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Simple formulas for the fuel of climbing propeller driven airplanes

Gilles Labonté*

Department of Mathematics and Computer Science and Department of Electrical Engineering and Computer Engineering, Royal Military College of Canada, Kingston, Ontario, Canada

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Abstract. Simple solutions are obtained for the fuel required by internal combustion engine airplanes on trajectories with a constant rate of climb or descent. Three modes of flight are considered: constant speed, constant Mach number and constant angle of attack. Starting from the exact solutions of the equations of motion for the modes of motion considered, approximate solutions are obtained that are much easier to compute while still being quite precise. Simpler formulas are derived for the weight of fuel, speed, altitude, horizontal distance, time to climb, and power required. These formulas represent a new important contribution since they are fundamental for the analysis of aircraft dynamics and thus have direct applications for the analysis of aircraft performances and mission planning.

Keywords: airplane fuel consumption; airplane equation of motion; climbing airplanes; descending airplanes

1. Introduction

Building on the work of Labonté (2011), this article presents simple formulas for the amount of fuel that propeller driven airplanes, with an internal combustion engine, will use on rectilinear climbing and descending trajectories. Our motivation for this study is the necessity of such formulas, which are preferably easy to calculate, for automatic mission planning of unmanned aerial vehicles (UAVs). Whereas the flight programs of passenger airplanes are fairly predetermined and constant from one flight to another, those of UAVs are generally much more complex, especially when they are flying in cluttered operational areas and also, they generally vary with each mission. Furthermore, it is desirable that UAVs be able to perform on board automatic trajectory re-planning in response to unforeseen events. We recall that one of the preferred approaches to automatic trajectory planning consists in starting by constructing a skeleton trajectory as a continuous sequence of rectilinear segments at various inclinations. In a second phase, the connections between these segments are rounded off so that the velocity of a vehicle traveling the trajectory is continuous; see for example Judd (2001), Anderson *et al*. (2005), Zheng *et al*. (2003), Yang and Sukkarieh (2010). Even after this process is completed, most of the trajectory will still consist of rectilinear segments. With uneven terrain, the trajectory will have

*Corresponding author, Ph.D., E-mail: labonte-g@rmc.ca

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many climbing and descending segments together with some horizontal segments. This is why formulas are needed for the power and the fuel required by airplanes to follow arbitrary rectilinear segments. These formulas are fundamental for determining whether or not the planned trajectory is compatible with the dynamics of the airplane in question.

An important side benefit of this study is that the formulas obtained are actually applicable to all propeller driven aircrafts with internal combustion engines, and thus constitute a valuable tool in the analysis of their performance. The importance of flight planning has to be appreciated, even for passenger line airplanes. Typical commercial flights are composed of many segments, some climbing and descending segments, and some longer essentially horizontal cruising segments. Most of the time, the inclined segments are decomposed in segments with different flight modes, and it also often happens that a long horizontal flight will be decomposed into segments flown at different altitudes and speeds, calculated to optimize resources following the changing weight of the fuel in the airplane. One can appreciate the importance of fuel management in commercial airlines when looking at the Airbus Customer Services (2004) brochure on fuel economy for the Airbus airplanes. It introduces the subject with the statement: “Fuel Consumption is a major cost to any airline, and airlines need to focus their attention on this in order to maintain their profitability”. It then discusses the many factors that affect fuel consumption, considering in particular the optimization of climb, cruise and descent techniques and the altitude and speed of cruise flight, which are precisely the subjects we discuss here.

In this study, we are not concerned with finding the values of the flight parameters that correspond to an “optimal trajectory” because each trajectory planning problem is an individual optimization problem in its own right. The notion of “best” trajectory depends entirely on the mission at hand. In one mission flying very fast may be preferred over saving fuel, if a danger zone has to be evaded. In another mission, saving fuel may be preferred. In automatic mission planning, trajectories are assigned a cost by a function that weights the length of the trajectory, the flying altitude, the traversing of danger zones or no-fly zones, etc. The relative costs of the variables considered can be adjusted according to the mission at hand. The formulas we presented in this article constitute basic tools that allow determining the feasibility of a trajectory and calculating its cost.

Level rectilinear flight is discussed in essentially all manuals on airplane performance; as, in particular, Hale (1984), Anderson (2000), Eshelby (2000), Yechout *et al*. (2003), Stengel (2004) and Filippone (2006). Fundamental formulas that are always given are the so-called Bréguet formulas for rectilinear flight at constant angle of attack. Yechout *et al*. (2003), Stengel (2004) also discuss flight at constant airspeed-constant lift coefficient (i.e., Cruise-climb). Hale (1984) presents a thorough analysis of the three flight modes with: constant altitude-constant lift coefficient, constant airspeed-constant lift coefficient, constant altitude-constant airspeed.

Because they are equally important, climbing and descending flights are also always discussed. However, in our survey of some classical manuals, we did not find any that provided a complete solution of the climbing flight equations even for simple flight modes. The discussions presented usually consider steady-state or quasi-steady-state motion and only deal with local aspects of a climbing flight, and never with the entire flight. For example Yechout *et al*. (2003) simply define the local climbing rate and, although Torenbeek (1976), Hale (1984), Anderson (2000), Eshelby (2000), Stengel (2004), Filippone (2006) all derive conditions for the climb angle or the climb rate to be maximum, these conditions are also only local in the sense that they depend on the specific altitude and weight of the airplane. Such local conditions are not very useful as such because conditions are needed for the entire trajectory, and implementing them requires the values of the weight of the airplane at each altitude. Before Labonté (2011), one could not find formulas for the time evolution of the speed, the altitude, the power and the fuel required for the airplane to travel an entire climbing or descending trajectory. This fact is recognized in Section 2.6 of Stengel (2004) and in Section 8.6 of Filippone (2006), where it is explicitly stated that steady state models cannot be correct because the climb rate and the optimal climb conditions change with changing altitude so that the airplane in fact accelerates. It was one of the main contributions of Labonté (2011) to derive complete exact solutions for the three different modes of climb and descent on a rectilinear trajectory with constant angle of attack, or constant speed, or constant Mach number.

We further note that in all studies of climbing or descending flight found in airplane dynamics manuals, the approximation is made that θ, the angle of the trajectory with the horizontal plane, is small so that $\cos(\theta)\approx 1$. Although this is true for most conventional propeller-driven airplanes for which θ is limited to roughly 10°-15° or less, it is not true for UAVs. These come in a wide range of sizes and agilities and they can fly much more daring manoeuvres as inhabited airplanes. For example, some commonly available radio-controlled planes can easily climb at 45° or steeper, as for example, the Carl Goldberg Falcon 56 described by Granelli (2007) and the Hangar 9 Twist 40 described by Horizon Hobby (2004). Thus, as in Labonté (2011), we do not make the approximation of small angles and all the formulas we derive are applicable for any angle of climb and descent.

We shall not consider horizontal trajectories in the present article. In Labonté (2011), the equations of motion were solved for two modes of flight at constant altitude, namely flight at constant angle of attack and flight at constant speed. The formulas obtained are relatively simple, containing only some square roots and the functions *In* and *tan*. We therefore see no need to try and obtain simpler ones. In the same document, formulas describing climbing or descending flights at constant angle of attack, at constant speed and at constant Mach number were also derived. These, on the other hand, involve rather complex functions to calculate and will gain much in the simplifications that we present hereafter. The simpler formulas we obtain are illustrated by explicit calculations of many trajectories with the following three very different hypothetical airplanes. We present measures of the precision of these formulas as the discrepancy between the values they yield and those of the exact formulas.

The first one is the CP-1 airplane, described in Section 6 of Anderson (2000). It is similar to the Cessna Skylane and has the characteristics shown in Table 1. Note that all the parameters used in this article can be found defined in a Nomenclature Section at the end.

The second one is a Silver Fox-like Unmanned Aerial Vehicle (UAV). Some specifications for the Silver Fox can be found at the Faculty of Engineering, University of Porto (2013). The power available $P_A(0)$ for the Silver Fox is only about 370 W, which allows it to climb only at low angles. Meanwhile, it is common for Radio Controlled (RC) airplanes to climb at very steep angles (See for example Bell (2005) and Granelli (2007)). Thus, upon taking advantage of motors that

Table 1 Characteristics of the CP-1 airplane from Anderson (2000). The parameters listed are W_1=the weight of the airplane without fuel, W_f=the initial weight of fuel, η=the propeller efficiency, c=the specific fuel consumption, b=the wingspan, S=the wing area, e=Oswald's efficiency factor, C_{D0}=the global drag coefficient at zero lift, $C_{L\max}$=the maximum global lift coefficient

W_1=9,454 N	W_f=1,343 N	$P_A(0)$=137,209 W	η=0.8
c=7.4475×10^{-7}	b=10.9118 m	S=16.1653 m^2	e=0.8
C_{D0}=0.025	$C_{L\max}$=2.10		

Table 2 Characteristics of the Silver Fox-Like airplane. The parameters are the same one as in Table 1

W_1=113 N	W_f=19 N	$P_A(0)$=1850 W	η=0.8
c=7.4475×10^{-7}	b=2.4 m	S=0.768 m^2	e=0.8
C_{D0}=0. 0251	$C_{L\max}$=1.26		

Table 3 Characteristics of the Hercules-Like airplane. The parameters are the same one as in Table 1

W_1=337,120 N	W_f=266,717 N	$P_A(0)$=11,113,200 W	η=0.81
c=7.4475×10^{-7}	b=40.4 m	S=162.1 m^2	e=0.92
C_{D0}=0.0138 (1)	$C_{L\max}$=2.7 (2)		

have been developed in this domain, a Silver Fox-like airplane could be endowed with much more power in order to improve considerably its manoeuvre envelope. One such motor is the O.S. 120AX 20cc that outputs 3.1hp, i.e., 2312 W, and weights only 650 g; so we shall consider a Silver Fox-like UAV with this particular motor.

The last one is a Lockheed C-130 Hercules-like airplane. Some of its specifications are those of the Hercules itself, as can be found in Lockheed Martin (2013), Stewart Air Force Base (2005) and Sadraey (2013) and some parameters have been set at plausible values, by comparison with other available transport airplanes Filippone (2000).

This article is organized as follows. It starts by recalling the equations of motion for an airplane traveling on an inclined rectilinear trajectory, and the concepts of power available, power required and fuel consumption that are fundamental to the subject studied. It then considers in turn the three climbing modes in which the speed, the Mach number or the angle of attack is constant. For each of these modes of flight, it recalls the exact solution of the equations of motion and presents an approximate formula for the speed of the airplane, its altitude, its weight, and the power it requires for the motion. In each case, many examples of trajectories are considered for the three reference airplanes and, for each one of them, the precision of the approximate formula is calculated. These test trajectories are the longest ones that are possible for the initial conditions considered. The proposed formulas are then expected to be also precise for all other trajectories that are necessarily shorter.

2. Flight at constant climb or descent angle

Let us consider an airplane that moves on a rectilinear trajectory making an angle θ with a horizontal plane. When the wind effects are neglected, the forces acting on that airplane have two components: a tangential component, along the unit tangent vector $\boldsymbol{\tau}$ to the trajectory, and a normal component, along the unit vector **N** normal to $\boldsymbol{\tau}$. The total force vector **F** can be written as

$$\mathbf{F} = [\mathrm{T_R} - \mathrm{D} - \mathrm{W}\sin(\theta)]\, \boldsymbol{\tau} + [\, \mathrm{L} - \mathrm{W}\cos(\theta)]\, \mathbf{N} \tag{1}$$

in which, T_R is the thrust that the propeller must provide, D is the total drag, W is the weight of the airplane and L is the lift. L and D are given by the equations

$$\mathrm{L} = \frac{1}{2}\rho_\infty \mathrm{S}\,\mathrm{C_L}\,\mathrm{V}_\infty^2 \qquad \text{and} \qquad \mathrm{D} = \frac{1}{2}\rho_\infty \mathrm{S}\,\mathrm{C_D}\,\mathrm{V}_\infty^2 . \tag{2}$$

In Labonté (2011), one can find the Newton equation of motion for an airplane, the weight of which changes due to the burning of fuel with the air to fuel ratio AFR. It is

$$M\frac{d\mathbf{v}}{dt} - AFR\left[\frac{dM}{dt}\right]\mathbf{v} = \mathbf{F} \tag{3}$$

in which M is the mass of the airplane and $\mathbf{v}$ is its velocity. Note that both the velocity and the acceleration are in the direction of the tangent vector $\boldsymbol{\tau}$. Thus, upon substituting the force vector given in Eq. (1) into Eq. (3), and projecting the resulting equation along the vectors $\boldsymbol{N}$ and $\boldsymbol{\tau}$, one obtains the following two equations.

$$L = W\cos(\theta) \tag{4}$$

$$M\frac{dV_\infty}{dt} - AFR\left[\frac{dM}{dt}\right]V_\infty = T_R - D - W\sin(\theta). \tag{5}$$

Upon multiplying Eq. (5) by $-\frac{c}{\eta}V_\infty$, it becomes

$$-\frac{c\,W\,V_\infty}{\eta g}\left[\frac{d\,V_\infty}{dt}\right] + \frac{c\,(AFR)\,V_\infty^2}{\eta g}\left[\frac{d\,W}{dt}\right] = \left[\frac{d\,W}{dt}\right] + \frac{c V_\infty}{\eta}\left[D + W\sin(\theta)\right] \tag{6}$$

so that Eqs. (4) and (6) can be taken to be the two fundamental equations of motion to solve.

For simplicity, in the present study, we consider only flights below 11 km, so that a_1, the rate of variation of the temperature with the altitude, is constant, with

$$T(h) = T_s - a_1\,h. \quad \text{with } a_1 = 6.5\times 10^{-3} \tag{7}$$

Our results can be readily generalized to flights at higher altitudes by solving the equations of motion inside each of the zones of the atmosphere that are traversed, in which the temperature gradients differ, and then requiring that their solutions match continuously at the zone boundaries. For such trajectories, the following equations for the altitude h, the horizontal distance traveled x, and the air density ρ_∞ will hold

$$\frac{dh}{dt} = V_\infty\sin(\theta) \qquad\qquad x = h\cot(\theta) \tag{8}$$

$$\frac{d\rho_\infty}{dt} = -4.2433\,a_1\sin(\theta)\frac{V_\infty\rho_\infty}{T(h)}, \quad \text{since} \quad \rho_\infty(h) = \rho_S\left[\frac{T(h)}{T_S}\right]^{4.2433} \tag{9}$$

in which ρ_s and T_s are the air density and the temperature at sea level.

3. Power available, power required and fuel consumption

Chapter 9 of Anderson (2000) explains that when the power P_P produced by an internal combustion engine is transferred to a propeller of efficiency η, the power available to move the airplane P_A will be: $P_A=\eta P_P$. When this power is produced, the rate of fuel usage is

$$\frac{dW}{dt} = -c\,P_P = -\frac{c}{\eta}\,P_A \tag{10}$$

in which c is the specific fuel consumption. We note that there is always an upper bound $P_{A\,\max}$ to the power an engine can generate and a lower bound $P_{A\,\min}$ below which it shuts down.

How much power should the engine produce depends on the motion that the airplane has to perform. From Classical Mechanics we know that to move a body with the velocity **v** under the action of a force **F**, the power that is required from the mechanism that causes this motion is $P_R=\mathbf{F}\cdot\mathbf{v}$, where " · " denotes the scalar product of two vectors. If this body is an airplane with a propeller that produces a thrust **T** along the direction of its motion, then **T** is parallel to **v** and the power that is required from the propeller action to move this airplane is $P_R=T\,V$ where T and V are the magnitudes of **T** and **v**. This airplane's engine should then provide the power $P_A=P_R$. Finally, we recall that the power produced by a combustion engine varies with the altitude according to the equation

$$P_A(h) = P_A(0)\;\frac{\rho(h)}{\rho_S}\,. \tag{11}$$

4. Formulas for climb or descent at constant speed

In a flight at constant speed, the rate of climb is constant and the altitude and temperature are linear functions of time

$$h(t) = h_i + v_3\,(t - t_i) \qquad \text{and} \qquad T(t) = T_i - a_1 v_3\,(t - t_i) \qquad \text{with} \qquad v_3 = V_\infty \sin(\theta). \tag{12}$$

where h_i is the altitude at the initial time t_i. Thus, in particular, the time required to climb from h_i to h_f is simply $t_c=(h_f - h_i)\,/v_3$.

Eq. (4) determines how the lift coefficient C_L should change in terms of W and the altitude, while Eq. (6) yields the weight W as a function of time. When the speed is constant, the latter equation becomes the following Riccatti equation

$$\frac{dW}{dt} = -\left[\alpha\,T^{4.2433} + \beta W + \delta\,T^{-4.2433}W^2\right] \tag{13}$$

in which the constants α, β, δ are defined as

$$\alpha = \frac{c\,g\,\rho_\infty(h_i)\,S\,C_{D0}V_\infty^3}{2T_i^{4.2433}\,G} \qquad\qquad \beta = \frac{c\,g\,v_3}{G}$$

$$\delta = \frac{2\,c\,g\cos^2(\theta)T_i^{4.2433}}{\pi\,e\,AR\,\rho_\infty(h_i)S\,V_\infty G} \qquad \text{with} \quad G = \left\lfloor \eta\,g - c\,(AFR)V_\infty^2 \right\rfloor. \tag{14}$$

In Labonté (2011), it was shown that the solution to Eq. (13) is

$$W(t) = -\frac{\beta T^d}{\delta}\left[\frac{\left(1-\sqrt{1-4A}\right)}{2} + \frac{y_1'(z)+\Gamma\, y_2'(z)}{y_1(z)+\Gamma\, y_2(z)}\right] \tag{15}$$

with

$$d = 4.2433, \qquad A = \frac{\alpha\delta}{\beta^2} \qquad z = \frac{\beta}{a_1 v_3} T(t)$$

$$\Gamma = -\frac{B y_1(z_0) + y_1'(z_0)}{B y_2(z_0) + y_2'(z_0)} \qquad B = \frac{\delta W_0}{\beta T_0^d} + \frac{1}{2} - \frac{\sqrt{1-4A}}{2} \tag{16}$$

with two different possibilities for the functions y_1 and y_2.

Case 1: A = 1/4

$$y_1(z) = z^{(1-d)/2}\ J_{d-1}\left(\sqrt{2dz}\right) \tag{17}$$

$$y_2(z) = z^{(1-d)/2}\ Y_{d-1}\left(\sqrt{2dz}\right) \tag{18}$$

where J_{d-1} and Y_{d-1} are Bessel functions.

Case 2: $A \neq 1/4$

$$y_1(z) = {}_1F_1\left(k_1, d, \sqrt{1-4A}\ z\right) \qquad \text{with} \qquad k_1 = \frac{d}{2}\left[1 - \frac{1}{\sqrt{1-4A}}\right] \tag{19}$$

$$y_2(z) = z^{1-d}\ {}_1F_1\left(k_1 + 1 - d,\ 2-d,\ \sqrt{1-4A}\ z\right) \tag{20}$$

in which ${}_1F_1$ is the confluent hypergeometric function (See Section 9.2 of Gradshteyn and Ryzhik 1965).

Fig. 1 illustrates the behavior of this solution. It shows the weight of fuel W_f as a function of time for the CP-1 airplane when it starts with 425 N of fuel, climbs at the constant speed of 25 m/s at an angle of 20°, until it reaches the altitude of 2190 m .

4.1 Power and lift constraints

According to Eqs. (10) and (13), the power required to climb at angle θ, at constant speed is

$$P_R = \frac{\eta}{c}\left[\alpha T^{4.2433} + \beta W + \delta T^{-4.2433} W^2\right] \tag{21}$$

For a flight to be possible, the power required P_R must be smaller or equal to the maximum power $P_{A\max}$ that can be produced by the motors and propellers. We recall that the power available is given by Eq. (11). Fig. 2 shows the graph of P_R as a solid line and $P_{A\max}$ as a dotted line, in the sample flight of the CP-1 airplane described above. The two curves intersect at about 2190 m, which indicates that the CP-1 airplane will not have enough power to climb above this altitude on this particular trajectory.

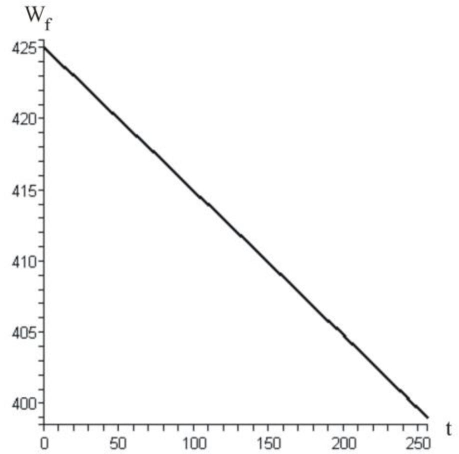

Fig. 1 Fuel of the CP-1 airplane as it climbs at 20°, at the speed of 25 m/s, up to 2190 m

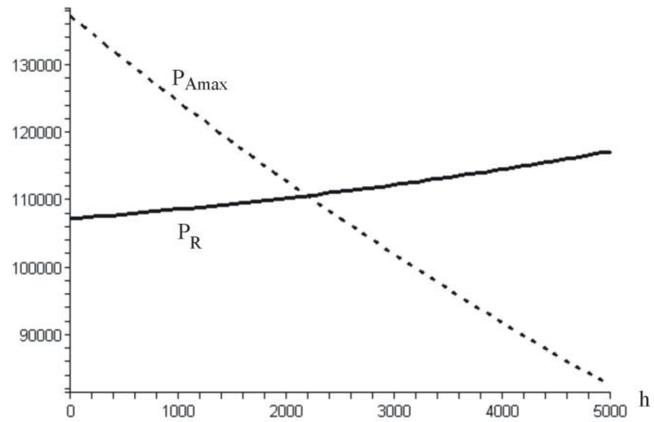

Fig. 2 Required power P_R and maximum available power $P_{A\max}$ as a function of altitude h, for the climbing airplane CP-1 at 20°, with a constant speed of 25 m/s

Another constraint is implied in Eq. (4). Since in the flight mode considered V_∞ is constant, while ρ_∞ and W change with the altitude, the angle of attack, i.e., the lift coefficient C_L must be constantly adjusted for Eq. (4) to keep on holding. Because the lift coefficient is bounded above by $C_{L\max}$, the following inequality must be satisfied at all times.

$$W\cos(\theta) \le L_{\max} = \frac{1}{2}\rho_\infty V_\infty^2 S C_{L\max} \tag{22}$$

Fig. 3 shows the graph of both sides of Inequality (22) as a function of the altitude h. The left-hand side is represented by the solid line and the right-hand side by the dotted line. The two curves intersect at about 3418 m, therefore from this altitude on there is not enough lift for the CP-1 airplane to climb higher.

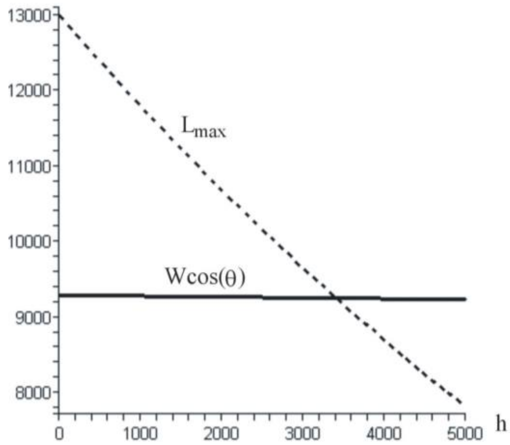

Fig. 3 Wcos(θ), represented by the solid line, and $L_{\max}$, represented by the dotted line, as a function of the altitude, for the climbing airplane CP-1 at 20°, with a constant speed of 25 m/s

Thus, the examination of the power and lift constraints, for the particular flight considered, indicates that the maximum altitude attainable is 2190 m.

4.2 Simpler formulas for the fuel used

Examination of the graph of W as a function of time for different trajectories, such as that shown in Fig. 1, indicates that W is almost a function linear in t. This suggests the possibility of finding a simpler expression for $W(t)$ than that given in Eq. (14), which would be much welcomed from the point of view of minimizing the calculation requirements. We present here two possibilities: one is a linear approximation and the other one is a Runge-Kutta one step approximation.

4.2.1 Linear formula

Let us approximate $W(t)$ by $W_a(t)=W_i+m(t-t_i)$, in which m is a constant. A value for the slope m can be obtained by requiring that W_a satisfies Eq. (13) at a particular instant of time. This can be realized as follows. Select an instant of time t_a, between t_i and t_f, and set $W(t_a)=W_a(t_a)$ in Eq. (13) that is evaluated at time t_a. There results a quadratic equation for the slope m that is readily solved. Experimentation suggested that a particularly good choice for t_a is at the mid-point, where $t_a=t_m=(t_f+t_i)/2$. We have tested this procedure for many trajectories (the set of test trajectories is described below) and found that, at the end of the trajectories, the maximum relative error in fuel used W_f, was never higher than −2.74% for the three airplanes: CP-1, Silver Fox-Like, and Hercules-Like.

More precision can be obtained with the linear approximation by performing more than one step, the two-step approximation being done as follows. A first linear approximation, as described above, with $t_a=(t_i+t_m)/2$, is used to obtain the intermediate value $W_a(t_m)$. In a second phase, this value of the weight is used as initial value for a second linear step that goes from t_m to t_f, with $t_a=(t_m+t_f)/2$. For all our test trajectories, the overall relative errors then fall below 1.4% for all the airplanes. Fig. 4 shows the relative error made by approximating $W(t)$ by a one-step and a two-step linear expression. The trajectories considered to produce this graph were for the Hercules-like

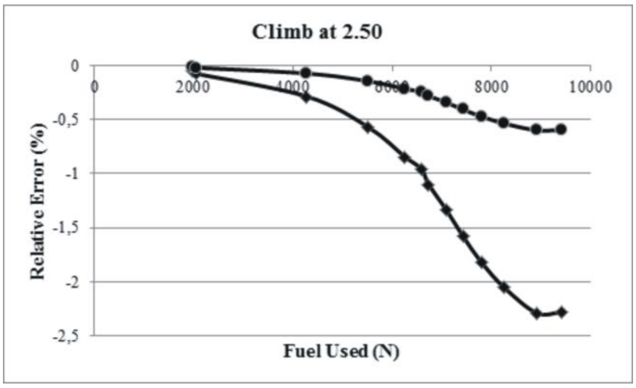

Fig. 4 Relative error in the approximation of the amount of fuel in % for a 2.5° climb-angle of the Hercules-Like airplane, in terms of the total fuel used in the trajectory. The curve with the square markers and that with the round markers correspond respectively to the one-step and the two-step linear approximation.

airplane, that climbs at 2.5°, starting with a weight of W_1+133,358 N, with different constant speeds with values from 45 m/s to 165 m/s, taken by steps of 10 m/s.

4.2.2 Runge-Kutta formula

The linear approximation procedure we have just described is essentially the midpoint or improved Euler numerical method for the solution of differential equations that can be found described in most textbooks on numerical methods. (See, for example, Chapter 19 of Kreyzig 1979). What is remarkable here is that the final solution is obtained with just one or two steps. This makes the method very efficient for fast and simple computations. A still better method than the Euler method is also described in most references on numerical methods: it is the Runge-Kutta method of order four. For the origin of this method, see Butcher (1996). This method gives much more precise results than the Euler method, and given the relative simplicity of the expression on the right-hand side of Eq. (13), it will also prove very easy to apply. We recall that the Runge-Kutta (R-K) method applies to ODE $dy/dx=f(x,y)$, for which f is a continuous function of x and y, and y is analytic, i.e., can be expanded in a Taylor series (see Lotkin (1951)). Given the explicit form of $W(t)$, it is easy to see that these two conditions are verified here so that the R-K procedure can be applied. According to this procedure, a sequence of points (t_n, W_n), for n=0,1,2... is constructed that lie close to the exact solution (t, $W(t)$) of the differential equation, as follows. Denote the right-hand side of Eq. (13) as $F(t, W)$

$$F(t,W) = -\left[\alpha T(h(t))^{4.2433} + \beta W + \delta T(h(t))^{-4.2433} W^2\right] \tag{23}$$

and select a time step Δt. Then, starting from the initial condition (t_0, W_0), the sequence of points is constructed as

$$A_n = \Delta t\, F(t_n, W_n), \qquad B_n = \Delta t\, F(t_n + \Delta t/2, W_n + A_n/2)$$

$$C_n = \Delta t\, F(t_n + \Delta t/2, W_n + B_n/2) \qquad D_n = \Delta t\, F(t_n + \Delta t, W_n + C_n)$$

$$t_{n+1} = t_n + \Delta t \qquad W_{n+1} = W_n + \frac{1}{6}\left[A_n + 2B_n + 2C_n + D_n\right] \tag{24}$$

What is remarkable with Eq. (13), and the values of its parameters in the problem we are considering, is that a single step of the R-K procedure yields an approximate solution W_a of remarkable precision. This can be seen by considering the estimate of the truncation error, given in Lotkin (1951), for a sequence of steps of the R-K method. To obtain this estimate, let $W_{a1}=W_N$ be the approximate value of $W(t+N\Delta t)$ constructed by a sequence of N steps of size Δt, and $W_{a2}=W_{2N}$ that obtained with $2N$ steps of size $\Delta t/2$. Then the truncation error $E_1=W(t_N)-W_{a1}$ is of the order of

$$E_1 \approx \frac{16}{15}\left[W_{a1}-W_{a2}\right]. \tag{25}$$

Consider, for example, the CP-1 airplane, that starts at t_i=0 with 425 N of fuel, at sea level and climbs at the constant speed of 25 m/s at an angle of 20°, until it reaches the altitude of 2190 m at time t_f=256.13 s. At the end of the trajectory, the estimate of W, given by the R-K method with a single step of $\Delta t=t_f-t_i$ is W_{a1}=9853.476394 N. That given by the R-K method with two steps of size $(t_f-t_i)/2$ is W_{a2}=9853.476404. Thus, $E_1\approx-0.00001$ N and consequently the one-step R-K estimate should be precise enough for all practical purposes. Upon rewriting equations (24) in a somewhat simpler notation, we propose to obtain the estimate for $W(t)$, at any time t, as

$$A(t) = \Delta t\, F(t_i, W_i), \qquad B(t) = \Delta t\, F(t_i+\Delta t/2, W_i + A(t)/2)$$

$$C(t) = \Delta t\, F(t_i+\Delta t/2, W_i + B(t)/2) \qquad D(t) = \Delta t\, F(t_i+\Delta t, W_i + C(t))$$

$$\Delta t = t - t_i \qquad W_a(t) = W_i + \frac{1}{6}\left[A(t)+2B(t)+2C(t)+D(t)\right] \tag{26}$$

Fig. 5 shows the difference between the exact value of $W(t)$ and the approximate value W_a, given by Eq. (26). One can see that this difference is always below 0.00005 N. Note that the number of points used in drawing these two graphs was limited because using many points yields a dense band of values instead of a discernible curve.

We have performed extensive testing to verify the precision of these formulas compared to the exact values given by Eq. (14). Our tests covered the three selected reference airplanes on various climbing trajectories. In each case, the climb starts at the sea level and the final altitude is the

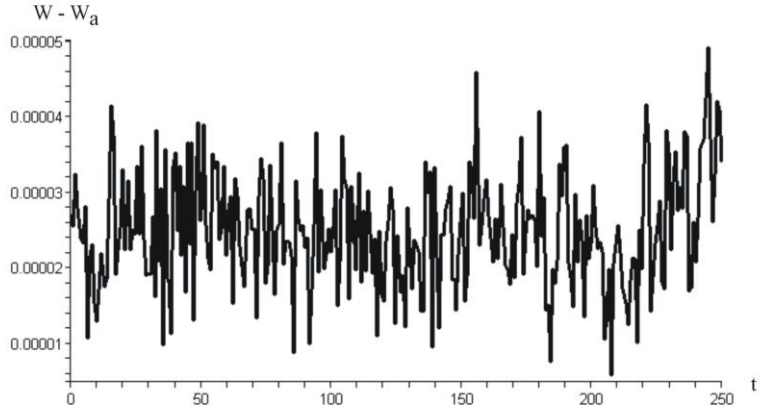

Fig. 5 Difference between the exact value of the weight $W(t)$ and the approximate value $W_a(t)$, for the trajectory of the CP-1 at 20° with the speed of 25 m/s

Table 4 Characteristics of the test set of trajectories for the CP-1 airplane. Each block of four lines corresponds to a same climb angle θ that is given in degrees. In the line headed by V_∞ are listed the constant speeds for the trajectory studied in m/s. In the line with the header $h_{\max}$ are the maximum altitude possible for the corresponding trajectory in m, and under these values are the amount of fuel used to reach that $h_{\max}$ in N

θ	25									
V_∞	25									
$h_{\max}$	811									
Fuel	9.03									
θ	20									
V_∞	25	30								
$h_{\max}$	2190	988								
Fuel	25.95	40.99								
θ	15									
V_∞	25	30	35	40						
$h_{\max}$	3152	2857	1785	511						
Fuel	40.99	34.79	21.06	6.01						
θ	10									
V_∞	25	30	35	40	45	50				
$h_{\max}$	2967	4925	4279	3402	2289	906				
Fuel	44.58	69.29	56.98	44.42	30.11	12.30				
θ	5									
V_∞	25	30	35	40	45	50	55	60	65	
$h_{\max}$	2870	6343	7066	6752	6220	5438	4343	2814	614	
Fuel	60.24	124.28	128.11	116.47	105.80	93.99	78.39	54.54	13.22	
θ	2.5									
V_∞	25	30	35	40	45	50	55	60	65	70
$h_{\max}$	2876	6379	8579	8592	8433	8078	7479	6540	5075	2648
Fuel	94.39	191.31	239.70	222.55	211.25	202.72	193.85	180.23	153.57	92.19

highest one possible, given the constraints on the power and the lift. The first trajectory we considered has close to the steepest angle and the lowest speed allowed. The last speed we considered is close to the upper bound speed allowed by the airplane available power. Our results clearly show that the difference between the exact value $W(t)$ and the approximate value $W_a(t)$ is of no practical significance.

For the CP-1 airplane, all our tests were made with the initial weight of W_1+425 N. Table 4 shows the characteristics of the 32 trajectories used in our tests. We note that these cover a wide range of speeds, of final altitudes and of amount of fuel used. In all these trajectories, the maximum percent relative difference between the exact value of fuel used and that produced by the R-K one-step method is less than 0.04%. This value occurs for the trajectory at 2.5° and speed of 35 m/s. This corresponds to a discrepancy of 13 ml of fuel over the 34 liters used.

For the Silver Fox-Like airplane, 32 trajectories were also examined: all trajectories started at sea level with an initial weight of W_1+19 N. Table 5 shows the characteristics of the six steepest trajectories. The other trajectories studied were for four climb-angles: 15°, 10°, 5° and 2.5°. For these angles, trajectories were respectively considered in the intervals of speeds between 15 m/s

Table 5 Characteristics of the 6 steepest trajectories of the Silver Fox-Like airplane. θ is the angle of climb in degrees, V_∞ is the constant speed in m/s, h_{max} is the maximum altitude attainable in m and Fuel is the amount of fuel used in this trajectory in *N*

θ	V_∞	h_{max}	Fuel
65	10	634	0.0810
55	10	847	0.109
45	15	2071	0.274
35	15	2160	0.296
25	15	5602	0.830
"	25	1305	0.188

Table 6 Characteristics of the 6 steepest trajectories of the Hercules-Like airplane. θ is the angle of climb in degrees, V_∞ is the constant speed in m/s, h_{max} is the maximum altitude attainable in m and Fuel is the amount of fuel used in this trajectory in *N*

θ	V_∞	h_{max}	Fuel
30	40	132	66
25	40	62	33
"	45	529	269
20	45	1965	1059
"	50	1342	698
"	55	676	343

and 35 m/s, 15 m/s and 40 m/s, 15 m/s to 45 m/s, and 15 m/s to 50 m/s, always by steps of 5 m/s. For all the trajectories studied, the maximum percent relative difference between the exact value of the fuel used and that calculated with the R-K one-step method is less than 0.047%. This value occurs with the trajectory at 2.5° and speed of 25 m/s in which the airplane reach the altitude of 9782 m. This corresponds to a discrepancy of 0.2 ml of fuel over the 463 ml used.

For the Hercules-Like airplane, 48 trajectories were examined: in all our tests the initial weight was W_1+133,358 N, which corresponds to about one half tank of fuel and no cargo; we considered these conditions because we wanted to test our formulas for climb angles that included rather steep angles. Table 6 shows the characteristics of the six steepest trajectories. The other trajectories studied were for four climb-angles: 15°, 10°, 5° and 2.5°. For these angles, trajectories were respectively constructed in the intervals of speeds between 45 m/s and 75 m/s, 45 m/s and 105 m/s, 45 m/s to 145 m/s, and 45 m/s to 165 m/s, by steps of 5 m/s in the first two sets and of 10 m/s in the last two sets. For all the trajectories studied, the maximum percent relative difference between the exact value of the fuel used and that produced by the R-K one-step method is less than 0.04%. This value occurs for the trajectory at 2.5° and speed of 75 m/s, where the airplane reaches a final altitude of 8884 m. This corresponds to a discrepancy of one half liter of fuel over the 1261 liters used.

5. Formulas for climb or descent at constant Mach number

In a flight at constant Mach number *M*, the speed of the aircraft varies as

$$V_\infty(h) = k\,T(h)^{1/2} \quad \text{with} \quad k = M\sqrt{\gamma R} \quad \text{being a constant.} \tag{27}$$

The equation for the altitude

$$\frac{dh}{dt} = V_\infty \sin(\theta) = k \sin(\theta)\sqrt{T_i - a_1(h - h_i)}\,. \tag{28}$$

can easily be solved and yields for *h* the following quadratic expression in *t*

$$h(t) = h_i + V_\infty(h_i)\sin(\theta)\,(t - t_i) - \frac{k^2 a_1}{4}\sin^2(\theta)\,(t - t_i)^2\,. \tag{29}$$

This expression is readily inverted to yield the time required to climb from h_i to h as

$$t = t_i + \frac{2\left[T(h_i)^{1/2} - T(h)^{1/2}\right]}{k\,a_1 \sin(\theta)} \tag{30}$$

Eqs. (27) and (28) imply that

$$\frac{dV_\infty}{dt} = -\frac{k\,a_1}{2}T(h)^{-1/2}\left[\frac{dh}{dt}\right] = -\frac{1}{2}k^2\,a_1 \sin(\theta) \quad = \text{a constant.} \tag{31}$$

Thus, the speed is a linear function of time

$$V_\infty(t) = V_\infty(t_i) - \frac{1}{2}k^2\,a_1 \sin(\theta)(t - t_i) \tag{32}$$

5.1 Fuel consumption

In Labonté (2011), it was shown that the weight of the airplane, when constant Mach number is considered, is given by the following equation

$$\left[\frac{dW}{dt}\right] = -\alpha T^{5.7433} - \beta T^{0.5}\,W - \delta T^{-4.7433}\,W^2 \tag{33}$$

with $\alpha = \dfrac{c\,S\,C_{D0}\,\rho_\infty(h_i)\,k^3}{2\,\eta\,T_i^{4.2433}}$, $\beta = \dfrac{c\,k \sin(\theta)}{\eta}\left[1 + \dfrac{k^2 a_1}{2g}\right]$, $\delta = \dfrac{2\,c\cos^2(\theta)\,T_i^{4.2433}}{\eta\,k\,\pi\,e\,AR\,S\,\rho_\infty(h_i)}$

Its solution is given by the same formula as that for the climb at constant speed, given in Eqs. (14) and (15), except that in the present case

$$d = 5.2433 \qquad A = \frac{\alpha\delta}{\beta^2} \qquad z = \frac{\beta}{a_1\,k \sin(\theta)}T\left[h(t)\right] \tag{34}$$

Fig. 6 shows the graph of the weight of fuel W_f as a function of time for the CP-1 airplane that starts with 425 N of fuel at sea level and climbs up to 2335 m, at an angle of 20°, while keeping a

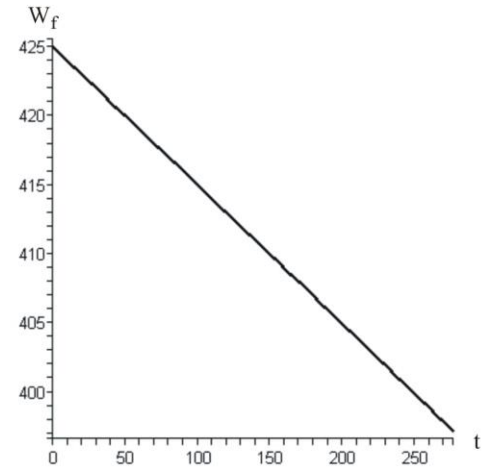

Fig. 6 Fuel of the CP-1 airplane as it climbs at 20°, with the initial speed of 25 m/s, up to 2335 m, with a constant Mach number of 0.0735.

constant Mach number, with initial speed of 25 m/s. Note that climbs at other angles yield a similarly looking curve.

5.2 Power and lift constraints

In all modes of climb, there will always be the same two constraints to satisfy for the power and the lift, as we discussed for climbs at constant speeds. For flights at constant Mach number, Eqs. (10) and (33) give the power required for a climb at angle θ, as

$$P_R = \frac{\eta}{c}\left[\alpha\, T^{5.7433} + \beta\, T^{0.5}\, W + \delta\, T^{-4.7433}\, W^2\right] \tag{35}$$

In order to visualize the constraint on the power, we can draw the graph that corresponds to that shown in Fig. 2 in which P_R is given by Eq. (35). In the case of a flight for the CP-1 airplane, that starts with 425 N of fuel, with the initial speed of 25 m/s at sea level and climbs at 20° with the constant Mach number of 0.0735, one then sees the two curves for P_r and or $P_{A\max}$ intersecting at about 2335 m, which indicates that the CP-1 airplane cannot climb higher that this altitude.

Just as in the climb at constant speed, there is also a constraint on the lift that comes from Eq. (4). With the flight mode presently considered V_{∞}, ρ_{∞} and W change with the altitude so that the angle of attack, i.e., the lift coefficient C_L must be constantly adjusted for Eq. (4) to keep on holding. The upper bound on the lift coefficient implies an upper bound on the lift that is translated into an upper bound on the altitude. For the particular flight of the CP-1 airplane mentioned above, the graph that corresponds to Fig. 3 will show that the curve for $W\cos(\theta)$ and that for $L_{\max}$ intersect at about 2781 m, indicating that, after this altitude, the airplane cannot provide enough lift to climb higher.

Thus, for the particular flight considered, the stronger constraint on the altitude comes from the upper bound on the power available. It indicates that the maximum altitude attainable is 2335 m. This altitude is reached after 4 minutes 37 seconds. The amount of fuel remaining at the end of this trajectory is 27.82 N.

5.3 Simpler formulas for the fuel used

Because the solution of the fuel equation for a climb at constant Mach number, Eq. (33), is the same as that for a climb at constant speed, Eq. (14), the formulas developed for the latter case will also apply to the present case. Thus, in particular, the Runge-Kutta formulas listed in Eq. (24) can be used to provide an approximate solution W_a, when using the function *F*, defined as

$$F(t, W) = -\alpha\, T(h(t))^{5.7433} - \beta\, T(h(t))^{0.5}\, W - \delta\, T(h(t))^{-4.7433}\, W^2 \qquad (36)$$

We have performed the same series of tests as for the climbs at constant speeds, in order to verify the precision of these formulas as compared to the exact values of $W(t)$. In each case, the climb starts at the sea level and the final altitude is the highest one possible, given the constraints on the power and the lift, mentioned in the above section on that subject. For the climbs at constant Mach number, the initial speeds considered were the same ones as the constant speeds used in the tests for climbs at constant speeds.

For the CP-1 airplane, in all the 32 trajectories examined, the maximum percent relative difference between the exact value of fuel used and that produced by the R-K one-step method is less than 0.04 %. This value occurs for the trajectory at 2.5° and speed of 40 m/s. This corresponds to a discrepancy of 13 ml of fuel over the 33 liters used.

For the Silver Fox-Like airplane, in all the 32 trajectories studied, the maximum percent relative difference between the exact value of W and that produced by the R-K one-step method is less than 0.067 %. This occurs for the trajectory at 2.5° and speed of 30 m/s in which the airplane reach the altitude of 10,000 m. This corresponds to a discrepancy of 0.3 ml of fuel over the 484 ml used. These results clearly show that the difference between the exact value of the fuel used and it approximate value is inconsequentially small.

For the Hercules-Like airplane, 48 trajectories were examined: in all our tests the maximum percent relative difference between the exact value of the fuel used and that produced by the R-K one-step method is less than 0.04%. This value occurs for the trajectory at 2.5° and speed of 75 m/s, where the airplane reaches a final altitude of 8830 m. This corresponds to a discrepancy of 541 ml of fuel over the 1311 liters used.

6. Formulas for climb or descent at constant angle of attack

In a flight at a constant angle of attack, Eq. (2) for *L* combined with Eq. (4) yield

$$V_\infty = k\left[\frac{W}{\rho_\infty}\right]^{1/2} \qquad \text{with} \qquad k = \left[\frac{2\cos(\theta)}{SC_L}\right]^{1/2} \quad \text{being a constant.} \qquad (37)$$

In Labonté (2011), Eq. (6) for the weight of the airplane was examined and it was shown to reduce to

$$\frac{dW}{dz} = KW \qquad (38)$$

with

$$z(t) = T[h(t)] \qquad \text{and} \qquad K = \frac{c}{\eta a_1}\left[\frac{C_D}{C_L}\cot(\theta) + 1\right] \text{ a constant.} \tag{39}$$

The solution of Eq. (38) is simply

$$W = W_i\, e^{K(z - z_i)}. \tag{40}$$

in which W_i is the initial weight of the airplane at time t_i, and $z_i = T[h_i]$ with h_i being the initial altitude. Since the rate of climb is $V_\infty \sin(\theta)$, Eqs (8), (37) and (40) imply

$$\frac{dh}{dt} = \frac{B}{a_1} T(h)^{-2.12165} \exp\left\{\frac{K\,T(h)}{2}\right\} \tag{41}$$

in which the constant B is defined as

$$B = a_1 V_\infty(t_i)\sin(\theta)\, T_i^{2.12165} \exp\left\{-\frac{KT_i}{2}\right\}. \tag{42}$$

Upon changing variables from h to $T(h)$, Eq. (41) becomes the following separable equation

$$T^{2.12165} \exp\left\{-\frac{KT}{2}\right\} dT = -B\, dt \tag{43}$$

This equation can be integrated to yield

$$\frac{1}{B}\left\{y(T_i) - y(T)\right\} = t \tag{44}$$

with $y(x) = \dfrac{x^{3.12165}}{3.12165} \exp\left\{-\dfrac{K\,x}{2}\right\} {}_1F_1\left(1, 4.12165, \dfrac{Kx}{2}\right).$

6.1 Power and lift constraints

From Eqs. (10), (38) and (39), one obtains the following expression for the power required for the motion

$$P_R = -\frac{\eta}{c}\frac{dW}{dt} = -\frac{\eta}{c}\left[\frac{dW}{dz}\right]\left[\frac{dz}{dt}\right] = \frac{\eta}{c} a_1 kK \sin(\theta)\left[\frac{W^3}{\rho_\infty}\right]^{1/2} \tag{45}$$

In order for a trajectory to be possible, the power required P_R must be smaller or equal to the maximum power the engine can provide for the airplane motion: P_{Amax}. Upon drawing the graph of both P_{Amax} and P_R as functions of h, as was done in Fig. 2, one can see how the available power limits the altitude. For example, in the case of the climbing airplane CP-1 at 10°, with a constant

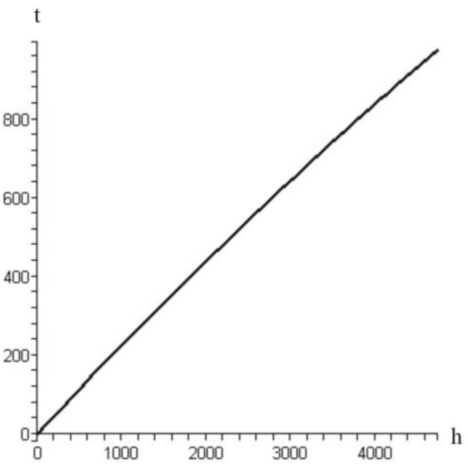

Fig. 7 Time as a function of altitude, for the CP-1 airplane climbing at 10°, starting at the speed of 25 m/s, up to 4748 m

angle of attack, and an initial speed of 25 m/s, P_R reaches $P_{A\max}$ at about 4748 m, so that the airplane cannot climb higher than this altitude. For this particular mode of flight, there is no explicit constraint on the lift to examine because Eq. (37) for the speed already insures that Eq. (4) will always hold.

6.2 Simpler formulas

Eq. (44) gives the time t in terms of the altitude h. However, it would be useful to have available the inverse relation that gives the altitude as a function of time. Unfortunately, Eq. (44) cannot be readily inverted. Furthermore, calculating t with Eq. (44) requires sophisticated computational power that would not presently be available on board smaller UAVs. For these reasons, we have endeavored to find a simpler formula than Eq. (44). We started by examining the graph of t as a function of h as produced with Eq. (44). Fig. 7 shows this graph for the CP-1 airplane climbing at 10° from sea level, with the initial speed of 25 m/s. We further observed that for all the other trajectories examined, the corresponding curve always had the same appearance as that in Fig. 7.

Given the behavior of this curve, one is inclined to try to approximate the value $h(t)$ by a quadratic expression, such as

$$h(t) = h_i + p(t - t_i) + q(t - t_i)^2 \tag{46}$$

in which p and q are constants. This formula already incorporates the initial condition that $h(t_i)=h_i$. Upon substituting this expression for $h(t)$ in the left-hand side of Eq. (41), one obtains

$$p + 2\, q\, (t - t_i) = F(h). \tag{47}$$

in which $F(h)$ represents the left-hand side of Eq. (41). The parameter p is readily determined by evaluating this equation at $t=t_i$, which yields

$$p = F(h_i) \tag{48}$$

Upon solving Eq. (46) for (t - t_i) in terms of h, and substituting the value obtained in Eq. (47), one obtains

$$q = \frac{F(h)^2 - F(h_i)^2}{4(h - h_i)} \tag{49}$$

Obviously, this equation cannot be satisfied for all values of h because its right-hand side is not constant. Thus a particular value has to be selected that will determine the value of the constant q. Unfortunately, there is no rule for selecting such a value. We have performed experiments that showed that if h_f is the final altitude at which Eq. (46) would be applied, the choice of the mid-point 0.5 (h_i+h_f) in Eq. (49) yields a very good value for q. Finally, we found that still better results can be obtained by taking q to be the average of the value Eq. (49) yields with h=0.5 (h_i+h_f) and h=0.75 (h_i+h_f). Once a value is obtained for q, Eq. (47) can be solved for the time in terms of the altitude, which yields

$$(t - t_i) = \frac{2(h - h_i)}{p + \sqrt{p^2 + 4q(h - h_i)}} \tag{50}$$

6.2.1 Example

Consider the trajectory described above for the CP-1 airplane that starts at sea level with the speed of 25 m/s, and climbs at 10° up to the limit imposed by its available power that is 4748 m. Its climb lasts 16 minutes and 16 seconds. The above procedure to compute the constant parameters a and b, at the end of the climb, yields:

p = 4.3412 and q = 0.5340e-3

Upon using Eq. (50) to calculate t_f, the time at which h_f=4748 m is reached, one obtains t_f=976.4248656 s, whereas, the exact value of t_f, according to Eq. (44) is 976.0306877 s. The difference between the two values is −0.3942 s and the relative error made in using the

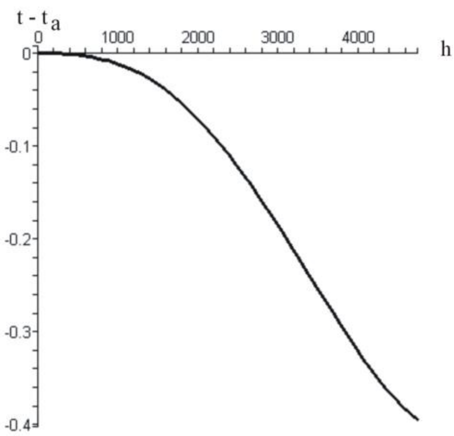

Fig. 8 The error $t(h)-t_a(h)$ as a function of the altitude h, for a climb of the CP-1 airplane at 10°, climbing from sea level with a speed of 25 m/s, up to 4748 m

approximate form is about −0.0404 %. Fig. 8 shows the graph of the difference $[t(h) - t_a(h)]$, where t is the exact value and t_a is the value given by Eq. (50)

We have performed equivalent series of tests as those for the climbs at constant speeds, and constant Mach number, to verify the precision of these formulas compared to the exact values. We have considered the three very different CP-1, Silver Fox-Like and Hercules-Like airplanes. In each case, the climb starts at the sea level and the final altitude is the highest one possible, given the constraints on the power and the lift, mentioned in the above section on that subject. For the climbs at constant Mach number, the initial speeds considered were the same ones as the constant speeds used in the tests for climbs at constant speeds. In all these tests, we have observed that the procedure described above produces a quadratic form for $h(t)$ that is accurate to a very good approximation and the resulting difference between the exact value $t(h)$ and the approximate value $t_a(h)$ should be negligible in most practical applications.

For the CP-1 airplane, in all the 32 trajectories examined, the maximum percent relative difference between the exact value of t_f and its approximate value is less than 0.084%. This occurs for the trajectory at 2.5° and speed of 25 m/s. This corresponds to a discrepancy of 5.4 seconds over the 1 hour, 47 minutes and 27 seconds that lasts the trajectory.

For the Silver Fox-Like airplane, in all the 32 trajectories studied, the maximum percent relative difference between the exact value of t_f and its approximate value is less than 0.227%. This occurs for the trajectory at 5° and speed of 15 m/s in which the airplane reach the altitude of 10,000 m. This corresponds to a discrepancy of 13.6 s over the 1 hour, 39 minutes and 57 seconds for the whole trajectory.

For the Hercules-Like airplane, 48 trajectories were examined: in all our tests the maximum percent relative difference between the exact value of t_f and its approximate value is less than 0.110 %. This occurs for the trajectory at 2.5° and speed of 55 m/s, where the airplane reaches a final altitude of 8830 m. This corresponds to a discrepancy of 3.3 s over the 49 minutes and 54 seconds for the whole trajectory.

7. Descending flights

The formulas obtained above, for the three flight mode considered, are also valid for descending flights, i.e., for trajectories for which the angle θ is negative. It can be verified that descents at any angle are possible in these flight modes. In such flights, gravity's pull has a component in the same direction as the engine thrust so it could eventually by itself cancel the drag. Because of this, care must be taken to keep the flight parameters in ranges such that the formula for the power required does not yield a negative value.

8. Conclusions

We have found approximations for the exact solutions to the equations of motion for airplanes that fly on rectilinear trajectories, inclined at an arbitrary angle, for three different modes of flight, namely flight at constant

- speed,
- Mach number,
- angle of attack.

For each of these modes, we have obtained formulas for the weight of fuel, speed, altitude, horizontal distance, and power required as a function of time, and the time to climb or descent at a given altitude. For the simplified formulas based on the Runge-Kutta method, we have shown their accuracy by calculating the Lotkin (1951) estimate for the truncation error. Furthermore, we have calculated the discrepancy in the values obtained with these simpler formulas and with the exact ones, for three very different reference airplanes on many representative trajectories with different speeds and inclinations. These discrepancies are small enough to be inconsequential; in actual situations, their deviation from the exact values will be smaller than those due to inhomogeneities in the atmosphere. These formulas are original, and constitute important tools for the analysis of airplane performances.

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EC

Nomenclature

a	=speed of sound in air. At altitude h,
$a(h)$	$=\sqrt{\gamma RT(h)}$. At sea level, $a(0)$=340.3029 m/s
a_1	=absolute value of the slope of the temperature as as function of altitude, below 11 km, $a_1=6.5\times10^{-3}$ K/m
AFR	=air fuel ratio (about 14.7)
AR	=aspect ratio=b^2/S
b	=wingspan
c	=specific fuel consumption in Newton per Watt-second, that is in m^{-1}
C_D	=global drag coefficient for the aircraft $= C_{D0}+\frac{C_L^2}{\pi eAR}$ (Drag polar)
C_{D0}	=drag coefficient at zero lift
C_L	=global lift coefficient for the global aircraft
D	=drag=$\frac{1}{2}\rho_\infty SC_D V_\infty^2$
e	=Oswald's efficiency factor
p	=power of the engine in Watt
R	=specific gas constant for air=287.058 J/(kg K)
S	=wing area
t	=time variable
T_s	=temperature at sea level=288.16 K
T	=temperature
v_3	=vertical component of airplane velocity
V_∞	=airplane speed with respect to the undisturbed air in front of it
W	=weight of the airplane
W_1	=weight of the airplane without fuel
W_f	=total weight of fuel at the time of departure
W_0	$=W_1+W_f$=total weight of the airplane at departure

g =gravitational constant=9.8 m/s^2

h =altitude of airplane

L =lift=$\frac{1}{2}\rho_\infty S C_L V_\infty^2$

M =Mach number=$V/a(h)$

γ =ratio of the constant pressure specific heat to the constant volume specific heat=c_p/c_v=1.4 for air

η =propeller efficiency

ρ_s =air density at sea level=1.225 kg/m^3

ρ_∞ =density of undisturbed air in front of airplane