

A multilevel framework for decomposition-based reliability shape and size optimization

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Abstract. A method for decoupling reliability based design optimization problem into a set of deterministic optimization and performing a reliability analysis is described. The inner reliability analysis and the outer optimization are performed separately in a sequential manner. Since the outer optimizer must perform a large number of iterations to find the optimized shape and size of structure, the computational cost is very high. Therefore, during the course of this research, new multilevel reliability optimization methods are developed that divide the design domain into two sub-spaces to be employed in an iterative procedure: one of the shape design variables, and the other of the size design variables. In each iteration, the probability constraints are converted into equivalent deterministic constraints using reliability analysis and then implemented in the deterministic optimization problem. The framework is first tested on a short column with cross-sectional properties as design variables, the applied loads and the yield stress as random variables. In addition, two cases of curvilinearly stiffened panels subjected to uniform shear and compression in-plane loads, and two cases of curvilinearly stiffened panels subjected to shear and compression loads that vary in linear and quadratic manner are presented.

Keywords: reliability analysis; shape optimization; reliability-based design optimization; stiffened panels

1. Introduction

In the future, the use of probabilistic methods both in structural design and during the certification process is expected to undergo a significant increase. There is an increasing realization in many quarters that designing a structure considering various uncertainties is a more rational approach than the current approach of using safety factors (Mohaghegh 2005). Therefore, several investigations have been performed using an uncertainty based approach.

The conventional approach to formulate the Reliability Based Design Optimization (RBDO) is to utilize a double-loop optimization and uncertainty analysis which are nested one into another to

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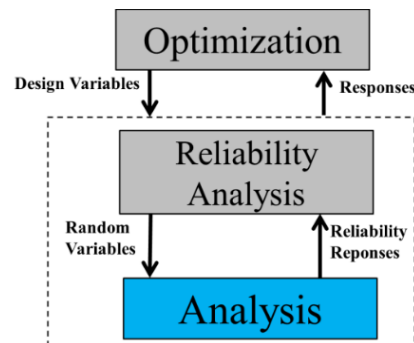


Fig. 1 The double-loop RBDO framework

minimize the objective function while satisfying the probability constraints, Fig. 1. The purpose of the optimization loop is to execute optimum search. The purpose of the uncertainty analysis loop is to evaluate the design and its uncertainty characteristics. At every iteration of the outer loop, the optimizer calls the uncertainty analysis, which executes many simulations, depending on the uncertainty analysis methods being used, e.g., the Monte Carlo Simulation and the First and Second Order Reliability Methods. The computational cost associated with the nested RBDO is very high due to the number of simulations required for each uncertainty analysis during every optimization iteration (Yao *et al.* 2011). Since the double loop procedure may be computationally impractical, researchers have studied several techniques to reduce the high computational expense of RBDO. To merge the double loop RBDO into one single level problem, Agarwal *et al.* (2004) proposed replacing the lower-level inverse reliability analysis optimization problem with the corresponding First Order necessary Karush-Kuhn-Tucker (KKT) optimality conditions at the upper level optimization. The proposed formulation is implemented in an augmented design space that consists of both the original design variables and the so-called most probable point (MPP) of failure corresponding to each critical constraint. This formulation is mathematically equivalent to solving the original nested optimization if the constraint qualification conditions are satisfied. A concern with Agarwal *et al.* (2004)'s formulation is that the number of design variables is increased with the number of critical constraints. This greatly increases the optimization computational cost. Chen *et al.* (1997) also developed another method to convert the double loop RBDO into a single loop procedure by approximately finding the MPP of each active constraint. The MPP is found by using the gradients of the constraints and the desired safety factor.

Another way of converting the double loop RBDO into a single loop procedure is to perform the optimization and uncertainty analysis sequentially. The double loop reliability constraints are formulated as deterministic constraints based on the uncertainty analysis. Then the equivalent constraints are used in the optimization to direct the optimal solution to the feasible region which satisfies the reliability requirement. Agarwal and Renaud (2006) developed a decoupled method for RBDO. The deterministic optimization loop is separated from the reliability analysis loop. The MPPs are updated during the deterministic optimization by using a first-order Taylor series expansion about the design point from the preceding cycle. The sensitivities required to update the MPP are obtained using a post-optimality analysis at the MPP optimal solution. Elishakoff and Chamis (2001) studied the relationship between the safety factor and reliability levels and showed that in many cases the safety factors can be directly expressed by the required reliability levels. However, since the value of the safety factor does not specify the reliability, the author introduced

the notion of probabilistic sufficiency factor (Qu and Haftka 2004), which has a high correlation with the target reliability. A probabilistic sufficiency factor of 1.0 represents that the achieved probability of failure is equal to the target one. If a probabilistic sufficiency factor is less than one, the probability of failure exceeds the target and the design is not safe, and a probabilistic sufficiency factor larger than one means that the probability of failure is less than the target probability. Qu and Haftka (2004) compared the performance of response surface approximations fitted for three approaches of describing failure in a non-deterministic manner, namely: (a) the probability of failure, (b) the safety index, and (c) the probabilistic sufficiency factor. They showed that the response surface approximation can have better accuracy when it is fitted to the probabilistic sufficiency factor than to either of the other two. Furthermore, they showed that the probabilistic sufficiency factor provides more information in regions of low probability than either of the other two.

Wu *et al.* (2001) also proposed a safety factor based RBDO by converting reliability constraints to the equivalent deterministic constraint with safety factor in the optimization cycle. Du and Chen (2004) developed a sequential RBDO methodology. In their framework, the optimization is conducted by first using the MPP of the previous design point and then performing a reliability analysis to update the MPP. The combined optimization and reliability analysis cycle is repeated until the objective convergence and the reliability requirement is achieved. Ba-abbad *et al.* (2006) improved the Du and Chen technique to distribute the reliability of the system over its components in an optimal way. In Ba-abbad *et al.* (2006)'s technique, at each iteration, the first-order reliability analysis is carried out to check if this design has an acceptable reliability. Then, the performance measure analysis is performed to calculate the MPPs of the various failure modes. Finally, the approximate deterministic optimization is conducted to find the optimum design and measure the maximum of the safety indices.

Another approach for improving the efficiency of RBDO is to use surrogate models such as response surfaces. Lopez *et al.* (2015) developed a methodology based on the response surface method and Firefly Algorithm to perform design optimization of truss structures. The response surface method is responsible for the reduction of the computational cost corresponding to the evaluation of the probabilistic constraints and the Firefly Algorithm addresses the issues related to the non-convexity and mixed-variables of the optimization problem.

As was discussed earlier, most of the reported single-loop RBDOs in the literature utilize an iterative procedure which finds the design variables while updating the MPPs and replacing the probabilistic constraints by equivalent deterministic constraints. Recently, some single-loop RBDO techniques have emerged that do not involve the iterative cycle. Shan and Wang (2008) proposed an approach by finding the feasible and reliable space of solutions in the first step and then performing an optimization in that space. Although the developed method eliminates the reliability analysis in the RBDO process, using the design point to calculate the gradient vector instead of using the MPPs may lead to an accuracy issue in some engineering applications (Li *et al.* 2013).

The robustness and efficiency problems in reliability based shape design optimization result in its limited range of applications (Yao *et al.* 2011). This research proposes a multilevel approach for decomposition-based reliability shape and size optimization. A sequential optimization and reliability analysis methodology is developed that utilizes the shape and size variables as design variables, and the applied loads and the Young's modulus as random variables. The proposed approach, first, conducts the reliability analysis to find MPPs and the probability of satisfying the given constraints. Next, each probability constraint in the multilevel shape and size optimization is

converted to an equivalent deterministic constraint by using its MPP obtained in the previous iteration. Since the changes in size and shape variables during the optimization process result in different kinds of changes in the structure's performance, the single step shape and size optimization may suffer from a lack of convergence and may lead to a sub-optimal solution. In addition, due to the large number of design variables, the combined shape and size optimization requires significant computational effort. One way to improve the efficiency and reach lower optimal design is breaking large structural optimization problems into multilevel optimizations (Wang *et al.* 1997, Hansen and Horst 2008, Montemurro *et al.* 2016). We propose a two-step optimization algorithm for decomposing the shape and size optimization problem. In the two-step optimization algorithm, the size and shape optimization process is divided into two parts; the first part consists of a sizing optimization, while keeping the structural layout unchanged to minimize the mass while satisfying the buckling, stress, and crippling constraints, and the second step involves calculating the best layouts for the maximum multiple objective functions that include buckling, stress and crippling.

The sequential multilevel RBDO framework employs *EBF3PanelOpt*, a Computational Design Environment for panel with curvilinear stiffeners, to analyze the structures. *EBF3PanelOpt* developed at Virginia Tech employs the PYTHON programming environment (Kapania *et al.* 2013, Tamijani *et al.* 2014). The finite element commercial software, MSC.PATRAN and MSC.NASTRAN are used to parametrically create and analyze a detailed finite element model of curvilinearly stiffened panels. The developed sequential RBDO framework also utilizes DAKOTA, Design Analysis Kit for Optimization and Terascale Applications, for reliability analysis and design optimization. The present framework is demonstrated on two set of examples: the first example relates to RBDO of a short column with rectangular cross section having uncertain material properties and subject to uncertain loads; the second set of examples includes the panel with two and four curvilinear stiffeners with 13 and 25 size and shape design variables, respectively, and up to seven random variables, depending on the loading condition. In these examples, various combinations of loading conditions, including uniform, linearly varying, and quadratically varying in-plane compression and shear loads, are taken into account as random variables.

2. Reliability based design optimization framework

For deterministic design optimization, all of the important parameters influencing the system are assumed to be well defined with known values. These parameters could include loading conditions and material properties. Traditionally, uncertainties are accounted for by using safety factors in the design process. This approach often leads to overdesigning the system. Thus, the need to include uncertainty in design process becomes important and reliability-based design optimization is being increasingly accepted by the industry. However, RBDO encounters computational issues when it is applied to a complex engineering design. Performing a reliability analysis for a given structure requires repeating the structural analysis for different sets of random variables, which can be computationally very expensive when using numerical methods such as finite element analysis. In order to reduce the computational time of RBDO, the framework can be reformulated by converting the probabilistic constraints into equivalent deterministic constraints. Then, the deterministic optimization and uncertainty analysis loop are conducted sequentially, which finally converge to an optimized design while satisfying the reliability constraints after a required number of iterations. In order to illustrate the sequential optimization and reliability

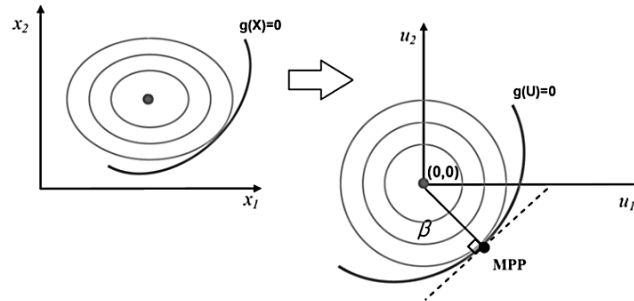


Fig. 2 Transformation and MPP (Choi *et al.* 2006)

analysis framework, first the double-loop RBDO, Fig. 1, is presented

$$\begin{cases} \text{find } d \\ \min f(d) \\ \text{s. t. } P[g_i(d, X) \leq 0] \geq R \quad i = 1, 2, \dots, m \\ d^L \leq d \leq d^U \end{cases} \quad (1)$$

where d is the design variables' vector and d^L and d^U are the design variables' lower and upper bounds. Here, X is the random variables vector, f is the objective function, g_i is the i th constraint function, m is the number of constraints, $Prob[g_i(d, X) \leq 0]$ is the probability of satisfying the i th constraint, and R is the specified reliability for the constraints. The RBDO goal is to minimize the structural weight subjected to given probabilistic constraints on buckling, von Mises stress, and crippling ($g_i, m=3$). The buckling factor is considered for the buckling constraint and is defined as the inverse of the fundamental eigenvalue. The von Mises stresses for all elements were aggregated using the Kreisselmeier and Steinhauser (KS) function and the crippling constraint is calculated as the ratio of maximum negative principal stress and the maximum allowable stress in the stiffener. The structure responses including the first buckling eigenvalue, von Mises stress, and the principal stress vector are calculated using finite element analysis.

In order to perform the RBDO efficiently rather than utilizing a double-loop framework (Eq. (1)) the deterministic optimization and the uncertainty quantification are decoupled from one another. Various techniques have been developed to decouple the optimization and the reliability analysis. One of these techniques is sequential optimization and reliability analysis (SORA). The main idea behind SORA is to perform optimization by applying the equivalent deterministic constraints, instead of using the reliability constraints. The constraints on the probability of satisfying constraints of a structure can be converted to the equivalent deterministic constraints by using the MPPs at the desired level of safety.

For MPP based sequential RBDO, the random variables are replaced with their MPPs; therefore, the deterministic constraints are shifted to meet the desired reliability level. The calculated MPP is improved after each iteration to provide an accurate MPP for the deterministic optimization. A multilevel MPP RBDO framework is developed utilizing the sequential optimization and reliability analysis (see Fig. 3). In the proposed approach, first the reliability analysis of initial design (d^0) is conducted to find the MPP corresponding to the desired reliability (R). MPP is the design point that has most significant contribution to the Probability of failure (Choi *et al.* 2006). The MPP is defined in a standardized and independent coordinate system. In the transformation procedure, the design vector X is transformed into the vector of standardized,

independent Gaussian variables, U (see Fig. 2). Generally, MPP calculation can be formulated as an optimization problem

$$\begin{cases} \text{find } U \\ \min \beta = (U^T U)^{1/2} \\ \text{s. t } g(U) = 0 \end{cases} \quad (2)$$

The shortest distance β from the origin to a point on the limit-state surface, $g(U)$, is called the reliability index (Hasofer and Lind 1974). Typical MPP-based reliability analysis methods include first and second order reliability methods (FORM/SORM). To calculate the probability of failure using FORM and SORM, first, the MPP needs to be found. After finding the MPP and reliability index using Eq. (2), FORM and SORM approximate the probability of failure by using first or second order Taylor series expansion of limit state function at the MPP. In the MPP approaches, first the Taylor series is constructed at the means of random variables, and then the Taylor series is evaluated at each MPP. The uncertainty analysis is performed using DAKOTA. An overview of the various MPP search algorithms available in DAKOTA are provided in its reference manual (Adams *et al.* 2009). We have tested the following MPP algorithms for the current research:

- first order and second order Taylor series centered at the uncertain variable means (x_taylor_mean)
- first order and second order Taylor series starts at the uncertain variable means and updates the Taylor series approximation at each MPP prediction (x_taylor_mpp)
- two-point adaptive nonlinear approximation (TANA)

Of these algorithms, the x_taylor_mpp yields to MPP results similar to TANA and converges in less computational time. Therefore, the x_taylor_mpp is selected for the MPP search and is implemented in the framework.

After performing the reliability analysis, each probability constraint $g_i(d, X_{MPP})$ in the multilevel shape and size optimization is converted to an equivalent deterministic constraint by using its MPP. By replacing the random variables with their MPPs, the current constraint is shifted to meet the desired reliability level. Then the deterministic multilevel optimization is performed to optimize the mass of structures while satisfying the equivalent constraints on buckling, stress, and crippling. Once the optimum is found, the reliability analysis is performed at current optimum to find the updated MPPs (X_{MPP}^j) and also to calculate the probability of satisfying constraints, P_s . If the objective function is not close to one obtained in the previous iteration, or some constraints are violated, the iterative procedure will continue until the objective function and MPPs converge, and the probability of satisfying constraints is larger than the desired system probability of safety. In order to reduce the number of optimization iterations and find the optimized design faster, the design obtained in the previous iteration is given as the initial design for the current optimization cycle.

3. EBF3PanelOpt

EBF3PanelOpt, a finite element framework for panels with curvilinear stiffeners, is utilized to analyze the structures. Detailed information concerning the framework can be found in (Mulani *et al.* 2013), however, a brief summary of it is given here.

EBF3PanelOpt is developed using an object oriented script written in Python that interfaces

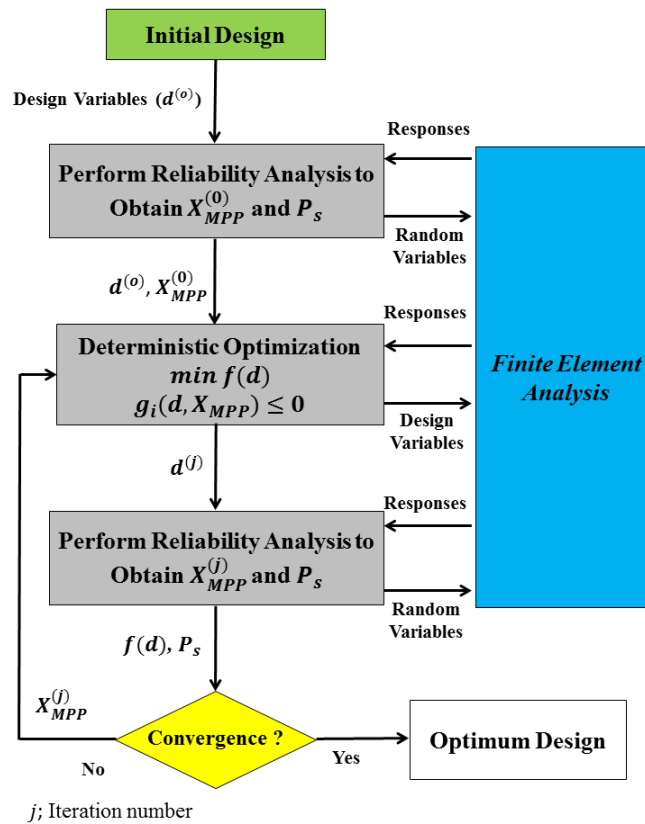


Fig. 3 The sequential framework for RBDO

MSC.PATRAN® and MSC.NASTRAN® to perform FEA on a panel with curvilinear, blade-type stiffeners and returns the mass of the panel and constraints on yielding, buckling, and crippling or local failure of the panel. In this framework, the user needs to create an initial Patran database using either EBF3PanelOpt or utilizing Patran directly.

After giving the input variables, *EBF3PanelOpt* writes a Patran session file in Python, launches Patran and executes this session file to update geometry of the stiffened panel. After successful execution of the session file, it writes the input file (bdf) for Nastran. The successful execution of this Nastran (bdf) file, the Nastran response file (f06) is read by Python; responses like buckling factor, von Mises stress and the crippling stress for a stiffener are calculated by Python. During the execution of Patran and Nastran, if any error occurs or if it takes more time than the allocated time, these processes will be terminated and default responses will be sent with “pass/fail” as an active constraint. The pass/fail response indicates a failure so that the design can be disregarded without discontinuities being introduced into the design space. During analysis, completion of bdf and session files closure is monitored for successful execution.

A key feature of *EBF3PanelOpt* is the ability to specify the geometry in a parametric fashion such that the optimizer fully specifies the panel shape and size. Sizing quantities such as panel thickness, stiffener thickness, and stiffener height are used as design variables to define the geometry. The stiffener curve is represented using a third order uniform rational B-spline using two end-points and a control point so the stiffener always remains in the panel area. The stiffener

Table 1 Description of the thirteen design variables

Design Variable	Meaning	Lower Limit	Upper limit
d_1	Starting point of first stiffener	0	1
d_2	Shape parameter (x-coordinate) for first stiffener	0	1
d_3	Shape parameter (y-coordinate) for first stiffener	0	1
d_4	Ending point of first stiffener	0	1
d_5	Starting point of second stiffener	0	1
d_6	Shape parameter (x-coordinate) for second stiffener	0	1
d_7	Shape parameter (y-coordinate) for second stiffener	0	1
d_8	Ending point of second stiffener	0	1
d_9 & d_{10}	Height of Stiffener 1 and 2	2 cm	6 cm
d_{11} , d_{12} & d_{13}	Thickness of Stiffener 1, 2 and Panel	2 mm	6 mm

end-points lie on the perimeter of the panel, the end-point design variable always lies between 0 and 1. The control-point is defined using interpolation of the panel surface, so the control-point's x co-ordinate and y co-ordinate have values between 0 and 1. For a uniform cross-section blade-stiffener, height and thickness are design variables apart from plate thickness as a design variable. Thus, for two and four stiffeners cases, a stiffened panel having uniform cross-section stiffeners has 13 and 25 design variables, respectively.

During the successful execution of *EBF3PanelOpt*, linear buckling analysis is carried out using the SOL-105 solver from Nastran that gives the lowest buckling eigenvalue or the critical buckling load factor, as well as von Mises stress distribution for the static analysis. Crippling criteria for the stiffeners is calculated using the formulation given in the page 444 of Niu (2011). Various responses are normalized before being reported to the user or optimizer. The buckling factor is defined as the inverse of the fundamental eigenvalue. The von Mises stress in the stiffened panel should be less than the allowed stress (yield stress) for safe design. In *EBF3PanelOpt*, during the optimization or to account artificial effect of boundary conditions, the von Mises stress, are averaged over all finite elements using Kreisselmeier-Steinhauser (KS) criteria. Hence, the successful *EBF3PanelOpt* evaluation provides the mass of the structure, the buckling factor, normalized von Mises stress and crippling stress, and pass/fail response as output.

4. Multilevel reliability based optimization

Using the combined shape and sizing optimization methods for optimizing problems with high number of shape variables may result in numerous complications. Since the optimizer must perform a large number of iterations to find the global optimum for shape and size variables, the computational cost is very high. Combined shape and sizing optimization methods may also fail because of numerical problems and the lack of convergence to a sub-optimal solution. Therefore, during the course of this research, new optimization methods have been developed that essentially divide the design domain into two sub-spaces: one of the sizing design variables, and the other of the shape design variables. The advantage of doing two-step optimization lies in the principle of the Divide and Conquer strategy. In two-step optimization, first size optimization is carried out for fixed values of shape design variables with minimization of mass as an objective and the

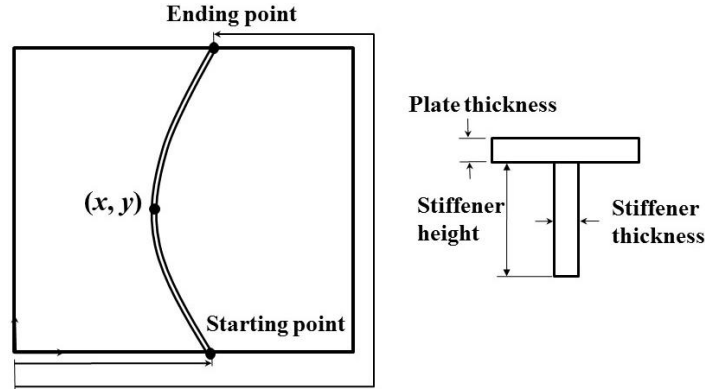


Fig. 4 Description of the design variables

constraints on buckling parameter, crippling, and von Mises stress. Next, with the size parameters fixed as obtained in size optimization, the shape optimization is performed. In shape optimization, the optimal shape (layout) is obtained which is best for resisting critical constraints with fixed values of size variables. These two sub-optimizations must be iterated, as size parameters obtained in size optimization may not be optimal for the shape variables obtained in shape optimization. For this iterative optimization procedure, the error in the mass of two successive iterations is observed to terminate the optimization procedure.

The RBDO problem can be formulated mathematically as follows

$$\left\{ \begin{array}{l} \text{find } d \text{ (size and shape design variables)} \\ \min \text{mass}(d) \\ \text{s. t. } P[\lambda(d, X) \leq 1] \geq R \\ \text{s. t. } P[KS(d, X) \leq 1] \geq R \\ \text{s. t. } P[\sigma_{cc}(d, X) \leq 1] \geq R \\ d^L \leq d \leq d^U \end{array} \right. \quad (3)$$

where λ , KS , and σ_{cc} are buckling parameter (1/buckling eigenvalue), Kreisselmeier and Steinhauser (KS) stress, and the crippling, respectively. The shape and size design variables (d) are listed in Table 1 and shown in Fig. 4.

Using the sequential RBDO, as explained in Fig. 3, the optimization and reliability assessment are separated, and the optimization is defined as

$$\left\{ \begin{array}{l} \text{find } d \text{ (size and shape design variables)} \\ \min \text{mass}(d) \\ \lambda(d, X_{MPP1}) \leq 1 \\ KS(d, X_{MPP2}) \leq 1 \\ \sigma_{cc}(d, X_{MPP3}) \leq 1 \\ d^L \leq d \leq d^U \end{array} \right. \quad (4)$$

The optimization problem, Eq. (4), is divided into two step optimization. The two-step optimization formulation of layout and size of structure can be expressed as follows. First, perform size optimization for fixed values of shape design variables. In this step, the optimizer optimizes the mass using size design variables while satisfying the buckling, stress, and crippling constraints

$$\left\{ \begin{array}{l} \text{find size design variables } (d_s) \\ \text{keep shape design variables } (d_v) \text{ unchanged} \\ \min \text{mass}(d_s) \\ \lambda(d_s, X_{MPP1}) \leq 1 \\ KS(d_s, X_{MPP2}) \leq 1 \\ \sigma_{cc}(d_s, X_{MPP3}) \leq 1 \\ d_s^L \leq d_s \leq d_s^U \end{array} \right. \quad (5)$$

Although the mass is optimized and optimal size variables are obtained after the size optimization, some constraints might become critical. The critical constraints can be improved by changing the structural layout. Thus, the framework moves onto the second step, shape optimization, which optimizes the layout with respect to the critical constraints. At this stage the critical constraints and the constraint on structural weight ratio ($W_g = \frac{W}{W_{OptSize}}, W_{OptSize}$: the optimal weight obtained in size optimization) are defined as the objective functions by using the weighted-sum or scalarization method (Wang *et al.* 1997, Marler and Arora 2010). The multiple objectives are represented as a single composite function

$$f_0 = \sum_{j=1}^n w_j f_c^j \quad f_c^j: \text{Critical constraints and weight constraint} \quad (6)$$

$n = \text{number of critical constraints} + 1$
 $w_j: \text{weighting coefficient for the } j^{\text{th}} \text{ function}$

The constraints ($\lambda(d_v, X_{MPP1}), (d_v, X_{MPP2}), KS(d_v, X_{MPP2}),$ and $\sigma_{cc}(d_v, X_{MPP3})$) that reach their critical values are included in f_c^j . The multiple objectives shape design optimization problem is stated as

$$\left\{ \begin{array}{l} \text{find shape design variables } (d_v) \\ \text{keep size design variables } (d_s) \text{ unchanged} \\ \min f_0 \\ d_v^L \leq d_v \leq d_v^U \end{array} \right. \quad (7)$$

With the improvement of critical constraints as the result of shape optimization, the structural weight can be further reduced in the following size optimization. The final optimal result is achieved by using an iterative size and shape optimization procedure. Using global optimization methods, such as genetic algorithms, for both shape and size optimization prevents the optimizer from getting stuck into a local optimum.

5. Results

In this section, results for a test case, optimum design of a short column, two cases of curvilinearly stiffened panels subjected to uniform shear and compression in-plane loads, and two cases of curvilinearly stiffened panels subjected to linearly and parabolically varying shear and compression are presented. The test case considers RBDO of a rectangular short column with cross-section design variables, and the applied loads and the yield stress as random variables.

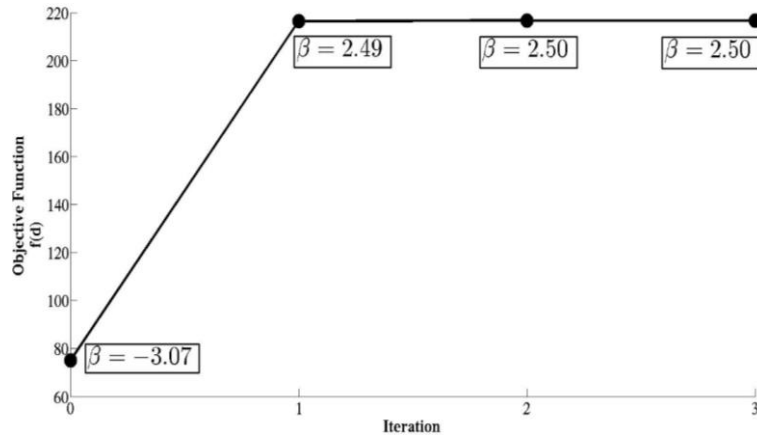


Fig. 5 Iteration history of the sequential optimization and reliability analysis of short column

Finally, the application of developed RBDO framework for curvilinearly stiffened panels is demonstrated. The design variables include the shape and size variable of stiffened panels, and the in-plane loads and young modulus are defined as random variables. To determine the effect of in-plane load variations on the optimal mass of the panel, four sets of in-plane load distributions are considered. The computations are performed on a computer with dual four-core 3.00 GHz Intel Xeon processors with 20 GB of RAM. Parallel processing is used for all cases, resulting in eight simultaneous analyses for each iteration of the optimization process.

6. Short column

This test problem involves the plastic analysis and design of a short column with rectangular cross section (width b and depth h) having uncertain material properties (yield stress Y) and subject to uncertain loads (bending moment M and axial force F) (Kuschel and Rackwitz 1997). The objective and limit state functions are defined as

$$\begin{cases} f(d) = b \cdot h \\ G(d, X) = 1 - \frac{4M}{bh^2Y} - \frac{F^2}{(bhY)^2} \end{cases} \quad (8)$$

The distributions for F , M , and Y are normal (500, 100), normal (2000, 400), and lognormal (5, 0.5), respectively, with a correlation coefficient of 0.5 between F and M . An objective function of cross-sectional area and a target reliability index of 2.5 (cumulative failure probability $P_f \leq 0.00621$) are used in the design problem

$$\begin{cases} \min f(d) \\ s. t \beta \geq 2.5 \\ 5.0 \leq b \leq 15.0 \\ 15.0 \leq h \leq 25.0 \end{cases} \quad (9)$$

First, the reliability analysis of the initial design is performed to find the MPP corresponding to the desired target reliability index of 2.5 (see iteration zero in Fig. 5). Starting from the initial

Table 2 Reliability design optimization of short column

	Objective function	Reliability index	Iter.num	*NFE
Outer Approximations RBDO (Kirjner-Neto <i>et al.</i> 1998, Cheng <i>et al.</i> 2006)	-	-	14	277
Bi-level Approach (Cheng <i>et al.</i> 2006)	216.82	2.503	5	136
Sequential Approximate Programming (Cheng <i>et al.</i> 2006)	216.83	2.503	9	116
Performance Measure Bi-level Approach (Yi <i>et al.</i> 2008)	216.69	-	5	154
Performance Measure Sequential Approximate Programming (Yi <i>et al.</i> 2008)	216.73	-	6	98
Sequential RBDO	216.75	2.501	3	72

*NFE is the number of function evaluation

variables $(b, h)=(5 \text{ mm}, 15 \text{ mm})$, the reliability index is -3.07 and the MPP corresponding to the desired reliability index of 2.5 is $(P, M, Y)=(697.2 \text{ MPa}, 2523 \text{ MNm}, 4.286 \text{ MPa})$. The random variables in $G(d, X) = 1 - \frac{4M}{bh^2Y} - \frac{F^2}{(bhY)^2}$ are replaced with the calculated MPP, and the optimization is performed to find the optimum b and h for the shifted constraint. The optimal sizes variables (b, h) are (8.65, 25). The reliability index at the current optimum (8.65, 25) is 2.491, and the MPP corresponding to the reliability index of 2.5 is (690.5 MPa, 2583 MNm, 4.278 MPa). Next, the constraints, $G(d, X)$ are evaluated at MPP, and used as equivalent deterministic constraints in optimization, $G(d, X_{MPP})$. Once the optimum is found $(b, h)=(8.67, 25)$, the reliability analysis is performed at the current optimum to find the updated MPP and to calculate the reliability index. The MPP and reliability index at second iteration are (690.4 MPa, 2583 MNm, 4.278 MPa) and 2.501, respectively. It is noted from Fig. 5 that the difference in the value of the reliability index is less than 0.4% after the second iteration, and the third iteration is conducted to guarantee the convergence of the objective function. The optimal sizes variables, MPP and reliability index at third iteration are (8.67, 25), (690.4 MPa, 2583 MNm, 4.278 MPa) and 2.501, respectively. The Sequential RBDO requires 3 iterations and 72 function evaluations. The optimal design from sequential reliability design optimization and the comparison of performance with other algorithms is shown in Table 2.

7. Curvilinearly stiffened panels subjected to uniform shear and compression in-plane loads

A simply supported rectangular plate of size $0.4064 \times 0.5080 \text{ m}$ with material properties listed in Table 3 is studied under combined uniform shear and compression. The baseline panel configuration, loading, material properties, and design constraints are representative of typical aircraft structure for this design optimization study. All panel analyses, with or without stiffeners, are performed with NASTRAN using *EBF3PanelOpt*. The stiffened panel geometry and mesh are regenerated for each design point analysis during optimization. Reliability based design minimizes the mass of curvilinearly stiffened panel subjected to the constraints on buckling (λ), Kreisselmeier and Steinhauser (KS) and the crippling (σ_{cc}). The desired probability of safety (R)

Table 3 Material properties of curvilinearly stiffened plate

Modulus of Elasticity	$73 \times 10^9 \text{ Pa}$
Density	2795 kg/m^3
Poisson's Ratio	0.33
Yield stress	427.4 MPa

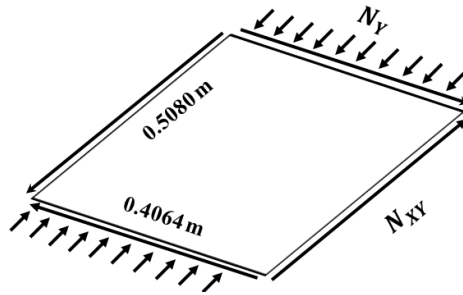


Fig. 6 Panel dimensions and loading conditions

for all cases is 0.9998.

The first load case (L1) studied is a stiffened panel under combined shear and compression with dominant compression ($N_Y/N_{XY}=4.36$), as shown in Fig. 6. N_Y and N_{XY} are normally distributed and uncorrelated with a mean of 308 kN/m and 71 kN/m, respectively and 15% coefficient of variation (COV). The distribution for E is lognormal with a mean of 73 GPa and 1% COV. The second load case (L2) has smaller ratio of shear and compressive load magnitudes ($N_Y/N_{XY}=1.13$). In this load case, N_Y and N_{XY} are normally distributed and uncorrelated with a mean of 152 kN/m and 134 kN/m, respectively, and 15% COV. The distribution, mean, and covariance for E are similar to the previous case.

Following the sequential RBDO scheme presented in the previous section, the iteration histories for load case one are shown in Fig. 7 (L1). It is shown that both the optimization and reliability analysis converges after two iterations. The panel's mass is 2.031 kg, which is slightly lighter than the deterministic optimum of 2.032 kg. The applied loads for deterministic optimization are obtained after applying a factor of safety of 1.5 to the limit loads and the panels are designed for that loads. The deterministic optimum configuration is shown in Fig. 8 (L1). The optimum objective, probability of safety, and shape and sizing design variable values for RBDO and deterministic optimization are shown in Table 4 (L1). A few observations are of interest here. First, since in the studied case the compression is the dominant load, using an appropriate safety factor (here it is 1.5) can give the desired probability of failure, as can be seen in Table 4. When there is only one important random variable, the safety factors can be directly expressed by the required reliability levels. However, in many cases, there does not exist a relationship between the safety factor and reliability levels. Furthermore, the buckling constraint is active for both configurations, which yield closely optimal results.

The optimum configuration and iteration histories for the second load case are shown in Fig. 7 (L2). The shape and sizing design variable values obtained from the sequential RBDO and the related final optimum mass, and the probability of safety for three constraints are shown in Table 4 (L2). The deterministic optimization result for the second load case using a factor of safety of 1.5 is presented in Fig. 8 (L2). The optimum mass of the panel is 1.746 kg, which is lighter than the

Table 4 Optimum mass, constraint and design variable obtained for (L1) $N_Y/N_{XY}=4.36$ (L2) $N_Y/N_{XY}=1.13$

Variable No.	Deterministic Optimization (L1)	Sequential RBDO (L1)	Deterministic Optimization (L2)	Sequential RBDO (L2)
x_1	0.0861	0.5806	0.6417	0.6393
x_2	0.2052	9.8569	0.0361	0.7125
x_3	0.2484	0.6736	0.2437	0.3899
x_4	0.6403	0.1479	0.0651	0.0480
x_5	0.1412	0.6481	0.1339	0.5619
x_6	0.6613	0.3642	0.8111	0.3642
x_7	0.7401	0.3123	0.7497	0.6698
x_8	0.5812	0.0725	0.5701	0.1392
x_9, m	0.0321	0.0318	0.0325	0.0318
x_{10}, m	0.0428	0.0324	0.0317	0.0390
x_{11}, m	0.0031	0.0031	0.0027	0.0026
x_{12}, m	0.0024	0.0025	0.0021	0.0021
x_{13}, m	0.0021	0.0021	0.0020	0.0020
Mass, kg	2.0324	2.0316	1.7941	1.7455
$Prob[\lambda(d,X)\leq 1]$	0.9998	0.9998	0.9999	0.9998
$Prob[KS(d,X)\leq 1]$	0.9999	0.9999	0.9999	0.9999
$Prob[(\sigma_{cc}(d,X)\leq 1]$	0.9999	0.9999	0.9999	0.9999
Number of evaluations	10447	20849	11637	33270

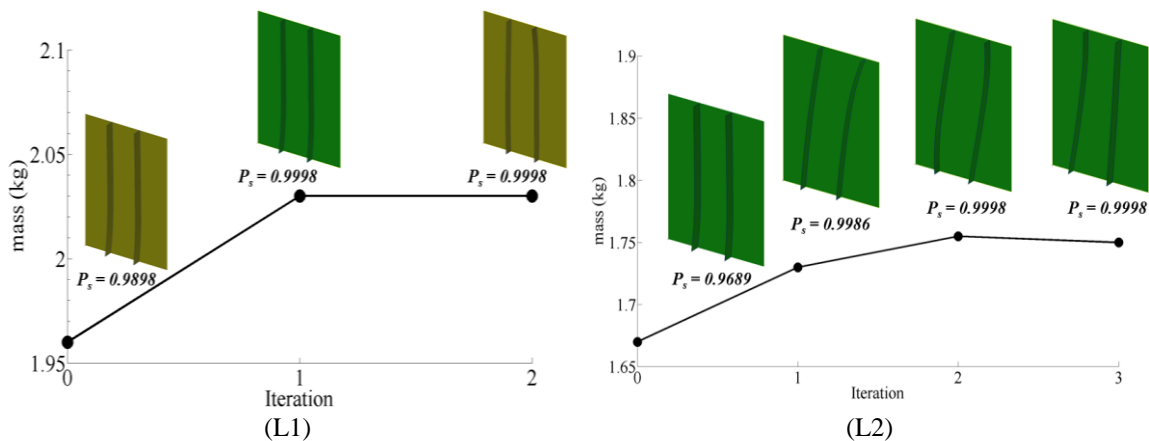


Fig. 7 Iteration history of mass and probability of safety (L1) $N_Y/N_{XY}=4.36$ (L2) $N_Y/N_{XY}=1.13$

deterministic optimum of 1.794 kg with the probability of safety 0.9999. For the second load case, the compression and shear are both important and neither one can be ignored. Therefore, using the same safety factor for both loads may not result in the desired probability of failure and optimum mass obtained using RBDO. By comparing the sequential RBDO Fig. 7 (L2) and deterministic optimization Fig. 8 (L2), it further becomes evident that the deterministic optimization using

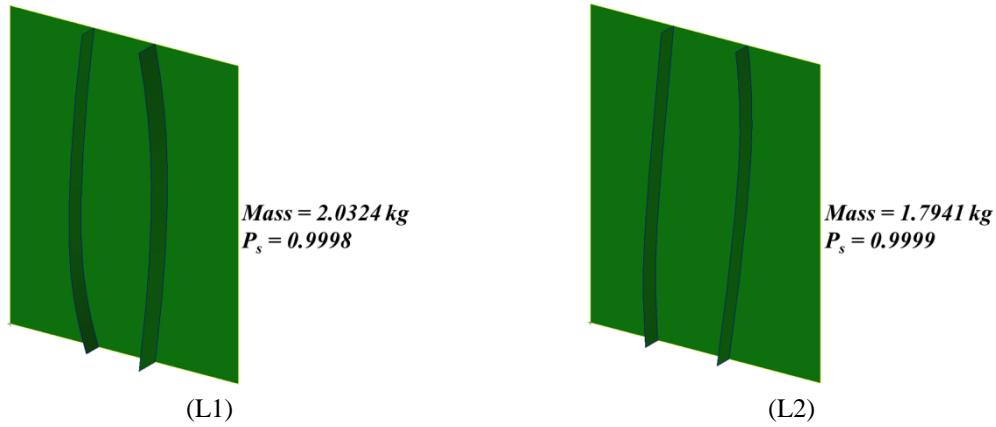


Fig. 8 Deterministic optimum configuration (L1) $N_Y/N_{XY}=4.36$ (L2) $N_Y/N_{XY}=1.13$

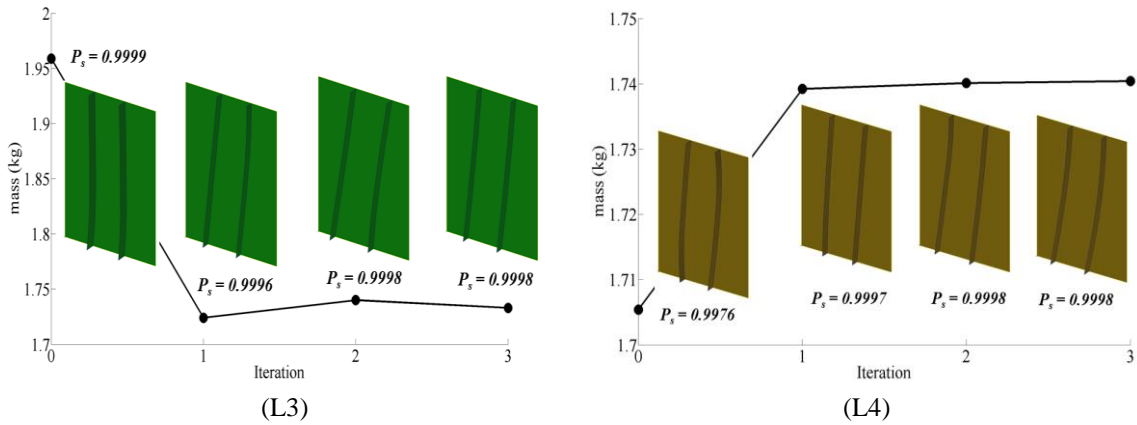


Fig. 9 Iteration history of mass and probability of safety for linearly (L3) and parabolically (L4) varying load cases

safety factor did not yield the RBDO final configuration. It is important to note that changing the safety factor would change the shear and compression loads, but it does not change their ratio, and consequently it only changes the size variables while having no effect on the shape variables. The final configuration of curvilinear stiffeners is governed by the ratio of the shear and compression loads, rather than their magnitudes.

8. Curvilinearly stiffened panels subjected to non-uniform shear and compression in-plane loads

It is seen from the results shown in the previous subsection that the influence of the ratio of shear and compression loads on the final results is substantial. Furthermore, it is also important to understand the influence of the additional random variables, such as the linearly and parabolic varying loads, and to study their effect on the optimal mass and probability of safety. In this subsection, the curvilinearly stiffened panels under shear and compression loads with linearly and

Table 5 Optimum mass, constraint and design variable obtained for linearly (L3) and parabolically (L4) varying load cases

Variable/Response	Sequential RBDO (L3)	Sequential RBDO (L4)	Variable/Response	Sequential RBDO (L3)	Sequential RBDO (L4)
x_1	0.6363	0.6337	x_{10}, m	0.0324	0.0322
x_2	0.6956	0.6258	x_{11}, m	0.00266	0.0027
x_3	0.3524	0.3727	x_{12}, m	0.0022	0.0020
x_4	0.0518	0.034	x_{13}, m	0.0020	0.0021
x_5	0.1339	0.1284	Mass, kg	1.7334	1.7404
x_6	0.4556	0.5768	$Prob[\lambda(d,X)\leq 1]$	0.9998	0.9998
x_7	0.6910	0.7586	$Prob[KS(d,X)\leq 1]$	0.9999	0.9999
x_8	0.5530	0.5550	$Prob[(\sigma_{cc}(d,X)\leq 1]$	0.9999	0.9999
x_9, m	0.0321	0.0325	Number of evaluations	33632	36557

parabolically varying random distributions is studied. The rectangular panel has the same dimensions and boundary conditions as discussed in the previous subsection, but is subjected to different loading condition. The linearly load distribution (L3) and parabolic load distribution (L4) are

$$\text{Linear Load Case: } \begin{cases} N_Y = N_{Y1} + N_{Y2} \left(1 - \frac{2x}{a}\right) \\ N_{XY} = N_{XY1} + N_{XY2} \left(1 - \frac{2x}{a}\right) \\ N_{XY} = N_{XY1} + N_{XY2} \left(1 - \frac{2y}{b}\right) \end{cases} \tag{10}$$

$$\text{Parabolic Load Case: } \begin{cases} N_Y = N_{Y1} + N_{Y2} \left(1 - \frac{2x}{a}\right) + 4N_{Y3} \frac{x}{a} \left(1 - \frac{x}{a}\right) \\ N_{XY} = N_{XY1} + N_{XY2} \left(1 - \frac{2x}{a}\right) + 4N_{XY3} \frac{x}{a} \left(1 - \frac{x}{a}\right) \\ N_{XY} = N_{XY1} + N_{XY2} \left(1 - \frac{2y}{a}\right) + 4N_{XY3} \frac{y}{a} \left(1 - \frac{y}{a}\right) \end{cases}$$

As can be seen in Eq. (10), in the linear load case, two random variables are defined for each of shear and compression loads. N_{Y1} and N_{XY1} are normally distributed with a mean of 152 kN/m and 134 kN/m, respectively, and 15% COV, and N_{Y2} and N_{XY2} are normally distributed with a mean of zero, and the standard deviation is 5% of means of N_{Y1} and N_{XY1} , respectively.

The distribution, mean, and covariance for E are similar to that in the previous subsection. Starting from an initial design that satisfies all the constraints, the sequential multilevel RBDO is carried out for the third load case and converges in three iterations with 33270 analyses. The history of the objective function and reliability constraints with respect to the iteration number are shown in Fig. 9 (L3), and the optimal design variable values and the mass of the structure along with the probability of safety are presented in Table 5 (L3).

By comparing Fig. 8 (L1) and Fig. 9 (L3), it is seen that the optimal design for linearly varying load case and those obtained for the uniform load case are similar to each other, due to the small influence of load variation on structural response.

Table 6 Optimum mass, constraint and design variable obtained for panel with four curvilinear stiffeners

Variable/Response	Variable/Response	Variable/Response	Variable/Response	Variable/Response	
x_1	0.0896	x_{11}	0.4527	x_{21}, m	0.0020
x_2	0.1980	x_{12}	0.0259	x_{22}, m	0.0022
x_3	0.0095	x_{13}	0.5827	x_{23}, m	0.0020
x_4	0.6652	x_{14}	0.3191	x_{24}, m	0.0020
x_5	0.1721	x_{15}	0.5516	x_{25}, m	0.0021
x_6	0.4365	x_{16}	0.1167	Mass, kg	1.4501
x_7	0.8158	x_{17}, m	0.0233	$Prob[\lambda(d,X)\leq 1]$	0.9999
x_8	0.5238	x_{18}, m	0.0220	$Prob[KS(d,X)\leq 1]$	0.9999
x_9	0.6117	x_{19}, m	0.0237	$Prob[(\sigma_{cc}(d,X)\leq 1]$	0.9999
x_{10}	0.4443	x_{20}, m	0.0228	Number of evaluations	62040

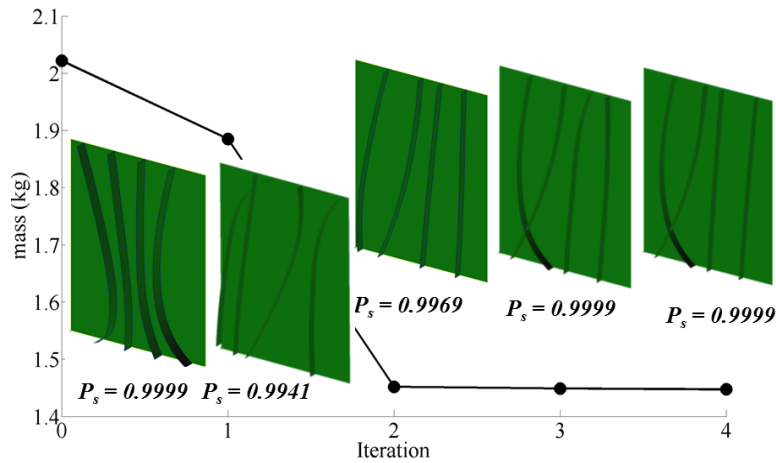


Fig. 10 Iteration history of mass and probability of safety for panel with four curvilinear stiffeners

As for the parabolic load distribution (Eq. (10)), three random variables are defined for each of the shear and compression loads. Both N_{Y1} and N_{XY1} are normally distributed and uncorrelated with a mean of 152 kN/m and 134 kN/m and 15% COV. Similarly, N_{Y2} and N_{XY2} are also normally distributed and uncorrelated with a mean of zero, and the standard deviation is 5% of means of N_{Y1} and N_{XY1} , and N_{Y3} and N_{XY3} are normally distributed and uncorrelated with a mean of zero, and the standard deviation is 2% of the mean values of N_{Y1} and N_{XY1} .

Table 5 compares the optimal design variables, the objective function, the probability constraints, and the number of iterations of two load cases (L3 and L4), all achieved by using sequential RBDO. Note that, as expected, the parabolically varying load case has objective function values larger than linearly varying load case, due to the effect of third variable in the parabolic load distribution. However as shown in Fig. 9, the stiffener layouts obtained for two load cases are similar to each other. This can be explained by the fact that the first variable in the compression and shear loads (N_{Y1} and N_{XY1}) are dominant, and they appear to be governing the final optimal layout while the other parameters (N_{Y2} , N_{XY2} , N_{Y3} , and N_{XY3}) change the size variables and consequently the weight of the stiffened panel.

9. Panels with four curvilinear stiffeners subjected to uniform shear and compression in-plane loads

The final test case is a panel with four curvilinear stiffeners with nine size and 16 shape design variables, under combined shear and compression ($N_Y/N_{XY}=1.13$). Both N_Y and N_{XY} are normally distributed and uncorrelated with mean values of 152 kN/m and 134 kN/m, respectively, and 15% COV. The distribution for E is lognormal with a mean of 73 GPa and 1% COV.

Using the sequential RBDO framework, the optimal design of panel and stiffeners is obtained and the final design and iteration histories are presented in Fig. 10. As can be seen in Fig. 10, the convergence is monotonic. Having 16 shape design variables makes the RBDO of stiffened panel a complex engineering design and a demanding computational problem. The successful convergence of the developed framework after four iterations of deterministic optimization clearly shows its capability in solving complex engineering uncertainty design problems.

The optimum weight, probability of safety, and shape and sizing design variable values for RBDO are shown in Table 6. The panel weight is 1.450 kg (Table 6), which is about 17% lighter than the two stiffeners case of 1.746 kg (Table 4), while the probability of safety of the former is slightly higher than the latter. As was discussed earlier, the rate of convergence highly depends on the initial design. Higher convergence rate is achieved by finding the MPP corresponding to the desired reliability and using it to find the equivalent deterministic constraints in the initial design optimization stage. Additionally, to reduce the number of optimization iterations, the optimum design obtained in the previous iteration is given as the initial design for the current optimization cycle.

Although not at a commercial level yet, curvilinear-stiffened structures such as those presented in Fig. 10 can be manufactured with emerging methodologies and techniques in manufacturing technologies, such as electronic beam freeform fabrication and friction stirwelding (Kapania *et al.* 2013).

10. Conclusions

An efficient reliability based design optimization framework is studied. A sequential optimization and reliability analysis methodology is developed. The sequential multilevel RBDO, first, conducts the reliability analysis to find MPPs and the probability of satisfying the given constraints. Next, each probability constraint is converted to an equivalent deterministic constraint by using its MPP of previous iteration. Since the changes in size and shape variables during the optimization process result in different kinds of changes to the structure's performance, a method for decomposing the shape and size optimization problem is utilized to improve the efficiency and accuracy of the developed framework. In the two-step optimization algorithm, the shape and size optimization process is divided into two parts; the first part involves a sizing optimization while keeping the shape variables unchanged to minimize the mass while satisfying the buckling, stress, and crippling constraints, and the second step consists of calculating the best structural layout that minimizes the constraints on buckling, stress, and crippling. The present study includes an evaluation case, the reliability based design optimization of a short column, and more complex cases related to stiffened panels subjected to uniform, linearly, and parabolically varying shear and compression in-plane loads. It is shown that the optimal configuration of curvilinear stiffeners is governed by the ratio of the shear and compression loads rather than their magnitudes, which

requires imposing various safety factors for shear and compression loads. This makes the requirement for an RBDO further evident. The stiffened panel test cases include up to 25 shape and size design variables, which makes them complex engineering designs and demanding computational problems. The successful convergence of the proposed methodology after a few iterations of deterministic optimization clearly shows its capability in solving complex engineering uncertainty design problems.

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