Modeling of supersonic nonlinear flutter of plates on a visco-elastic foundation

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Abstract.   Numerical study of the flutter of a plate on a viscoelastic foundation is carried out in the paper. Critical velocity of the flutter of a plate on an elastic and viscoelastic foundation is determined. The mathematical model for the investigation of viscoelastic plates is based on the Marguerre's theory applied to the study of the problems of strength, rigidity and stability of thin-walled structures such as aircraft wings. Aerodynamic pressure is determined in accordance with the A.A. Ilyushin's piston theory. Using the Bubnov - Galerkin method, the basic resolving systems of nonlinear integro-differential equations (IDE) are obtained. At wide ranges of geometric and physical parameters of viscoelastic plates, their influence on the flutter velocity has been studied in detail.

Keywords: mathematical modeling; numerical algorithms; supersonic flutter; viscoelastic foundation

1. Introduction

At present, composite materials with pronounced viscoelastic properties are widely used in the aviation industry and many other branches of engineering. These industries have obtained light, elegant and economical thin-walled structures, for which the role of stability calculation in general cycle of strength calculations increased sharply. In connection with this, the hereditary theory of viscoelasticity attracts more and more attention. This is evidenced by the publication in recent years of a number of scientific papers in which the latest achievements of the theory of viscoelasticity are reflected. The growing interest in this theory is due to the development of computer technology, which makes it possible to reliably compare the computational experiment obtained on the basis of mathematical models to a full-scale experiment.

The basis for studying the processes of strain of composite materials is the hereditary theory of viscoelasticity; its specific application depends on the parameters of materials, the shape of the product and the range of variation in environmental conditions. At the same time, significant difficulties in obtaining "good" models arise in connection with the consideration of viscoelasticity properties and nonlinear effects. It should be noted that the use of traditional materials in aircraft construction allowed the use of mathematical models, which can now be called simplified, not fully considering the properties of viscoelasticity and other effects. These effects are most pronounced under conditions of supersonic air or fluid flows, i.e., at high velocities, which lead to

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the occurrence of the flutter effect.

Thus, the previously obtained scientific results in the field of modeling the processes of aircraft elements behavior under high-speed conditions cannot be directly applied to the problems under consideration, which proves the urgency of the problem of obtaining the adequate mathematical models for the dynamics of aircraft elements constructed from materials with pronounced viscoelastic and non-linear properties and functioning in flutter modes.

The noted properties of construction materials and the above factors increase the complexity of research and lead to the need to develop the computational methods for studying the stability of viscoelastic elements of thin-walled structures. Therefore, the development of effective computational algorithms for solving nonlinear integro-differential equations for dynamic problems of viscoelastic elements of thin-walled structures with weakly singular kernels of heredity is an urgent aspect.

A flutter of plates and shallow shells on an elastic foundation, was considered by a number of authors (Bolotin 1961, Rao and Rao 1984, Chai *et al*. 2017, Hasheminejad*et al*. 2013, Nguyen *et al*. 2014, Pagani *et al*. 2014, Zenkour 2017). In Bolotin (1961) an infinite plate, which lies on an elastic foundation and is flowed by a gas flow was considered.

Rao *et al*. (1984) have presented the results of the investigation of the effect of elastic sealing on the conditions for flutter occurrence in skin elements of supersonic aircraft in the longitudinal flow regime. The equations of motion are obtained by the energy method with allowance for the shear strain and are written in matrix form. Aerodynamic pressure is taken in accordance with the piston theory at Mach numbers $M_\infty > 1,5$. To find the eigenvalues and modes of flutter oscillations, the finite element method is used. The effect of the elastic foundation coefficient on the stability of panels with various geometric, mechanical and weight characteristics is estimated.

Chai *et al*. (2017) have studied the effect of an elastic foundation on the flutter of three-layer panels. The equations of motion are obtained on the basis of the Hamilton principle. Aerodynamic pressure is determined in accordance with the piston theory. The influence of geometric parameters and elastic foundation on the flutter of three-layer panels is analyzed.

In Hasheminejad *et al*. (2013), on the basis of classical theory of thin plates, a supersonic flutter of a three-layer plate on an elastic foundation is investigated. Foundation response is described by the Winkler and Pasternak models. The solution is carried out by the widely used Galerkin approximate method, taking into account the polynomial approximation of the deflection. The system of ordinary differential equations in time is solved by the Runge-Kutta method of the fourth order of accuracy.

Nguyen *et al*. (2014) have carried out numerical studies of nonlinear oscillations of the flutter of layered circular cylindrical shells in a supersonic flow. Nonlinear aerodynamic pressure is determined in accordance with Ilyushin's piston theory. Using the Galerkin method, with two-term approximation of the deflection, the nonlinear ordinary integral-differential equations are obtained. Numerical integration is obtained by the Runge-Kutta method of the fourth order. The influence of material and geometric properties, imperfections, initial conditions and elastic foundations on supersonic characteristics of the flutter of cylindrical shells is analyzed.

At present an account of viscoelastic properties in dynamic strain of plates and shells is one of the most relevant tasks in the mechanics of deformable bodies. Its solution is an effective application of the theory of viscoelasticity to real processes. Therefore, the methods and problems of the theory of hereditary elasticity attract great attention of researchers. There is a significant number of publications devoted to solving problems of calculating the characteristics of viscoelastic thin-walled structures (Wang *et al.* 2017, Merrett 2016, Librescu and Chandiramani

1989, Teifouet *et al.* 2016, Badalov *et al.* 2007a, b, Khudayarov and Bandurin 2007, Pouresmaeeli *et al.* 2013, Mahmoudkhani *et al.* 2016). General theoretical bases and methods for solving problems of determining the stress-strain state and analyzing the dynamic properties of load-bearing structures with regard to the rheological behavior of their material have already been developed.

Wang *et al.* (2017) have investigated a nonlinear flutter of viscoelastic panels under supersonic gas flow. To construct a mathematical model of viscoelastic panels, the Karman theory is used. Aerodynamic pressure is determined in accordance with the piston theory. To describe the viscoelastic properties of materials, the model of a standard viscoelastic body (the Kelvin theory) is used.

Librescu *et al.* (1989) based on the Boltzmann theory have considered the dynamic stability of viscoelastic isotropic plates. Transverse strain of shear and rotational inertia are taken into account. To solve the linear dynamic problems of viscoelasticity, the method of integral transformations is applied.

Teifouet *et al.* (2016) have investigated viscoelastic rectangular plates under various boundary conditions. To describe the deformation processes of viscoelastic materials, the Kelvin-Voigt theories are used. The obtained numerical results are compared with the known results.

Pouresmaeeli *et al.* (2013) have investigated the natural frequency of orthotropic viscoelastic nanoplates lying on an elastic foundation.

Three-layer shells have been widely used in various fields of industry, aviation and shipbuilding. Mahmoudkhani *et al.* (2016) have investigated the problem of the flutter of viscoelastic three-layer cylindrical shells, which are flown by a supersonic gas flow. Numerical investigation of the effect of geometric parameters, the viscoelastic damping parameter and temperature on the flutter boundaries of the shell has been conducted.

Liao and Sun (1993); Eshmatov *et al.* (2013), Song and Li (2012), Song *et al.* (2018), Zhao and Cao (2013), Pacheco *et al.* (2017), Singha and Mandal (2008), Chen and Li (2017), Yazdi (2017) are devoted to solving concrete problems of the flutter of composite laminated plates, panels, and shells. Pacheco *et al.* (2017) have studied numerically the nonlinear oscillations of the flutter of composite panels (multilayer panels) on several supports in supersonic flow. In Yazdi (2017), a nonlinear flutter of composite plates on an elastic foundation is studied.

Dixon *et al.* (1993) have investigated the aeroelastic stability of a composite multilayer rectangular panel in a supersonic gas flow. The velocity vector of the unperturbed flow is assumed to be directed parallel to the middle plane of the panel. The equations of motion of the structure are derived on the basis of the principle of virtual work and are written in matrix form. The aerodynamic forces and moments are calculated in accordance with the quasi-stationary piston theory of the first order. The relationship between strains and displacements is given by nonlinear Karman relations. Numerical integration of the equations is carried out by the finite element method. Limit cycles of oscillations are constructed and their amplitudes for graphite-epoxy and boron-epoxy panels are calculated depending on the number of layers, fiber orientation, damping coefficient and boundary conditions. The accuracy of the proposed method is estimated by comparison with the known theoretical solutions.

Shiau (1992) has investigated the aeroelastic dynamic stability of two-dimensional honeycomb panels used as elements of the supersonic aircraft skin. The panel represents two symmetrical multilayer plates with cellular orthotropic filler. Structural strains in a plane problem are considered as a superposition of bending strains of plates and shear strain of the filler. Membrane stresses are assumed to be small. Critical values of the aeroelastic system parameters are defined in

the class of harmonic functions of time. The complex eigenvalues satisfy the polynomial characteristic equations obtained from the condition for the existence of a nontrivial solution of the homogeneous boundary value problem for an ordinary differential equation of the fourth order. The dependence of the flutter velocity on fiber orientation under different mechanical and geometric characteristics of the layers is estimated.

Mathematical and computer modeling of the flutter of viscoelastic elements and structural units of an aircraft is an actual scientific problem; its study is stimulated by the failure of aircraft structures, parts of space and jet engines.

Due to complexity of the flutter phenomenon of aircraft elements, simplifying assumptions are used in studies. However, these assumptions, as a rule, turn out to be so restrictive that the mathematical model ceases to reflect the real conditions with sufficient accuracy. Therefore, the results of theoretical and experimental studies are still in bad agreement.

As seen from the above review, the problem of aeroelastic oscillations and stability of viscoelastic plates, panels and shells is far from complete realization. Despite a significant number of publications, comparatively little research has been done on nonlinear flutter of plates and panels on elastic and viscoelastic foundations. Therefore, this article is aimed to investigate a nonlinear flutter of a plate on a viscoelastic foundation.

So, a theoretical study of nonlinear flutter of viscoelastic plates is given in this paper. Based on the Bubnov-Galerkin method using quadrature formulas and the method of eliminating weakly singular operators, an effective computational algorithm has been developed that makes it possible to investigate the problems of a nonlinear flutter of viscoelastic plates flown by a supersonic gas flow.

2. Mathematical model

Consider the nonlinear problem of plate flutter on a viscoelastic foundation. Let the plate with sides a and b, and thickness h, supported with hinges along the entire contour, is flown with a supersonic gas flow on one side (Fig. 1). Aerodynamic pressure is calculated in accordance with the piston theory (Ilyushin 1957). Under the assumption adopted in (Bolotin 1961, Volmir 1972, Ilyushin 1956, Khudayarov 2010), the equation of oscillations of a viscoelastic plate on a foundation has the form

$$\frac{D}{h}\left(1-R^*\right)\nabla^4 w = L\left(w,\Phi\right)-k(1-\Gamma^*)w-$$
$$-\rho\frac{\partial^2 w}{\partial t^2}-\frac{B}{h}\frac{\partial w}{\partial t}-\frac{BV}{h}\frac{\partial w}{\partial x}-\frac{B_1 V^2}{h}\left(\frac{\partial w}{\partial x}\right)^2,$$
$$\frac{1}{E}\nabla^4\Phi = -\left(1-R^*\right)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left[\frac{\partial^2 w}{\partial x\partial y}\right]^2\right\} \tag{1}$$

where $D=\dfrac{Eh^3}{12(1-\mu^2)}$ – rigidity; ρ – material density; $\nabla^4=\left(\nabla^2\right)^2=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)$; h – is a plate thickness; E - is the modulus of elasticity; μ - is the Poisson ratio; w - deflection of the plate; V is the flow velocity; R^*- is an integral operator with a relaxation kernel $R(t)$, having a weakly singular feature of Abel type
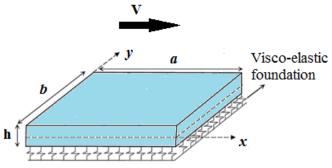
Fig. 1 Plate lying on a viscoelastic foundation

$$R^*\varphi(t) = \int_0^t R(t-\tau)\varphi(\tau)d\tau \quad \left(R(t) = A\cdot\exp(-\beta t)\cdot t^{\alpha-1}, \; A>0, \; \beta>0, \; 0<\alpha<1\right)$$

A – parameter of viscosity, β - damping parameter; α - the singularity parameter determined by the experiment; t - time; τ - the time preceding the moment of observation;

L – is a differential operator

$$R^*\varphi(t) = \int_0^t R(t-\tau)\varphi(\tau)d\tau \quad L(w,\Phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2 \Phi}{\partial x\partial y};$$

Φ – is a stress function; κ – is a bed coefficient of viscoelastic foundation; Γ^* - is an integral operator with relaxation kernel $\Gamma(t)$ $\left(\Gamma^*\varphi(t) = \int_0^t \Gamma(t-\tau)\varphi(\tau)d\tau, \quad \Gamma(t) = A_0 t^{\alpha_0-1}\exp(-\beta_0 t)\right)$; A_0, β_0 and

α_0 – are the rheological parameters of viscoelastic foundation; $B = \frac{\aleph p_\infty}{V_\infty}$, $B_1 = \frac{\aleph(\aleph+1)p_\infty}{4V_\infty^2}$, $\aleph$- is

the polytropic gas index; p_∞, V_∞ are pressure and sound speed in the unperturbed gas flow, respectively.

In accordance with the boundary conditions:

G_1) - hinged support on all edges

At $x=0$, $x=a$

$$w=0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \frac{\partial^2 \Phi}{\partial y^2} = 0, \; \vartheta = 0,$$

At $y=0$, $y=b$

$$w=0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \frac{\partial^2 \Phi}{\partial y^2} = 0, \; u = 0,$$

G_2) - hinged support on two edges and fixing on the other two:

At $x=0$, $x=a$

$$w=0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \frac{\partial^2 \Phi}{\partial y^2} = 0, \; u = 0,$$

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At $y=0, y=b$

$$w=0, \quad \frac{\partial w}{\partial y}=0, \quad u=0, \quad \vartheta =0,$$

G_3) - rigid fixing on all edges:
At $x=0, x=a$

$$w=0, \quad \frac{\partial w}{\partial x}=0, \quad u=0, \quad \vartheta =0,$$

At $y=0, y=b$

$$w=0, \quad \frac{\partial w}{\partial y}=0, \quad u=0, \quad \vartheta =0,$$

Solution of Eq. (1) is taken in the form

$$w(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm}\varphi_{nm}(x,y),$$

$$\Phi(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Phi_{nm}\varphi_{nm}(x,y),$$

(2)

where $w_{nm}=w_{nm}(t)$ and $\Phi_{nm}=\Phi_{nm}(t)$ are the sought for time functions; $\varphi_{nm}(x,y)$ – known functions, depending on boundary conditions

$$G_1: \quad \varphi_{nm}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b};$$

$$G_2: \quad \varphi_{nm}(x,y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b};$$

$$G_3: \quad \varphi_{nm}(x,y) = (\cos\frac{(n-1)\pi x}{a} - \cos\frac{(n+1)\pi x}{a})(1-\cos\frac{2m\pi\pi}{b}).$$

Completing the Bubnov-Galerkin procedure, the system of integro-differential equations (IDE) with respect to $w_{nm}(t)$ and $\Phi_{nm}(t)$ is obtained; excluding $\Phi_{nm}(t)$ from this system, the following nonlinear IDE with respect to the sought for function $w_{nm}(t)$ is written

$$\ddot{w}_{kl} + \lambda^4\Omega^2\left[\left(\frac{k}{\lambda}\right)^2 + l^2\right]^2(1-R^*)w_{kl} + \frac{16}{\pi^2}g_k g_l k(1-\Gamma^*)w_{kl}$$

$$+\frac{12\lambda^4\left(1-\mu^2\right)\Omega^2}{\pi^2}\sum_{n,i,j=1}^{N}\sum_{m,r,s=1}^{L}a_{k\ln mirjs}w_{nm}\left(1-R^*\right)w_{ir}w_{js} + M\,\dot{w}_{kl}$$

$$+2MM^*\sum_{n=1}^{N}\gamma_{kl}w_{nl} + M_1 M^{*2}\sum_{n,i}^{N}\sum_{m,r}^{L}F_{k\ln mir}w_{nm}w_{ir} = 0,$$

(3)

$$w_{kl}(0)=w_{0kl}, \quad \dot{w}_{kl}(0)=\dot{w}_{0kl}, \quad k=\overline{1,N}; \quad l=\overline{1,L};$$

where $\Omega^2 = \dfrac{\pi^4}{12(1-\mu^2)} M_E^2 \left(\dfrac{h}{a}\right)^2$, $M = \aleph M_p^2 \lambda_1$, $M_1 = \aleph(\aleph+1)\dfrac{M_p^2}{4}$; $\lambda = \dfrac{a}{b}$; $\lambda_1 = \dfrac{a}{h}$; $M^* = \dfrac{V}{V_\infty}$ - is a

Mach number; $M_E = \sqrt{\dfrac{E}{\rho V_\infty^2}}$; $M_p = \sqrt{\dfrac{P_\infty}{\rho V_\infty^2}}$; g_k, γ_{kl}, F_{klnmir}, a_{klnmir} - are the dimensionless coefficients.

3. Numerical results

Systems of nonlinear IDE (3) are solved numerically using the method proposed in (Badalov 1987, Badalov *et al*. 1987, Verlan *et al*. 2004). To do this, the system is written in integral form and, using a rational transformation, the weakly singular features of the integral operator R^* are excluded. Assuming that $t=t_i$, $t_i=i\Delta t$, $i=1,2,\dots$ ($\Delta t=const$) and replacing the integrals with some quadrature formulas for computing $w_{nm}= w_{nm}(t)$, the following recurrence relationships are obtained

$$
\begin{aligned}
w_{ikl} = \frac{1}{1+A_i M}\Bigg\{ & w_{0kl} + \left(\dot{w}_{0kl} + Mw_{0kl}\right)t_i - \sum_{j=0}^{i-1} A_j \Bigg(M\, w_{jkl} \\
& -(t_i - t_j)\Bigg\langle -2MM^* \sum_{n=1}^{N} \gamma_{kn} w_{jnl} - \lambda^4 \Omega^2 \left[\left(\frac{k}{\lambda}\right)^2 + l^2\right]^2 \left(w_{jkl} - \frac{A}{\alpha}\times\right. \\
& \times \sum_{s=0}^{j} B_s e^{-\beta t_s} w_{j-skl} \Bigg) - \frac{12\lambda^4\left(1-\mu^2\right)\Omega^2}{\pi^2} \sum_{n,i_1,j_1=1}^{N}\sum_{m,r,s_1=1}^{L} a_{k\ln mi_1 rj_1 s_1} w_{jnm} \times \\
& \times\left(w_{ji_1 r} w_{jj_1 s_1} - \frac{A}{\alpha}\sum_{s=0}^{j} B_s e^{-\beta t_s} w_{j-si_1 r} w_{j-sj_1 s_1}\right) - \frac{16}{\pi^2} g_k g_l k\Bigg(w_{jkl} - \frac{A_ц}{\alpha_ц}\times \\
& \sum_{s=0}^{j} B_s e^{-\beta t_s} w_{j-skl}\Bigg) - M_1 M^{*2} \sum_{n,i_1=1}^{N}\sum_{m,r=1}^{L} F_{k\ln mi_1 r} w_{jnm} w_{ji_1 r} \Bigg\rangle\Bigg)\Bigg\}
\end{aligned}
\tag{4}
$$

$i = 1,2,\dots$; $n = \overline{1,N}$; $m = \overline{1,L}$; where A_j, B_s - are the numerical coefficients applied to the quadrature formulas of the trapezoid

$$
A_0 = \frac{\Delta t}{2}; \quad A_j = \Delta t, \quad j = \overline{1,i-1}; \quad A_i = \frac{\Delta t}{2}; B_0 = \frac{\Delta t^\alpha}{2}; \quad s = j, \quad B_j = \frac{\Delta t^\alpha(j^\alpha - (j-1)^\alpha)}{2};
$$

$$
B_s = \frac{\Delta t^\alpha((s+1)^\alpha - (s-1)^\alpha)}{2}.
$$

Owing to the proposed approach in the algorithm for numerical solution of the problem in formula (4), the multiplier t_i-t_j at $j = i$ acquires a zero value, i.e., the last summand of the sum is zero. Therefore, the summation is from zero to i -1 ($j = \overline{0,i-1}$).

Thus, according to numerical method with respect to the unknowns, a system of linear algebraic equations is obtained.

Table 1 Comparison of exact and approximate solutions of IDE

t	Solution		Δh
	Exact	Approximate	
0	1.000000	1.000000	-
1	0.970445	0.970373	$0.7 \cdot 10^{-4}$
2	0.941764	0.941622	$1.4 \cdot 10^{-4}$
3	0.913931	0.913644	$2.8 \cdot 10^{-4}$
4	0.886920	0.886569	$3.5 \cdot 10^{-4}$
5	0.860707	0.860271	$4.3 \cdot 10^{-4}$
6	0.835270	0.834855	$4.1 \cdot 10^{-4}$
7	0.810584	0.810278	$3 \cdot 10^{-4}$
8	0.786627	0.786113	$5.1 \cdot 10^{-4}$
9	0.763379	0.763126	$2.5 \cdot 10^{-4}$
10	0.740818	0.740509	$3 \cdot 10^{-4}$

3.1 Example of test solutions

Verification of efficiency of the proposed numerical method and programs, based on the solution of test cases, is a necessary stage to confirm the reliability of research results obtained in solving specific problems. The problems for which an exact solution is known (Badalov 1987) have been considered as test cases. Table 1 show a satisfactory agreement of approximate solutions with exact ones; this shows the reliability and high accuracy of calculation results.

Test Example 1. Consider a non-linear integro-differential equation of the form

$$\ddot{w} + \lambda_0 \dot{w} + \omega^2 w = q$$

$$- \lambda_1 \int_0^t R(t-\tau)w(\tau)d\tau - \lambda_2 w \int_0^t R(t-\tau)w(\tau)d\tau - \lambda_3 \int_0^t R(t-\tau)w^2(\tau)d\tau; \tag{5}$$

$$w(0)=1, \quad \dot{w}(0) = -\beta,$$

where

$$R(t) = A\exp(-\beta t)t^{\alpha-1}, 0<\alpha<1; \quad q = \left[\beta^2 + \omega^2 - \lambda_0\beta - \frac{At^\alpha}{\alpha}(\lambda_1 + [\lambda_2 + \lambda_3]\exp(-\beta t))\right]\exp(-\beta t) \;.$$

Eq. (5) has an exact solution $w=\exp(-\beta t)$, which satisfies the initial conditions. According to Eq. (5), the approximate values $w_n=w(t_n)$ ($t=t_n=n\Delta t$, $n=0, 1, 2, \dots$) are found from the relationships

$$w_n = \frac{1}{1+\lambda_0 A_n}\left\{ 1-(\beta-\lambda_0)t_n - \sum_{i=0}^{n-1} A_i\left\langle \lambda_0 w_i - (t_n - t_i)[q(t_i) - \omega^2 w_i + \frac{A\lambda_2}{\alpha}w_i \times \right.\right.$$

$$\left.\left. \times \sum_{s=0}^{i} B_s \exp(-\beta t_s)w_{i-s} + \frac{A}{\alpha}\sum_{s=0}^{i} B_s \exp(-\beta t_s)w_{i-s}\left(\lambda_1 + \lambda_3 \exp(-\beta t_s)w_{i-s}\right)] \right\rangle\right\} \tag{6}$$

$n=1,2,\ldots,$; where A_i, B_s – are the coefficients of the quadrature formula of trapezoids.

Table 1 gives approximate results of calculations by formulas (6) within the interval from 0 to 1 with $\Delta t=0.01$ step, and exact solutions. The following initial data have been used: $\lambda_0=1.1$; $\lambda_1=1.2$; $\lambda_2=1.3$; $\lambda_3=1.4$; $A=0.01$; $\beta=0.03$; $\alpha=0.01$. It follows from the table that the maximum error Δ of calculations performed by described method represents the value $const \cdot \Delta t^2$. The efficiency of this numerical method and programs is shown in other test cases as well.

From the table it follows that the error Δh of calculations performed by described method coincides with the error of the quadrature formulas used and has the same order of smallness relative to the interpolation step (for the trapezoid formula the error of the method with respect to the interpolation step is of second-order, for the Simpson formula – of third order, etc.).

2. Test case 2. Consider the problem of vibrations and stability of viscoelastic strip in a gas flow (I.A. Kiiko, V.V. Pokazeev 2005, 2013) and present a comparative analysis of the results of solution with the ones obtained by the proposed method.

In a rectangular coordinate system, the strip occupies region $0 \leq y \leq l$, $x \geq 0$. On one side it is flown over by a gas flow with velocity vector $V=V \cdot \vec{n}_0$, $\vec{n}_0 = \{\cos\theta, \sin\theta\}$. Here θ – angle of flow; l – width strip.

Strip vibrations are described by equation (I.A. Kiiko, V.V. Pokazeev, 2005, 2013)

$$D(1-R^*)\nabla^4 w + \rho\, h \frac{\partial^2 w}{\partial t^2} + B \frac{\partial w}{\partial t} + BV\vec{n}_0 \cdot grad w = 0 \tag{7}$$

Assuming that $R(t) = A \cdot \exp(-\beta\, t) \cdot t^{\alpha-1}$ and introducing the dimensionless coordinates $\dfrac{x}{l}, \dfrac{y}{l}$, and time βt given in (Kiiko and Pokazeev 2005, 2013), and retaining the previous notations, Eq. (7) is written in the form

$$(1-\lambda_0 R_1^*)\nabla^4 w + a_3 \frac{V}{V_\infty} \vec{n}_0 \cdot grad w + a_2 \frac{\partial w}{\partial t} + a_1 \frac{\partial^2 w}{\partial t^2} = 0 \tag{8}$$

Here the notations $\lambda_0 = \dfrac{A}{\beta}$, $R_1 = \exp(-t) \cdot t^{\alpha-1}\beta^{1-\alpha}$, $a_1 = 12(1-\mu^2)\dfrac{\beta^2 l^2 \rho}{h^2 E}$, $a_2 = 12(1-\mu^2)\dfrac{\beta\, l^4 \aleph p_\infty}{h^3 V_\infty E}$,

$a_3 = 12(1-\mu^2)\dfrac{\aleph p_\infty}{E}\left(\dfrac{l}{h}\right)^3$ are introduced, and other notations correspond to the ones assumed in (Kiiko and Pokazeev 2005, 2013); The biharmonic operator is defined as

$$\nabla^4 = \left(\nabla^2\right)^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2.$$

Solution of Eq. (8) is chosen in the form

$$w = w(t) \cdot \exp(-\alpha_1 x) \cdot \sin \pi y \tag{9}$$

Substituting Eq. (9) into Eq. (8) at $\theta=0$ we get

$$a_1 \ddot{w} + a_2 \dot{w} + \mu_0(1-\lambda_0 R_1^*)w - \alpha_1 a_3 \frac{V}{V_\infty} w = 0 \tag{10}$$

Table 2 Dependence of critical velocity of strip flutter on the parameters of the kernel of heredity

λ_0	α	V^*_{cr} (results given in Kiiko and Pokazeev 2005, 2013)	V^*_{cr} (Present study)
0	-	0.096029	0.096027
0.1	1	0.096065	0.096025
0.1	0.1	-	0.093745
0.1	0.03	-	0.0919745
0.3	0.005	-	0.079220

$$w(0)=S_1, \quad \dot{w}(0) = S_2, \tag{11}$$

where $\mu_0 = (\alpha_1^2 - \pi^2)^2$; S_1, S_2 – are the known constants.

Integrating system (10) twice with respect to *t*, it can be written in integral form; with rational transformation we eliminate the singular features of integral operator R_1^*. Then, assuming that $t=t_i$, $t_i=i\cdot\Delta t$, $i=1,2, \dots$ *(Δt =const)* and replacing the integrals with quadrature formulas of trapezoids for the calculation of $w_{ik}=w_k(t_i)$, the recurrence formulas for the Koltunov-Rzhanitsyn kernel are obtained

$$w_p = \frac{1}{1+B_p(a_2/a_1)}\left\{ w_0 + \left(\dot{w}_0 + (a_2/a_1)w_0 \right)t_p - \right.$$
$$\left. -\frac{1}{a_1}\sum_{j=0}^{p-1}B_j\left(a_2 w_j - (t_p - t_j)\left[\alpha_1 a_3 \frac{V}{V_\infty}w_j + \mu_0 (w_j - \lambda_0 \sum_{s=0}^{j}C_s \exp(-t_s)\beta^{1-\alpha}w_{j-s}) \right] \right) \right\} \tag{12}$$

Thus, according to numerical method with respect to the unknowns, a system of algebraic equations is obtained. The Gauss method is used to solve the system. Based on the developed algorithm, a package of applied computer programs is created. Results of calculations are given in Tables 2.

Table 2 shows the results of specific calculations for the parameters values (Kiiko and Pokazeev 2005, 2013)

$$\frac{p_\infty}{E} = 5\cdot10^{-7}; \rho = 8\cdot10^3 \,\text{kg/m}^3 \; ; \; \aleph = 1.4; \; \mu = 0.3; \; V_\infty = 330 \; m/s; \; \frac{l}{h} = 3\cdot10^2 \quad .$$

The third column shows the results (Kiiko and Pokazeev 2005, 2013) when dimensionless critical flutter velocities (V^*_{cr}) are determined by a numerical-analytical method.

As seen from the results obtained, for ideally elastic and viscoelastic strips (in the case of exponential kernel of heredity) the critical velocities of the flutter exactly coincide with the results given in (Kiiko and Pokazeev 2005, 2013). For viscoelastic strip with a weakly singular heredity kernel, this velocity decreases.

3.2 Results and discussion

Results of calculations are presented in the table and are shown by the graphs in Figs. 2-3 for $N=5$, $L=2$. On the basis of Eq. (4), the critical velocity of the flutter of viscoelastic plates is

determined. As a criterion determining the critical velocity V_{cr}, the condition is assumed that at this velocity there is an oscillatory motion with rapidly increasing amplitudes, which can lead to structure damage. In the case $V < V_{cr}$ the flow velocity is less than the critical one, the amplitude of viscoelastic plate oscillations is damping (Badalov 2007, 1987, Khudayarov 2007, 2008, Verlan 2004).

To determine $V = V_{cr}$, consider the values V_1 and V_2 located on the interval (V_0, V_n) in such a way that $V_0 < V_1 < V_2 < V_n$. Comparing the law of variation of w at $V = V_1$ and $V = V_2$, the following conclusions can be drawn:

a) if, at $V < V_1$, the law of variation of the function w is close to a harmonic one, then V_{cr} cannot be in the interval (V_0, V_1); that is V_{cr} lies in the interval (V_1, V_n);

b) if, at $V > V_1$, a rapid growth of the function w with time is observed, then V_{cr} lies in the interval (V_0, V_1).

Processes a) and b), i.e., the processes of excluding the intervals that do not give rise to undesirable phenomena is repeated for (V_0, V_1) or (V_1, V_n), etc. The search ends when the remaining sub-interval is reduced to a sufficiently small size.

The Table 3 shows the critical values of the flutter velocity depending on physical-mechanical and geometric characteristics of the plate.

The effect of viscoelastic properties of the plate material on the critical values of flutter velocity is investigated. Calculation results presented in the table show that the solutions of elastic *(A=0)* and viscoelastic *(A>0)* problems differ significantly. For example, as the parameter A increases from zero to 0.1, the critical flutter velocity decreases by 27.7%.

Next, the effect of the singularity parameter α on the critical flutter velocity is investigated. With increasing of α, this velocity increases. For example, the difference between the values of the critical velocity at $\alpha = 0.1$ and $\alpha = 0.4$ is 53%.

It is seen from the table above that the effect of damping parameter β of the heredity kernel on the flutter velocity of the plate compared with the effect of viscosity parameter A and the singularity α is insignificant, which confirms that the exponential relaxation kernel is unable to completely describe the hereditary properties of structure material.

The effect of the relative thickness parameter of the plate λ_1 on the critical velocity V_{cr} of the flutter is studied. The calculations are carried out at $\lambda_1 =$ 220, 280, 300, and 350. The results obtained show that as the plate thickness decreases (with increasing parameter λ_1), the critical velocity of the flutter of viscoelastic plate decreases.

The effect of elongation parameter of plate λ on the critical flutter velocity is investigated. As λ increases, the critical velocity increases, which is explained by the fact that the size of the plate decreases perpendicular to the flow direction with increasing λ (at constant λ_1) and, consequently, the relative rigidity of the system increases.

From the tables it follows that, with account of viscoelastic foundation, the flutter velocity increases compared to the cases when viscoelastic foundations are not considered. For great values of the coefficients of bed, the flutter velocity increases markedly.

The effect of viscoelastic properties of material on the oscillation amplitudes of the plate is shown in Fig. 2. As seen from the figure, with increasing parameter A, the amplitude and frequency of oscillations decrease.

Fig. 3 shows the curves of deflection w – time dependence of the viscoelastic plate at different values of the parameter $\lambda_1 = a/h$. The calculation was carried out for viscoelastic plates with a relative thickness $\lambda_1 = a/h$, varying in the range from 200 to 310. Analysis of these curves suggests
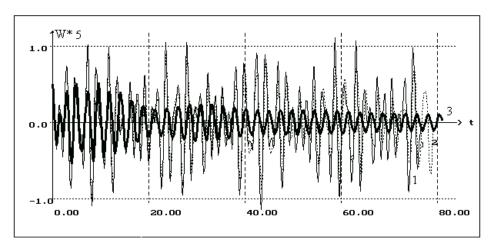
Fig. 2 Dependence of the viscoelastic plate deflection on time t at A=0 (1); A=0.005 (2); A=0.1 (3); k=0.0001; α=0.25; β=0.05; λ=2.5; A_0=0.1; α_0=0.25; β_0=0.02; λ_1=250; N=5; L=2
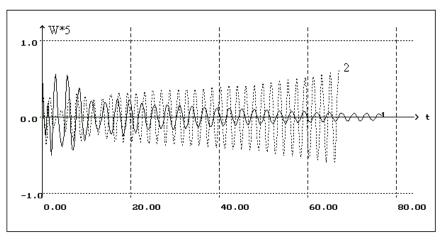

Fig. 3 Dependence of the viscoelastic plate deflection on time t at λ_1=200 (1); λ_1=310 (2); A=0,1; α=0,25; β=0,05;   k=0,0001; λ=2,5; A_0=0,11; α_0=0,2; β_0=0,02; N=5; L =2; V=875 m/s

that a decrease in the thickness of the plate leads to an increase in the frequency of oscillations. In Fig. 3 there is a noticeable increase in the deflection amplitude of the plate at λ_1=a/h (curve 2).

3.2.1 Effect of boundary conditions

The study of boundary conditions on flutter velocity of plate given. Results of calculations are given in Table 4. Comparison of different cases of plate fixation shows that with an increase in a number of fixed sides of the plate, flutter critical velocity increases. For elastic plate flutter velocity is 990 m/s (G_1), 1535 m/s (G_2), and 1688 m/s (G_3). These results practically coincide with the values obtained by analytical method in (A.A. Movchan 1956, 1957) (G_1: V_{cr} = 969 m/s, A_1 = 513; G_2: V_{cr} = 1537 m/s; A_1 = 814 m/s; G_3: V_{cr} = 1542 m/s; A_1 = 842). For viscoelastic plate with regular kernel of heredity, this velocity is 935 m/s (G_1), 1442 m/s (G_2) and 1605 m/s (G_3), respectively.

Table 3 Dependences of the critical velocities of the flutter of a plate on a viscoelastic foundation on physical-mechanical and geometric parameters

A	α	β	A_o	α_o	β_o	λ	λ_1	k	V_{cr} (m/s)
0.0 0.001 0.01 0.1	0.25	0.05	0.02	0.3	0.01	2.5	250	0.0001	1620 1595 1492 1170
0.1	0.1 0.4 0.7	0.05	0.02	0.3	0.01	2.5	250	0.0001	865 1325 1479
0.1	0.25	0.01 0.1	0.02	0.3	0.01	2.5	250	0.0001	1186 1167
0.1	0.25	0.05	0.0 0.05 0.3	0.3	0.01	2.5	250	0.0001	1219 1158 1137
0.1	0.25	0.05	0.02	0.1 0.5 0.9	0.01	2.5	250	0.0001	1162 1211 1230
0.1	0.25	0.05	0.02	0.3	0.1 0.5	2.5	250	0.0001	1165 1162
0.1	0.25	0.05	0.02	0.3	0.01	2 2.2 2.7	250	0.0001	718 887 1467
0.1	0.25	0.05	0.02	0.3	0.01	2.5	220 280 300 350	0.0001	1740 846 674 412
0.1	0.25	0.05	0.02	0.3	0.01	2.5	250	0.0 0.0002 0.0004 0.0006	1047 1244 1322 1439

Table 4 Effect of boundary conditions on flutter velocity of plate

Boundary conditions	A	α	β	V_{cr} (m/s)
G_1	0 0,01 0,05	- 1 0,5	- 0,1 0,1	990 935 536
G_2	0 0,01 0,05	- 1 0,5	- 0,1 0,1	1535 1442 871
G_3	0 0,01 0,05	- 1 0,5	- 0,1 0,1	1688 1605 915

It can be seen from the obtained results that if the exponential kernel ($\alpha = 1$) is used, flutter velocity decreases by approximately 5%, and when the Koltunov-Rzhanitsyn kernel is used this velocity decreases by 45% relative to critical velocity of the flutter of ideally elastic plates. Therefore, when using exponential kernels, flutter velocity of viscoelastic plate practically

coincides with critical flutter velocity for ideally elastic plates. These conclusions and results fully agree with the conclusions and results given in (I.A. Kiiko, V.V. Pokazeev 2005, 2013), where critical flutter velocities are determined by a numerical-analytical method.

4. Conclusion

Thus, we may conclude that the singularity parameter α affects not only the viscoelastic system oscillations, but also the critical flutter velocity. Consequently, consideration of this effect in aviation structures design is important, since the smaller the singularity parameter of structure material, the more intensive the dissipation processes in these structures.

On the basis of the results obtained, it can be concluded that an account of viscoelastic properties of plate material leads to a decrease in the critical flutter velocity V_{cr}, which is the cause of flutter phenomenon.

A number of new dynamic effects have been studied in the modeling of nonlinear problems:

• it is found that an account of viscoelastic properties of thin-walled structures of an aircraft leads to 40-60% decrease in the critical flutter velocity;

• it is established that an account of nonlinear effects in solving the flutter problem of viscoelastic elements of an aircraft leads to 15-20% increase in the critical velocity.

It should be also noted that at a flow velocity less than V_{cr}, the effect of viscoelastic property of material reduces the amplitude and the frequency of oscillations. If the flow velocity exceeds V_{cr}, the viscoelastic property of material exerts a destabilizing effect.

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