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Sensitivity analysis based on complex variables in FEM for linear structures

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Abstract. One of the efficient and useful tools to achieve the optimal design of structures is employing the sensitivity analysis in the finite element model. In the numerical optimization process, often the semi-analytical method is used for estimation of derivatives of the objective function with respect to design variables. Numerical methods for calculation of sensitivities are susceptible to the step size in design parameters perturbation and this is one of the great disadvantages of these methods. This article uses complex variables method to calculate the sensitivity analysis and combine it with discrete sensitivity analysis. Finally, it provides a new method to obtain the sensitivity analysis for linear structures. The use of complex variables method for sensitivity analysis has several advantages compared to other numerical methods. Implementing the finite element to calculate first derivatives of sensitivity using this method has no complexity and only requires the change in finite element meshing in the imaginary axis. This means that the real value of coordinates does not change. Second, this method has the lower dependency on the step size. In this research, the process of sensitivity analysis calculation using a finite element model based on complex variables is explained for linear problems, and some examples that have known analytical solution are solved. Results obtained by using the presented method in comparison with exact solution and also finite difference method indicate the excellent efficiency of the proposed method, and it can predict the sustainable and accurate results with the several different step sizes, despite low dependence on step size.

Keywords: complex variables method (CVM); discrete sensitivity method (DSM); linear structures; semi-analytical method

1. Introduction

Sensitivity is a calculation of the derivative of the dependent variable with respect to another in a problem. Application of sensitivity quantity is when the designer is looking for an optimal design for a problem using gradient reduction methods, and in practice, it determines the degree of importance of objective function change of the problem respect to design variables change (Choi 2005, van Keulen *et al.* 2005). There are many efficient algorithms such as SQP method needing calculation of derivatives of objective functions and constraints respect to design parameters for

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finding direction search in each step of the optimization process. Error in the calculation of the derivatives reduces the efficiency of the optimization algorithm and can lead to convergence difficulties. Derivatives of objective functions and constraints depend on the sensitivity of structural response (such as displacement, strain, stress) with respect to design variables. Therefore, using an accurate and efficient solution to perform sensitivity analysis is very important for the structural optimization process. In addition, sensitivity analysis has other important applications such as parameter identification and analysis of structure reliability.

The sensitivity calculation methods can be divided into two categories: Analytical methods and numerical methods (Hassanzadeh 2016, 2005, Ghoddosian and Sheikhi 2013). Among the numerical methods, the most famous of them is the finite difference method. Although this method is simple, it requires high computational cost, and it is subject to errors arising from step size selection, and it has high complexity for geometrical variables (Vatsa 2000, Anderson *et al*. 2001, Lai and Crassidis 2008).

Numerical methods that have less sensitivity to step size is complex variables method (Gomez-Farias *et al*. 2015, Voorhees *et al*. 2012). When complex variables method is used properly, it calculates design space numerically and accurately. Unfortunately, this method has high computational cost, as the finite difference method. Its computational cost is almost the same as the analysis of the problem for each design variable (Voorhees *et al*. 2011).

Discrete sensitivity analysis method and adjoint variable method are equivalent to differentiation from discrete equations for production the equations system to calculate required sensitivities (Chung *et al*. 2009). These two methods are strong since they generate the calculation code of sensitivities automatically. In many of the cases, implementation requires human intervention to supply effectiveness to final code for sensitivity.

In semi analytical methods, there are two steps for analysis of problems such as calculating of the displacement of the main problem (u) and sensitivity of displacement. A Semi-analytical method has been proposed as a tradeoff between accuracy, efficiency, and simplicity because this method can be easily implemented and has the accuracy of analytical methods (van Keulen *et al*. 2005). To implement a sensitivity analysis in finite element code, sensitivity analysis includes calculating derivatives of stiffness matrix, mass matrix, and force vector to design variables of the problem. In the analytical method, these derivatives are calculated analytically, but in many cases, especially for calculating the derivative relative to geometric variables, it is difficult to calculate derivatives.

In the semi-analytical method, stiffness matrix derivatives, force vector, and other parameters are calculated numerically using finite difference, but the final solution is performed by an analytical procedure (Ghoddosian and Sheikhi 2010). Thus, we can easily implement the finite difference method and achieve desired results with the same accuracy of the analytical method. However, the semi-analytical method as the finite difference method is not free from cutting and rounding errors, and we should be careful in selecting the step size (Cho and Jung 2003).

To improve the accuracy of the semi-analytical method, several studies were performed. (Olhoff *et al*. 1993) and (Cheng and Olhoff 1993) remove the error by means of correction factors in the numerical derivative. (Cheng and Olhoff 1993) work on the rigid body motion to reduce the error. (Oral 1996) used the Neumann series to improve the accuracy of the semi-analytical method. (De Boer and van Keulen 2000) introduced modified semi-analytical design sensitivities and used it for structural linear, linear buckling problems and geometric non-linear problems (Deboer and van Keulen 2000). This modification is based on accurate differentiation of the rigid body mode. (Lund and Olhoff 1994) used accurate numerical differentiation for sensitivity analysis of eigenvalue problems. (Cho and Kim 2005) presented iterative procedure that is the combination of mode separation technique and series expansion to eliminate cutting errors caused by large step and rounding errors caused by small step.

Complex variables method (CVM) was developed for the first time by (Lyness and Moler 1967). Then, (Lyness 1967) used it to determine the derivatives of some complex functions. (Squire and Trapp 1998) determined the derivatives of real functions using CVM. (Anderson *et al*. 2001) used CVM to determine the sensitivity derivatives for turbulent flows. (Rodriguez 2000) used complex variables method to obtain gradients for optimization algorithm for nonlinear optimization algorithm coupled with Navier-Stokes equations of flow to design the aircraft inlet valve. To date, in finite element structural analysis, CVM has not been used extensively. (Deng *et al*. 2014, Deng and Suresh 2016, Deng and Suresh 2017a, b) were studied topology sensitivity analysis for thermo-elastic problems.

In this paper, an improved semi-analytical method is proposed for calculating the sensitivity analysis that in addition to high accuracy and performance, it can be easily implemented, and it is not dependent on the size of the step size. In the current study, discrete sensitivity method has been combined with complex variables method to reduce the cutting and rounding errors. Accordingly, the advantage of analytical method efficiency and accuracy of complex variables method has been used so that a simple and reliable method to be obtained in calculating the sensitivity analysis with help of finite element code. Then we prove mathematically that this semi-analytical method and complex variables method are equivalent. To implement the mentioned method, finite element code along with sensitivity analysis was written in MATLAB software, and its accuracy and efficiency were applied to several examples that have an analytical solution. Then, it was compared with the finite difference method and analytical method.

2. Discrete sensitivity analysis

Discrete sensitivity analysis estimates the design space derivatives using the residual vector of governing equations, R(u, h). In this expression, u is the vector of field variables and h is the vector of design variables.

The response ϕ is a function of the design parameters h_p (p = 1,...,P), and is dependent both explicitly and through the displacement (Choi 2005):

$$\phi = \phi\left(u\left(h_p\right), h_p\right) \tag{1}$$

Where P is the number of total design variables. The dependency of $u(h_p)$ is only implicit. In other words, it is dependent on design dependency in the coefficients of the equilibrium equations system that it is the solution is u.

The derivative of response ϕ to design parameters h_p is defined as:

$$\frac{d\phi}{dh_p} = \frac{\partial\phi}{\partial h_p} + \frac{\partial\phi}{\partial u}\frac{du}{dh_p} \tag{2}$$

All quantities of the above equation, except for $du/(dh_p)$ that is unknown, can be explicitly calculated. To calculate this expression, additional equations system should be solved.

Equilibrium equation in the structure design problem assuming linear elastic in finite element model is considered:

$$Ku = F \tag{3}$$

Where K is the global stiffness matrix, u is the nodal displacement vector and F is the nodal force vector.

By implicit differentiation of Equation (3) relative to design variables (hp) and arranging the equation, we will have (Bakshi and Pandey 2000):

$$K\frac{\partial u}{\partial h_p} = -\frac{\partial K}{\partial h_p}u + \frac{\partial F}{\partial h_p} \tag{4}$$

Equation (4) is similar to Equation (3). In this equation, only the right side of the vector called as a pseudo-force vector should be calculated.

Stiffness matrix element is given by the following formula (Lund and Olhoff 1994).

$$k = \int_{\Omega} B^T E\, B|J|d\Omega \tag{5}$$

Where E and Ω are modulus of elasticity and the domain of finite element in the coordinates of the curve and without dimension ξ-η-ζ respectively, | J | is determinant of the Jacobian J which at each point defines the transformation of differentials dξ, dη, dζ to dx, dy, dz, B is strain-displacement matrix defined as follows.

$$B = [b_1\, b_2\, \dots\, b_i\, \dots\, b_n] \tag{6}$$

Where,

$$b_i = \begin{bmatrix} N_{i,x} & 0 & 0 \\ 0 & N_{i,y} & 0 \\ 0 & 0 & N_{i,z} \\ N_{i,y} & N_{i,x} & 0 \\ 0 & N_{i,z} & N_{i,y} \\ N_{i,z} & 0 & N_{i,x} \end{bmatrix}. \quad i = 1, \dots, n \tag{7}$$

Where N and n are the shape functions and the number of nodes in each element respectively. Finite element vector f is given by the following equation.

$$f = \int_{\Omega} N^T F_B |J| d\Omega + \int_{\omega} N^T F_S |J| d\omega \tag{8}$$

Where F_B indicates that the body forces, ω is the surface described in the coordinates of the curve and without dimensions of ξ-η, η-ζ or ξ-ζ for the element that forces F_S have been applied to it, N includes shape functions N_i and J calculate on surface ω.

The greatest challenge here is to calculate the stiffness matrix derivatives and force vector relative to the design parameters.

Derivatives $\partial F/\partial h_p$ and $\partial K/\partial h_p$ are generally calculated on each element (Bakshi and Pandey 2000).

$$\left\{\frac{\partial F}{\partial h_p}\right\} = \sum_{e=1}^{NE}\left\{\frac{\partial F^e}{\partial h_p}\right\} \tag{9}$$

$$\left\{\frac{\partial K}{\partial h_p}\right\} = \sum_{e=1}^{NE}\left[\frac{\partial K^e}{\partial h_p}\right] \tag{10}$$

In the above equations, K^e is the element stiffness matrix and NE is the total number of elements. Therefore, the right side of equation (4) is calculated on each element as Equation (11):

$$-\frac{\partial K}{\partial h_p}u+\frac{\partial F}{\partial h_p}=\sum_{e=1}^{NE}\left(-\frac{\partial K^e}{\partial h_p}u^e+\frac{\partial F^e}{\partial h_p}\right) \tag{11}$$

3. Differential of stiffness matrix

Derivatives required for discrete sensitivity analysis are calculated analytically or numerically. In an analytical method, finite element equations are differentiated exactly. The implementation of this method is difficult but very efficient and calculated sensitivities are exact. On the other hand, in the numerical methods, some or all derivatives required are calculated by numerical techniques such as finite difference. Numerical methods itself can be subdivided into local finite difference method and semi-analytical method.

In the local finite difference method, the response of sensitivities to special design parameter is obtained by changing that design parameter several times (depending on the finite difference technique), and equilibrium analysis is performed for any change. Responses to each analysis are kept and they are subtracted so that response sensitivity to be obtained. Although this method is implemented easily, it is costly computationally since the total equilibrium problem should be solved for each change.

In the conventional semi-analytical method that it can be considered as the tradeoff between analytical method and finite difference methods, pseudo-force vectors are obtained by the finite difference method. However, the final solution is obtained by analytical equation (Equation 2) similar to an analytical method of discrete sensitivity analysis solution.

$$\frac{\partial F}{\partial h_p}=\frac{F(h_p+\Delta h_p)-F(h_p-\Delta h_p)}{2\Delta h_p} \tag{12}$$

$$\frac{\partial K}{\partial h_p}=\frac{K(h_p+\Delta h_p)-K(h_p-\Delta h_p)}{2\Delta h_p} \tag{13}$$

The advantage of the semi-analytical method over the analytical method is that it can be easily implemented, and it has higher efficiency than the finite difference method. However, results are sensitive to step size still, and they are not free from cutting and rounding errors. However, as this paper uses the complex variables for numerical calculation of these parameters, the sensitivity of results with respect to step size is eliminated.

In complex variables method (CVM), the calculation of total variables such as global stiffness matrix and global force vector is performed in the complex numerical system. So, this method is required to high the occupied memory and time solving the problem (CVM is the high computational method). In the proposed method, complex numerical system is used only at the local stiffness matrix and don’t require to high the occupied memory and time solving the problem. The advantage of CVM is the calculation of the derivatives of displacement, strain and, stress simultaneously. But if these parameters weren’t required, this case can be considered to the weakness of CVM. The advantage of the proposed method, the design derivatives are calculated exactly where it is needed.

In fact, this paper combines the semi-analytical method with the complex variable method to achieve the advantages and efficiency of semi-analytical methods and accuracy of the complex variables method in the finite element method.

To extract finite difference approximation to calculate the derivatives, Taylor's series of function can be expanded at the point x using forward step and backward step. Then, by subtracting them, the following formula is obtained.

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{14}$$

In equation (14), the order error of first derivative calculation is two (O (Δx^2)). The disadvantage of this method is its high computational time and the possible imprecision of derivatives.

The first disadvantage is since equations (12) and (13) need two solutions with good conversion to calculate its functions. The second disadvantage is because derivatives are sensitive to step size. To minimize the cutting error, step size should be selected small. However, the too small step may lead to the error of meaningful figures removing. The optimal value for step size is not pre-determined, and it may vary from one function to the other one and from one design variable to another one.

Conversely, if the Taylor series of the function is expanded using a complex step in the form of Equation (15) (Voorhees, Millwater and Bagley 2011) the above-mentioned disadvantages are removed.

$$f(x+i\Delta x) = f(x) + i\Delta x\frac{df}{dx} - \frac{\Delta x^2}{2}\frac{d^2 f}{dx^2} - \frac{i\Delta x^3}{6}\frac{d^3 f}{dx^3} + \frac{\Delta x^4}{24}\frac{d^4 f}{dx^4} + \cdots \tag{15}$$

Resolving this equation result for the imaginary part of a function and three first terms of equation (15) give:

$$\frac{df}{dx} \approx \frac{Im[f(x+i\Delta x)]}{\Delta x} \tag{16}$$

Symbol "Im" is the imaginary part of each component. This expression has the second-order error for the derivative (O (h2)) too. Therefore, by calculation of function through complex argument, its function and derivative can be obtained without subtraction error. In this equation, a real part is function value. The disadvantage of complex variable approximation is that runtime program increases by calculation of the program through complex argument. However, as this article limits the use of complex variables only to calculate ∂F/ (∂hp) and ∂K/(∂hp), the runtime does not increase so much. Calculation of ∂F/(∂hp) and ∂K/(∂hp) is according to following equations.

$$\frac{\partial F}{\partial h_p} = \frac{Im\left[F\left(h_p + i\Delta h_p\right)\right]}{\Delta h_p} \tag{17}$$

$$\frac{\partial K}{\partial h_p} = \frac{Im\left[K\left(h_p + i\Delta h_p\right)\right]}{\Delta h_p} \tag{18}$$

In other words, it is enough that design parameter to be changed in an imaginary way as much as step size. When displacement the sensitivity was calculated, the sensitivity of main function dφ/(dhp) is obtained using equation (2) or the following formula.

$$\frac{d\phi}{dh_p} = \frac{Im\left[\phi\left(u + i\Delta u, h_p + i\Delta h_p\right)\right]}{\Delta h_p} \tag{19}$$

Where,

$$\Delta u = \frac{du}{dh_p}\Delta h_p \tag{20}$$

The results of the proposed method and CVM are similar. To prove this, CVM can apply to solve systems of partial differential equations. In CVM imaginary perturbation is applied in global code and after the solution of the equations system, the real part is the field responded, and the imaginary part is related to sensitivities. For this purpose, solving of a problem by combining the finite element method and CVM is presented. After consideration of the imaginary step in equation (3), equation (21) is obtained.

$$\left[K\left(h_p + \mathrm{i}\Delta h\right)\right](u_R + \mathrm{i}u_I) = F\left(h_p + \mathrm{i}\Delta h\right) \tag{21}$$

Where Δh input data, u_I and u_R are obtained. These expressions are exactly used in coding. K stiffness matrix is n × n where n is the number of degrees of freedom. The real and imaginary part of a matrix is assembled thus:

$$[K_R + \mathrm{i}K_I](u_R + \mathrm{i}u_I) = (F_R + \mathrm{i}F_I) \tag{22}$$

Where subscribe R and I are represented of real and imaginary part of each component. By extending the above equations and considering the real and imaginary expressions by apart we obtain:

$$[K_R]u_R - [K_I]u_I = F_R \tag{23}$$

$$[K_I]u_R + [K_R]u_I = F_I \tag{24}$$

By consideration of equation (16), $[K_I]$, $u_I$and $F_I$are equivalent to equation (25-27)

$$[K_I] = \left[\frac{\partial K_R}{\partial h}\right]\Delta h \tag{25}$$

$$u_I = \frac{du}{dh}\Delta h \tag{26}$$

$$F_I = \left(\frac{\partial F}{\partial h}\right)_R \Delta h \tag{27}$$

By substituting the equations (25-27) into equations (23) and (24), we have

$$[K_R]u_R - \left[\frac{\partial K_R}{\partial h}\right]\frac{du}{dh}\Delta h^2 = F_R \tag{28}$$

$$\left[\frac{\partial K_R}{\partial h}\right]\Delta h u_R + [K_R]\frac{du}{dh}\Delta h = \left(\frac{\partial F}{\partial h}\right)_R \Delta h \tag{29}$$

In equation (28), if Δh was considered small, $\left[\frac{\partial K_R}{\partial h}\right]\frac{du}{dh}\Delta h^2$ can neglect thus we have:

$$[K_R]u_R = F_R \tag{30}$$

In the other words, we obtained equation (3) and the real part gives the response of problem. Equation (29) was simplified as below.

$$[K_R]\frac{du}{dh} = \left(\frac{\partial F}{\partial h}\right)_R - \left[\frac{\partial K_R}{\partial h}\right]u_R \tag{31}$$

Equation (31) is similar to equation (4). We can prove that equations (4) and (22) are equivalent to and get the same answer.

4. Numerical examples

In this section, some examples that have analytical solutions are used to validate the method mentioned throughout this article to calculate the sensitivity of linear structures.

As rounding error for calculating, the sensitivity analysis has the direct relationship with the rigid rotation of an element to reveal the efficiency of the presented method, the examples were used in which element rotation value was significant. To implement the mentioned method, two-dimensional finite element program is written with formulation and assumptions mentioned in previous sections for linear problems analysis in MATLAB software. Shape function's equation can be found in almost all of the finite element references. Problem geometry and its meshing are created in ABAQUS software and then imported into MATLAB software. MATLAB's solver was used for solving the equations systems. In the finite element program, four-nodal linear square shape functions and three-nodal shape functions are used for the two-dimensional mesh. After solving the system of equations for obtaining the displacements, stresses in each node element are calculated in the form of element-to-element and by using the elasticity equations, values of shape functions and displacement of nodes. As the calculated stresses of one element may not match with stresses calculated in other elements, averaging was used for smoothing the stress field at the nodal points. For calculating of shape sensitivity, the small step should be applied at appropriate nodal coordinates, while this step should be imaginary. For this purpose, the first node placed in the considered parameter is selected. Relative perturbation for X and Y coordinates depends upon the sensitivity that is supposed to be calculated. All other nodes in the model remain unchanged (the similar process that is applied for the finite difference method, but perturbation is the image not real). If the purpose is the calculation of a parameter sensitivity respect to material properties, for the same studied property, change step is considered. To calculate the sensitivity, the equation (19) has been used due to the simplicity. The obtained results are compared with the sensitivity obtained from analytical solutions. Additionally, sensitivities were compared with central finite difference method. In the finite difference method for analyzing the problems, finite element program without the complex variables was used.

Table 1 problem parameters of the cantilever beam

Problem parameters	Input data
Moment M	100 units
Beam length L	10 units
Height of beam section w	1 unit
With of beam section t	0.5 unit
Young's modulus E	2000 units

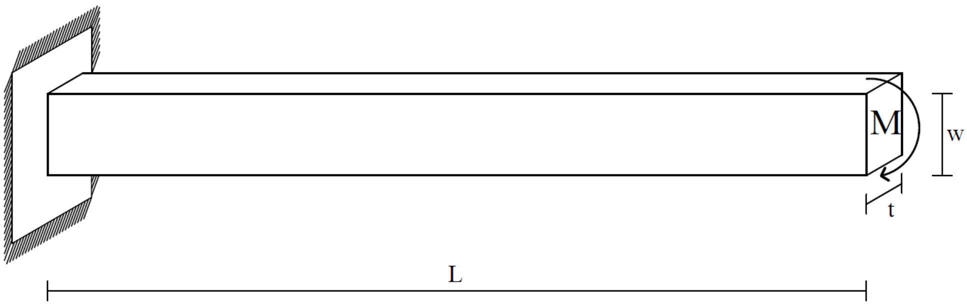

Fig. 1 Cantilever beam with the rectangular section

4.1 Cantilever beam

In this example, a cantilever beam with the rectangular section under moment at the free end that is shown in fig 1. has been considered. The characteristics of it presented in Table 1. Using Euler's theory, we can calculate the beam tip displacement according to Equation (32) (Beer and DeWolf 2002).

$$u = \frac{ML^2}{2EI} \tag{32}$$

Where M is applied moment, L length of the beam, E is the modulus of elasticity, and I is bending moment of beam cross-section. By differentiating from the above expression with respect to L, the amount of displacement sensitivity to beam length is obtained according to Equation (33).

$$\frac{du}{dL} = \frac{ML}{EI} = \frac{ML}{E\left(\frac{1}{12}w^3t\right)} \tag{33}$$

Where w is the width and t is the thickness of the beam. This equation is used to compare the results obtained from the semi-analytical method based on complex variables and finite difference method by applying the finite element method. Clearly, the objective function φ, is the vertical displacement of beam tip (u) and design parameters (h_p) is the beam length (L).

In this example, beam meshing has been considered for two types of the triangle and square elements. For triangular mesh, 2356 elements were used and 1000 elements were considered for square mesh (100 × 10). In Tables 2 and 3, sensitivity values of displacement for various values of step size have been presented using semi-analytical and finite difference method. Numerical results for the triangle and square element have been presented respectively in Figures 2 and 3. As

Table 2 Calculations of displacement sensitivity respect to the length for triangular elements

Step size (h)	$\frac{du}{dL}$ (FDM)	$\frac{du}{dL}$ (SAM)
10^{-2}	11.6845462590199	-22.3371517514557
10^{-3}	11.6851673008718	11.3415818970966
10^{-4}	11.6851779533533	11.6817367355095
10^{-5}	11.6851763568349	11.6851386278959
10^{-6}	11.6851043934219	11.6851726411320
10^{-7}	11.6741590971969	11.6851729843137
10^{-8}	11.6478314993174	11.6851729816150
10^{-9}	12.1984129464181	11.6851729856973
10^{-10}	12.0473586662229	11.6851729837083
10^{-11}	46.7210270471696	11.6851729864456
10^{-12}	-151.860746200327	11.6851729843966
10^{-13}	-5007.65651167967	11.6851729857735
10^{-14}	-20128.9651613479	11.6851729864039
10^{-15}	-158767.221591916	11.6851729790985
10^{-16}	0	11.6851729851719
10^{-17}	0	11.6851729849670
10^{-18}	0	11.6851729860968
10^{-19}	0	11.6851729820022
10^{-20}	0	11.6851729846447
exact	12	

it can be seen in these figures, a good efficiency is seen to calculate the sensitivity design using the combined method of semi-analytical and complex variables. The results show that there is good convergence regardless of the size of the step. There are similar results from both types of elements, triangle, and square. However, the performance of the finite difference method is highly dependent on the selected step size. In the step size less than 10^{-9}, the dramatic increase is seen in the error. In addition, the average time to implement the program (T_{av}) is less for semi-analytical methods compared to the finite difference.

The percentage of relative error of displacement sensitivity obtained by the presented method is very small compared to its exact value at measure steps size 10^{-4} and smaller than it, and it is about 2.65% for triangular element and 0.28% for the square element. However, this error value is not caused by differentiation method, and it is due to the error arising from the finite element model solving (Voorhees, Millwater and Bagley 2011). The presented method is sensitive to greater step size (cutting error) since calculating sensitivity in the method is dependent on function (in this example, displacement) and as the step is greater, the calculated displacement would have the error. Therefore, it causes the error in the displacement sensitivity. The reason for the dependency of displacement to step is that the program code written on this paper, stiffness matrix and its derivation are calculated simultaneously due to reduced program runtime.

Table 3 Calculations of displacement sensitivity respect to the length for square elements

Step size (h)	$\frac{du}{dL}$ (FDM)	$\frac{du}{dL}$ (SAM)
10^{-2}	11.9701739802025	-24.4536719441518
10^{-3}	11.9701749737189	11.6023302825188
10^{-4}	11.9701947586037	11.9664955415641
10^{-5}	11.9701120588900	11.9701375486135
10^{-6}	11.9724634828344	11.9701739721297
10^{-7}	11.9727328495856	11.9701743304791
10^{-8}	11.7898792950655	11.970174337300
10^{-9}	8.77438566249111	11.9701743389902
10^{-10}	-0.264677169070637	11.9701743375128
10^{-11}	-167.503344528086	11.9701743387048
10^{-12}	3751.23931917187	11.9701743409468
10^{-13}	-758.859641791786	11.9701743382964
10^{-14}	-575216.319020910	11.9701743432354
10^{-15}	0	11.9701743423267
10^{-16}	0	11.9701743410797
10^{-17}	0	11.9701743353410
10^{-18}	0	11.9701743457464
10^{-19}	0	11.9701743428501
10^{-20}	0	11.9701743387371
Exact	12	

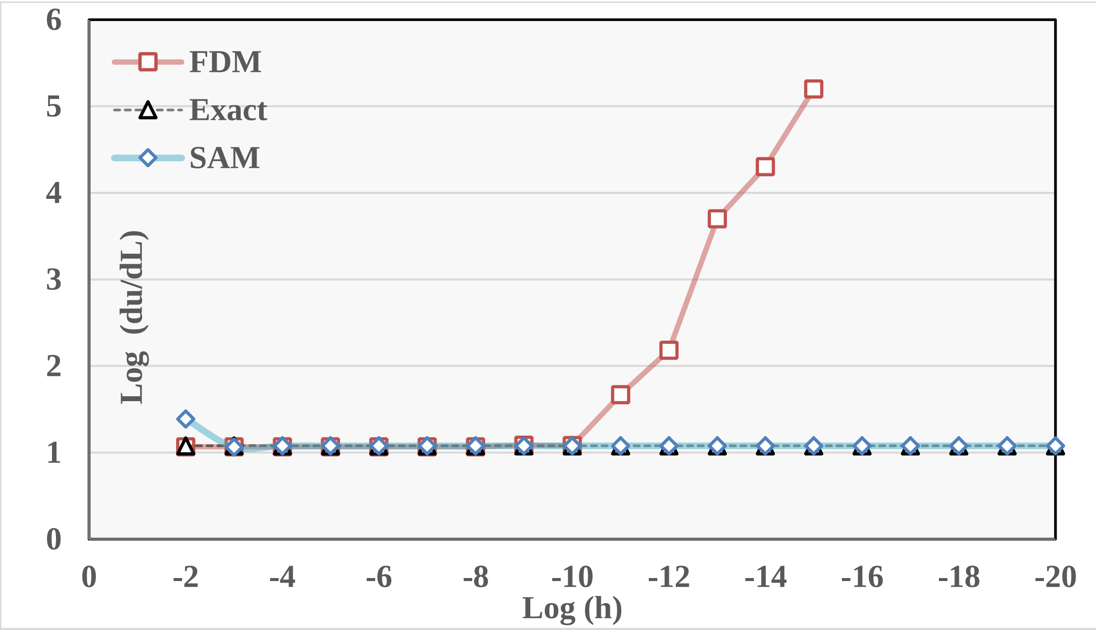

Fig. 2 Vertical displacement sensitivity of the beam tips for triangular elements

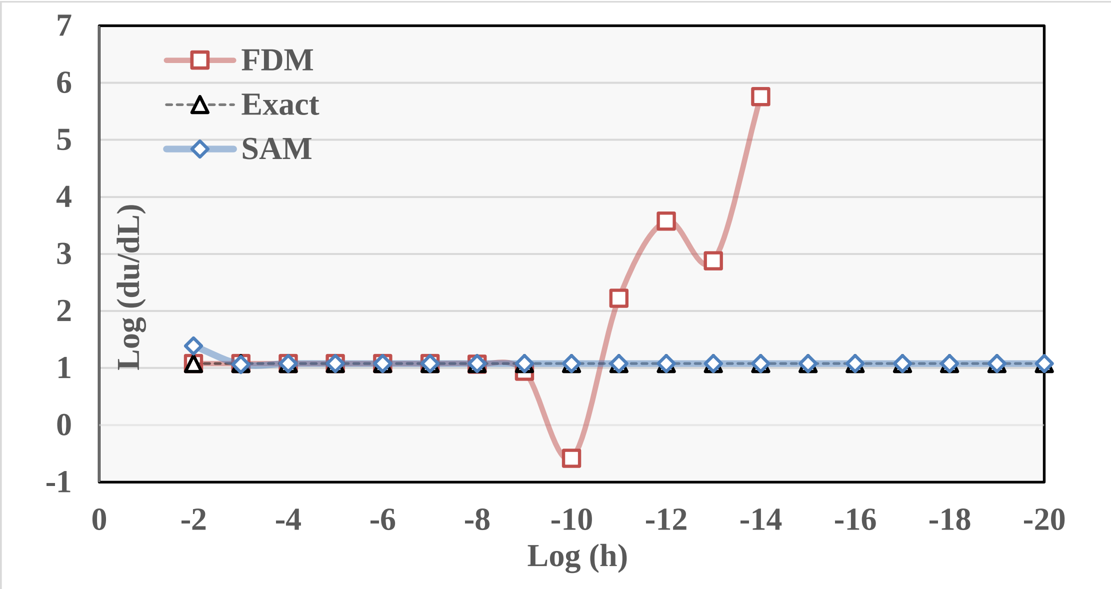

Fig. 3 Vertical displacement sensitivity of the beam tips for square elements

Table 4 Problem parameters in finite element modeling for the curved cantilever beams

Problem parameters	Input data
Force F	10 units
Inner radius R_i	90 units
outer radius R_o	100 units
Beam thickness t	5 units
Beam width w	10 units
Poisson's ratio v	0.3 unit
Young's modulus E	2000 units

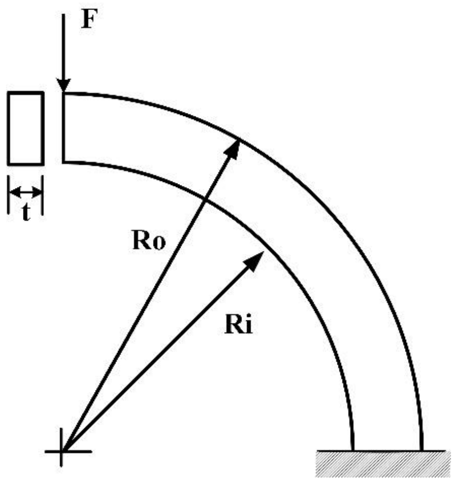

Fig. 4 Curved cantilever beam

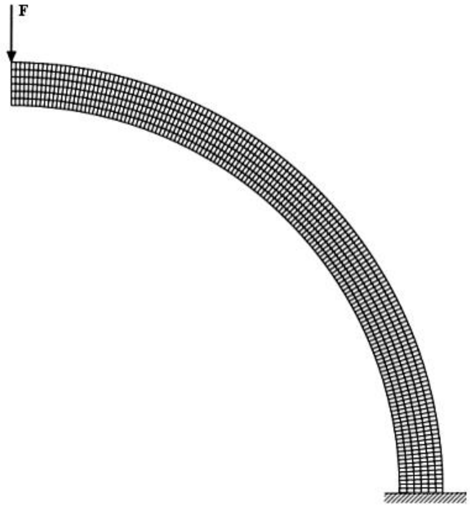

Fig. 5 Meshing of the curved cantilever beam

4.2 Curved cantilever beam

In this section, a curved beam has been considered as shown in Figure 4. Parameters of finite element modeling are shown in Table 4. In this example, the objective is calculating the sensitivity of vertical and horizontal displacement of the beam tip respect to its thickness. Using the classical theory of elasticity for curved beams, horizontal and vertical displacement of values' beam tip can be obtained. Then, they can be differentiated with respect to thickness. Horizontal and vertical displacement (respectively u and v) are analytically obtained along with its derivatives as Equation (34) and (35) (Beer and DeWolf 2002).

$$u = \frac{1}{2}\frac{FR^3}{EI} = \frac{1}{2}\frac{FR^3}{E\left(\frac{1}{12}tw^2\right)} \Rightarrow \frac{\partial u}{\partial t} = -\frac{1}{2}\frac{FR^3}{E\left(\frac{1}{12}t^2w^2\right)} \tag{34}$$

$$v = \frac{\pi}{4}\frac{FR^3}{EI} = \frac{\pi}{4}\frac{FR^3}{E(\frac{1}{12}tw^2)} \Rightarrow \frac{\partial v}{\partial t} = -\frac{\pi}{4}\frac{FR^3}{E(\frac{1}{12}t^2w^2)} \tag{35}$$

In the aforementioned equations; F is applied force, R is the average radius of the beam; E is the modulus of elasticity, I the moment inertia of the beam cross-section, w is the width of the beam, and t is beam thickness. The above equations are used to calculate the sensitivity of the analytical method. To analyze the problem, 894 square elements (149 × 6) are used (Figure 5).

In Tables 5 and 6, results related to sensitivity calculation using semi-analytical and finite difference methods has been provided. Numerical results of vertical and horizontal displacement sensitivity calculation to beam thickness is shown in Figures 6 and 7, respectively. As it can be seen in these Figures, using a combination of the semi-analytic method with complex variables, the sensitivity of design parameters can be calculated with high efficiency. The results show that there is good convergence regardless of the step size and performance of finite difference is highly dependent on the selected step size. There is the sharp increase in error at the step size than 10^{-9}. In addition, program runtime for the semi-analytical method is less than finite difference.

Table 5 Calculations of vertical displacement sensitivity respect to the thickness

Step size (h)	$\frac{du}{dL}$ (FDM)	$\frac{du}{dL}$ (SAM)
10^{-2}	1.60641458426127	1.60640817996036
10^{-3}	1.60640815829005	1.60640817995965
10^{-4}	1.60640723506411	1.60640817996744
10^{-5}	1.60639448063549	1.60640817990789
10^{-6}	1.60641987889676	1.6064081799321
10^{-7}	1.60498026424705	1.6064081800330
10^{-8}	1.60558011330636	1.6064081799450
10^{-9}	1.60953383954165	1.60640817991017
10^{-10}	2.11750617040707	1.60640817983094
10^{-11}	-18.7790227812457	1.60640817990765
10^{-12}	-196.472171865025	1.60640817994555
10^{-13}	-1039.35526851728	1.60640817994889
10^{-14}	6518.16378649528	1.60640817995388
10^{-15}	-216330.9531511	1.60640817996601
10^{-16}	0	1.60640817988663
10^{-17}	0	1.60640817998013
10^{-18}	0	1.60640817993936
10^{-19}	0	1.60640817990344
10^{-20}	0	1.60640817988717
Exact	1.62	

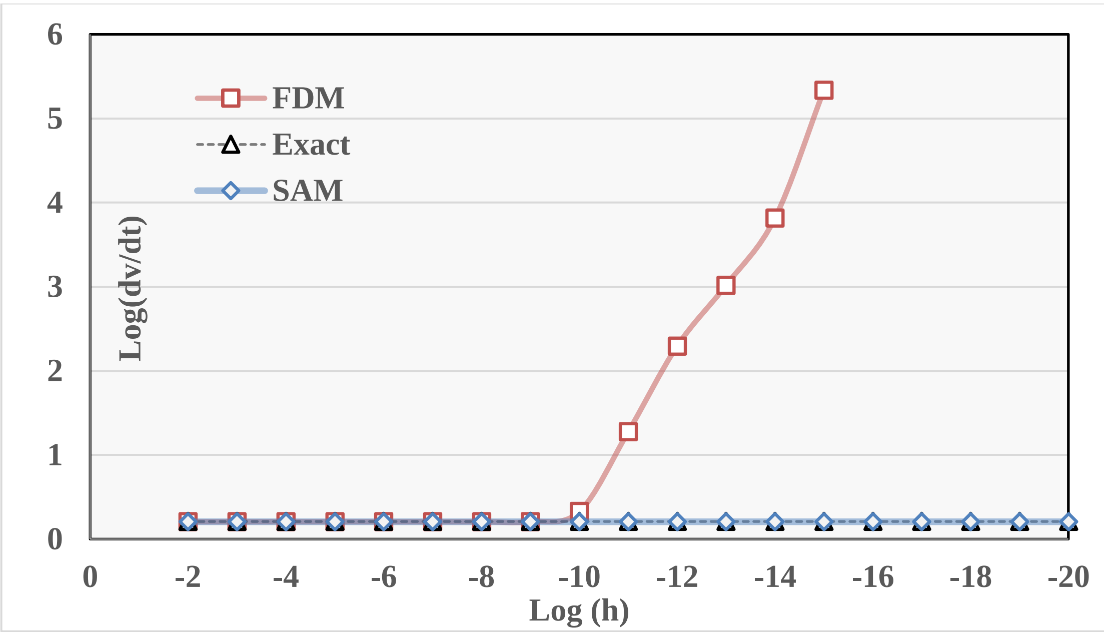

Fig. 6 Calculations of vertical displacement sensitivity respect to the thickness

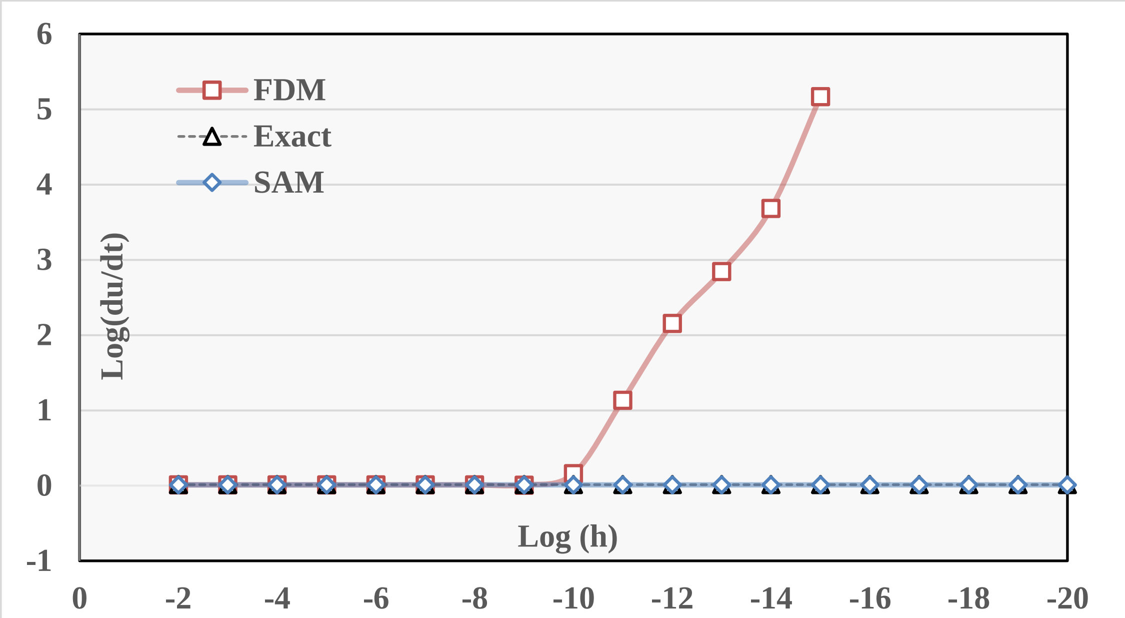

Fig. 7 Horizontal displacement sensitivity respect to the beam thickness

Table 6 Calculation of horizontal displacement sensitivity respect to the thickness

Step size (h)	$\frac{du}{dL}$ (FDM)	$\frac{du}{dL}$ (SAM)
10^{-2}	1.02278621492213	1.02278213856483
10^{-3}	1.02278212932916	1.02278213856101
10^{-4}	1.02278142379841	1.02278213856685
10^{-5}	1.02277216753599	1.02278213852045
10^{-6}	1.02279012059725	1.02278213853955
10^{-7}	1.02170208737817	1.02278213861228
10^{-8}	1.02382169409054	1.02278213854993
10^{-9}	1.01917096984039	1.02278213852365
10^{-10}	1.43828948750979	1.02278213846956
10^{-11}	-13.671730414444	1.02278213852174
10^{-12}	-143.014933229324	1.02278213855027
10^{-13}	-696.624979923399	1.02278213855312
10^{-14}	4808.77559994042	1.02278213855639
10^{-15}	-147414.969120518	1.02278213856389
10^{-16}	0	1.02278213850949
10^{-17}	0	1.02278213857545
10^{-18}	0	1.02278213854806
10^{-19}	0	1.02278213852158
10^{-20}	0	1.02278213850857
Exact	1.03	

Relative error percentage of vertical and horizontal displacement sensitivity obtained by the presented method to its exact value is very little for steps size of 10^{-2} and smaller, and it is approximately %0.84 and %0.70, respectively.

5. Conclusions

This paper provides a robust and accurate method to calculate sensitivity using the combination of discrete sensitivity method with complex variables method in structural problems. Discrete sensitivity method is an efficient method that saves time in calculation compared to other methods, and thereby it is a fast method, as it calculates the sensitivity values only where analysts wanted to calculate. However, common methods use an accurate procedure for stiffness matrix and mass matrix differentiation. Differentiation in this procedure in a precise way for any finite element type can be cumbersome.

To improve the efficiency of the discrete sensitive method, exact differentiation can be replaced by a finite difference method. However, conventional sensitivity calculation methods are highly dependent on step size values. Therefore, this paper provides a new computational procedure to calculate numerical sensitivity based on discrete sensitivity method by using advantages of discrete sensitivity method and complex variables that are not affected by the step size.

In this paper, by solving several examples, advantage and superiority of the proposed method compared to the conventional computational methods have been shown. If only step size is selected small, rounding error would be negligible and this method would provide the stable solution for structure sensitivity. Advantages of the proposed method can be listed as follows:

- In larger steps, calculation of sensitivity analysis error by the proposed method is a little more compared to finite difference method, but it includes the small range and by choosing the smaller step as much as you like, it can be achieved with a high precision sensitivity analysis.
- The proposed method is not sensitive to small steps size, and it means that step size can be considered as small as you want, while the finite difference method is highly dependent on step size. Therefore, there is a limit to select the step size in this method.
- Program runtime for the proposed method is almost half of program runtime for the finite difference method.
- The proposed method is stable at the very high range, and it reaches to convergence with high accuracy quickly.

Finally, sensitivity errors generated in conventional the computational methods can be reduced significantly by using complex variables. Numerical sensitivity in the framework of discrete sensitivity method can be improved by using complex variables. Compared with conventional methods, the computational method presented in this paper has high precision and accuracy and ensures good performance at all step size values.

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