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The effect of finite strain on the nonlinear free vibration of a unidirectional composite Timoshenko beam using GDQM

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Abstract. In this manuscript, free vibrations of a unidirectional composite orthotropic Timoshenko beam based on finite strain have been studied. Using Green-Lagrange strain tensor and comprising all of the nonlinear terms of the tensor and also applying Hamilton's principle, equations of motion and boundary conditions of the beam are obtained. Using separation method in single-harmonic state, time and locative variables are separated from each other and finally, the equations of motion and boundary conditions are gained according to locative variable. To solve the equations, generalized differential quadrature method (GDQM) is applied and then, deflection and cross-section rotation of the beam in linear and nonlinear states are drawn and compared with each other. Also, frequencies of carbon/epoxy and glass/epoxy composite beams for different boundary conditions on the basis of the finite strain are calculated. The calculated frequencies of the nonlinear free vibration of the beam utilizing finite strain assumption for various geometries have been compared to von Karman one.

Keywords: finite strain; composite beam; Timoshenko model; nonlinear free vibration; GDQM

1. Introduction

Today, due to the promotion of computational facilities, using nonlinear vibrations in many sections of the mechanical engineering have been increased. One of the reasons of nonlinear vibration behavior is vibration with large deformation. The nonlinear vibrations based on the geometry are divided to two main groups: 1) vibration with large deformations and small strains, and 2) vibration with large deformations and no infinitesimal strains which is denominated finite strain. Based on the finite strain assumption, not only are deformations large, but also the strains are not limited to infinitesimal strain.

The first theory about vibration of a beam was expressed by Euler and Bernoulli. Rayleigh (1877) incorporated the effect of rotary inertia in beam. Using this assumption in vibration of beams, created error into natural frequencies was corrected. One of the most accurate theories for vibration of thick beams in high frequencies was suggested by Timoshenko. In Timoshenko theory (Timoshenko 1921), transverse shear deformation and the effect of rotary inertia were included. Singh *et al*. (1990) studied free vibration of the Euler-Bernoulli beam with large amplitude using finite element method (FEM). Lewandowski (1994) investigated nonlinear free vibration of the

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Euler-Bernoulli beam using the von Karman assumption and applying the FEM. In his research, bifurcation points into Backbone curves were indicated for the first time. Foda (1999) considered nonlinear free vibration of Timoshenko beam using multiple scales method subjected to simply supported boundary condition. He demonstrated that the effect of shear deformation and rotary inertia in large amplitude vibration behavior for thick and short beams were remarkable. Yardimoglu and Yildirim (2004) investigated the influence of pre-twisted on vibration of a Timoshenko beam by applying FEM. The results indicated that the natural frequencies of the beam had appropriate agreement with previous experimental and theoretical results. Free and forced vibrations of a laminated FGM Timoshenko beam were analyzed by Xiang and Yang (2008). Not only was the beam thickness variable, but also it was under thermally induced initial stresses. The numerical results of their research showed that the effects of thickness variation, temperature change, slenderness ratio, volume fraction index, thickness of the functionally graded (FG) layer and the end support conditions on vibration frequencies, mode shapes and dynamic response were considerable. Using the dynamic stiffness method, the free vibration and buckling of axially loaded laminated composite beams (LCBs) were analyzed by Jun *et al*. (2008). The governing equations of motion were derived based on the first order shear deformation theory and applying Hamilton's principle. The results indicated that the influence of axial force on the natural frequencies and mode shapes was remarkable. Gunda *et al*. (2010) studied large amplitude nonlinear vibration of isotropic Timoshenko beam using a relatively simple finite element formulation subjected to different boundary conditions. They illustrated the nonlinearity was obtained independent for each mode by the finite element formulation. Jafari-Talookolaei *et al*. (2012) studied the free vibration of laminated composite Timoshenko beams using Lagrange multipliers method. The results showed that the natural frequencies of the bam declined by growing the material anisotropy. Also, maintaining the longitudinal and torsional deformations in some cases had significant effect on vibration of the LCB. Rahimi *et al*. (2013) analyzed post-buckling of FG Timoshenko beams using an exact solution method. They presented closed-form solution method to study the buckled configuration of the beam subjected to different boundary conditions. In addition, the influence of FG material power law index and geometrical parameters of the beam on free vibration frequencies and static deflection were investigated. Also, the natural frequencies of the beam were compared with the Euler-Bernoulli beam and the results showed that accuracy of the Timoshenko beam was more than the Euler-Bernoulli one. Asadi and Aghdam (2014) analyzed large amplitude nonlinear vibration and post-buckling of variable cross-section composite beams using GDQM. The Euler-Bernoulli beam was with symmetric and asymmetric lay-ups and rested on nonlinear elastic foundation. Nonlinear forced vibration of nanocomposite beam reinforced by single-walled carbon nanotube was analyzed by Ansari *et al*. (2014). In their research were considered the forced vibrations of beams based on the Timoshenko theory and was used the von Karman assumption for deriving the equations. The nonlinear free vibration of a composite Euler-Bernoulli beam for different boundary conditions was analyzed by Ghasemi *et al*. (2016). They applied the finite strain assumption to investigate vibration of the beam. Also, Mohandes and Ghasemi (2016) compared the finite strain assumption with the von Karman hypothesis to analyze nonlinear free vibration of the beam. Unlike the obtained results of the von Karman assumption, in their study was demonstrated that for simply-simply supported beam, difference between the linear and nonlinear mode shapes were remarkable.

In this research, the free vibration of composite Timoshenko beam based on the finite strain is considered and its results are compared to the von Karman assumption. Deflection and cross section rotation of the beam in the linear and nonlinear states are drawn and compared with each other. Also, the effect of thickness on the nonlinear vibration of the beam undergoing finite strain is investigated and compared to the von Karman hypothesis. Finally, to study the effect of materials on the vibration, the natural frequencies of carbon/epoxy and glass/epoxy is compared to each other.

2. Governing equations of motion

In this section, the equations of motion and boundary conditions of the unidirectional composite Timoshenko beam are calculated. Based on the Timoshenko beam theory, displacement field can be defined as follows (Dong *et al*. 2005)

$$\begin{aligned} u(x,z,t) &= u(x,t) - z\varphi(x,t) \\ w(x,z,t) &= w(x,t) \end{aligned} \tag{1}$$

where u and w are longitudinal and transverse displacements of the beam, respectively. Also, φ is cross-section rotation of the beam. As the effect of longitudinal displacement in the free vibration analysis of the beam is very less than transverse one, longitudinal displacement of the beam can be ignored against transverse one. In this research, Hamilton's principle is used to obtain the governing equations of motion and boundary conditions. According to this principle, in the absence of external forces can be written as (Reddy 2003)

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} (\delta T - \delta U) dt = 0 \tag{2}$$

where t_1 and t_2 are two given arbitrary time, δT and δU are virtual kinetic and strain energies, respectively, and L is the Lagrangian. Using this principle can be obtained the equations of motion and boundary conditions together. The variations of kinetic δT and strain energies δU can be expressed as (Reddy 2003)

$$\delta T = \frac{1}{2}\int_V \rho v_i \delta v_i dV \tag{3}$$

$$\delta U = \frac{1}{2}\int_V S_{ij} \delta \varepsilon_{ij} dV \tag{4}$$

where ρ, v, V, S and ε are density, speed vector of environment particles, volume of non deformation, stress and strain tensors, respectively. The von Karman hypothesis which is applied in the most studies is valid for large transformations and small strains, while according to the finite strain assumption (Attard 2003) not only transformations are large, but also the strains can be greater than the unit. Therefore, none of the transformation terms are not eliminated in Green-Lagrange strain tensor on the basis of the finite strain assumption. So, by substitution of the Timoshenko displacement field into the Green-Lagrange strain tensor, the finite strain type nonlinear strain-displacement relations of the beam can be written as following for the axial and shear strains

$$\varepsilon_{xx} = \frac{1}{2}\left[-2z\frac{\partial\varphi}{\partial x} + \left(-z\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{5a}$$

$$\varepsilon_{xz} = \frac{1}{2}\left[-\varphi + \frac{\partial w}{\partial x} + z\varphi \frac{\partial \varphi}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} - z\frac{\partial \varphi}{\partial z} \right] \tag{5b}$$

By using Eqs. (5a) and (5b) into Eq. (4), the strain energy variations can be obtained as

$$\delta U = \frac{1}{2}\int_V \left[\sigma_{xx}(-z\frac{\partial \delta\varphi}{\partial x} + z^2 \frac{\partial \varphi}{\partial x}\frac{\partial \delta\varphi}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}) + \tau_{xz}(-\delta\varphi + \frac{\partial \delta w}{\partial x} + z\varphi\frac{\partial \delta\varphi}{\partial x} + z\frac{\partial \varphi}{\partial x}\delta\varphi) \right] dV \tag{6}$$

The stress resultants (Yang *et al*. 2010) for composite beam can be defined as

$$\begin{Bmatrix} N_x \\ M_x \\ P_x \\ Q_x \\ R_x \end{Bmatrix} = \int_A \begin{Bmatrix} \sigma_{xx} \\ Z\sigma_{xx} \\ Z^2\sigma_{xx} \\ \sigma_{xz} \\ Z\sigma_{xz} \end{Bmatrix} dA \tag{7}$$

where N_x, M_x and Q_x are the in-plane force, bending moment and shear force resultants, respectively. Also, P_x are the high order resultants of normal stress and R_x are the high order resultants of shear stress. Stiffness coefficients are defined as the following (Yas and Samadi 2012)

$$(A_{ij}, D_{ij}, F_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z^2, z^4)dz \tag{8}$$

where A_{ij} are extensional stiffnesses, D_{ij} are bending stiffnesses and F_{ij} are additional stiffnesses coefficient matrix. By substituting Eq. (5) into Eq. (7), the stress resultants are calculated based on the cross section rotation and displacement.

$$N_x = \int_A \sigma_{xx} dA = \int_A Q_{11}\varepsilon_{xx} dA = \frac{1}{2}D_{xx}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}A_{xx}\left(\frac{\partial w}{\partial x}\right)^2 \tag{9a}$$

$$M_x = \int_A z\sigma_{xx} dA = -D_{xx}\frac{\partial \varphi}{\partial x} \tag{9b}$$

$$P_x = \int_A z^2\sigma_{xx} dA = \frac{1}{2}F_{xx}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}D_{xx}\left(\frac{\partial w}{\partial x}\right)^2 \tag{9c}$$

$$Q_x = \int_A \sigma_{xz} dA = \int_A G_{12}\gamma_{xz} dA = A_{xz}\left[-\varphi + \frac{\partial w}{\partial x} \right] \tag{9d}$$

$$R_x = \int_A z\sigma_{xz} dA = \int_A zG_{12}\gamma_{xz} dA = D_{xz}\left[\varphi\frac{\partial \varphi}{\partial x} \right] \tag{9e}$$

By using Eq. (9) into Eq. (6), the variations of strain energy are obtained according to the stress resultants. Substituting the variations of kinetic and strain energies into Hamilton’s principle, the equations of motion and boundary conditions can be expressed as

$$-\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}\right)-\frac{\partial Q_x}{\partial x}+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{10a}$$

$$\frac{\partial M_x}{\partial x}-\frac{\partial}{\partial x}\left(P_x\frac{\partial \varphi}{\partial x}\right)-Q_x-\frac{\partial}{\partial x}(R_x\varphi)+R_x\frac{\partial \varphi}{\partial x}+\rho\frac{h^3}{12}\frac{\partial^2\varphi}{\partial t^2}=0 \tag{10b}$$

$$N_x\frac{\partial w}{\partial x}+Q_x=0 \qquad \text{or} \quad w=0 \tag{11a}$$

$$-M_x+P_x\frac{\partial \varphi}{\partial x}+R_x\varphi=0 \qquad \text{or} \quad \varphi=0 \tag{11b}$$

For a single orthotropic generally layer, the stiffness coefficients according to transformed coefficients Q_{ij} and thickness of the beam h can be obtained as

$$A_{xx}=Q_{11}h \qquad D_{xx}=Q_{11}\frac{h^3}{12} \qquad F_{xx}=Q_{11}\frac{h^5}{80} \qquad A_{xz}=Q_{12}h \qquad D_{xz}=Q_{12}\frac{h^3}{12} \tag{12}$$

Substituting Eqs. (9a) to (9e) into Eqs. (10) and (11) and using Eq. (12), the governing equations of motion and boundary conditions of the unidirectional composite Timoshenko beam are expressed as

$$\begin{gathered}-Q_{11}\frac{h^3}{12}\left[\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}\frac{\partial w}{\partial x}+\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}\right]-Q_{11}h\left[\frac{3}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2}\right]-Q_{12}h\left[-\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right]\\ +\rho h\frac{\partial^2 w}{\partial t^2}=0\end{gathered} \tag{13a}$$

$$\begin{gathered}-Q_{11}\frac{h^3}{12}\left(\frac{\partial^2\varphi}{\partial x^2}\right)-Q_{11}\frac{h^5}{80}\left[\frac{3}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}\right]-Q_{12}\frac{h^3}{12}\left[\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial\varphi}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2\varphi}{\partial x^2}\right]\\ +Q_{12}h\left[\varphi-\frac{\partial w}{\partial x}\right]-Q_{12}\frac{h^3}{12}\left[\frac{\partial^2\varphi}{\partial x^2}\varphi^2+2\varphi\left(\frac{\partial\varphi}{\partial x}\right)^2-\varphi\left(\frac{\partial\varphi}{\partial x}\right)^2\right]+\rho\frac{h^3}{12}\frac{\partial^2\varphi}{\partial t^2}=0\end{gathered} \tag{13b}$$

Boundary conditions in x=0 and x=L are as

$$Q_{11}\frac{h^3}{12}\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\frac{\partial w}{\partial x}\right]+Q_{11}h\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^3\right]+Q_{12}h\left[-\varphi+\frac{\partial w}{\partial x}\right]=0 \qquad \text{or } w=0 \tag{14a}$$

$$\begin{gathered}Q_{11}\frac{h^3}{12}\left(\frac{\partial\varphi}{\partial x}\right)+Q_{11}\frac{h^5}{80}\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^3\right]+Q_{11}\frac{h^3}{12}\left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial\varphi}{\partial x}\right]\\ +Q_{12}\frac{h^3}{12}\left[\frac{\partial\varphi}{\partial x}\varphi^2\right]=0\end{gathered} \qquad \text{or } \varphi=0 \tag{14b}$$

In this section, the governing equations of motion and boundary conditions for the unidirectional composite beam based on the finite strain were obtained.

3. Free vibration of a unidirectional composite Timoshenko beam based on the finite strain

3.1 Harmonic response

In this section, equations responses of free vibration of the beam using harmonic solution are obtained. Utilizing single-harmonic solution, time and locative parameters into the equations of motion and boundary conditions are separated and then the time parameter is eliminated. Single-harmonic solution (Zhong and Guo 2003) for the variables $\varphi(x,t)$ and $w(x,t)$ are assumed as

$$\begin{aligned}\varphi(x,t) &= \varphi(x)\cos(\omega t)\\ w(x,t) &= w(x)\cos(\omega t)\end{aligned} \tag{15}$$

where ω is natural frequency. By substituting Eq. (15) into Eqs. (13) and (14), the following harmonic equations are obtained

$$\begin{aligned}&-Q_{11}\frac{h^3}{16}\frac{d\varphi(x)}{dx}\frac{d^2\varphi(x)}{dx^2}\frac{dw(x)}{dx}-Q_{11}\frac{h^3}{32}\left(\frac{d\varphi(x)}{dx}\right)^2\frac{d^2w(x)}{dx^2}\\ &-Q_{11}\frac{9h}{8}\left(\frac{dw(x)}{dx}\right)^2\frac{d^2w(x)}{dx^2}+Q_{12}h\frac{d\varphi(x)}{dx}-Q_{12}h\frac{d^2w(x)}{dx^2}-\omega^2\rho h w(x)=0\end{aligned} \tag{16a}$$

$$\begin{aligned}&-Q_{11}\frac{9h^5}{160}\left(\frac{d\varphi(x)}{dx}\right)^2\frac{d^2\varphi(x)}{dx^2}-Q_{11}\frac{h^3}{16}\frac{dw(x)}{dx}\frac{d^2w(x)}{dx^2}\frac{d\varphi(x)}{dx}-Q_{11}\frac{h^3}{32}\left(\frac{dw(x)}{dx}\right)^2\frac{d^2\varphi(x)}{dx^2}\\ &-Q_{12}\frac{h^3}{16}(\varphi(x))^2\frac{d^2\varphi(x)}{dx^2}-Q_{12}\frac{h^3}{8}\varphi(x)\left(\frac{d\varphi(x)}{dx}\right)^2+Q_{12}\frac{h^3}{16}\varphi(x)\left(\frac{d\varphi(x)}{dx}\right)^2\\ &-Q_{11}\frac{h^3}{12}\frac{d^2\varphi(x)}{dx^2}+Q_{12}h\varphi(x)-Q_{12}h\frac{dw(x)}{dx}-\omega^2\rho\frac{h^3}{12}\varphi(x)=0\end{aligned} \tag{16b}$$

Boundary conditions in x=0 and x=L are as

$$Q_{11}\frac{h^3}{32}\left(\frac{d\varphi(x)}{dx}\right)^2\frac{dw(x)}{dx}+Q_{11}\frac{3h}{8}\left(\frac{dw(x)}{dx}\right)^3-Q_{12}h\varphi(x)+Q_{12}h\frac{dw(x)}{dx}=0 \quad \text{or } w=0 \tag{17a}$$

$$\begin{aligned}&Q_{11}\frac{3h^5}{160}\left(\frac{d\varphi(x)}{dx}\right)^3+Q_{11}\frac{h^3}{32}\left(\frac{dw(x)}{dx}\right)^2\frac{d\varphi(x)}{dx}+Q_{12}\frac{h^3}{16}\frac{d\varphi(x)}{dx}(\varphi(x))^2\\ &+Q_{11}\frac{h^5}{12}\frac{d\varphi(x)}{dx}=0 \qquad \text{or } \varphi=0\end{aligned} \tag{17b}$$

Introducing the dimensionless parameters as follows

$$W=\frac{w}{h} \qquad X=\frac{x}{L} \qquad \varphi=\phi \qquad a=\frac{Q_{12}}{Q_{11}} \tag{18}$$

Using the dimensionless parameters into Eqs. (16) and (17), the equations of motion and boundary conditions are turned to the dimensionless form. For simplicity purpose in communication of this research, the dimensionless quantities W, X and ϕ are replaced by w, x and φ, respectively.

$$\begin{aligned}&-\frac{3}{4}\frac{d\varphi}{dx}\frac{d^2\varphi}{dx^2}\frac{dw}{dx}-\frac{3}{8}\left(\frac{d\varphi}{dx}\right)^2\frac{d^2w}{dx^2}-\frac{27}{2}\left(\frac{dw}{dx}\right)^2\frac{d^2w}{dx^2}+12a\left(\frac{L}{h}\right)^3\frac{d\varphi}{dx}-12a\left(\frac{L}{h}\right)^2\frac{d^2w}{dx^2}\\&-\frac{12\rho L^4\omega^2}{Q_{11}h^2}w=0\end{aligned} \tag{19a}$$

$$\begin{aligned}&-\frac{81}{40}\left(\frac{d\varphi}{dx}\right)^2\frac{d^2\varphi}{dx^2}-9\frac{dw}{dx}\frac{d^2w}{dx^2}\frac{d\varphi}{dx}-\frac{9}{2}\left(\frac{dw}{dx}\right)^2\frac{d^2\varphi}{dx^2}-9\left(\frac{L}{h}\right)^2a\varphi^2\frac{d^2\varphi}{dx^2}-9\left(\frac{L}{h}\right)^2a\varphi\left(\frac{d\varphi}{dx}\right)^2\\&-12\left(\frac{L}{h}\right)^2\frac{d^2\varphi}{dx^2}+144a\left(\frac{L}{h}\right)^4\varphi-144a\left(\frac{L}{h}\right)^3\frac{dw}{dx}-\frac{12\rho L^4\omega^2}{Q_{11}h^2}\varphi=0\end{aligned} \tag{19b}$$

Boundary conditions in x=0 and x=1 are as

$$\frac{3}{4}\frac{h^4}{L^3}\left(\frac{d\varphi}{dx}\right)^2\frac{dw}{dx}+9\frac{h^4}{L^3}\left(\frac{dw}{dx}\right)^3-24ah\varphi+24a\frac{h^2}{L}\frac{dw}{dx}=0 \quad \text{or } w=0 \tag{20a}$$

$$\frac{3}{4}\frac{h^5}{L^3}\left(\frac{d\varphi}{dx}\right)^3+5\frac{h^5}{L^3}\left(\frac{dw}{dx}\right)^2\frac{d\varphi}{dx}+10\frac{h^3}{L}a\varphi^2\frac{d\varphi}{dx}+\frac{40}{3}\frac{h^3}{L}\frac{d\varphi}{dx}=0 \quad \text{or } \varphi=0 \tag{20b}$$

Dimensionless frequency λ has relation with natural frequency ω as the following

$$\lambda=\sqrt{\frac{12\rho \mathrm{L}^4}{\mathrm{Q}_{11}h^2}}\omega \tag{21}$$

By applying Eq. (21) into Eqs. (19) and (20), the governing equations of motion and boundary conditions are obtained according to the dimensionless frequency.

$$\begin{aligned}&-\frac{3}{4}\frac{d\varphi}{dx}\frac{d^2\varphi}{dx^2}\frac{dw}{dx}-\frac{3}{8}\left(\frac{d\varphi}{dx}\right)^2\frac{d^2w}{dx^2}-\frac{27}{2}\left(\frac{dw}{dx}\right)^2\frac{d^2w}{dx^2}+12a\left(\frac{L}{h}\right)^3\frac{d\varphi}{dx}\\&-12a\left(\frac{L}{h}\right)^2\frac{d^2w}{dx^2}-\lambda^2w=0\end{aligned} \tag{22a}$$

$$-\frac{81}{40}\left(\frac{d\varphi}{dx}\right)^2\frac{d^2\varphi}{dx^2}-9\frac{dw}{dx}\frac{d^2w}{dx^2}\frac{d\varphi}{dx}-\frac{9}{2}\left(\frac{dw}{dx}\right)^2\frac{d^2\varphi}{dx^2}-9\left(\frac{L}{h}\right)^2 a\varphi^2\frac{d^2\varphi}{dx^2}$$

$$-9\left(\frac{L}{h}\right)^2 a\varphi\left(\frac{d\varphi}{dx}\right)^2-12\left(\frac{L}{h}\right)^2\frac{d^2\varphi}{dx^2}+144a\left(\frac{L}{h}\right)^4\varphi-144a\left(\frac{L}{h}\right)^3\frac{dw}{dx}-\lambda^2\varphi=0 \tag{22b}$$

Boundary conditions in x=0 and x=1 are as

$$\frac{3}{4}\left(\frac{d\varphi}{dx}\right)^2\frac{dw}{dx}+9\left(\frac{dw}{dx}\right)^3-24\left(\frac{L}{h}\right)^3 a\varphi+24\left(\frac{L}{h}\right)^2 a\frac{dw}{dx}=0 \quad \text{or } w=0 \tag{23a}$$

$$5\left(\frac{dw}{dx}\right)^2\frac{d\varphi}{dx}+10\left(\frac{L}{h}\right)^2 a\varphi^2\frac{d\varphi}{dx}+\frac{3}{4}\left(\frac{d\varphi}{dx}\right)^3+\frac{40}{3}\left(\frac{L}{h}\right)^2\frac{d\varphi}{dx}=0 \quad \text{or } \varphi=0 \tag{23b}$$

To obtain the frequency, deflection and cross section rotation of the beam, the equations with boundary conditions are solved by applying the GDQM.

3.2 Generalized differential quadrature method

In this paper, the GDQM has been applied to solve the nonlinear equations of the finite strain vibration of composite beam. In the GDQM, differential function and its derivatives at all grid point in the whole domain of spatial coordinate are demonstrated as a weighted linear sum of all the functional values. In other words, governing differential equations using weighting coefficients change to the first order algebraic equations (Bert and Malik 1996). In the present study is used the GDQM that was derived by Du *et al*. (1995). In this method, the first order derivative of function $f(x)$ can be approximated as linear sum of weighting coefficients and function values for all grid point in the x domain.

$$\frac{df(x_i)}{dx}=\sum_{j=1}^{N}C_{ij}^{(1)}f(x_j) \qquad i=1,2,3,..,N \tag{24}$$

where N is the number of grid point in the x domain, $f(x_i)$ is function in the point of x_j and C_{ij}^{1} is weighting coefficient of the first order derivate. Weighting coefficient for the first order derivate is expressed as following

$$C_{ij}^{(1)}=\begin{cases}\dfrac{M^{(1)}(x_i)}{(x_i-x_j)M^{(1)}(x_j)} & i\neq j\\ -\sum\limits_{\substack{k=1\\k\neq i}}^{N}C_{ik}^{(1)} & i=j\end{cases} \tag{25}$$

$$M(x_i)=\prod_{k=1,k\neq i}^{N}(x_i-x_k) \qquad i,j=1,2,..,N$$

The rth-order approximate of function $f(x)$ into the GDQM for x domain is given as following (Du *et al* 1995)

$$\frac{d^r f(x_i)}{dx^r} = \sum_{j=1}^{N} C_{ij}^{(r)} f(x_j) \tag{26}$$

$$C_{ij}^{(r)} = \begin{cases} r\left(C_{ij}^{(1)} C_{ii}^{(r-1)} - \dfrac{C_{ij}^{(r-1)}}{x_i - x_j}\right) & (i, j = 1,2,3,\ldots,N;\ r = 2,3,\ldots,N-1; i \neq j) \\ -\sum\limits_{\substack{j=1 \\ i \neq j}}^{N} C_{ij}^{(r)} & (i = j = 1,2,3,\ldots,N; r = 1,2,\ldots,N-1) \end{cases} \tag{27}$$

In this research has been used the Chebyshev-Guass-Lobatto sample points (Wu and Shu 2002) to calculate the weighting coefficients.

$$x_i = \frac{1-\cos\left[\dfrac{(i-1)\pi}{(N-1)}\right]}{2} L \qquad (i = 1,2,3,..,N) \tag{28}$$

4. Numerical results

In this section, the results of the free vibration of the unidirectional composite Timoshenko beam based on finite strain assumption by using the GDQM have been obtained and compared with the Euler-Bernoulli beam. Also, the finite strain assumption has been compared with the von Karman hypothesis and difference between two assumptions has been analyzed. In addition, the effects of length to thickness ratio (*L/h*) on the free vibration of the beam undergoing finite strain have been investigated. The frequencies of the beam have been obtained for carbon/epoxy and glass/epoxy subjected to various boundary conditions.

4.1 Validation of the numerical procedure

In this section, frequencies of the unidirectional composite Timoshenko beam according to the finite strain are compared with the composite Euler-Bernoulli beam (Ghasemi *et al*. 2016). The calculated frequencies are shown in Table 1 for glass/epoxy, *L/h*=10, *N*=35 nodes and clamp-free boundary condition. The elastic properties of these two materials (Tsai 1980) are indicated in Table 2. As depicted, the frequencies of the Timoshenko beam are less than the Euler-Bernoulli beam and difference between frequencies of the two beams for high modes increases. Also, ratio of the nonlinear to linear frequency of the unidirectional composite Timoshenko beam is compared with the Euler-Bernoulli beam presented by Guo and Zhong (2004). The results have been calculated for *N*=75 nodes and clamp-clamp and clamp-simply boundary conditions and have been shown in

Table 1 Dimensional frequencies of nonlinear vibration of the composite Timoshenko and Euler-Bernoulli C-F beams

frequency	λ_1	λ_2	λ_3	λ_4	λ_5
(Ghasemi *et al*. 2016)	3.9448	22.9683	62.5786	121.7854	200.7538
Present	3.8056	20.2790	48.7735	82.6461	119.2119

Table 2 Properties of carbon/epoxy and glass/epoxy

Material	E_x (GPa)	E_y (GPa)	E_x (GPa)	υ_x	ρ (kg/m^3)
Carbon/epoxy	181	10.3	7.17	0.28	1600
Glass/epoxy	38.6	8.27	4.14	0.26	1800

Table 3 Ratio of the nonlinear to linear frequency

Boundary conditions	present	(Guo and Zhong 2004)		
		SBDQM	GFEM	FEM
Clamp-clamp	1.0255	1.0296	1.0295	1.0295
Clamp-simply	1.0442	1.0592	1.0641	1.0641

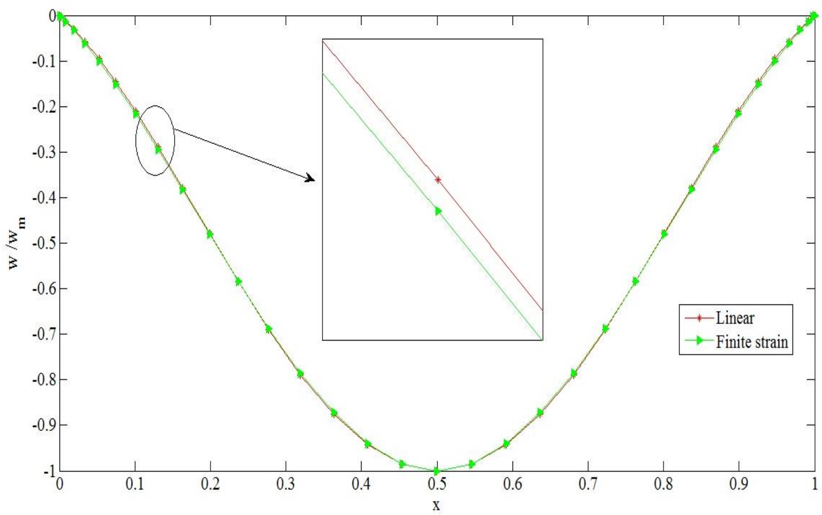

Fig. 1 Deflection of the first mode of the clamp-clamp supported carbon/epoxy composite beam

Table 3. The results demonstrate that they have good agreement with the results of the finite element method (FEM), the generalized finite element method (GFEM) and the spline-based differential quadrature method (SBDQM).

4.2 Vibration analysis of the beam subjected to clamp-clamp boundary condition

The clamp-clamp boundary conditions at two ends of the beam are given as

$$w = \varphi = 0 \tag{29}$$

Deflection of the first three modes of the composite Timoshenko beam based on the finite strain assumption for carbon/epoxy material in the dimensionless form are shown in Figs. 1, 2 and 3. In the figures, the amplitude is made dimensionless through dividing by maximum amplitude. In these figures are seen that the most difference between the linear and nonlinear states occur between maximum and minimum of the amplitude. Also, the slope of nonlinear curve in the clamp end is more than the linear one.

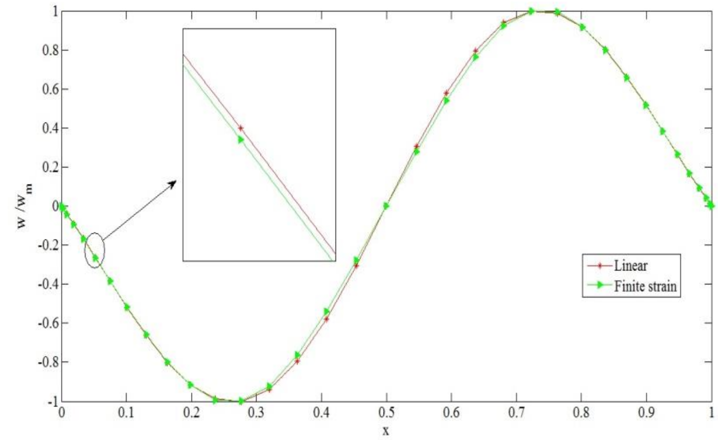

Fig. 2 Deflection of the second mode of the clamp-clamp supported carbon/epoxy composite beam

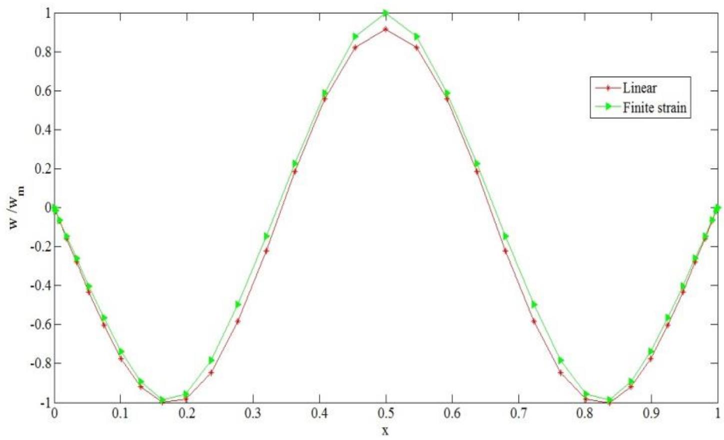

Fig. 3 Deflection of the third mode of the clamp-clamp supported carbon/epoxy composite beam

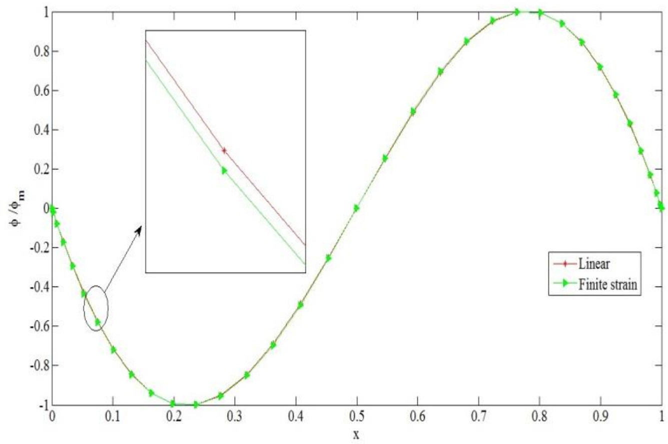

Fig. 4 Cross section rotation of the first mode of the clamp-clamp supported carbon/epoxy composite beam

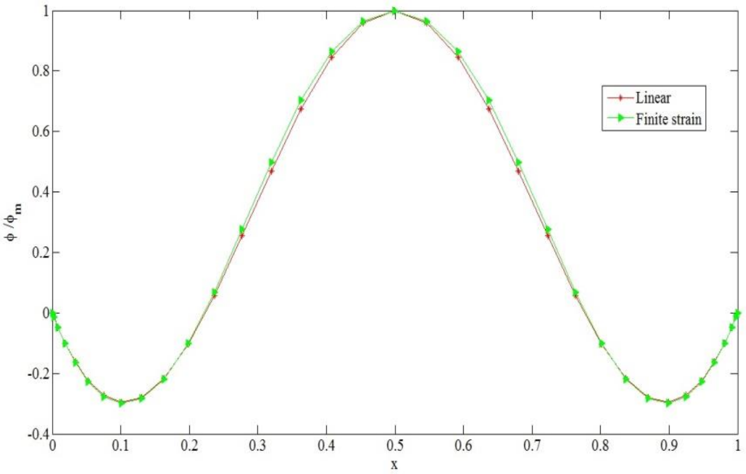

Fig. 5 Cross section rotation of the second mode of the clamp-clamp supported carbon/epoxy composite beam

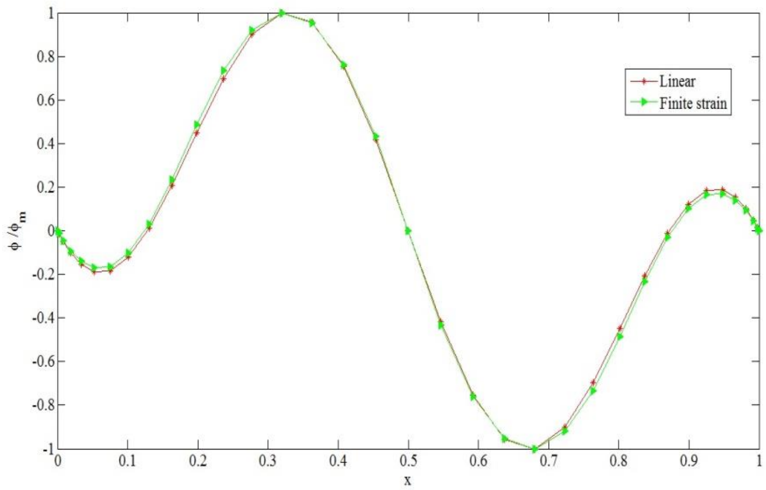

Fig. 6 Cross section rotation of the third mode of the clamp-clamp supported carbon/epoxy composite beam

In Figs. 4 to 6 have drown cross-section rotation of the carbon/epoxy composite beam for the first to third modes, respectively. In this figures are shown that the slope of nonlinear curve in the clamp end is more than the linear one.

As depicted in the figures, the difference between nonlinear and linear states is small because the carbon/epoxy material is stiff and nonlinear deflection and cross-section rotation due to nonlinear terms cannot create a significant difference. To demonstrate the remarkable difference between linear and nonlinear states, the deflections of the beam have been drawn for glass/epoxy material as shown in Figs. 7 to 9. As shown, the difference between linear and nonlinear cases is considerable, because the glass/epoxy material is more flexible than the carbon/epoxy one.

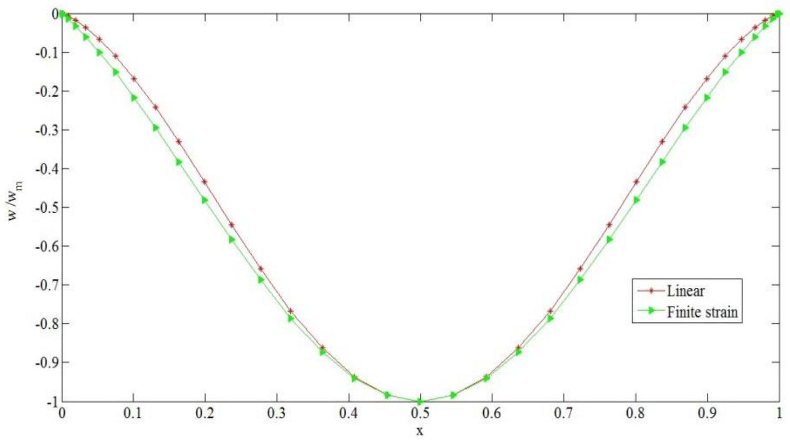

Fig. 7 Deflection of the first mode of the clamp-clamp supported glass/epoxy composite beam

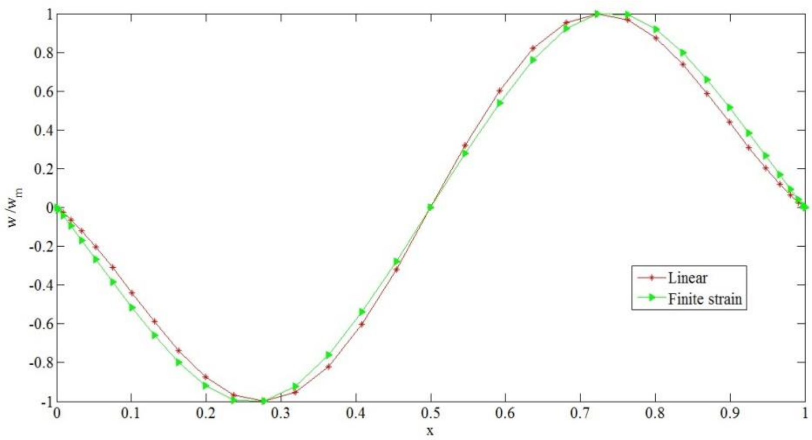

Fig. 8 Deflection of the second mode of the clamp-clamp supported glass/epoxy composite beam

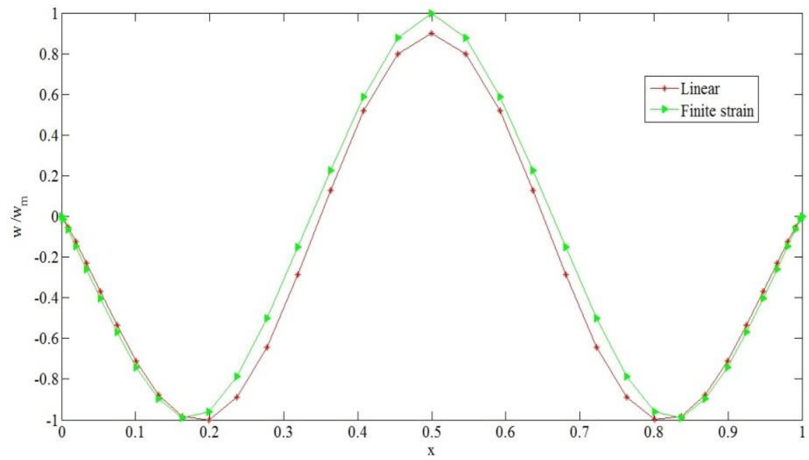

Fig. 9 Deflection of the third mode of the clamp-clamp supported glass/epoxy composite beam

4.3 Vibration analysis of the beam subjected to clamp-free boundary condition

The clamp-free boundary conditions at two ends of the beam are given as

$$w=\varphi=0 \qquad \text{in } x=0 \tag{30a}$$

$$\begin{aligned} &5\left(\frac{dw}{dx}\right)^2\frac{d\varphi}{dx}+10\left(\frac{L}{h}\right)^2 a\varphi^2\frac{d\varphi}{dx}+\frac{3}{4}\left(\frac{d\varphi}{dx}\right)^3+\frac{40}{3}\left(\frac{L}{h}\right)^2\frac{d\varphi}{dx}=0 \\ &\frac{3}{4}\left(\frac{d\varphi}{dx}\right)^2\frac{dw}{dx}+9\left(\frac{dw}{dx}\right)^3-24\left(\frac{L}{h}\right)^3 a\varphi+24\left(\frac{L}{h}\right)^2 a\frac{dw}{dx}=0 \end{aligned} \qquad \text{in } x=1 \tag{30b}$$

4.4 Vibration analysis of the beam subjected to simply-simply boundary condition

The simply-simply boundary conditions at two ends of the beam are given as

$$w=0 \qquad \text{in } x=0,1 \tag{31a}$$

$$5\left(\frac{dw}{dx}\right)^2\frac{d\varphi}{dx}+10\left(\frac{L}{h}\right)^2 a\varphi^2\frac{d\varphi}{dx}+\frac{3}{4}\left(\frac{d\varphi}{dx}\right)^3+\frac{40}{3}\left(\frac{L}{h}\right)^2\frac{d\varphi}{dx}=0 \qquad \text{in } x=0,1 \tag{31b}$$

As depicted, simply-simply supported boundary conditions are nonlinear, which these nonlinear terms are related to the finite strain assumption however, the obtained boundary conditions by using the von Karman assumption are linear. The deflection and cross-section rotation of the simply supported beam for carbon/epoxy material have been shown in Figs. 10 to 12 and 13 to 15, respectively. As seen, the least difference between linear and nonlinear states occurs in the simply supported beam. Unlike the von Karman assumption, which the difference between the linear and nonlinear vibrations subjected to simply-simply boundary condition is very little, the difference the linear and nonlinear states on the basis of the finite strain is remarkable. Also, unlike the clamp-clamp boundary condition, the curves slope of the nonlinear vibration in the simply-simply state is less than the linear one.

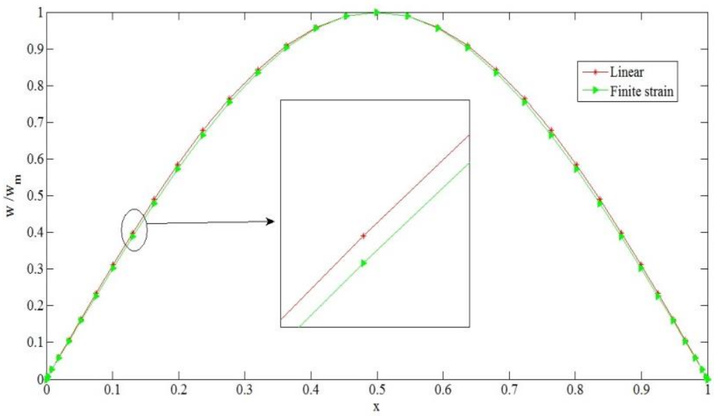

Fig. 10 Deflection of the first mode of the simply-simply supported carbon/epoxy composite beam

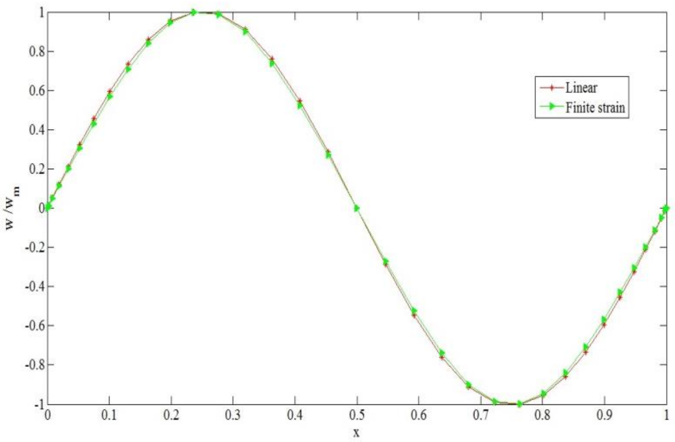

Fig. 11 Deflection of the second mode of the simply-simply supported carbon/epoxy composite beam

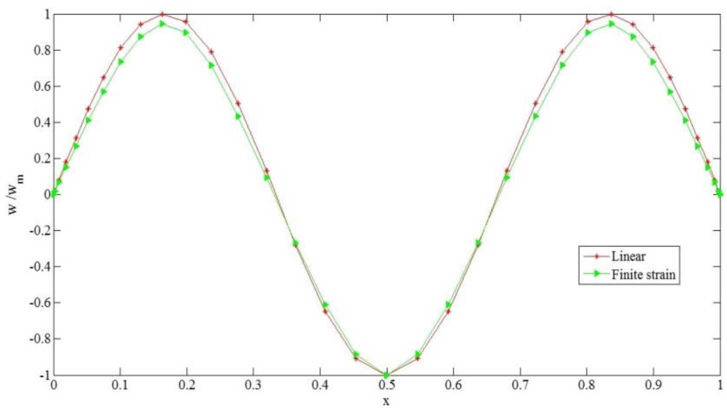

Fig. 12 Deflection of the third mode of the simply-simply supported carbon/epoxy composite beam

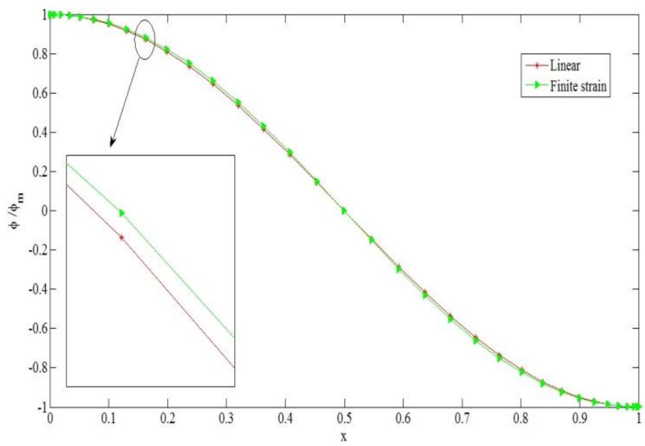

Fig. 13 Cross section rotation of the first mode of the simply-simply supported carbon/epoxy composite beam

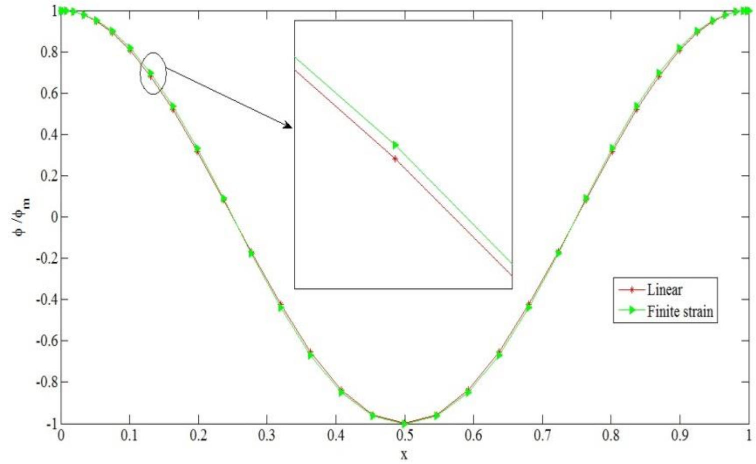

Fig. 14 Cross section rotation of the second mode of the simply-simply supported carbon/epoxy composite beam

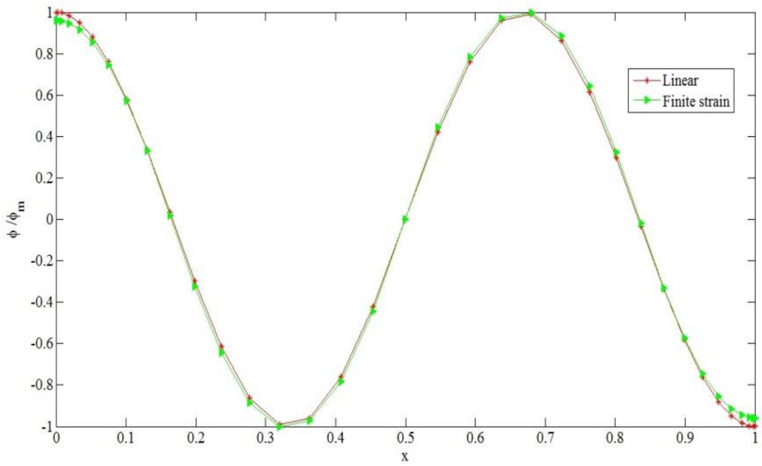

Fig. 15 Cross section rotation of the third mode of the simply-simply supported carbon/epoxy composite beam

4.5 The effect of thickness on the finite strain vibration

In this section, the influence of thickness on vibration of the beam based on the finite strain is studied and compared with the von Karman assumption. The equations of the von Karman dimensionless don't depend to geometry of the beam, but vibration equations of the beam undergoing the finite strain depend to geometry of the beam. First five dimensionless frequencies of the beam for N=35 nodes and L/h=5,8 and 10 have been calculated based on the finite strain. Also, they have been compared with the von Karman assumption subjected to clamp-clamp boundary condition and have been shown in Table 4. As seen, the frequencies of the finite strain

Table 4 Dimensionless frequencies of the von Karman and finite strain

von Karman	finite strain with *L/h*=10	finite strain with *L/h*=8	finite strain with *L/h*=5
19.6868	19.7012	18.3278	14.7259
48.2608	48.3050	43.4187	33.0169
80.1115	80.1632	70.3702	51.6315
114.9642	115.0152	99.3291	70.9550
151.4435	151.5194	129.2385	90.5447
188.7734	188.8581	159.5251	109.2585

Table 5 Dimensionless frequencies of carbon/epoxy and glass/epoxy of the composite Timoshenko beam

Boundary conditions	clamp-clamp		simply-simply		clamp-free	
Material	carbon/epoxy	glass/epoxy	carbon/epoxy	glass/epoxy	carbon/epoxy	glass/epoxy
λ_1	16.4190	19.7012	10.2165	10.6533	3.6762	3.8057
λ_2	37.7371	48.3050	32.2116	36.8207	17.6972	20.2790
λ_3	59.7990	80.1632	56.1147	70.2548	39.1653	48.7735
λ_4	83.0509	115.0152	80.4803	107.4104	62.2634	82.6461
λ_5	106.6421	151.5194	104.6198	145.8656	85.8634	119.2119

with *L/h*=10 are close to the von Karman, nevertheless with decreasing the *L/h* ratio, the difference between two assumptions increases.

4.6 The effect of material properties

In this section, the dimensionless frequencies of carbon/epoxy and glass/epoxy for various boundary conditions are calculated. First five frequencies of the beam for *N*=35 nodes and *L/h*=10 are shown in Table 5. The results illustrate that the dimensionless frequencies of carbon/epoxy are less than glass/epoxy for different boundary conditions.

5. Conclusions

In this research, the free vibration of the unidirectional composite Timoshenko beam based on the finite strain was studied. The equations and boundary conditions were obtained using Hamilton's principle. In the first step, the equations were separated using single-harmonic solution and then they were solved by applying the GDQM for different boundary conditions. The results demonstrated that the curves slopes of deflection and cross-section rotation of the beam for the nonlinear vibration in the clamp end are more than the linear one and by increasing number of modes, this slope increases. Also, the curve slope of deflection and cross-section rotation of the beam subjected to simply-simply boundary condition for the nonlinear state is less than the linear one. The results for simply-simply beam showed that the difference between linear and nonlinear states was remarkable. In this article was showed that the frequencies of the finite strain assumption with *L/h*=10 were close to the von Karman and also by decreasing *L/h*, the difference between the finite strain and von Karman assumptions increased. Also, the results indicated that the dimensionless frequencies for glass/epoxy were more than carbon/epoxy for various boundary conditions.

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