Advances in Aircraft and Spacecraft Science, *Vol. 4, No. 3 (2017) 335-351*
DOI: http://dx.doi.org/10.12989/aas.2017.4.3.335

Attitude estimation: with or without spacecraft dynamics?

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(Received March 8, 2016, Revised June 21, 2016, Accepted November 1, 2016)

Abstract. Kalman filter based spacecraft attitude estimation has been used in many space missions and has been widely discussed in literature. While some models in spacecraft attitude estimation include spacecraft dynamics, most do not. To our best knowledge, there is no comparison on which model is a better choice. In this paper, we discuss the reasons why spacecraft dynamics should be considered in the Kalman filter based spacecraft attitude estimation problem. We also propose a reduced quaternion spacecraft dynamics model which admits additive noise. Geometry of the reduced quaternion model and the additive noise are discussed. This treatment is easier in computation than the one with full quaternion. Simulations are conducted to verify our claims.

Keywords: extended Kalman filter; linearization; spacecraft attitude estimation

1. Introduction

The Kalman filter found its earliest applications in some high-profile missions in the aerospace industry, such as the Apollo project McGee *et al*. (1985). Spacecraft attitude estimation has been a major research area since the Kalman filter was invented Lefferts *et al*. (1982). Although many different methods have been proposed, most models suggest using only quaternion kinematics equations of motion for the attitude estimation without considering spacecraft dynamics. See, for example, some widely cited survey papers Lefferts *et al*. (1982), Crassidis *et al*. (2007) and references therein. This model reduces the problem size but discards useful spacecraft attitude information available in the spacecraft dynamics equation. The drawbacks of this simplified model are (a) when gyros measurements have significant noise, the spacecraft dynamics information is not used to prevent the degradation of the attitude estimation, and (b) when gyro measurements are not available (as a matter of fact, gyros are not used in most small spacecraft, for example, Stoltz *et al*. (1998)), the simplified model cannot be used to estimate the spacecraft attitude. Moreover, the simplified model without the spacecraft dynamics cannot estimate the attitude rates even through the gyros measurements are used in the estimation. In contrast, the attitude rates can be estimated directly by the Kalman filter with the spacecraft dynamics. There are papers that consider models including the spacecraft dynamics in Kalman filter designs, for example, Lovera

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http://www.techno-press.org/?journal=aas&subpage=7 ISSN: 2287-528X (Print), 2287-5271 (Online)

et al. (2002), Khan *et al*. (2001). But to our best knowledge, there is no discussion of which model is a better fit to the application of spacecraft attitude estimation, and there is no performance comparison for Kalman filters using the two different models. In this paper, we will discuss the importance of the spacecraft dynamics to the attitude estimation problem and examine the performance difference between models that incorporate spacecraft dynamics and models that do not. As it is well-known that the models for the attitude estimation and for spacecraft dynamics are nonlinear, some natural choices for solving the estimation problem are either extended Kalman filter (EKF) or unscented Kalman filter (UKF).

We recognize the recent trend of using an unscented Kalman filter instead of the extended Kalman filter in spacecraft attitude estimation problem Julier *et al*. (2000), Cheon *et al*. (2007), Crassidis *et al*. (2003). However we are also aware of some simulation comparison between the two methods and different opinions about the potential advantages of unscented Kalman filter performed by LaViola (2003). Given the facts that (a) it is not clear which filter is better and (b) EKF is computational cheaper than UKF, we will consider only the extended Kalman filter in this paper.

A special feature of the spacecraft attitude estimation problem is that the quaternion has a norm constraint, and many methods have been proposed to deal with this constraint Markley (2003), Zanetti *et al*. (2009), Persson *et al*. (2013), Forbes *et al*. (2014). These methods are more complicated in concept and more expensive in computation than traditional EKF without the norm constraint. Therefore, we suggest using a reduced quaternion model which does not need the norm constraint Yang (2010, 2014). The drawback of using reduced quaternion is that it has a singular point in this reduced quaternion model. Since this singular point is the farthest point from the equilibrium point Yang (2010), the reduced quaternion model should be a good choice for normal mode control system design which controls the attitude to align with a reference frame.

The remainder of the paper is organized as follows. Section 2 provides a description of the extended Kalman filter for spacecraft attitude estimation that follows common practice, i.e., using a model without spacecraft dynamics. Section 3 provides a parallel description of the extended Kalman filter for spacecraft attitude estimation that is our vision, i.e., using a model with spacecraft dynamics. The merits of the proposed model over commonly used models are discussed. Simulations and results for these two methods are presented in Section 4 to demonstrate the superiority of using a model with spacecraft dynamics. The conclusions are summarized in Section 5.

2. Extended Kalman filter without spacecraft dynamics

This type of model is widely used in literature (see Lefferts (1982)) for spacecraft attitude estimation and can be expressed as follows. Let $q_0 = \cos\left(\frac{\alpha}{2}\right)$, $q^T = [q_1, q_2, q_3]^T = \hat{e}^T \sin\left(\frac{\alpha}{2}\right)$, and

$$\bar{q} = [q_0 \quad q^T]^T \tag{1}$$

be the quaternion that represents the rotation of the body frame relative to the inertial frame, where $\hat{e}$ is the unit vector of the rotational axis and α is the rotational angle; the rate of change of the quaternion is given by Wertz (1978)

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{2}$$

where $\omega_I=[\omega_1 \quad \omega_2 \quad \omega_3]^T$ is the body rotational rate with respect to the inertial frame represented in the body frame. However, using this full quaternion model introduces a singularity in the covariance matrix (see Lefferts *et al*. (1982)). Therefore, we suggest using a reduced representation derived in Yang (2014) given as follows.

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_1) \tag{3}$$

where ϕ_1 is the process noise, and Ω is a matrix given by

$$\Omega = \begin{pmatrix} g(q) & -q_3 & q_2 \\ q_3 & g(q) & -q_1 \\ -q_2 & q_1 & g(q) \end{pmatrix} \tag{4}$$

with $g(q) = \sqrt{1 - q_1^2 - q_2^2 - q_3^2}$. The reduced model embeds the unit length requirement in $g(q)$ which means that there is no need to consider the unit length constraint in EKF as it was treated in Zanetti (2009). This model therefore significantly simplifies the problem. In Yang (2014), it is shown that a reduced quaternion model has other merits: it admits an analytic LQR design, and the LQR design globally stabilizes the original nonlinear spacecraft system. Assuming that three rate gyros and quaternion measurement sensors are installed on board, the measurement equation can be written as Crassidis *et al*. (2003)

$$\begin{aligned} \dot{\beta} &= \phi_2 \\ \omega_y &= \omega + \beta + \psi_2 \\ q_y &= q + \psi_1 \end{aligned} \tag{5}$$

where β is a drift in the angular rate measurement, ϕ_2 is the process noise, ω_y is the angular rate measurement obtained from gyros, q_y is the quaternion measurement (which can be obtained by using QUEST method by Shuster (1981) or analytic method by Yang *et al*. (2014) for measurements of astronomical vectors, such as sun sensor, magnetometer, gravitometer, and star trackers), and ψ_1 and ψ_2 are measurement noise.

The reduced quaternion geometry of q_y can be seen from the following argument. The noise ψ_2 can be viewed as a reduced rotational quaternion whose rotational axis is $\psi_2/\ \|\psi_2\|$ and rotational angle meets the condition $\sin\left(\frac{\delta}{2}\right) = \|\psi_2\|$. For small noise quaternion ψ_2 and the quaternion $q = \hat{e}\sin\left(\frac{\alpha}{2}\right)$ which is bounded away from the singular point $\alpha=\pi$ ($\|q\|<1$), we can see that

$q_y = q + \psi_2 = \frac{q_y}{\|q_y\|}\sin\left(\frac{\alpha}{2}+\Delta\right)$ is a quaternion whose rotational axis is $\frac{q_y}{\|q_y\|}$, a perturbation ψ_2 over q satisfying $\| q_y \| \le \| q \| + \| \psi_2 \|$ and $\|q_y\| \le 1$ (where $\|\psi_2\|$ is small), and the rotational angle around q_y is $\left(\frac{\alpha}{2}\right)+\Delta$ and Δ is small because $\|\psi_2\|$ is small. Therefore, the mathematical treatment for this model is much easier than the multiplicative perturbation model.

Let $x = \begin{bmatrix} q \\ \beta \end{bmatrix}$ and $u = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$. We can rewrite the system in a compact form

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0.5\Omega\omega \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5\Omega\phi_1 \\ \phi_2 \end{bmatrix} := f(x,u) + g(x,u)\phi \tag{6a}$$

$$y = \begin{bmatrix} q_y \\ \omega_y \end{bmatrix} = \begin{bmatrix} q \\ \omega+\beta \end{bmatrix} + \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} := h(x,u) + \psi \tag{6b}$$

where

$$f(x,u) = \begin{bmatrix} 0.5\Omega\omega \\ 0 \end{bmatrix}, \quad g(x,u) = \begin{bmatrix} 0.5\Omega & 0 \\ 0 & I_3 \end{bmatrix}, \quad h(x,u) = \begin{bmatrix} q \\ \omega+\beta \end{bmatrix}.$$

The simplest discrete version of (6) can be obtained by explicit Euler's method. However, the discrete formula obtained by this method as pointed in Stoer *et al*. (1993) is normally not stable for stiff differential equations. In Zanetti *et al*. (2009), the trapezoidal implicit method was proposed. But this method as pointed in Stoer *et al*. (1993) involves the solution of nonlinear system of equations which can be very expensive in computation. We suggest using the linearly implicit Euler method described in Sanda *et al*. (2013), Hairer *et al*. (2014). Let dt be the sampling time period and

$$X = \left(I - dt \begin{bmatrix} -\frac{q_1\omega_1}{2g(q)} & \frac{\omega_3}{2} - \frac{q_2\omega_1}{2g(q)} & -\frac{\omega_2}{2} - \frac{q_3\omega_1}{2g(q)} \\ -\frac{\omega_3}{2} - \frac{q_1\omega_2}{2g(q)} & -\frac{q_2\omega_2}{2g(q)} & \frac{\omega_1}{2} - \frac{q_3\omega_2}{2g(q)} \\ \frac{\omega_2}{2} - \frac{q_1\omega_3}{2g(q)} & -\frac{\omega_1}{2} - \frac{q_2\omega_3}{2g(q)} & -\frac{q_3\omega_3}{2g(q)} \end{bmatrix} \right) \tag{7}$$

The discrete version of (6) is therefore given as follows

$$q_{k+1} = q_k + X^{-1}\left(.5\Omega_k\omega_k + .5\Omega_k\phi_{1k}\right)dt \tag{8a}$$

$$\beta_{k+1} = \beta_k + \phi_{2k}dt \tag{8b}$$

$$y_k = \begin{bmatrix} q_{yk} \\ \omega_{yk} \end{bmatrix} = \begin{bmatrix} q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} 0_1 \\ \omega_k \end{bmatrix} + \begin{bmatrix} \psi_{1k} \\ \psi_{2k} \end{bmatrix} = x_k + u_k + \psi_k = H(x_k, u_k) + \psi_k \tag{8c}$$

where $H(x_k, u_k) = x_k + u_k$.

As always, we assume that ϕ_k and ψ_k are white noise signals and the following relations hold

$$E(\phi_k) = 0\,,\quad E(\psi_k) = 0\,,\quad \forall k\,; \tag{9a}$$

$$E(\phi_k \phi_k^T) = Q_k\,,\quad E(\psi_k \psi_k^T) = R_k\,,\; E(\phi_i \psi_j^T) = 0 \;\; \forall i, j, k\,; \tag{9b}$$

$$E(\phi_j \phi_k^T) = 0\,,\quad E(\psi_j \psi_k^T) = 0\,,\quad \forall j \neq k\,; \tag{9c}$$

We need some explicit expression of (8a) to obtain the formulas of the extended Kalman filter. Note that (7) can be simplified as

$$X(q,\omega,dt) = dt \begin{bmatrix} \frac{1}{dt} + \frac{q_1\omega_1}{2g(q)} & -\frac{\omega_3}{2} + \frac{q_2\omega_1}{2g(q)} & \frac{\omega_2}{2} + \frac{q_3\omega_1}{2g(q)} \\ \frac{\omega_3}{2} + \frac{q_1\omega_2}{2g(q)} & \frac{1}{dt} + \frac{q_2\omega_2}{2g(q)} & -\frac{\omega_1}{2} + \frac{q_3\omega_2}{2g(q)} \\ -\frac{\omega_2}{2} + \frac{q_1\omega_3}{2g(q)} & \frac{\omega_1}{2} + \frac{q_2\omega_3}{2g(q)} & \frac{1}{dt} + \frac{q_3\omega_3}{2g(q)} \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} & \bar{c} \\ \bar{d} & \bar{e} & \bar{f} \\ \bar{g} & \bar{h} & \bar{i} \end{bmatrix} dt$$

Therefore,

$$X^{-1}(q,\omega,dt) = \frac{1}{dt(\bar{a}\bar{A} + \bar{b}\bar{B} + \bar{c}\bar{C})} \begin{bmatrix} \bar{A} & \bar{B} & \bar{C} \\ \bar{D} & \bar{D} & \bar{F} \\ \bar{G} & \bar{H} & \bar{I} \end{bmatrix}.$$

where

$$\bar{A} = (\bar{e}\bar{i} - \bar{f}\bar{h}),\quad \bar{B} = (\bar{f}\bar{g} - \bar{d}\bar{i}),\quad \bar{C} = (\bar{d}\bar{h} - \bar{e}\bar{g}),$$

$$\bar{D} = (\bar{c}\bar{h} - \bar{b}\bar{i}),\quad \bar{E} = (\bar{a}\bar{i} - \bar{c}\bar{g}),\quad \bar{F} = (\bar{b}\bar{g} - \bar{a}\bar{h}),$$

$$\bar{G} = (\bar{b}\bar{f} - \bar{c}\bar{e}),\quad \bar{H} = (\bar{c}\bar{d} - \bar{a}\bar{f}),\quad \bar{I} = (\bar{a}\bar{e} - \bar{b}\bar{d}),$$

which leads to

$$\frac{1}{2} X^{-1}(q,\omega,dt)\Omega\omega dt = \begin{bmatrix} \bar{w}(q,\omega) \\ \bar{u}(q,\omega) \\ \bar{v}(q,\omega) \end{bmatrix} \tag{10}$$

$$= \frac{1}{2(\bar{a}\bar{A} + \bar{b}\bar{B} + \bar{c}\bar{C})} \begin{bmatrix} (\bar{A}g + \bar{D}q_3 - \bar{G}q_2)\omega_1 + (-\bar{A}q_3 + \bar{D}g + \bar{G}q_1)\omega_2 + (\bar{A}q_2 - \bar{D}q_1 + \bar{G}g)\omega_3 \\ (\bar{B}g + \bar{E}q_3 - \bar{H}q_2)\omega_1 + (-\bar{B}q_3 + \bar{E}g + \bar{H}q_1)\omega_2 + (\bar{B}q_2 - \bar{E}q_1 + \bar{H}g)\omega_3 \\ (\bar{C}g + \bar{F}q_3 - \bar{I}q_2)\omega_1 + (-\bar{C}q_3 + \bar{F}g + \bar{I}q_1)\omega_2 + (\bar{C}q_2 - \bar{F}q_1 + \bar{I}g)\omega_3 \end{bmatrix}.$$

and

$$\frac{1}{2}X^{-1}(q,\omega,dt)\phi_1\omega dt = \begin{bmatrix} \bar{x}(q,\omega) \\ \bar{y}(q,\omega) \\ \bar{z}(q,\omega) \end{bmatrix} \tag{11}$$

$$= \frac{1}{2(\bar{a}\bar{A}+\bar{b}\bar{B}+\bar{c}\bar{C})}\begin{bmatrix} (\bar{A}g+\bar{D}q_3-\bar{G}q_2)\phi_{11}+(-\bar{A}q_3+\bar{D}g+\bar{G}q_1)\phi_{12}+(\bar{A}q_2-\bar{D}q_1+\bar{G}g)\phi_{13} \\ (\bar{B}g+\bar{E}q_3-\bar{H}q_2)\phi_{11}+(-\bar{B}q_3+\bar{E}g+\bar{H}q_1)\phi_{12}+(\bar{B}q_2-\bar{E}q_1+\bar{H}g)\phi_{13} \\ (\bar{C}g+\bar{F}q_3-\bar{I}q_2)\phi_{11}+(-\bar{C}q_3+\bar{F}g+\bar{I}q_1)\phi_{12}+(\bar{C}q_2-\bar{F}q_1+\bar{I}g)\phi_{13} \end{bmatrix}.$$

Let

$$F_{k-1} = \begin{bmatrix} I_3 + \frac{\partial}{\partial q}\begin{bmatrix} \bar{w}(q,\omega) \\ \bar{u}(q,\omega) \\ \bar{v}(q,\omega) \end{bmatrix} & 0_3 \\ 0_3 & I_3 \end{bmatrix}_{\hat{x}_{k-1|k-1}} \tag{12}$$

$$L_{k-1} = \begin{bmatrix} \frac{\partial}{\partial \phi_1}\begin{bmatrix} \bar{x}(q,\omega) \\ \bar{y}(q,\omega) \\ \bar{z}(q,\omega) \end{bmatrix} & 0_3 \\ 0_3 & dtI_3 \end{bmatrix}_{\hat{x}_{k-1|k-1},u_{k-1}} = \begin{bmatrix} \frac{1}{2}X^{-1}(q,\omega,dt)\Omega_{k-1}dt & 0_3 \\ 0_3 & dtI_3 \end{bmatrix}_{\hat{x}_{k-1|k-1},u_{k-1}} \tag{13}$$

and

$$H_k = \left.\frac{\partial H}{\partial x}\right|_{\hat{x}_{k-1|k-1},u_{k-1}} = I\,. \tag{14}$$

The extended Kalman filter iteration is as follows

$$\hat{x}_{k|k-1} = F(\hat{x}_{k-1|k-1},u_{k-1}) \tag{15a}$$

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^T + L_{k-1}Q_{k-1|k-1}L_{k-1}^T \tag{15b}$$

$$\tilde{y}_k = y_k - H(\hat{x}_{k|k-1}) \tag{15c}$$

$$S_k = P_{k|k-1} + R_k \tag{15d}$$

$$K_k = P_{k|k-1}S_k^{-1} \tag{15e}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k\tilde{y}_k \tag{15f}$$

$$P_{k|k} = (I-K_k)P_{k|k-1} \tag{15g}$$

Since u_k is not available, it is suggested in Lefferts *et al*. (1982) to set $u_k = \hat{\omega}_{k|k} = \omega_{y_k} - \hat{\beta}_{k|k}$

and $\widehat{\omega}_{k|k-1} = \widehat{\omega}_{k-1|k-1}$.

Clearly, the extended Kalman filter using this model cannot be updated without three dimensional gyro measurements ω_{y_k}. Moreover, the simplified model without the spacecraft dynamics cannot estimate the attitude rates because the rates are not in the state variable of the filter. In the next section, we will show that even if the gyro measurements are available, using this model is not as good as using a model which incorporates the spacecraft dynamics. In section 4, we will use simulation to compare the performance of two different methods to support our claim.

To improve the estimation accuracy of $\widehat{x}_{k|k-1}$, we can reduce the step size of *dt*. But in some applications, the measurements may be available only after several sampling period. In this case, a multi-rate Kalman filter should be considered (see Yang 2006), which is beyond the scope of this paper.

3. Extended Kalman filter with spacecraft dynamics

Using the method in Yang (2010, 2014), we can write this type of model as follows.

$$\dot{\omega} = -J^{-1}\omega \times J\omega + J^{-1}u + \phi_1 \tag{16a}$$

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2) \tag{16b}$$

$$y = h(\omega, q) + \psi \tag{16c}$$

where $[\omega^T, q^T]^T$ is the state vector, u is the control torque, y is the measurement vector, $\phi = [\phi_1^T, \phi_2^T]^T$ is the process Gaussian noise which models various disturbance torques, ψ is the measurement Gaussian noise, J is the inertia matrix of the spacecraft, and Ω is defined in (4). The control torques are in general known, for example, given the measured geomagnetic vector m and the current applied to the magnetic torque rods, the control torque can be calculated by the method of Shinde *et al*. (2016). Depending on the design, we may have angular rate measurements ω_y and quaternion measurement q_y; or we may have only quaternion measurement q_y. Assuming that three gyros and quaternion measurement sensors are installed on board, then the measurement equation can be written as in Crassidis *et al*. (2003)

$$\begin{aligned} \dot{\beta} &= \phi_3 \\ \omega_y &= \omega + \beta + \psi_1 \\ q_y &= q + \psi_2 \end{aligned} \tag{17}$$

where β is a drift in the angular rate measurement, ϕ_3 is the process noise, ω_y is the angular rate measurement, q_y is the quaternion measurement, and ψ_1 and ψ_2 are measurement noise. The overall system equations are given as follows

$$\dot{\omega} = -J^{-1}\omega \times J\omega + J^{-1}u + \phi_1 \tag{18a}$$

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2) \tag{18b}$$

$$\dot{\beta} = \phi_3 \tag{18c}$$

$$\omega_y = \omega + \beta + \psi_1 \tag{18d}$$

$$q_y = q + \psi_2 \tag{18e}$$

which can be rewritten as a standard state space model as follows

$$\dot{x} = f(x,u) + \phi \tag{19a}$$

$$y = Hx + \psi \tag{19b}$$

where $x = [\omega^T, q^T, \beta^T]^T$, $y = [\omega_y^T, q_y^T]^T$, $\phi = [\phi_1^T, \phi_2^T, \phi_3^T]^T$, $\psi = [\psi_1^T, \psi_2^T]^T$, and

$$H = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \end{bmatrix}.$$

The discrete version of (18) is given by

$$\begin{bmatrix} \omega_{k+1} \\ q_{k+1} \\ \beta_{k+1} \end{bmatrix} = \left(\begin{bmatrix} \omega_k \\ q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} -J^{-1}\omega_k \times J\omega_k + J^{-1}u_k \\ \frac{1}{2}\Omega_k \omega_k \\ 0 \end{bmatrix} dt \right) + \begin{bmatrix} \phi_{1k} \\ \frac{1}{2}\Omega_k \phi_{2k} \\ \phi_{3k} \end{bmatrix} dt \tag{20a}$$

$$:= F(x_k, u_k) + G(x_k, u_k)\phi_k$$

$$\begin{bmatrix} \omega_{yk} \\ q_{yk} \end{bmatrix} = \begin{bmatrix} I & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \omega_k \\ q_k \\ \beta_k \end{bmatrix} + \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} := H(x_k) + \psi_k \tag{20b}$$

where

$$\Omega_k = \begin{pmatrix} g(q_k) & -q_{3k} & q_{2k} \\ q_{3k} & g(q_k) & -q_{1k} \\ -q_{2k} & q_{1k} & g(q_k) \end{pmatrix} \tag{21}$$

Note that for two vectors $w=[w_1, w_2, w_3]^T$ and $v=[v_1, v_2, v_3]^T$, the cross product of $w \times v$ $w \times v$ can be written as the product of matrix $w^\times$ and vector v where

$$w^\times = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

We also assume ϕ_k and ψ_k are white noise signals satisfying Eq. (9). For

$$F_1(x,u) = (-J^{-1}\omega_k \times J\omega_k + J^{-1}u_k)dt + \omega_k,$$

we have

$$\frac{\partial}{\partial x}F_1(x,u)=\left[I-J^{-1}(\omega_k^{\times}J-J\omega_k^{\times})dt \quad 0 \quad 0\right].$$

For $F_2(x,u)=\frac{1}{2}\Omega_k\omega_k dt+q_k$, we have

$$\frac{\partial}{\partial x}F_2(x,u)=\begin{bmatrix}\frac{\partial}{\partial\omega}F_2 & \frac{\partial}{\partial q}F_2 & 0\end{bmatrix},$$

with

$$\frac{\partial}{\partial\omega}F_2=\frac{1}{2}\Omega dt, \tag{22}$$

and

$$\frac{\partial}{\partial q}F_2=\begin{bmatrix}\frac{1}{dt}-\frac{q_1\omega_1}{2g(q)} & \frac{\omega_3}{2}-\frac{q_2\omega_1}{2g(q)} & -\frac{\omega_2}{2}-\frac{q_3\omega_1}{2g(q)}\\ -\frac{\omega_3}{2}-\frac{q_1\omega_2}{2g(q)} & \frac{1}{dt}-\frac{q_2\omega_2}{2g(q)} & \frac{\omega_1}{2}-\frac{q_3\omega_2}{2g(q)}\\ \frac{\omega_2}{2}-\frac{q_1\omega_3}{2g(q)} & -\frac{\omega_1}{2}-\frac{q_2\omega_3}{2g(q)} & \frac{1}{dt}-\frac{q_3\omega_3}{2g(q)}\end{bmatrix}dt. \tag{23}$$

For $F_3(x,u)=\beta_k$, we have

$$\frac{\partial}{\partial x}F_3(x,u)=\left[0 \quad 0 \quad I_3\right].$$

Therefore

$$F_{k-1}:=\frac{\partial}{\partial x}F_1(x,u)\Big|_{\hat{x}_{k-1|k-1}}=\begin{bmatrix}I-J^{-1}(\omega_k^{\times}J-J\omega_k^{\times})dt & 0 & 0\\ \frac{\partial}{\partial\omega}F_2 & \frac{\partial}{\partial q}F_2 & 0\\ 0 & 0 & I_3\end{bmatrix}_{\hat{x}_{k-1|k-1}} \tag{24}$$

Let

$$L_{k-1}:=\frac{\partial}{\partial\phi_k}G(x,u)\Big|_{\hat{x}_{k-1|k-1},u_{k-1}}=\begin{bmatrix}I & 0 & 0\\ 0 & \frac{1}{2}\Omega_{k-1} & 0\\ 0 & 0 & I_3\end{bmatrix}_{\hat{x}_{k-1|k-1}} \tag{25}$$

The extended Kalman filter iteration is as follows

$$\hat{x}_{k|k-1}=F(\hat{x}_{k-1|k-1},u_{k-1}) \tag{26a}$$

$$P_{k|k-1}=F_{k-1}P_{k-1|k-1}F_{k-1}^T+L_{k-1}Q_{k-1|k-1}L_{k-1}^T \tag{26b}$$

$$\tilde{y}_k = y_k - H\hat{x}_{k|k-1} \tag{26c}$$

$$S_k = HP_{k|k-1}H^T + R_k \tag{26d}$$

$$K_k = P_{k|k-1}H^T S_k^{-1} \tag{26e}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k \tag{26f}$$

$$P_{k|k} = (I - K_k H)P_{k|k-1} \tag{26g}$$

The beauty of the Kalman filter using spacecraft dynamics can be seen from (26f). The best estimation is composed of two parts. The first part is a prediction $\hat{x}_{k|k-1}$ which is based on the spacecraft dynamics and the inertia matrix information for the specific spacecraft. The second part is a correction $\tilde{y}_k$ which is based on observations. The filter gain K_k is constantly adjusted such that (a) if the measurement noise is higher, the gain is reduced so that the estimation depends more on the system dynamics model, and (b) if measurement noise is lower, the gain is increased so that the estimation depends more on the measurement. That is the reason why spacecraft dynamics should be included in the attitude estimation problem even if angular rate measurements are available. The attitude rates can be estimated directly by the Kalman filter with the spacecraft dynamics.

As mentioned before, the Kalman filter with spacecraft dynamics works without the (gyro) measurement of spacecraft angular velocity vector with respect to the inertial frame. In this case, gyro measurement drift β does not exist. Therefore, the continuous system (18) is reduced to

$$\dot{\omega} = -J^{-1}\omega \times J\omega + J^{-1}u + \phi_1 \tag{27a}$$

$$\dot{q} = \frac{1}{2}\Omega(\omega + \phi_2) \tag{27b}$$

$$q_y = q + \psi \tag{27c}$$

We still use (19) for this system but $x=[\omega^T, q^T]^T$, $y=q_y$, $\phi = [\phi_1^T, \phi_2^T]^T$, $\psi=\psi$ and $C=[0\ I]$. The discrete version of (27) is given by

$$\begin{aligned}\begin{bmatrix}\omega_{k+1}\\ q_{k+1}\end{bmatrix} &= \left(\begin{bmatrix}\omega_k\\ q_k\end{bmatrix} + \begin{bmatrix}-J^{-1}\omega_k \times J\omega_k + J^{-1}u_k\\ \frac{1}{2}\Omega_k\omega_k\end{bmatrix}dt\right) + \begin{bmatrix}\phi_{1k}\\ \frac{1}{2}\Omega_k\phi_{2k}\end{bmatrix}dt \\ &:= F(x_k, u_k) + G(x_k, u_k)\phi_k\end{aligned} \tag{28a}$$

$$q_{yk} = \begin{bmatrix}0 & I\end{bmatrix}\begin{bmatrix}\omega_k\\ q_k\end{bmatrix} + \psi_k := Hx_k + \psi_k \tag{28b}$$

where Ω_k is the same as in (21). We also assume ϕ_k and ψ_k are white noise signals satisfying equations (9). For

$$F_1(x,u) = (-J^{-1}\omega_k \times J\omega_k + J^{-1}u_k)dt + \omega_k,$$

we have

$$\frac{\partial}{\partial x}F_1(x,u)=\left[I-J^{-1}(\omega_k^{\times}J-J\omega_k^{\times})dt \quad 0\right].$$

For $F_2(x,u)=\frac{1}{2}\Omega_k\omega_k dt+q_k$, we have

$$\frac{\partial}{\partial x}F_2(x,u)=\begin{bmatrix}\frac{\partial}{\partial \omega}F_2 & \frac{\partial}{\partial q}F_2\end{bmatrix},$$

with $\frac{\partial}{\partial \omega}F_2$ and $\frac{\partial}{\partial q}F_2$ the same as (22) and (23). Therefore

$$F_{k-1}:=\frac{\partial}{\partial x}F_1(x,u)\bigg|_{\hat{x}_{k-1|k-1}}=\begin{bmatrix}I-J^{-1}(\omega_k^{\times}J-J\omega_k^{\times})dt & 0\\ \frac{\partial}{\partial \omega}F_2 & \frac{\partial}{\partial q}F_2\end{bmatrix}_{\hat{x}_{k-1|k-1}} \tag{29}$$

Let

$$L_{k-1}:=\frac{\partial}{\partial \phi_k}G(x,u)\bigg|_{\hat{x}_{k-1|k-1},u_{k-1}}=\begin{bmatrix}I & 0\\ 0 & \frac{1}{2}\Omega_{k-1}\end{bmatrix}_{\hat{x}_{k-1|k-1}} \tag{30}$$

The extended Kalman filter will be the same as (26).

4. Simulation test

The extended Kalman filters with and without spacecraft dynamics have been implemented in Simulink to assess their performances. The inertia matrix J of the spacecraft in the simulation has the following values taken from Zhou (2005):

$$J=\begin{pmatrix}1200 & 100 & -200\\ 100 & 2200 & 300\\ -200 & 300 & 3100\end{pmatrix}$$

The unit of J is kg-m^2. The state and measurement noise variance matrices Q_k and R_k are positive definite and represent the noise magnitudes of the angular and angular rate in state dynamics and measurement instruments. While the dimensions of Q_k in the extended Kalman filters (with or without spacecraft dynamics) are different, R_k is the same for both filters and given by

$$R_k=0.1I_6$$

where I_6 is a 6×6 identity matrix. State dynamics noise Q_k for the filter without spacecraft dynamics is given by

$$Q_k=\begin{bmatrix}0.4I_3 & -0.004I_3\\ -0.004I_3 & 0.4I_3\end{bmatrix}$$

For the filter with spacecraft dynamics, Q_k is given by a similar but different dimensional matrix

$$Q_k = \begin{bmatrix} 0.00005 I_3 & 0 & 0 \\ 0 & 0.4 I_3 & -0.004 I_3 \\ 0 & -0.004 I_3 & 0.4 I_3 \end{bmatrix}$$

The initial values of the states $\hat{x}_{0|0}$ and the covariance $P_{0|0}$ are set to zeroes. The true and estimated quaternions for the Kalman filters with and without spacecraft dynamics are shown in Fig. 1 through Fig. 4.

The test case shown in Fig. 1 through Fig. 4 represents the filter performance with the spacecraft undergoing a fast attitude slew. The torque for the maneuver is a sine wave as a

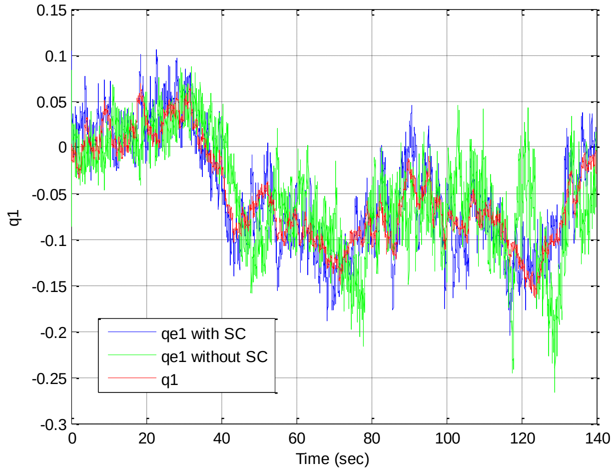

Fig. 1 The first component of the estimated and true quaternion

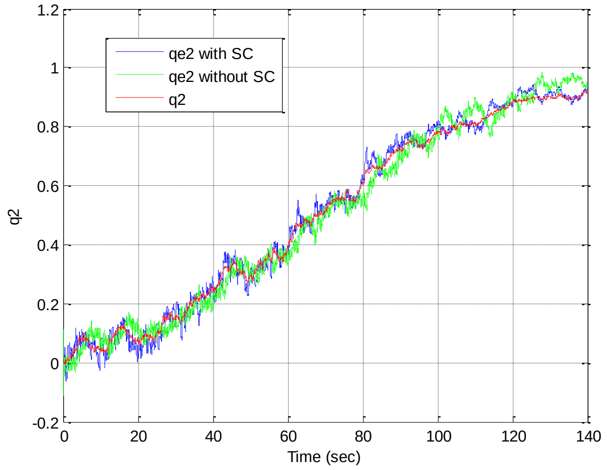

Fig. 2 The second component of the estimated and true quaternion

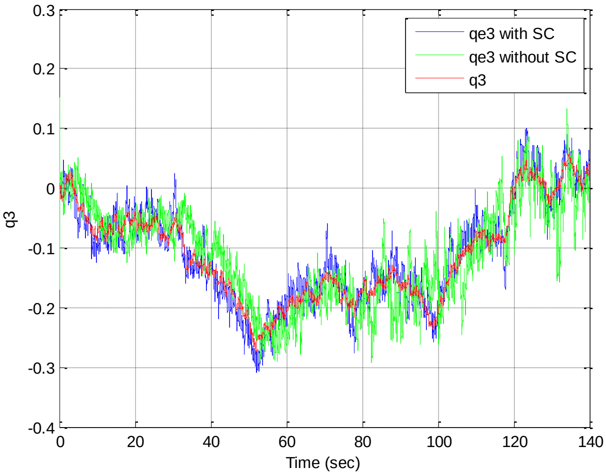

Fig. 3 The third component of the estimated and true quaternion

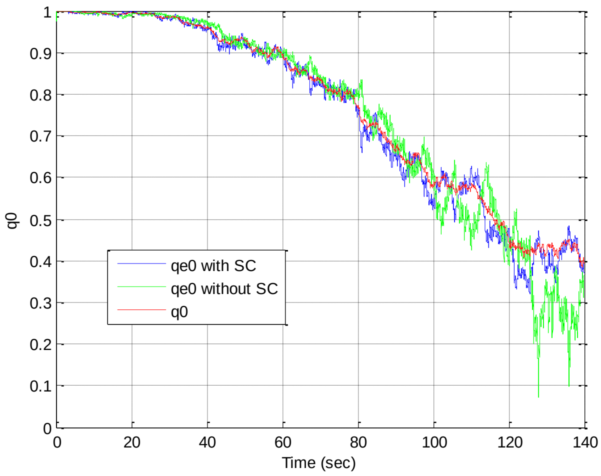

Fig. 4 The scalar component of the estimated and true quaternion

function of time. The largest roll, pitch, and yaw angles during the slew are about 178, 76, and 177 degrees, which can be calculated based on the quaternion shown in Fig. 1 through Fig. 4. The test case is a very aggressive maneuver with the most active spacecraft dynamics. In such an aggressive scenario the filter with the spacecraft dynamics performs better.

Figs. 1-4 show that the estimated attitudes for both filters follow the true attitude, but the estimation using spacecraft dynamics is clearly better than the estimation without using spacecraft dynamics. The attitude errors between the estimated and true attitude are represented by the Euler angles, roll, pitch, and yaw angle errors. The attitude errors of the extended Kalman filters with and without spacecraft dynamics are compared. The mean and standard deviation of the attitude errors with and without SC dynamics are summarized in Table 1.

Although the test mentioned above shows that using spacecraft model in Kalman filter is a better strategy, we would like to examine a case where the spacecraft model is inaccurate, for

Table 1 Mean and standard deviation of the attitude errors with and without SC dynamics

	Attitude error mean (deg)	Attitude error standard deviation (deg)
Roll with SC dynamics	-0.1573	2.6827
Roll without SC dynamics	0.4145	4.8599
Pitch with SC dynamics	1.0122	3.4061
Pitch without SC dynamics	1.1230	7.7963
Yaw with SC dynamics	0.0675	2.7551
Yaw without SC dynamics	0.3741	4.9706

Table 2 Means of the attitude error mean and standard deviation for 300 Monte-Carlo runs

Euler angles	Mean of the attitude error mean (deg)	Mean of the attitude error standard deviation (deg)
Roll	-0.1622	2.6794
Pitch	1.0087	3.4046
Yaw	0.0595	2.7610

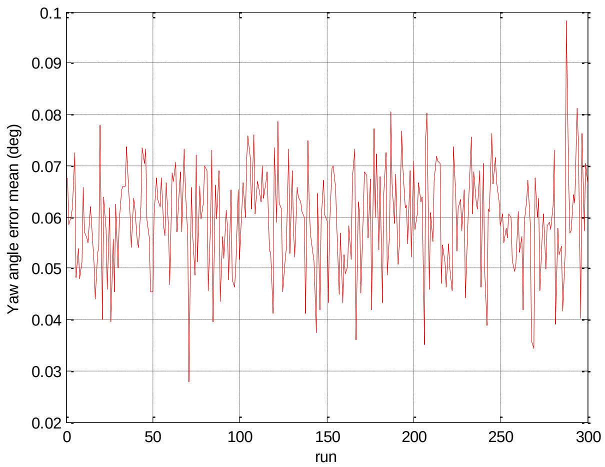

Fig. 5 The variation of the yaw angle error mean in the Monte-Carlo runs

example, in the inertia matrix. The impact of the uncertainties and variations of the inertia matrix J to the performance of the estimation is investigated by a set of Monte Carlo runs. The six different elements J_{11}, J_{22}, J_{33}, J_{12}, J_{13}, and J_{23}, of the inertia matrix are varied uniformly and randomly between 50% and 150% of their nominal values in the Monte Carlo runs. J_{21} is equal to J_{12}. J_{31} is equal to J_{13}. J_{32} is equal to J_{23}. Therefore, the inertia tensor is symmetric when the six different elements change. The uncertainties are 50% of the nominal values. For example, the nominal value of J_{11} is 1200 kg.m^2. In the Monte- Carlo runs, J_{11} will vary uniformly and randomly between 600 and 1800 kg.m^2.

The six different elements J_{11}, J_{22}, J_{33}, J_{12}, J_{13}, and J_{23}, of the inertia matrix are varied uniformly and randomly between 50% and 150% of their nominal values. The changes do not make the inertia tensor negative definite. The reason is as follows. In the worst case, the three diagonal terms J_{11}, J_{22}, and J_{33} are 50% of their nominal values and the three off-diagonal terms J_{12}, J_{13}, and

J_{23} are 150% of their nominal values. The inertia tensor becomes:

$$J = \begin{pmatrix} 600 & 150 & -300 \\ 150 & 1100 & 450 \\ -300 & 450 & 1550 \end{pmatrix}$$

The above inertia tensor is still a diagonally dominant matrix and is positive definite.

Fig. 5 through Fig. 7 show the variation of the attitude error mean for 300 Monte-Carlo runs. The attitude error mean and attitude error standard deviation are shown in Table 2. The impact of the uncertainties of the inertia matrix J to the performance of the Kalman filter estimation is small.

The filter with spacecraft dynamics needs more computations. However, with current flight computer's capability the CPU usage of the filter's computation is not a major concern.

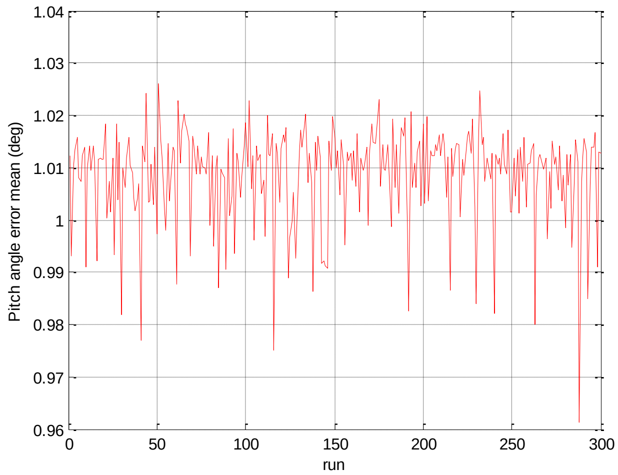

Fig. 6 The variation of the pitch angle error mean in the Monte-Carlo runs

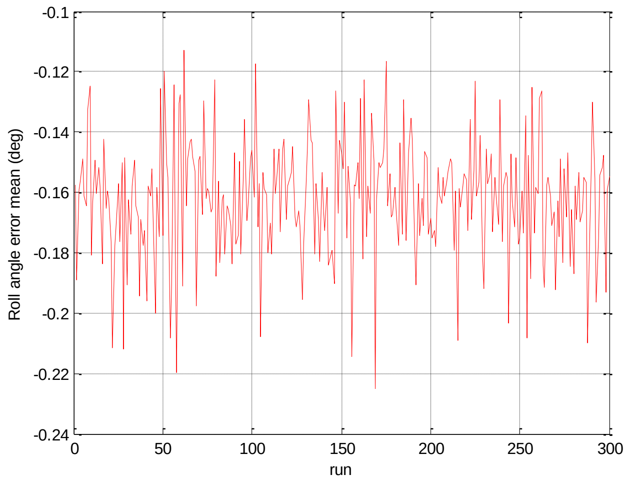

Fig. 7 Rhe variation of the roll angle mean in the Monte-Carlo runs

5. Conclusions

In this paper, we compared two different models that can be used for spacecraft attitude estimation. One model does not use spacecraft dynamics and is more popular in the guidance, navigation, and control community; the other model includes the spacecraft dynamics and has not been investigated as much as the first model. We adopted a reduced quaternion spacecraft dynamics model which admits additive noise. Geometry of the reduced quaternion model and the additive noise was discussed. This treatment is easier in computation. The simplified model without the spacecraft dynamics cannot estimate the attitude rates even through the gyros measurements are used in the estimation. In contrast, the attitude rates can be estimated directly by the Kalman filter with the spacecraft dynamics. Our analysis and simulation results show that the second model and the corresponding extended Kalman filter is a better choice in attitude determination because the method uses more information and gives more accurate attitude estimation.

Acknowledgments

The authors would like to thank Carey Buttrill, Suresh M. Joshi, and Steve Derry of the NASA Langley Research Center for their feedback on an earlier version of the manuscript, which helped them to improve the presentation of this paper.

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