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# Frequency analysis of liquid sloshing in prolate spheroidal containers and comparison with aerospace spherical and cylindrical tanks

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**Abstract.** Free surface fluid oscillation in prolate spheroidal tanks has been investigated analytically in this study. This paper aims is to investigate the sloshing frequencies in spheroidal prolate tanks and compare them with conventional cylindrical and spherical containers to select the best tank geometry for use in space launch vehicles in which the volume of fuel is very high. Based on this, the analytical method (Fourier series expansion) and potential fluid theory in the spheroidal coordinate system are used to extract and analyze the governing differential equations of motion. Then, according to different aspect ratios and other parameters such as filling levels, the fluid sloshing frequencies in the spheroidal prolate tank are determined and evaluated based on various parameters. The natural frequencies obtained for a particular tank are compared with other literature and show a good agreement with these results. In addition, spheroidal prolate tank frequencies have been compared with sloshing frequencies in cylindrical and spherical containers in different modes. Results show that when the prolate spheroidal tank is nearly full and in the worst case when the tank is half full and the free fluid surface is the highest, the prolate spheroidal natural frequencies are higher than of spherical and cylindrical tanks. Therefore, the use of spheroidal tanks in heavy space launch vehicles, in addition to the optimal use of placement space, significantly reduces the destructive effects of sloshing.

**Keywords:** fourier series expansion; free surface oscillation; potential fluid theory; sloshing frequencies; spheroidal containers

## 1. Introduction

The movement of fluid in a free surface under external motion is called sloshing. This phenomenon is very important in many vehicles, trains carrying chemicals, ships carrying gas, and spacecraft (Leonard and Walton 1961). In the condition that the frequency of the external force approaches the natural frequency, the sloshing can be strong and magnified, causing the system to

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become unstable and the system cannot reach its final state. Among them, spacecraft are of special interest due to their large volume of fluid and operation in an environment full of vibration frequencies. It makes up about 90 percent of the spacecraft's weight, which can affect stability and performance when affected by acceleration. In many cases, the sloshing caused by the forces entering the tank causes instability and reduction of guidance and control in the device, as a result, the assigned mission may be executed correctly or a project may fail. This issue is of particular importance in space transportation due to the choice of high costs, fuel volume, technology, high construction, electronic electronics, cargo, etc. Therefore, they are looking for a way to reduce the harmful consumption of sloshing or increase its frequency and damping. Here is only a brief overview of the most important works that are directly relevant to the present research. Leonard and Walton (1961) researched the natural frequencies of different types of liquids in prolate spheroidal-horizontal tanks. Experimental research was done by them to understand some characteristics of free vibration of liquids in ellipsoidal tanks that can be used in rocket systems and space vehicles. Measured natural frequencies for three or four asymmetric modes of oscillation as a function of liquid depth were obtained for three directions from each of several tanks of different sizes and diameters. In the same year, Stephens *et al.* (1961) conducted research on the damping of liquid fluctuations in an prolate spheroidal tank with and without baffles. They conducted many studies to determine the damping of the asymmetric mode of fluid oscillations in an elliptical tank with and without baffles for a wide range of fluid depths. In investigating the effects of baffles, ring and cruciform baffles were installed in different sizes and locations inside the tank. The presented data shows the change of the damping coefficient with the height of the fluid and the type, width, location, and orientation of the baffle, as well as the insignificant effect of the viscosity of the fluid on the damping. Sumner (1965) conducted a pendulum experiment to determine the properties of sloshing in horizontal spherical and elliptical tanks. He conducted this research to determine the general characteristics of fluid sloshing (such as fundamental frequencies, horizontal and lateral forces of sloshing, and damping ratio) and also to determine values for the analogy of a pendulum that effectively represents the fundamental frequency of sloshing in spherical tanks and horizontal prolate spheroidal tanks in a certain range of the height is fluid, he did. Seide *et al.* (1968) presented research related to the design of space vehicles. In this article, three criteria are mentioned regarding this issue, which are environmental structures, guidance and control, and propulsion. The stability studies in this research showed that if an equivalent mechanical system is used to represent the fluid sloshing, the dynamic response of the device can be calculated. Such mechanical systems consist of fixed masses and oscillating masses that are connected to the tank by springs and pendulums. Concus *et al.* (1969) presented a study titled lateral sloshing in prolate spheroidal-ring tanks with small fluctuations and under low gravity conditions. Their goal is to calculate the sloshing fluctuations in prolate spheroidal tanks with eccentricity ratios of 0, 0.5, 0.68, and 0.8 and for the filling volume of the tank from 1⁄7 to 1⁄8. Coney and Salzman (1971) presented a paper entitled Lateral sloshing in horizontal prolate spheroidal tanks under low gravity conditions as well as natural gravity. The result was that the highest percentage change in the natural frequency for the off-center tank is when the Bond number parameter is low and the tank is relatively empty. Howard (1972) conducted research on sloshing modeling for fluid movement in the prolatespheroidal tank. He proposed a mathematical model to analyze the dynamic effects of the internal motion of a fluid with a large oscillation amplitude attached to a rigid body. Albright (1977) conducted a study titled sloshing with a small amplitude of fluid fluctuation in a vertical cylindrical tank with a concave end. In this research, they studied the effects of sloshing at a high height of the fluid level in the tank. Dodge and Kana

(1987) presented an article on fluid sloshing dynamics in vertical and inverted tanks using flexible membranes. In this study, fluid sloshing was investigated in tanks that use a flexible and also non-flexible membrane. Bauer and Eidel (1989) investigated the free and forced oscillations of a frictionless fluid in a prolate spheroidal tank geometry. In the same year, Eidel (1989) presented an article titled “Nonlinear Fluid Oscillations in prolate spheroidal Tanks”. He characterized the nonlinear behavior of an incompressible and frictionless fluid in a spheroidal tank. He showed that the sloshing fluid exhibits nonlinear characteristics depending on the height and the way of excitation. Additionally, increasing the amplitude decreases the natural frequency relative to the linear natural frequency typically found for higher fluid heights. This nonlinear behavior is more pronounced at higher altitudes. Bao (1994) presented a paper on the numerical calculation of the fluid meniscus in a slowly rotating tank placed in a low gravity field. In this study, a fluid that fills a height of a tank with slow rotation, under conditions such as low gravity, small centrifugal force, and high surface drag force, was investigated. He showed that the free surface fluid changes its shape in the shape of a meniscus under these conditions. Such a form is solved by an ordinary differential equation in which there are two parameters and one uncertain boundary, and they are determined by the iteration method. The conventional iteration adopted in solving this problem is called the “simple throw method”, but it was found that the use of this method causes the degree of divergence to increase. In this paper, a two-way launch method is proposed and several experiments show that it converges relatively well with the experimental results. Utsumi (2008) did a paper titled sloshing Analysis for Drop Shape Tank. In this paper, he analyzed the sloshing in a drop-shaped reservoir with a new semi-analytical method for a fluid with asymmetric amplitude. In this research, two points were investigated. First, due to the acceleration of the engine, the sloshing is subjected to the acceleration force of the excitation. Secondly, the resonant frequencies of the sloshing are excited by the Coriolis force. It was shown that mechanical models in two orthogonal directions may show a difference for high levels of fluid height. To solve this problem, a method was presented by expanding two-way orthogonal mechanical models under gravity and centrifugal forces, to simulate the frequency of sloshing. Yang and Peugeot (2010) made a method to extract the propulsion sloshing parameter using computational fluid dynamics analysis. This study represents a continuous effort in validating CFD technology in the n sloshing dynamics modeling of space vehicles. This research clearly shows the correctness of the CFD approach in modeling fluid sloshing equations, to extract dynamic characteristics in different geometries of the tank and at different height levels. Mavrakos and Chatjigeorgiou (2012) presented a method to calculate the hydrodynamic forces acting on prolate spheroidal tanks immersed in water. This study aimed to provide an analytical solution for the problem of hydrodynamic diffraction in a prolate spheroidal tank submerged in water depth. In addition, the analytical calculation of hydrodynamic forces and the presentation of the solution method based on multipole expansion were of interest. He showed that multipole potential functions satisfy free surface conditions and boundary conditions and the main challenge in the analytical solution is to calculate the multipole potential function according to the reservoir geometry. However, for spherical reservoirs, the multipole potential must be converted to ellipsoidal coordinates, and this requires the assumption of additional and appropriate theorems. Turner and Bridges (2013) investigated the dynamic interaction between fluid sloshing and the motion of a tank containing fluid. It also specified a model that described the energy exchange between the movement of the reservoir and the movement of the fluid. Chatjigeorgiou and Miloh (2014) presented the results of a research entitled hydrodynamics of the free surface of a prolate spheroidal tank in shallow water depth based on the polygon expansion method. In this study, he investigated the hydrodynamics of an ellipsoidal free surface immersed in shallow water using ellipsoidal harmonics and the multipole expansion method and provided semi-analytical solutions to solve the basic problems of wave diffraction modeling. Yang *et al.* (2016) investigated liquid propellant sloshing that can significantly affect the stability of space vehicles. sloshing equations are typically represented by a mechanical model with a mass damper and spring. This mechanical model is included in the equation of motion of the whole vehicle for the analysis of guidance, navigation, and control. In addition, the usual parameters required by the mechanical model such as natural frequency, fluid mass, the center of fluid mass, and critical damping ratio were calculated. Yang derived an asymptotic damping equation as well as CFD simulation to complement the Miles equation (Yang *et al.* 2016) which is used in different flow regimes. This study aims to further expand the semi-empirical damping equations related to the spherical part of the propulsion tanks. Storey *et al.* (2020) presented an article titled sloshing Test in a Spherical Tank and its Expansion with CFD Validation. He showed that computational fluid dynamics simulations can be used to predict slosh fluid behavior, but these programs require extensive validation. Storey and Kirk (2020) used computational fluid dynamics to model sloshing in the tanks of liquid propellant space launchers. In addition, in this study, the results of testing a spherical tank without a ring baffle and also with a ring baffle using water and nitrogen were presented. He used constant stimulation with constant amplitude and variable frequencies to measure the forces. The natural frequency and damping parameters of the presented analytical models were compared with the experimental results. Coogan and Green 2019 presented methods to predict sloshing damping in tanks with annular baffles. He showed that the fluid mass in motion introduces forces and torques to the tank structure, which, if not properly controlled, can cause the device to become unstable. In addition, circular baffles have the highest amount of damping, and analytical methods can be used to calculate the size and spacing of baffles in propulsion tanks to achieve maximum damping. Also, the Miles equation (Yang *et al.* 2016) accurately predicts the damping for a single baffle, and the damping in spherical parts can be predicted through an equivalent cylinder. Zang *et al.* (2021) and his friends investigated the entropy production in prolate spheroidal tanks. They used a numerical model to simulate the sloshing phenomenon by applying the Reynolds and Navier-Stokes numerical solution methods, as well as the fluid volume method and some parameters such as the maximum free surface displacement, the maximum horizontal force applied to the environment of the tank, the total entropy production, etc., are set as design criteria. The results show that the relationship between the maximum displacement of the free surface and the maximum horizontal force applied to the tank environment is almost linear concerning the amplitude of the oscillating movement, and also, increasing the lean ratio reduces the maximum displacement of the free surface and the maximum horizontal force applied to the tank environment, which The relationship between these two parameters is nonlinear concerning the angular frequency of the oscillating motion. The review of the above research shows that although a large number of studies have been conducted in the field of sloshing in various tanks, there is no detailed report on the analysis of sloshing in prolate spheroidal tanks and comparison with conventional tanks such as cylindrical and spherical. The main goal of the current study is to investigate and calculate the sloshing frequencies in prolate spheroidal tanks in different modes and compare the results with spherical and cylindrical tanks, which leads to the selection of the best tank for the large volume of fluid in space launchers.

The most important innovations of this article can be summarized as follows:

Development of a three-dimensional computational model based on Fourier series expansion to calculate natural frequencies in prolate spheroidal tanks.

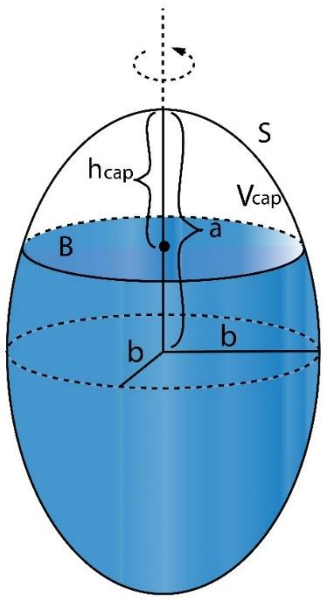


Fig. 1 Geometry of prolate spheroidal tank

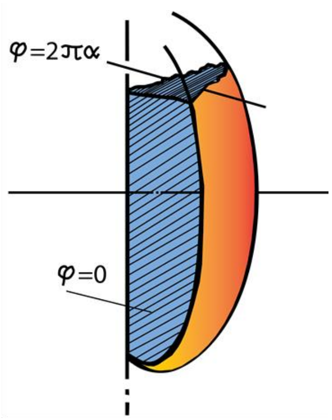


Fig. 2 A cut view of an prolate spheroidal tank and the display of the spheroidal coordinate system

Validation of the model by comparison with existing results.

Analysis of fluid fluctuations in prolate spheroidal and comparing the results with conventional spherical and cylindrical containers.

To achieve this goal, first, the equations governing the fluid sloshing are written in elliptic coordinates and the appropriate boundary conditions are extracted on the free surface of the fluid and the solid walls of the tank. Then, by expanding special functions in elliptic coordinates, matrix relationships have been obtained to extract frequencies and modes of fluid oscillation in the tank. Finally, by choosing prolate spheroidal tanks with specific dimensions, sloshing frequencies in four different and important modes have been obtained. In addition, the obtained frequencies have been compared and analyzed with the results of frequency analysis in spherical and cylindrical tanks of the same volume.

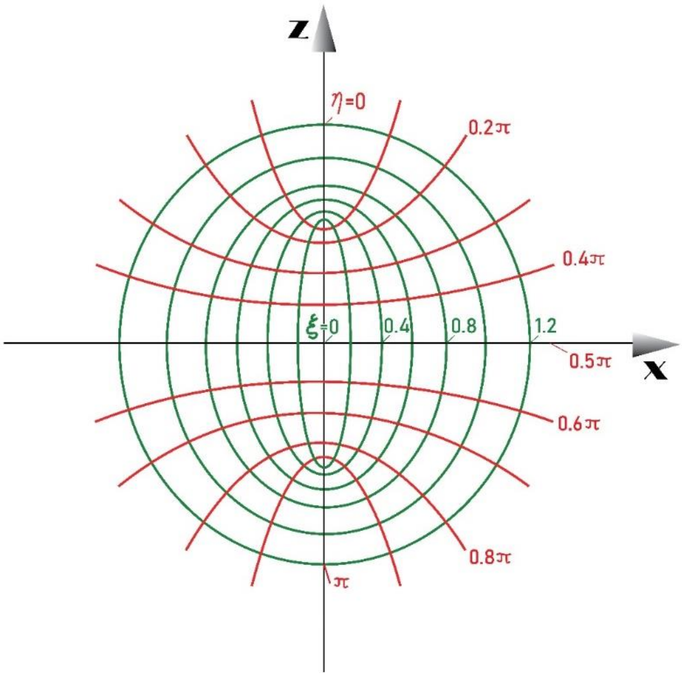


Fig. 3 The geometry of liquid containers and coordinate system

## 2. Formulation

The geometry of the prolate spheroidal tank is shown in Fig. 1. The tank is filled with fluid to the desired height of 2a-h. The effect of fluid fluctuations with a free surface in the tank can be described by considering a non-viscous and incompressible liquid.

The selected coordinate system for the expansion of the velocity potential function is shown in Fig. 2.

The flow velocity vector $v$ is a gradient of the velocity potential, which for incompressible fluids (by satisfying the continuity equation) creates Laplace's equation as follows

$$\sin\eta \frac{\partial}{\partial \xi}\left(\sinh\xi \frac{\partial \Phi}{\partial \xi}\right) + \sinh\xi \frac{\partial}{\partial \eta}\left(\sin\eta \frac{\partial \Phi}{\partial \xi}\right) + \frac{\sinh^2\xi + \sin^2\eta}{\sinh\xi \sin\eta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0 \tag{1}$$

Since we are dealing with fluid fluctuations in a spheroidal tank (Fig. 1), Laplace's Eq. (1) must satisfy the boundary conditions of the tank walls. Thus, for a fixed reservoir in the annular walls, the boundary conditions are written as follows

$$\frac{\partial \Phi}{\partial \xi} = 0 \Big|_{\xi=\xi_0} \tag{2a}$$

In addition, in the case of harmonic forced excitation $x_0\, e^{i\Omega t}$ in the transverse direction, the boundary condition on the tank walls at the boundaries of $\xi_0$ is written as follows.

$$\frac{\partial \Phi}{\partial \xi} = c\, x_0\, i\, \Omega\, e^{i\Omega t} \cosh\xi \sin\eta \cos\varphi \Big|_{\xi=\xi_0} \tag{2b}$$

$c$=2$a$ is the scale factor between Cartesian and elliptical coordinate systems. In addition, the potential function must satisfy the boundary conditions of the side walls of the tank, which are written as follows

$$\frac{\partial \Phi}{\partial \varphi} = 0 \quad \Big|_{\varphi=0.2\pi\alpha} \tag{3a}$$

The above boundary condition in harmonic excitation for free oscillations is rewritten as

$$\frac{\partial \Phi}{\partial \varphi} = -c\, x_0\, i\, \Omega\, e^{i\Omega t} \sinh \xi \;\; \sin \eta \sin \varphi \quad \Big|_{\varphi=0.2\pi\alpha} \tag{3b}$$

In addition, the boundary condition of the free surface of the fluid is written as Eq. (4), which is obtained from Bernoulli's equation (dynamic relationship) and the kinematic relationship between the free surface of the fluid and the potential function.

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{g \cosh \xi \sin \eta}{c(\sinh^2 \xi + \sin^2 \eta)} \frac{\partial \Phi}{\partial \eta} = 0 \quad \Big|_{\eta=\eta_0} \tag{4}$$

In the following, we examine the free oscillations of the sloshing fluid in the prolate spheroidal tank. Investigation of free oscillation in a stationary tank results in the fundamental natural frequencies and mode shapes of the fluid sloshing. The answer to Laplace's Eq. (1) under the boundary conditions of rigid walls, i.e., relations (2a), (3a) is obtained in the form of relation (5).

$$\Phi(\xi.\eta.\varphi.t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \mathrm{A_{mn}} \mathrm{e}^{\mathrm{i}\omega_{\mathrm{mn}}\mathrm{t}} P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\cosh \xi)\, P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos \eta)\cos\frac{m\varphi}{2\alpha} \tag{5}$$

In Eq. (5), $P_{\nu}^{m}$ is the generalized Legendre function of type 1, which is written as Eq. (6) based on hypergeometric functions.

$$P_{\nu}^{m}(\mathrm{x}) = \left(\frac{1+x}{1-x}\right)^{m/2} F\left(\nu+1.-\nu; 1-m; \frac{1}{2} - \frac{1}{2}x\right) \qquad -1 < x < +\infty \tag{6}$$

The following relationship can be used to determine the approximate natural frequencies.

$$\omega_{\mathrm{mn}} = -\frac{g}{c} * \frac{P'^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos \eta_0)}{P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos \eta_0)} \tag{7}$$

where $\lambda_{mn}$ is the roots of $P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\cosh \xi_1) = 0$ and must be obtained numerically. Also, the values of $A_{mn}$ are constants that can be calculated from the initial conditions of the problem. By placing the expansion (5) in the boundary conditions of the free surface (4), the equation governing the shape of the modes and fundamental frequencies of the sloshing is obtained as Eq. (7).

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} -\omega_{mn}^2\, A_{mn} e^{i\omega_{mn}t} P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\cosh \xi) P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos \eta_0) cos\frac{m\varphi}{2\alpha} =$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{g\sin\eta_0\cosh\xi}{c\,(\sinh^2\xi+\sin^2\eta_0)}A_{mn}e^{i\omega_{mn}t}\,P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\cosh\xi)\;P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos\eta_0)\cos\frac{m\varphi}{2\alpha}\varphi \quad (8)$$

To determine the natural frequencies $\omega_{mn}$, using the orthogonal relation (8), we expand the right side of (7) to a set of $P^{m/2\alpha}_{\lambda_{mn}}$ functions.

$$\int_1^{\xi_0} P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\xi)\,P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\xi)\,d\xi=\begin{cases}0 & for\ n\neq\nu\\ \dfrac{(1-\xi_0^2)}{(2\lambda_{mn}+1)}\;\dfrac{\partial P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\xi_0)}{\partial\lambda_{mn}}\;\dfrac{\partial P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\xi_0)}{\partial\xi} & for\ n=\nu\end{cases} \quad (9)$$

With this method, an infinite system of homogeneous linear equations is obtained in the form of an Eq. (9).

$$\det\left[\alpha_{n\nu}\left(m.\frac{g}{c}.\ \xi_0.\xi_1.\eta_0.\eta_1\right)\right]A_{mn}=0 \qquad (m=0.1.2.3.\dots) \quad (10)$$

where the $\alpha_{n\nu}$ matrix constants are calculated from the Eq. (10).

$$\alpha_{n\nu}\left(m.\frac{g}{c}.\ \xi_0.\xi_1.\eta_0.\eta_1\right)=\gamma_{n\nu}\left(m.\frac{g}{c}.\ \xi_0.\xi_1.\eta_0.\eta_1\right)+\omega_{mn}^2\delta_{n\nu}$$
$$(n=0.1.2.3.\dots;\nu=0.1.2.3.\dots;m=0.1.2.3.\dots) \quad (11)$$

In the above relation, $\delta_{n\nu}$ is Kronecker's delta function and the $\gamma_{n\nu}$ matrix elements are calculated from Eq. (11).

$$\gamma_{n\nu}\left(m.\frac{g}{c}.\ \xi_0.\xi_1.\eta_0.\eta_1\right)=\frac{g\sin\eta_0\,P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{m\nu}}(-\cos\eta_0)\,\mu^{(1)}_{n\nu}(m.\xi_0.\xi_1.\eta_0)}{c\,P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(-\cos\eta_0)} \quad (12)$$

in which $\mu^{(1)}_{n\nu}$ is obtained from Eq. (12).

$$\mu^{(1)}_{n\nu}(m.\xi_0.\xi_1.\eta_0)=\frac{\int_{\xi_0}^{\xi_1}\frac{\sinh\xi\cosh\xi}{c\,(\sinh^2\xi+\sin^2\eta_0)}\,P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{mn}}(\cosh\xi)}{\int_{\xi_0}^{\xi_1}P^{m/2\alpha}_{\lambda_{mn}}(\cosh\xi)\sinh\xi\,d\xi}P^{m/2\alpha}_{-\frac{1}{2}+i\lambda_{m\nu}}(\cosh\xi)\,d\xi \quad (13)$$

Based on this, by solving Eq. (10), the shape of modes and fundamental frequencies of sloshing in the prolate spheroidal tank is extracted.

## 3. Numerical results

Mathematica software was used to calculate the sloshing frequencies. Based on this, the sloshing frequencies have been investigated in terms of different parameters such as the height of the free surface of the fluid (fluid filling ratio) and the thinness ratio of the tank. In addit, the theoretical results have been compared with the numerical results available in the references to validate the extracted relationshipsces. Finally, the sloshing frequencies in the prolate spheroidal tank are compared with the frequencies of cylindrical and spherical tanks of the same volume in

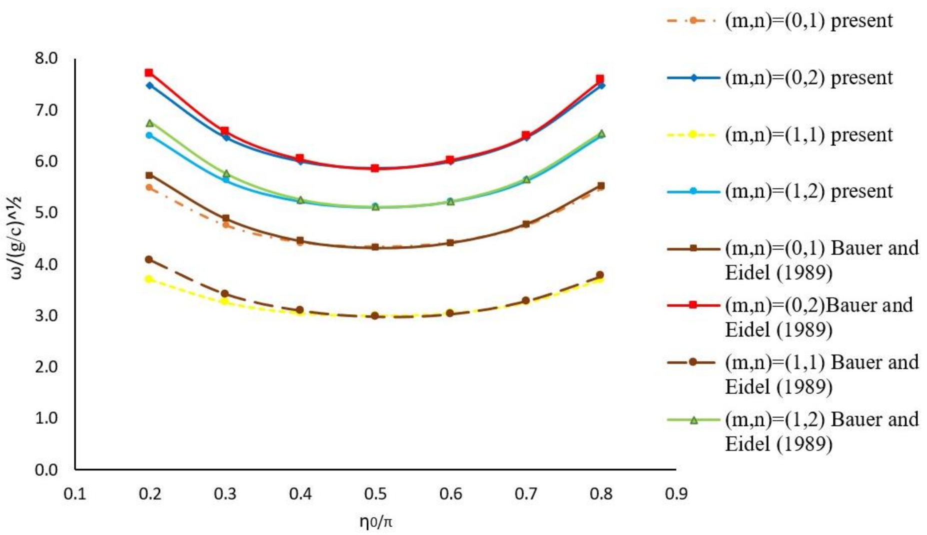


Fig. 4 Theoretical and experimental curves (Bauer and Eidel 1989) of the natural frequency ratio of fluid sloshing for a tank with a lean ratio of 0.2 according to the height of the fluid and in four different modes

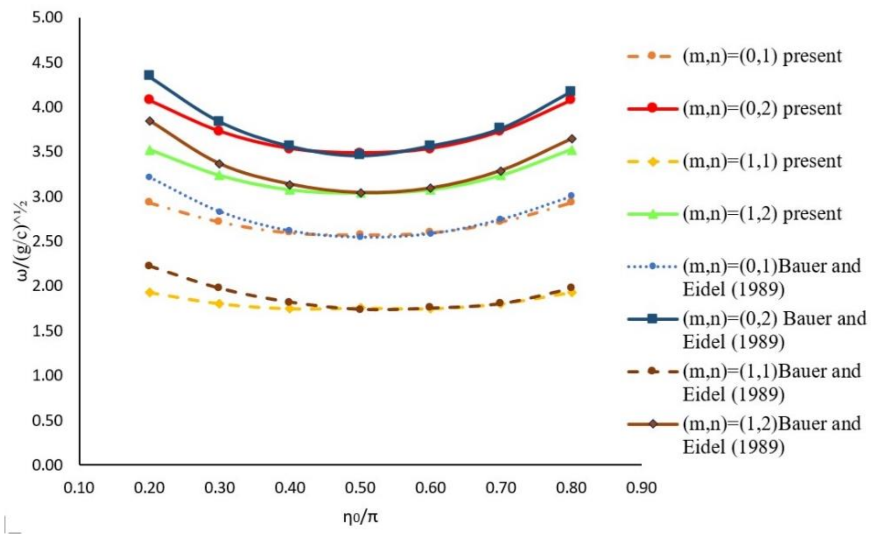


Fig. 5 Theoretical and experimental curves (Bauer and Eidel 1989) of the natural frequency ratio of fluid sloshing for a tank with a lean ratio of 0.5 according to the height of the fluid and in four different modes

different modes.The natural frequencies calculated for the tank containing the water fluid with thinness ratios ($\frac{b}{a} = 0.2$ and $0.5$) and according to the fluid height parameter $\frac{\eta_0}{\pi}$ are shown in Figs. 4 and 5. In these figures, the frequencies calculated by Bauer and Eidel (1989) can also be seen.

As can be seen, there is a good agreement between the analytical frequencies and the experimental frequencies calculated by Bauer and Eidel (1989). In addition, the natural frequencies in all modes decrease with the increase of the fluid height up to half of the tank $\frac{\eta_0}{\pi} = 1$. This is due to the increase in the free surface of the fluid along with the increase in the volume

Table 1 Dimensionless analytical frequencies for a prolate spheroidal tank with 0.1 filling percentage and different lean ratios

| $(b/a)$ | Spheroidal $(m.n)$ | | | |
|---|---|---|---|---|
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 6.8776 | 9.3282 | 4.7368 | 8.1244 |
| 0.2 | 4.4845 | 6.0855 | 3.0807 | 5.2992 |
| 0.3 | 3.7785 | 5.1506 | 2.5577 | 4.4773 |
| 0.5 | 2.6778 | 0.6687 | 1.7839 | 3.1826 |
| 0.76 | 1.7770 | 2.4565 | 1.1765 | 2.1281 |
| | cylindrical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 0.3639 | 6.1898 | 4.2593 | 7.3016 |
| 0.2 | 0.1824 | 4.3507 | 2.7838 | 5.1588 |
| 0.3 | 0.1216 | 3.4571 | 2.0341 | 4.1789 |
| 0.5 | 0.0730 | 2.4299 | 1.2944 | 3.0805 |
| 0.76 | 0.0479 | 1.7163 | 0.8679 | 2.2639 |
| | Spherical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 1.9296 | 2.8313 | 1.0503 | 7.4718 |
| 0.2 | 1.9296 | 2.8313 | 1.0503 | 7.4718 |
| 0.3 | 1.9296 | 2.8313 | 1.0503 | 7.4718 |
| 0.5 | 1.9296 | 2.8313 | 1.0503 | 7.4718 |
| 0.76 | 1.9296 | 2.8313 | 1.0503 | 7.4718 |

of the fluid inside the tank, which leads to a drop in frequency with a relatively high slope. By increasing the height of more than half of the tank, the frequency increases again due to the reduction of the free surface of the fluid. Although the increase in frequency occurs with a slower slope due to the increase in fluid mass. Next, to compare the sloshing frequencies in prolate spheroidal tanks with cylindrical and spherical tanks of the same volume, a prolate spheroidal tank with a height of 2 meters ($2b$=2) is considered. For the similarity of the free surface in spheroidal, cylindrical, and spherical tanks, the radius of spheroidal and cylindrical tanks is considered equal. As a result, by changing the height of the fluid in this tank, the same volume of fluid as the prolate spheroidal tank has been obtained. In the spherical tank, the radius is calculated so that the volume of fluid in this tank is equal to the volume of fluid in the prolate spheroidal tank, on the other hand, the free surface for the same volume of fluid in the spherical tank is much more than in the prolate spheroidal tank, which is the reason for the intensification of sloshing. For prolate spheroidal tanks with different slenderness ratios, the results of frequency analysis have been compared with cylindrical and spherical tanks of the same volume. The results of the frequency analysis of the first four sloshing frequencies for a filling ratio of 10% are shown in Table 1.

As can be seen, with the increase in the slenderness ratio in the prolate spheroidal and cylindrical tanks, due to the increase in the free surface of the fluid, the natural frequencies decrease in all modes, while in the spherical tank it becomes larger homogeneously (the *b/a* ratio is always constant in the sphere). And so the natural frequencies remain constant. In addition, the

Table 2 Dimensionless analytical frequencies for a prolate spheroidal tank with 0.5 filling percentage and different lean ratios

| $(b/a)$ | Spheroidal | | | |
|---|---|---|---|---|
| | $(m.n)$ | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 6.1740 | 8.3547 | 4.2772 | 7.2830 |
| 0.2 | 4.3288 | 5.8539 | 2.9951 | 5.1097 |
| 0.3 | 3.4878 | 4.7221 | 2.4044 | 4.1156 |
| 0.5 | 2.5591 | 3.4601 | 1.7348 | 3.0352 |
| 0.76 | 1.7951 | 2.4396 | 1.1945 | 2.1239 |
| | Cylindrical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 0.7631 | 6.1901 | 4.2909 | 7.3016 |
| 0.2 | 0.4008 | 4.3770 | 3.0341 | 5.1630 |
| 0.3 | 0.2698 | 3.5738 | 2.4767 | 4.2156 |
| 0.5 | 0.1627 | 2.7682 | 1.9048 | 3.2654 |
| 0.76 | 0.1070 | 2.2403 | 1.4941 | 2.6456 |
| | Spherical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 1.9352 | 2.6413 | 1.2491 | 7.1903 |
| 0.2 | 1.9352 | 2.6413 | 1.2491 | 7.1903 |
| 0.3 | 1.9352 | 2.6413 | 1.2491 | 7.1903 |
| 0.5 | 1.9352 | 2.6413 | 1.2491 | 7.1903 |
| 0.76 | 1.9352 | 2.6413 | 1.2491 | 7.1903 |

fundamental frequency in the cylindrical tank occurs in the mode (m,n)=(0,1), while the fundamental frequency in the spherical and prolate spheroidal tanks is in the mode (m,n)=(1,1). To investigate the effect of the degree of filling, the analysis results for filling ratios of 50% and 90% are shown in Tables 2 and 3 for different lean ratios.

As the results of Table 2 show, with the increase of the filling ratio from 10% to 50%, in prolate spheroidal and spherical tanks, with the increase of the free surface of the fluid, the natural frequencies in the modes decrease, while in the cylindrical tank, due to the constant free surface of the fluid, the natural frequencies Increased. In addition, according to Table 3, with a further increase in the filling ratio to 90%, the natural frequency of spherical and prolate spheroidal tanks increases again due to the reduction of the free surface of the fluid. To more clearly investigate the effect of filling on the sloshing frequencies and compare in different tanks, the analysis results for the first two natural frequencies according to the filling ratio and lean ratios of 0.1, 0.3, and 0.5 are shown in Figs. 6-8 respectively.

As Fig. 6 shows, at the lean ratio of 0.1, with the increase of the filling ratio up to 50%, the natural frequencies in the prolate spheroidal tank decrease, and with the further increase of the fluid height, the frequencies will increase. While in spherical and cylindrical tanks, the trend of frequency changes is almost increasing. In addition, the natural frequencies in the prolate spheroidal tank in all the corresponding modes are higher than the natural frequencies of spherical and cylindrical tanks.

Table 3 Dimensionless analytical frequencies for a prolate spheroidal tank with 0.9 filling percentage and different lean ratios

| $(b/a)$ | Spheroidal | | | |
|---|---|---|---|---|
| | $(m.n)$ | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 6.8808 | 9.3325 | 4.7389 | 8.1282 |
| 0.2 | 4.7694 | 6.4850 | 3.2579 | 5.6427 |
| 0.3 | 3.7785 | 5.1506 | 2.5577 | 4.4773 |
| 0.5 | 2.6770 | 3.6674 | 1.7835 | 3.1826 |
| 0.76 | 1.7770 | 2.4565 | 1.1765 | 2.1281 |
| | Cylindrical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 0.9815 | 6.1901 | 4.2908 | 7.3016 |
| 0.2 | 0.5180 | 4.3770 | 3.0341 | 5.1630 |
| 0.3 | 0.4405 | 3.5738 | 2.4773 | 2.4774 |
| 0.5 | 0.2169 | 2.7682 | 1.9187 | 3.2651 |
| 0.76 | 0.1432 | 2.2430 | 1.5501 | 2.6458 |
| | Spherical | | | |
| | (0.1) | (0.2) | (1.1) | (1.2) |
| 0.1 | 6.8808 | 9.3325 | 4.7389 | 8.3116 |
| 0.2 | 6.8808 | 9.3325 | 4.7389 | 8.3116 |
| 0.3 | 6.8808 | 9.3325 | 4.7389 | 8.3116 |
| 0.5 | 6.8808 | 9.3325 | 4.7389 | 8.3116 |
| 0.76 | 6.8808 | 9.3325 | 4.7389 | 8.3116 |

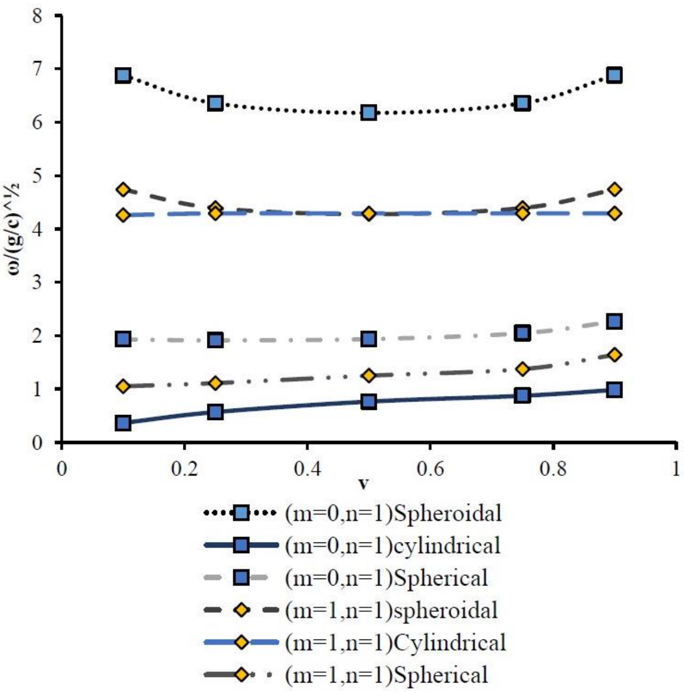


Fig. 6 Theoretical and experimental curves of fluid sloshing frequency for tanks with 0.1 lean ratios and in two critical modes

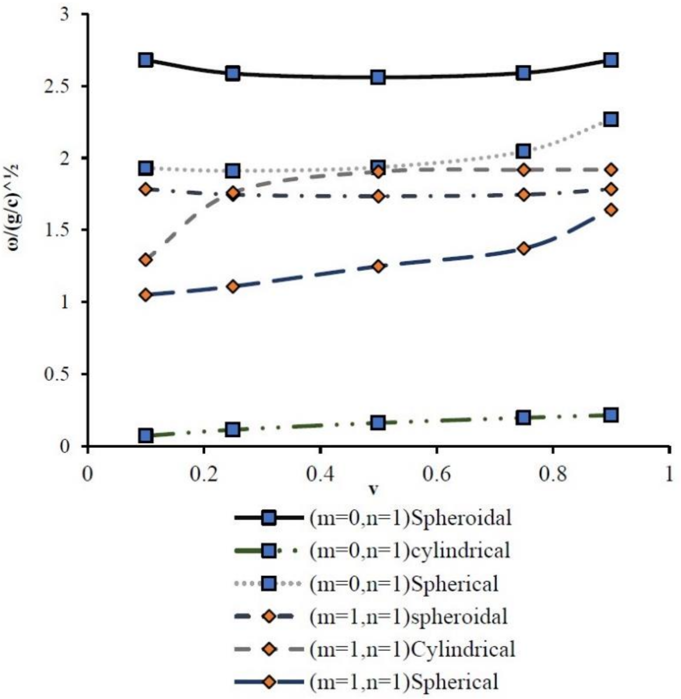


Fig. 7 Theoretical and experimental curves of fluid sloshing frequency for tanks with 0.3 lean ratios and in two critical modes

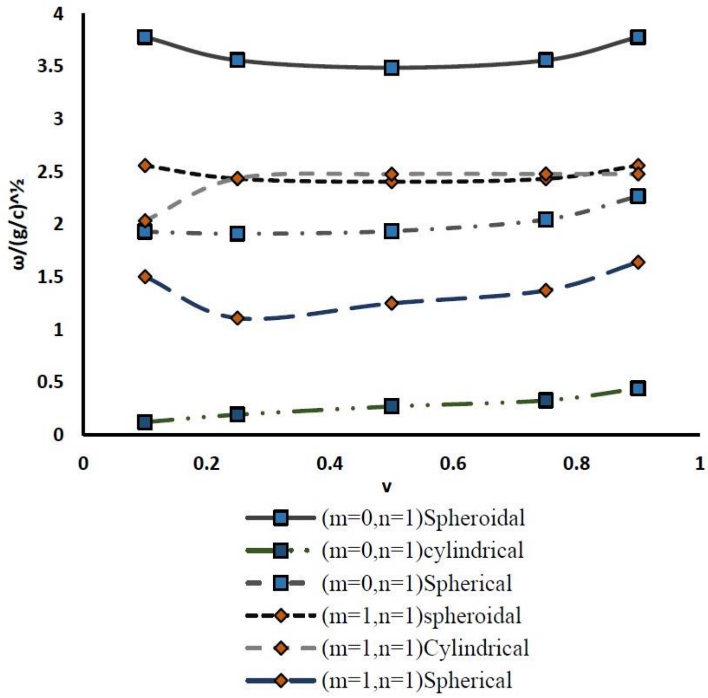


Fig. 8 Theoretical and experimental curves of fluid sloshing frequency for tanks with 0.5 lean ratios and in two critical modes

As Fig. 7 shows, with the increase of the slenderness ratio to 0.3, the natural frequencies of the cylindrical tank in the mode (*m,n*) =(1.1) increased with the increase of the depth ratio, so that from the depths close to half full and more, from The corresponding frequency in the prolate spheroidal tank is slightly higher. However, the frequency of the fundamental mode in the cylindrical tank is still much lower than the fundamental frequency of the prolate spheroidal and even spherical tank. In addition, with the increase of the filling ratio, the fundamental frequency of the spherical tank in the mode (*m,n*)=(1,1) continuously increases with a greater slope than the other two tanks, but it is always lower than the fundamental frequency of the prolate spheroidal tank.

In the following, the natural frequency of sloshing for the slenderness ratio of 0.5 is compared with each other in Fig. 8. As Fig. 8 shows, in this lean ratio, the fundamental frequency of sloshing in the prolate spheroidal tank is much higher than in the other two tanks. This is due to the changes of the free surface along the height compared to the cylindrical tank and the free surface is less in the same volume compared to the spherical tank. In addition, although in large lean ratios, the frequency of the prolate spheroidal tank in the mode (m,n)=(1,1) is slightly lower than the frequency of the cylindrical tank in some filling ratios, the existence of the mode (m,n)=(0,1) in the cylindrical tank, especially in space applications in which transient stimulations enter the tank in different directions, the tank puts the cylinders in a more critical situation. Therefore, in addition to having a more suitable placement space in the body of the space launcher than the spherical tank, the prolate spheroidal tank has less destructive effects of sloshing in dynamic stimulations than cylindrical and spherical tanks.

## 4. Conclusions

Analyzing, predicting, and controlling fluid motion is very important to improve the performance of space missions because a significant percentage of the mass of the spacecraft is covered by propellant fluid. In this article, the analytical investigation of sloshing in a prolate spheroidal tank was discussed. Also, the calculated answers were compared with the results found in other sources. The most important observed results are summarized as follows:

- In prolate spheroidal tanks, the natural frequencies in all modes decrease with increasing fluid height up to half of the tank $\eta_0/\pi = 0.5$. This is due to the increase in the free surface of the fluid along with the increase in the volume of the fluid inside the tank, which leads to a frequency drop with a relatively high slope. By increasing the height of more than half of the tank, the frequency increases again due to the reduction of the free surface of the fluid. Although the increase in frequency occurs with a slower slope due to the increase in fluid mass.
- By increasing the slenderness ratio in prolate spheroidal and cylindrical tanks, due to the increase in the free surface of the fluid, the natural frequencies decrease in all modes, while in the spherical tank it increases homogeneously and therefore the natural frequencies remain constant. In addition, the fundamental frequency in the cylindrical tank occurs in the mode (m,n)=(0,1), while the fundamental frequency in the spherical and prolate spheroidal tanks is in the (m,n)=(1,1) mode.
- By increasing the filling ratio from 10% to 50%, in prolate spheroidal and spherical tanks with the increase of the free surface of the fluid, the natural frequencies decrease in all modes, while in the cylindrical tank due to the constant free surface of the fluid, the natural frequencies have increased. In addition, with a further increase in the filling ratio to 90%, the natural

friction of spherical and prolate spheroidal tanks increases again due to the reduction of the free surface of the fluid.

• At the lean ratio of 0.1, with the increase of the filling ratio up to 50%, the natural frequencies in the prolate spheroidal tank decrease, and with the further increase of the fluid height, the frequencies will increase. While in spherical and cylindrical tanks, the trend of frequency changes is almost decreasing.

• By increasing the slenderness ratio to 0.3, the natural frequencies of the cylindrical tank in the mode (m,n)=(1,1) have increased more strongly with the increase in the depth ratio, so that from the depths close to half full and more, the corresponding frequency in the prolate spheroidal tank is slightly higher. has increased However, the frequency of the fundamental mode in the cylindrical tank is still much lower than the fundamental frequency of the prolate spheroidal and even spherical tank. In addition, with the increase of the filling ratio, the fundamental frequency of the spherical tank in the mode (m,n)=(1,1) continuously increases with a greater slope than the other two tanks, but it is always lower than the fundamental frequency of the prolate spheroidal tank.

• In the slenderness ratio of 0.5, the fundamental frequency of sloshing in the prolate spheroidal tank is much higher than in the other two tanks. This is due to the changes of the free surface along the height compared to the cylindrical tank and the free surface is less in the same volume compared to the spherical tank. Although in large lean ratios, the frequency difference of the prolate spheroidal tank in the (m,n)=(1,1) mode is slightly less than the frequency of the cylindrical tank in some filling ratios, the existence of the (m,n)=(0,1) mode in the cylindrical tank, especially in space applications that In that, transient excitations enter the tank in different directions, so the prolate spheroidal tank, in addition to having a more suitable placement space in the body of the space launcher than the spherical tank, but also in dynamic stimulations, the destructive effects of sloshing in it compared to both cylindrical and spherical tanks Lower.

• With the increase of the slenderness ratio from 0.1 to 0.3, the natural frequencies of the cylindrical tank in the mode (m,n)=(1,1) increased with the increase of the depth ratio, so that from the depths close to half-full and more, the corresponding frequency It has increased slightly in the prolate spheroidal tank. However, the frequency of the fundamental mode in the cylindrical tank is still much lower than the fundamental frequency of the prolate spheroidal and even spherical tank. In addition, with the increase of the filling ratio, the fundamental frequency of the spherical tank in the mode (m,n)=(1,1) continuously increases with a greater slope than the other two tanks, but it is always lower than the fundamental frequency of the prolate spheroidal tank.

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*EC*

## Nomenclature

| | |
|---|---|
| $g$ | acceleration due to gravity $(\mathrm{m/s^2})$ |
| $h$ | tank height (m) |
| $H$ | fluid height (m) |
| $a$ | radius of the tank (m) |
| $t$ | time (s) |
| $c$ | The conversion factor of spheroid coordinates to Cartesian coordinates |
| $z$ | complex variables |
| $\gamma$ | Sloshing damping rate |
| $\eta.\xi.\varphi$ | prolate spheroid coordinates |
| $\xi_0.\xi_1$ | prolate spheroidal surfaces |
| $\eta_0.\eta_1$ | hyperboloid surfaces |
| $\lambda$ | root of determinants see in (8) |
| $\rho$ | fluid density $(kg/m^3)$ |
| $\Phi$ | total velocity potential $(\mathrm{m^2 s^{-1}})$ |
| $b/a$ | fill ratio |
| $\delta_{nv}$ | Kronecker delta function |
| $\Omega$ | normalized frequency parameter |
| $m.n.v$ | dummy indices |
| N | truncation size |