beam as long as it is located on the beam span and nullifies its effect otherwise. A moving oscillator subsystem, itself, adds an extra constraint equation as given in Eq. (3). Finally, Eq. (4) is used to couple the vibration of the oscillator (Eq. (3)) and the plate dynamics (Eq. (1)). It should be noted that the coupled behaviour can be also obtained based on Eq. (3):

$$P_m=-M_m[g+\frac{\mathrm{d}^2}{\mathrm{d}t^2}(v_m(t)+w(X_m(t),Y_m(t),t))] \tag{5}$$

While Eqs. (4) and (5) both lead to the same final results, it will be discussed in following sections that the selection between these two equations can greatly affect the computational efficiency. An analytical-numerical solution of the previously mentioned constitutive equations could be easily obtained using modal superposition method, which is also known as eigenfunction expansion method. In this method, the response of the system is expressed in the modal coordinates. However, to solve the Partial Differential Equation (PDE) in Eq. (1), it should be assured that the shape functions satisfy the following requirement:

$$D\nabla^4\Phi_j(x,y)=\mu\omega_j^2\Phi_j(x,y) \tag{6}$$

Since Eq. (6) is a self-adjoint operator, the plate natural shape functions can be expressed by the following expansion series:

$$w(x,y,t)=\sum_{j=1}^{n}\tau_j(t)\Phi_j(x,y) \tag{7}$$

The orthogonality of the shape functions can be easily proved by using the principle of the virtual work. Therefore, the following equation is valid for the normalized mode shapes:

$$\iint_{A_{\text{Plate}}}\mu\Phi_i(x,y)\Phi_j(\mathrm{x,y})\mathrm{dA}=\delta_{ij} \tag{8}$$

Replacing the series expansion of the plate dynamic deformation in Eq. (7) into Eq. (1) leads to:

$$\sum_{j=1}^{n}(\tau_j(t)D\nabla^4\Phi_j(x,y)+\mu\Phi_j(x,y)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau_j(t))=B(x,y)\sum_{m=1}^{\mathrm{N}}P_m\delta(x-X_m(t))\delta(y-Y_m(t)) \tag{9}$$

To reduce the above problem to time domain alone, both sides of Eq. (9) should be multiplied by $\Phi_i(x,y)$. The resulting equation is then integrated on the whole spatial domain (A_{Plate}) as follows:

$$(\omega_j^2\tau_j(t)+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau_j(t))\delta_{ij}=\sum_{m=1}^{n}B(X_m,Y_m)P_m\Phi_i(X_m,Y_m) \tag{10}$$

The constraint equations of the oscillator can be also expressed in terms of the modal shape functions:

$$\begin{aligned}&M_m[\frac{\mathrm{d}^2}{\mathrm{d}t^2}v_m(t)+\sum_{i=1}^{n}B(X_m,Y_m)\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\tau_j(t)\Phi_j(X_m,Y_m))]+P_m+M_m g=0\\&m=1,2,\ldots,\mathrm{N}\end{aligned} \tag{11}$$

In this study, the following notations are used for the total derivative with respect to time in Eq. (11):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\tau_j(t)\Phi_j(X_m,Y_m))=\chi_j^{1m}(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau_j(t)+\chi_j^{2m}(t)\frac{\mathrm{d}}{\mathrm{d}t}\tau_j(t)+\chi_j^{3m}(t)\tau_i(t) \tag{12}$$

where

$$\chi_j^{1m}(t)=\Phi_j(X_m(t),Y_m(t)) \tag{13}$$

$$\chi_j^{2m}(t)=2[(\frac{\partial\Phi_j(x,y)}{\partial x})\frac{\mathrm{d}X_m}{\mathrm{d}t}+(\frac{\partial\Phi_j(x,y)}{\partial y})\frac{\mathrm{d}Y_m}{\mathrm{d}t}]_{\substack{x=X_m\\y=Y_m}} \tag{14}$$

and

$$\begin{aligned}\chi_j^{3m}(t)=[&(\frac{\partial^2\Phi_j(x,y)}{\partial x^2})(\frac{\mathrm{d}X_m}{\mathrm{d}t})^2+(\frac{\partial^2\Phi_j(x,y)}{\partial y^2})(\frac{\mathrm{d}Y_m}{\mathrm{d}t})^2+2(\frac{\partial^2\Phi_j(x,y)}{\partial x\partial y})(\frac{\mathrm{d}X_m}{\mathrm{d}t})(\frac{\mathrm{d}Y_m}{\mathrm{d}t})\\&+(\frac{\partial\Phi_j(x,y)}{\partial x})(\frac{\mathrm{d}^2X_m}{\mathrm{d}t^2})+(\frac{\partial\Phi_j(x,y)}{\partial y})(\frac{\mathrm{d}^2Y_m}{\mathrm{d}t^2})]_{\substack{x=X_m\\y=Y_m}}\end{aligned} \tag{15}$$

Subsequently, Eqs. (10) and (11) can be represented in an equivalent matrix form as shown below:

$$\mathbf{M}(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{T}(t)+\mathbf{C}(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t)+\mathbf{K}(t)\mathbf{T}(t)=\mathbf{F} \tag{16}$$

where

$$\mathbf{T}(t)=\begin{Bmatrix}\boldsymbol{\tau}(t)\\ \mathbf{V}(t)\end{Bmatrix}_{(n+N)\times 1} \tag{17}$$

$$\mathbf{M}(t)=\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21}(t) & \mathbf{M}_{22}\end{bmatrix}_{(n+N)\times(n+N)} \tag{18}$$

$$\mathbf{C}(t)=\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}(t)\\ \mathbf{C}_{21}(t) & \mathbf{C}_{22}\end{bmatrix}_{(n+N)\times(n+N)} \tag{19}$$

$$\mathbf{K}(t)=\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}(t)\\ \mathbf{K}_{21}(t) & \mathbf{K}_{22}\end{bmatrix}_{(n+N)\times(n+N)} \tag{20}$$

$$\mathbf{F}=\begin{Bmatrix}\{0\}_{n\times 1}\\ \{-M_m g\}_{\mathrm{N}\times 1}\end{Bmatrix}_{(n+N)\times 1} \tag{21}$$

$$\boldsymbol{\tau}(t)=\{\tau_i(t)\}_{n\times 1} \tag{22}$$

and

$$\mathbf{V}(t)=\{v_i(t)\}_{\mathrm{N}\times 1} \tag{23}$$

The components of the time-varying matrices in Eqs. (18)-(21) can be presented in the following forms

$$\mathbf{M}_{11}=\left[\delta_{ij}\right]_{n\times n} \tag{24}$$

$$\mathbf{M}_{12}=[0]_{n\times \mathrm{N}} \tag{25}$$

$$\mathbf{M}_{21}(t)=\left[B(X_i,Y_i)M\chi_j^{1i}(t)\right]_{\mathrm{N}\times n} \tag{26}$$

$$\mathbf{M}_{22}=\left[M\delta_{ij}\right]_{N\times N} \tag{27}$$

$$\mathbf{C}_{11}=[0]_{n\times n} \tag{28}$$

$$\mathbf{C}_{12}(t) = [0]_{n\times \mathrm{N}} \tag{29}$$

$$\mathbf{C}_{21}(t) = \left[MB(X_i, Y_i)\chi_j^{2i}(t) \right]_{\mathrm{N}\times n} \tag{30}$$

$$\mathbf{C}_{22} = [0]_{\mathrm{N}\times \mathrm{N}} \tag{31}$$

$$\mathbf{K}_{11} = \left[\omega_i^2 \delta_{ij} \right]_{n\times n} \tag{32}$$

$$\mathbf{K}_{12}(t) = \left[-kB(X_j, Y_j)\Phi_i(X_j, Y_j) \right]_{n\times N} \tag{33}$$

$$\mathbf{K}_{21}(t) = \left[MB(X_i, Y_i)\chi_j^{3i}(t) \right]_{N\times n} \tag{34}$$

$$\mathbf{K}_{22} = \left[k\delta_{ij} \right]_{N\times N} \tag{35}$$

$$\mathbf{F}_1 = \{0\}_{n\times 1} \tag{36}$$

$$\mathbf{F}_2 = \{-M_i g\}_{N\times 1} \tag{37}$$

The analytical eigensolutions of a thin rectangular plate is discussed in detail by Leissa (1973). For a single-span plate, as illustrated in Fig. 1, the two constraints defining the simply supported edges at $x = 0$ and $x = a$ can be expressed as:

$$\begin{gathered} \Phi(x, y) = 0 \\ \frac{\partial^2 \Phi(x, y)}{\partial y^2} + \nu \frac{\partial^2 \Phi(x, y)}{\partial x^2} = 0 \end{gathered} \tag{38}$$

Similarly, the constraints for the two free edges can be presented by the following equations

$$\begin{gathered} \frac{\partial^2 \Phi(x, y)}{\partial y^2} + \nu \frac{\partial^2 \Phi(x, y)}{\partial x^2} = 0 \\ \frac{\partial^3 \Phi(x, y)}{\partial y^3} + (2-\nu) \frac{\partial^3 \Phi(x, y)}{\partial y \partial x^2} = 0 \end{gathered} \tag{39}$$

Since the shape functions of a plate with simple boundary conditions (i.e. Eq. (38)) are sinusoidal, a Fourier Sine transform in x direction is used for direct separation of variables. Subsequently, the following equations are obtained by considering the free vibration equation of the plate given in Eq. (6):

If $\kappa^2 > \alpha^2$:

$$\begin{gathered} \Phi(x, y) = [A \sin \sqrt{\kappa^2 - \alpha^2}\, y + B \cos \sqrt{\kappa^2 - \alpha^2}\, y \\ + C \sinh \sqrt{\kappa^2 + \alpha^2}\, y + D \cosh \sqrt{\kappa^2 + \alpha^2}\, y] \sin \alpha x \end{gathered} \tag{40}$$

If $\kappa^2 < \alpha^2$:

$$\Phi(x,y) = [A\sinh\sqrt{\alpha^2-\kappa^2}\,y + B\cosh\sqrt{\alpha^2-\kappa^2}\,y \\ +C\sinh\sqrt{\kappa^2+\alpha^2}\,y + D\cosh\sqrt{\kappa^2+\alpha^2}\,y]\sin\alpha x \tag{41}$$

where

$\kappa^4 = \rho\omega^2/D$, $\alpha = m\pi/a, m = 1,2,\ldots$ and *A, B, C* and *D* are integration constant parameters. Finally, by replacing Eqs. (40) and (41) in the boundary conditions of the free edges (Eq. (39)), the following equation is derived:

If $\kappa^2 > \alpha^2$:

$$2\varphi_1\varphi_2(\lambda^2 - m^4\pi^4(1-\nu)^2)^2(\cos\varphi_1\cosh\varphi_2 - 1) \\ +[\varphi_1^2(\lambda + m^2\pi^2(1-\nu))^4 - \varphi_2^2(\lambda - m^2\pi^2(1-\nu))^4]\sin\varphi_1\sinh\varphi_2 = 0 \tag{42}$$

If $\kappa^2 < \alpha^2$:

$$2\eta_1\eta_2(\lambda^2 - m^4\pi^4(1-\nu)^2)^2(\cos\eta_1\cosh\eta_2 - 1) \\ +[\eta_1^2(\lambda + m^2\pi^2(1-\nu))^4 - \eta_2^2(\lambda - m^2\pi^2(1-\nu))^4]\sinh\eta_1\sinh\eta_2 = 0 \tag{43}$$

where

$$\lambda = \omega a^2\sqrt{\rho/D},\ \varphi_1 = \frac{b}{a}\sqrt{\lambda - m^2\pi^2}, \varphi_2 = \frac{b}{a}\sqrt{\lambda + m^2\pi^2}$$

$$\eta_1 = \frac{b}{a}\sqrt{m^2\pi^2 - \lambda}\ , \eta_2 = \frac{b}{a}\sqrt{m^2\pi^2 + \lambda}$$

2.2 A discussion on the moving force and moving mass methods

In common practice, moving vehicular loads are usually represented by a moving mass or a moving force to simplify the complex vibration analyses and reduce the associated computational costs. The moving mass consists of a solid mass (usually a lumped mass) interacting with the base structure to account for the inertia of the traversing load. On the other hand, in the moving force approach an equivalent load (equal to the weight of the moving object) in utilized to simplify the dynamic effects of the vehicular loads. By using the moving load approach, Eq. (10) can be re-written as follows:

$$(\omega_j^2\tau_j(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\tau_j(t))\delta_{ij} = \sum_{m=1}^{\mathrm{N}} -B(X_m, Y_m)M_m g\Phi_j(X_m, Y_m) \tag{44}$$

The main advantage of this approach is that the dynamic response of the system can be easily obtained based on the following closed-form solution of Eq. (44):

$$\tau_{hj}(t) = \frac{1}{\omega_j}\dot{\tau}_{0j}\sin(\omega_j t) + \tau_{0j}\cos(\omega_j t) \tag{45}$$

$$\tau_{pj}(t)=\sum_{m=1}^{\mathrm{N}}\frac{M_m gB(X_m,Y_m)}{\mathrm{W}(\mathrm{Sin}(\omega_j t),\mathrm{Cos}(\omega_j t))}$$

$$\times[\mathrm{Cos}(\omega_j t)\int_0^t \mathrm{Sin}(\omega_j\zeta)\Phi_j(X_m(\zeta),Y_m(\zeta))\mathrm{d}\zeta-\mathrm{Sin}(\omega_j t)\int_0^t \mathrm{Cos}(\omega_j\zeta)\Phi_j(X_m(\zeta),Y_m(\zeta))\mathrm{d}\zeta]$$

$$\tau_j(t)=\tau_{hj}(t)+\tau_{pj}(t)$$

in which

$$\mathrm{W}(\mathrm{Sin}(\omega_j t),\mathrm{Cos}(\omega_j t))=\begin{vmatrix}\mathrm{Sin}(\omega_j t) & \mathrm{Cos}(\omega_j t)\\ \omega_j\mathrm{Cos}(\omega_j t) & -\omega_j\mathrm{Sin}(\omega_j t)\end{vmatrix}=-\omega_j \tag{46}$$

where τ_{0j} and $\dot{\tau}_{0j}$ can be obtained based on the initial conditions.

The static deformation of the plate under a force, p, applied at the coordinates of (x, y), can be then calculated by removing the terms with time derivatives in Eq. (44), arriving at:

$$T_j=\frac{1}{\omega_j^2}p\Phi_j(X,Y) \tag{47}$$

The moving mass solution can be also obtained by replacing

$$P_m=-M_m[g+\frac{\mathrm{d}^2}{\mathrm{d}t^2}w(X_m(t),Y_m(t))] \tag{48}$$

in Eq. (10), leading to the following the matrix equation:

$$\mathbf{m}(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\tau}(t)+\mathbf{c}(t)\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\tau}(t)+\mathbf{k}(t)\boldsymbol{\tau}(t)=\mathbf{f}(t) \tag{49}$$

where

$$\mathbf{m}(t)=\left[\delta_{ij}\right]_{n\times n}+\sum_{m=1}^{\mathrm{N}}B(X_m,Y_m)\left[\Phi_i(X_m,Y_m)\chi_j^{1m}(t)\right]_{n\times n} \tag{50}$$

$$\mathbf{c}(t)=\sum_{m=1}^{\mathrm{N}}B(X_m,Y_m)\left[\Phi_i(X_m,Y_m)\chi_j^{2m}(t)\right]_{n\times n} \tag{51}$$

$$\mathbf{k}(t)=\left[\omega_i^2\delta_{ij}\right]_{n\times n}+\sum_{m=1}^{\mathrm{N}}B(X_m,Y_m)\left[\Phi_i(X_m,Y_m)\chi_j^{3m}(t)\right]_{n\times n} \tag{52}$$

$$\mathbf{f}(t)=\sum_{m=1}^{\mathrm{N}}B(X_m,Y_m)\left\{-M_m g\Phi_i(X_m,Y_m)\right\}_{n\times 1} \tag{53}$$

3. Parametric study

3.1 Case study parameters

In this section, the efficiency of the proposed method in predicting the dynamic response of a rectangular plate subjected to a series of moving sprung masses (see Fig. 1) is demonstrated

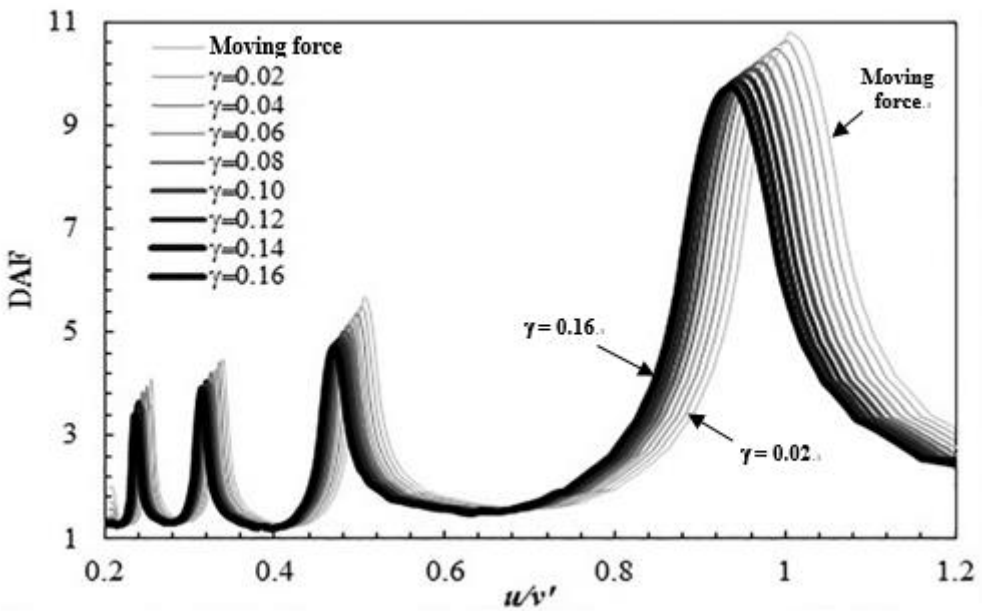

Fig. 2 The resonance velocity for different values of inertia (N=10)

compared to the conventional rigid moving mass models. In the numerical examples, the followings design values are considered for the rectangular plate:

$$b = 7\text{m},\ a = 3\text{b},\ v = 0.3$$

$$L = 0.9\text{a},\ D = 1.33516\times10^{9}\text{N.m}$$

For better comparison, the following normalized parameters are used to discuss the results (see Table of Notations):

$$\gamma = \frac{M}{\mu ab},\ T_1 = \frac{2\pi}{\omega_1},\ v' = \frac{L}{T_1}, \Omega = \frac{\sqrt{k/M}}{\omega_1}$$

$$w_{\text{stat}} = \sum_{j=1}^{n} \frac{-Mg}{\omega_j^2} \Phi_j(X,Y)\Phi_j(0.5a, 0.5b)$$

$$W_N = w(0.5a, 0.5b, t) / w_{stat}$$

$$\text{DAF} = \max(|w(0.5a, 0.5b, t)|) / w_{\text{stat}}$$

In the above equations, γ and Ω represent the mass ratio and the natural frequency ratio of the moving mass system, respectively. V' is a reference speed defined as the distance between consecutive loads divided by the fundamental period of the beam (L / T_1).

The maximum dynamic response of the base structure under moving loads can be sought through obtaining the Dynamic Amplification Factor (DAF) spectra versus the velocity of the moving load. The peak values of DAF spectra take place at the resonant states of the structure, at which the velocity of the moving load is called resonant velocity (Khoraskani *et al*. 2016).

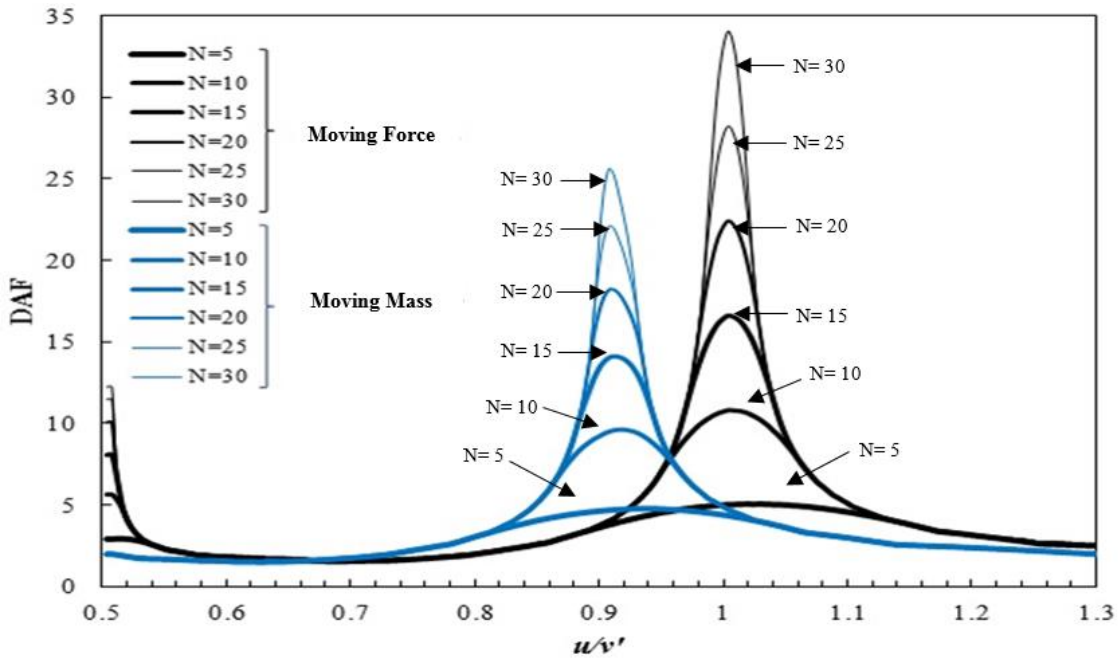

Fig. 3 Fig.3 The resonance velocity for different number of loads; $\gamma = 0.2$

The DAF spectra of the case study example under the selected moving loads (N=10) is illustrated in Fig. 2 using the moving force model by considering the first 20 mode shapes. In this figure, u is the absolute speed of the moving loads. It can be seen that the moving force model results in resonant velocity values of $u/v' = 1/i, i = 1, 2, 3, \ldots$, which can be also rearranged as $L/u = iT_1$. The latter equation indicates that resonance will occur when the repetition time of the successive loads is an integer coefficient of the fundamental period of the beam. This can be reasonably justified as the first natural mode is expected to have the most significant effect in the total response of the structure. Besides, a repetitive loading scheme with the period of $iT_1, i = 1, 2, \ldots$ can excite the first natural mode of the plate at the resonant state. In this case, the plate would vibrate with the maximum amplitude. It can be noted from Fig. 2 that the peak values corresponding to decrease as i increases.

The DAF spectra corresponding to the moving mass models with different mass ratios (γ ranging from 0.02 to 0.16) are also compared in Fig. 2. It is shown that by increasing the mass ratio (or inertia of the loads), the resonance velocity decreases. This conclusion can be explained with regard to the influence of the added masses (due to the moving loads) to the plate. The moving masses-plate system exhibits a higher fundamental period, since the mass of the system is increased for a fixed flexural rigidity. This implies that if the vehicle-bridge interaction is taken into account, which is a more realistic representation of the moving loads, the resonant speeds shift towards smaller values in comparison with those observed under the action of a series of moving forces (vehicle-bridge interaction being ignored). Moreover, it is shown in Fig. 2 that the maximum DAF of the system decreases as the mass of the moving objects increases. Therefore, the moving force model results in conservative (upper bound) values in this case.

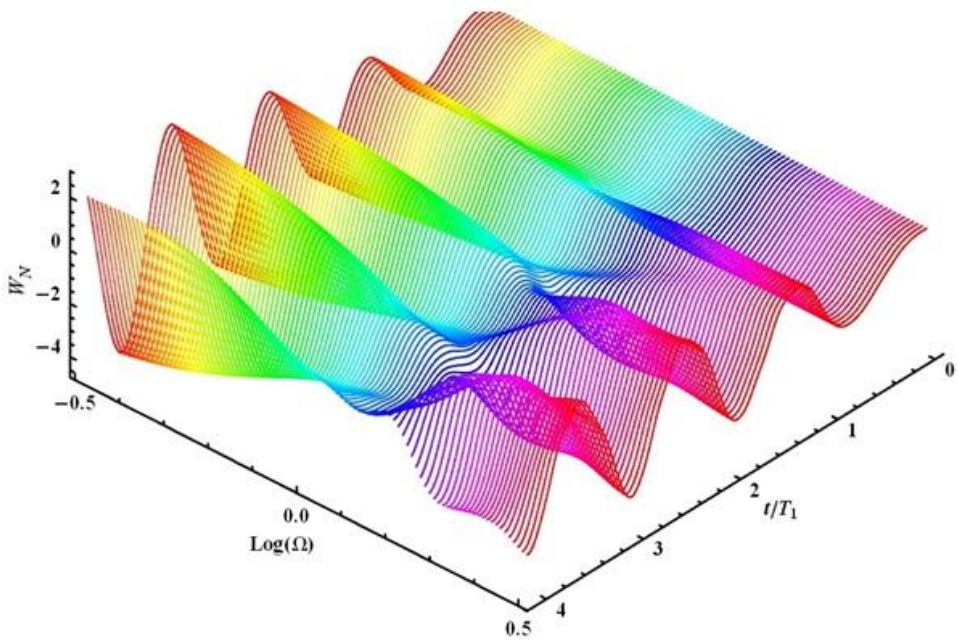

Fig. 4 Time history of plate normalized deflection versus the oscillator frequency; N =4, $\gamma = 0.25$, $u = v'$,

Fig. 3 investigates the influence of the load number (N) on the resonance velocity using moving force and moving mass methods. It is clear from the figure that increasing the number of loads at the resonant state mainly amplifies the vibration, while it does not considerably affect the value of the resonance velocity. However, by increasing the number of loads, N , the results tend to the steady state loading condition.

3.2 Contribution of the suspension system

Moving force/mass could be intuitively regarded as the asymptotic states of a moving oscillator problem. The dynamic behaviour of a moving oscillator with an extremely flexible suspension system is similar to that of a moving force, and a rigid suspension system results in a dynamic behaviour identical to that observed under a moving mass. However, it has been highlighted that the moving mass can be different from rigid moving oscillator model in terms of the contact force and stresses (Pesterev *et al*. 2003). To give a clear visual sense of the moving oscillator transition between the asymptotic states, the time history results of the plate normalized deflection versus the oscillator frequency is depicted in Fig. 4. It is shown that the trend of the dynamic response of the system with springs of infinitesimal stiffness is towards the moving force model. However, the response of the structure under the sequential moving oscillators having springs with infinite stiffness corresponds to the moving mass model.

To assess the influence of the stuffiness of the moving sprung masses on the dynamic response of the system, Fig. 5 shows the effects of using different frequency ratios on the resonance velocity of the system. The results in general indicate that at the intermediate values of moving oscillator eigenfrequencies, the dynamics of the structure would be substantially altered. This can significantly underline the contribution of the suspension systems.

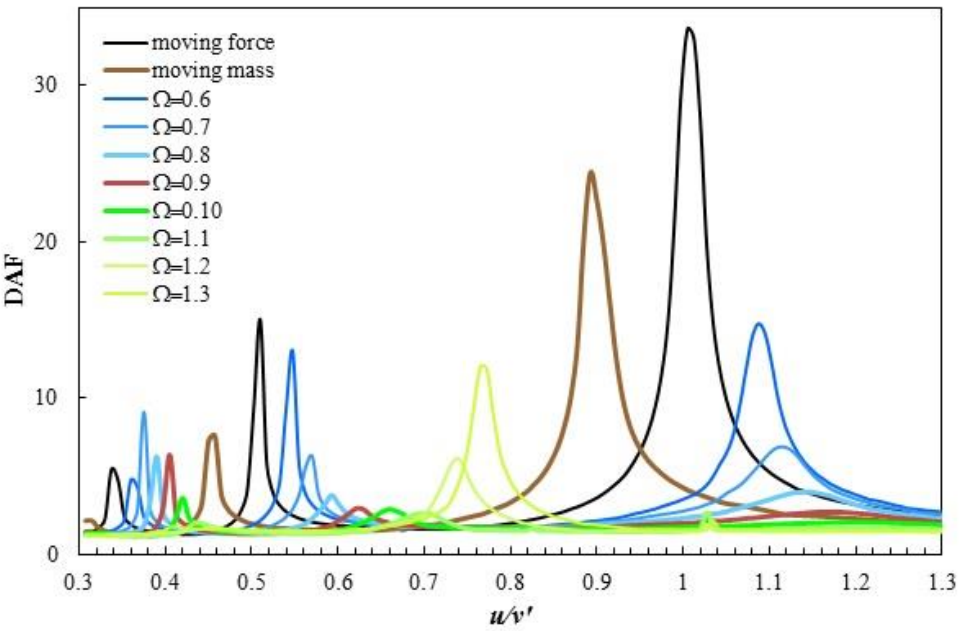

Fig. 5 The resonance velocity for different oscillator frequency values; $\gamma = 0.25$, $N = 30$

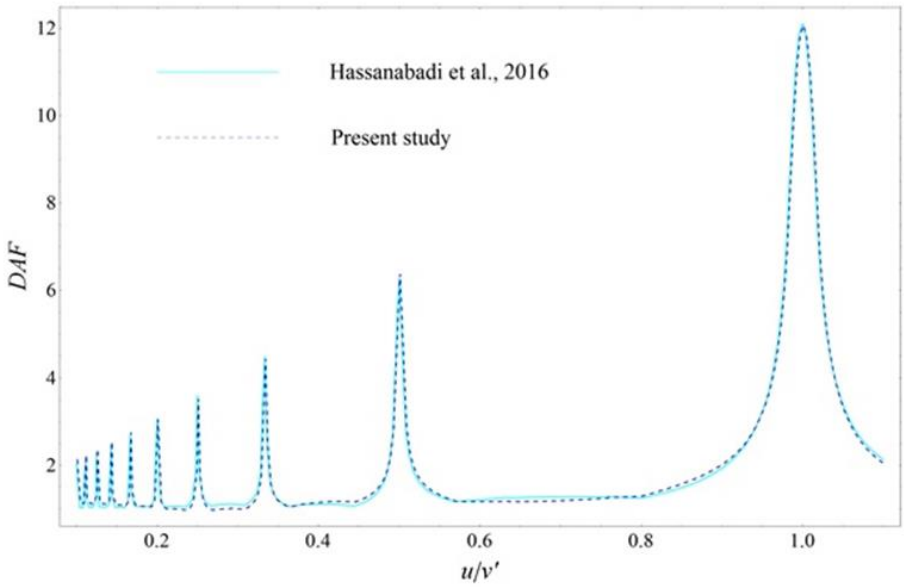

Fig. 6 Dynamic Amplification Factor (DAF) versus dimensionless speed for 30 equidistant moving masses traversing a plate calculated based on the proposed method and the results presented by Hassanabadi *et al*. (2016)

The spectral curves in Figs. 4 and 5 also denote that the dynamic response of the plate tends to a moving mass and a moving force system for larger and smaller spring stiffness values,

Table 1 Runtime comparison: $\mathrm{N}=5$, $L=0.9a$, $u=v'$. The values are normalized by the runtime of the proposed method with 5 modes

Method	Number of modes				
	5	10	15	20	25
Modal expansion technique (Nikkhoo *et al*. 2014)	4.7	16.1	37.89	65.62	100.05
The proposed method	1	2.22	2.98	3.71	4.95

respectively. Moreover, it can be seen that the envelope of the maximum DAF is notably underestimated when moving force/mass is used within a wide range of relative frequency components of the moving sprung masses.

4. Merits of the proposed methodology

In this section, the reliability and computational efficiency of the method presented in this paper is demonstrated using benchmark examples. Fig. 6 compares the Dynamic Amplification Factor (DAF) versus dimensionless speed for 30 equidistant moving masses traversing a plate calculated using the proposed method with the results of an improved semi analytical technique presented by Hassanabadi *et al*. (2016). It can be seen that there is an excellent agreement between the dynamic amplification spectra using these two different methods.

The calculation of the time varying coefficients of the coupled ODEs set in moving mass and moving oscillator solutions is a major factor which governs the total computational time (Hassanabadi *et al*. 2015). To assess the computational efficiency of the proposed method, the required computational time of the introduced solution for the moving mass problem in time domain is compared with the conventional modal expansion technique (Nikkhoo *et al*. 2014) as a function of the number of involved mode shapes. As discussed before, the time-varying matrices of the conventional moving mass solution in Eq. (49) are time dependent, and therefore, can significantly increase the required computational time. While in the presented formulation of the moving oscillator problem the dimensions of the state-space matrices are larger than the moving mass relations, the time varying components in the coefficient matrixes are largely eliminated. This can considerably improve the computational efficiency of the proposed method. Table 1 compares the total runtime of the moving oscillator problem solved with the proposed method and the conventional modal expansion technique (Nikkhoo *et al*. 2014) using $\mathrm{N}=5$, $L=0.9a$, $u=v'$. The results are normalized by the runtime of the proposed method with 5 modes. It is shown that the proposed method can reduce the required computational time by up to 20 times. Also it can be seen that the efficiency of the method increases almost linearly by increasing the number of modes considered in the analyses.

5. Summary and conclusions

In this study, a semi-analytical method based on eigenfunction expansion was developed to tackle the vibration of a thin rectangular plate under a series of moving sprung masses. An efficient

analytical-numerical solution based on modal expansion method was introduced, which could considerably reduce the computational costs (by up to 20 times) of the conventional methods by eliminating a large number of time-varying components in the coupled Ordinary Differential Equations (ODEs) matrixes. The efficiency of the proposed method was especially evident when higher number of modes were considered in the analyses. A comprehensive parametric study was then performed to investigate the effects of the key design parameters of the moving oscillator on the Dynamic Amplification Factor (DAF) of the system. It was shown that in general the moving force and moving mass models can represent the asymptotic states of the moving oscillator. However, for the frequency ratios in the range of $0.3 \le \Omega \le 1.3$, the results were very sensitive to the variation of the moving oscillator's frequency components. Moreover, it was shown that the moving force and moving mass models can substantially underestimate the maximum DAF of the structure by disregarding the flexibility of the moving sprung masses, which highlights the importance of using non-rigid connections for accurate estimation of the plate maximum dynamic response.

References

Baeza, L. and Ouyang, H. (2009), "Vibration of a truss structure excited by a moving oscillator", *J. Sound Vib.*, **321**(3-5), 721-734. https://doi.org/10.1016/j.jsv.2008.09.049.

Celebi, E. (2006), "Three-dimensional modelling of train-track and sub-soil analysis for surface vibrations due to moving loads", *Appl. Math. Comput.,* **179**(1), 209-230. https://doi.org/10.1016/j.amc.2005.11.095.

Chang, T.P. (2014), "Stochastic dynamic finite element analysis of bridge-vehicle system subjected to random material properties and loadings", *Appl. Math. Comput.*, **242**, 20-35. https://doi.org/10.1016/j.amc.2014.05.038.

Dimitrovová, Z. (2018), "Complete semi-analytical solution for a uniformly moving mass on a beam on a two-parameter visco-elastic foundation with non-homogeneous initial conditions", *Int. J. Mech. Sci.*. **144,** 283-311. https://doi.org/10.1016/j.ijmecsci.2018.05.055.

Eftekhari, S. (2016), "A modified differential quadrature procedure for numerical solution of moving load problem", *Proceedings of the Institution of Mechanical Engineers Part C,* **230**(5), 715-731. https://doi.org/10.1177/0954406215584630.

Frýba, L. (2013), *Vibration of Solids and Structures under Moving Loads*, Springer Science and Business Media, Germany.

Gbadeyan, J. and Dada, M. (2011), "A comparison of dynamic responses of three versions of moving load problem involving elastic rectangular plates", *J. Vib. Control*, **17**(6), 903-915. https://doi.org/10.1177/1077546310377910.

Ghafoori, E., Kargarnovin, M.H. and Ghahremani, A.R. (2011), "Dynamic responses of a rectangular plate under motion of an oscillator using a semi-analytical method", *J. Vib. Control*, **17**(9), 1310-1324. https://doi.org/10.1177/1077546309358957.

Ghazvini, T., Nikkhoo, A., Allahyari, H. and Zalpuli, M. (2016), "Dynamic response analysis of a thin rectangular plate of varying thickness to a traveling inertial load", *J. Brazilian Soc. Mech. Sci. Eng.*. **38**(2), 403-411. https://doi.org/10.1007/s40430-015-0409-2.

Hassanabadi, M.E., Amiri, J.V. and Davoodi, M. (2014), "On the vibration of a thin rectangular plate carrying a moving oscillator", *Scientia Iranica. Transaction A*, **21**(2), 284.

Hassanabadi, M.E., Attari, K., Nikkhoo, A. and Mariani, S. (2016), "Resonance of a rectangular plate influenced by sequential moving masses", *Coupled Syst. Mech.*, **5**(1), 87-100. http://dx.doi.org/10.12989/csm.2016.5.1.087.

Hassanabadi, M.E., Attari, N.K., Nikkhoo, A. and Baranadan, M. (2015), "An optimum modal superposition approach in the computation of moving mass induced vibrations of a distributed parameter system",

Proceedings of the Institution of Mechanical Engineers, Part C, **229**(6), 1015-1028. https://doi.org/10.1177/0954406214542968.

Jiang, J.Q. (2011), "Transient responses of Timoshenko beams subject to a moving mass", *J. Vib. Control,* **17**(13), 1975-1982. https://doi.org/10.1177/1077546310382808.

Khoraskani, R.A., Mofid, M., Azam, S.E. and Hassanabadi, M.E. (2016), "A new simplified formula in prediction of the resonance velocity for multiple masses traversing a thin beam", *Scientia Iranica Transaction A,* **23**(1), 133.

Klasztorny, M. and Langer, J. (1990), "Dynamic response of single-span beam bridges to a series of moving loads", *Earthq. Eng. Struct. Dynam.,* **19**(8), 1107-1124. https://doi.org/10.1002/eqe.4290190803.

Lee, K. and Chung, J. (2014), "Dynamic models for the contact analysis of a tensioned beam with a moving oscillator", *Proceedings of the Institution of Mechanical Engineers, Part C,* **28**(4), 676-689. https://doi.org/10.1177/0954406213489785.

Leissa, A.W. (1973), "The free vibration of rectangular plates", *J. Sound Vib.,* **31**(3), 257-293. https://doi.org/10.1016/S0022-460X(73)80371-2.

Lotfollahi-Yaghin, M.A., Kafshgarkolaei, H.J., Allahyari, H. and Ghazvini, T. (2015), "On the absolute maximum dynamic response of a beam subjected to a moving mass", *Struct. Eng. Mech.,* **54**(1), 55-57. http://dx.doi.org/10.12989/sem.2015.54.1.055.

Lu, Z. and Liu, J. (2013), "Parameters identification for a coupled bridge-vehicle system with spring-mass attachments", *Appl. Math. Comput.,* **219**(17), 9174-9186. https://doi.org/10.1016/j.amc.2013.03.047.

Luo, W.L., Xia, Y. and Zhou, X.Q. (2018), "A general closed-form solution to a Timoshenko beam on elastic foundation under moving harmonic line load", *Struct. Eng. Mech.,* **66**(3), 387-397. http://dx.doi.org/10.12989/sem.2018.66.3.387.

Malekzadeh, P. and Monajjemzadeh, S. (2015), "Nonlinear response of functionally graded plates under moving load", *Thin-Walled Struct*., **96,** 120-129. https://doi.org/10.1016/j.tws.2015.07.017.

Mamandi, A., Mohsenzadeh, R. and Kargarnovin, M.H. (2015), "Nonlinear dynamic analysis of a rectangular plate subjected to accelerated/decelerated moving load", *J. Theoretic. Appl. Mech.,* **53**(1), 151-166.

Michaltsos, G. and Kounadis, A. (2001), "The effects of centripetal and Coriolis forces on the dynamic response of light bridges under moving loads", *J. Vib. Control*., **7**(3), 315-326. https://doi.org/10.1177/107754630100700301.

Mohebpour, S., Malekzadeh, P. and Ahmadzadeh, A. (2011), "Dynamic analysis of laminated composite plates subjected to a moving oscillator by FEM", *Compos. Struct.*, **93**(6), 1574-1583. https://doi.org/10.1016/j.compstruct.2011.01.003.

Nikkhoo, A., Hassanabadi, M.E., Azam, S.E. and Amiri, J.V. (2014), "Vibration of a thin rectangular plate subjected to series of moving inertial loads", *Mech. Res. Communications*, **55,** 105-113. https://doi.org/10.1016/j.mechrescom.2013.10.009.

Nikkhoo, A. and Rofooei, F.R. (2012), "Parametric study of the dynamic response of thin rectangular plates traversed by a moving mass", *Acta Mechanica*, **223**(1), 15-27. https://doi.org/10.1007/s00707-011-0547-2.

Ouyang, H. (2011), "Moving-load dynamic problems: A tutorial (with a brief overview)", *Mech. Syst. Signal Processing,* **25**(6), 2039-2060. https://doi.org/10.1016/j.ymssp.2010.12.010.

Pesterev, A., Bergman, L., Tan, C., Tsao, T.C. and Yang, B. (2003), "On asymptotics of the solution of the moving oscillator problem", *J. Sound Vib.,* **260**(3), 519-536. https://doi.org/10.1016/S0022-460X(02)00953-7.

Pesterev, A., Yang, B., Bergman, L. and Tan, C.A. (2001), "Response of elastic continuum carrying multiple moving oscillators", *J. Eng. Mech.*, **127**(3), 260-265. https://doi.org/10.1061/(ASCE)0733-9399(2001)127:3(260).

Piccardo, G., Tubino, F.J.S.E. and Mechanics (2012), "Dynamic response of Euler-Bernoulli beams to resonant harmonic moving loads", *Struct. Eng. Mech.,* **44**(5), 681-704. http://dx.doi.org/10.12989/sem.2012.44.5.681.

Raftoyiannis, I.G., Avraam, T.P. and Michaltsos, G.T. (2012), "A new approach for loads moving on infinite beams resting on elastic foundation", *J. Vib. Control*, **18**(12), 1828-1836.

https://doi.org/10.1177/1077546311426440.

Rofooei, F.R., Enshaeian, A. and Nikkhoo, A. (2017), "Dynamic response of geometrically nonlinear, elastic rectangular plates under a moving mass loading by inclusion of all inertial components", *J. Sound Vib,.* **394**, 497-514. https://doi.org/10.1016/j.jsv.2017.01.033.

Roshandel, D., Mofid, M. and Ghannadiasl, A. (2015), "Dynamic response of a non-uniform Timoshenko beam, subjected to moving mass", *Proceedings of the Institution of Mechanical Engineers, Part C,* **229**(14), 2499-2513. https://doi.org/10.1177/0954406214561049.

Shadnam, M., Mofid, M. and Akin, J. (2001), "On the dynamic response of rectangular plate, with moving mass", *Thin-walled Struct*., **39**(9), 797-806. https://doi.org/10.1016/S0263-8231(01)00025-8.

Şimşek, M. (2015), "Bi-directional functionally graded materials (BDFGMs) for free and forced vibration of Timoshenko beams with various boundary conditions", *Compos. Struct.*, **133**, 968-978. https://doi.org/10.1016/j.compstruct.2015.08.021.

Şimşek, M., Aydın, M., Yurtcu, H. and Reddy, J. (2015), "Size-dependent vibration of a microplate under the action of a moving load based on the modified couple stress theory", *Acta Mechanica*, **226**(11), 3807-3822. https://doi.org/10.1007/s00707-015-1437-9.

Sorrentino, S. and Catania, G. (2018), "Dynamic analysis of rectangular plates crossed by distributed moving loads", *Math. Mech. Solids.,* **23**(9), 1291-1302. https://doi.org/10.1177/1081286517719120.

Takabatake, H. (2013), "Effects of dead loads on dynamic analyses of beams subject to moving loads", *Earthq. Struct.,* **5**(5), 589-605. http://dx.doi.org/10.12989/eas.2013.5.5.589.

Thai-Hoang, C., Nguyen-Thanh, N., Nguyen-Xuan, H. and Rabczuk, T. (2011), "An alternative alpha finite element method with discrete shear gap technique for analysis of laminated composite plates", *Appl. Math. Comput.*, **217**(17), 7324-7348. https://doi.org/10.1016/j.amc.2011.02.024.

Torkan, E., Pirmoradian, M. and Hashemian, M. (2018), "On the parametric and external resonances of rectangular plates on an elastic foundation traversed by sequential masses", *Arch. Appl. Mech.*, **88**(8), 1411-1428. https://doi.org/10.1007/s00419-018-1379-5.

Yamchelou, M.T., Nikbin, I.M., Zareian, H. and Charkhtab, S. (2017), "Assessing absolute maximum vibration amplitude of a rectangular plate subjected to a moving mass", *Iranian J. Sci. Technol., Trans. Civil Eng.*, **41**(2), 135-147.

Yamchelou, M.T. and Nouri, G. (2016), "Spectral analysis of dynamic response of a thin beam subjected to a varying speed moving mass", *J. Mech. Sci.Technol.,* **30**(7), 3009-3017. https://doi.org/10.1007/s12206-016-0609-4.

WT

Appendix

Notation

w	*Plate deflection*
t	*Time*
D	*Plate flexural rigidity*
H	*Heaviside step function*
N	*Number of moving loads*
μ	*Plate mass per unit of area*
γ	*Inertia*
δ	*Dirac delta*
a, b	*Plate dimensions*
g	*Gravitational acceleration*
ν	*Poisson's ratio*
Ω	*Frequency ratio*
L	*Distance between successive moving loads*
u	*Absolute speed of the moving loads*
$M(t)$	*Time-dependent mass matrix*
$C(t)$	*Time-dependent damping matrix*
$K(t)$	*Time-dependent stiffness matrix*
$T(t)$	*Modal coordinate matrix*
v'	*Reference speed*
X_m, Y_m	*Trajectory of the moving load*
ν_m	*Oscillator degree of freedom*
ν_{0m}	*Initial sag of the spring*
K_m	*Oscillator spring stiffness*
Φ_j	*Plate natural shape function*
ω_j	*Plate natural frequency*
δ_{ij}	*Kronecker delta*
τ_j	*Modal amplitude*
T_1	*Fundamental period of vertical vibration of the plate*
B_m	*Window function*
M_m	*The mass of the each moving load*
w_{stat}	*Static deflection at the center point of the plate*

W_N	*Normalized deflection at the center point of the plate*
P_m	*Contact force*