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Strong formulation finite element method for arbitrarily shaped laminated plates – Part II. Numerical analysis

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Abstract. The results of a series of numerical experiments are presented to verify some of the important developments made in the first part of this paper. Firstly, the static solution of an algebraic system obtained through Strong Formulation Finite Element Method (SFEM) is presented. Secondly, the stress and strain recovery procedure is descripted for the present technique. It will be clear that the present approach is suitable for any strong formulation finite element methodology, due to the presented general approach based on the unknown displacements and on the elasticity equations. Thirdly, the numerical solutions for some classical and other numerical results found in literature are exposed. Finally, an arbitrarily shaped composite plate is solved and good agreement is observed for all the presented cases.

Keywords: static analysis; arbitrarily shaped plates; stress and strain recovery; generalized differential quadrature; strong formulation finite element method

1. Introduction

Some laminated composite and cross-ply plates (Reddy 2003 and Natarajana *et al.* 2013) are studied in the following. In order to investigate the accuracy and reliability of the present technique several examples are presented. The implementation and the numerical results of a strain and stress recovery procedure applied to the Strong Formulation Finite Element Method (SFEM) are illustrated. The cited technique is based on the cohesion between mapping technique, also used in the well-known Finite Element Method (FEM), and a numerical approximation of the derivatives that can be based on Generalized Differential Quadrature (GDQ) method, orthogonal collocation or Spectral Method (SM). In this way, for instance, arbitrarily shaped plates can be solved even though geometric discontinuities are present. It is recalled that the GDQ method (Shu 2000, Tornabene and Viola 2007, 2008, 2009a, b, Viola and Tornabene 2005, 2006, 2009, and Viola *et al.* 2007) started from the early works on the Differential Quadrature (DQ) by (Bellman and Casti 1971 and Bellman *et al.* 1972) and SMs approximate a function by a linear combination of suitable basis functions as stated in the works (Boyd 2001, Gottlieb and Orszag 1977, and Orszag 1969, 1980). Furthermore, further studies on DQM showed that the two approaches are

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analogous under certain assumptions (Chen *et al*. 1997, 2000 and Jang *et al.* 1989). Although the theoretical developments of the SFEM were presented in Part I by (Fantuzzi and Tornabene 2014), it is better to recall that the major steps on a domain decomposition technique using DQM were reported in the works (Chen 1999a, b, 2000a, b, Shu *et al.* 1995, Shu and Chew 1997, 1999, Xing and Liu 2009, 2010, Zong and Zhang 2009 and Viola *et al.* 2013d, e, f). The generality of the present formulation will be demonstrated, in fact, it is possible to evaluate a partial derivative using several basis functions and weighting coefficient calculations (exact solutions or inversion of the coefficient matrix). In the present part II of this paper reliability and accuracy of the SFEM is investigated and the numerical results are compared with those found in literature and obtained through a finite element code, in the final section.

2. Strain and stress recovery procedure

The following strain and stress recovery procedure has been already presented in literature (Tornabene 2012, Tornabene *et al.* 2012a, b, c, and Tornabene and Ceruti 2013a, b), where the authors studied the recovery procedure with GDQ method for single domain cases. For the present technique, the mathematical developments follow the same guidelines of the previous papers (Tornabene 2012, Tornabene *et al.* 2012a, b, c, Tornabene and Reddy 2013, Tornabene and Viola 2013, Viola *et al.* 2013b, Tornabene *et al.* 2014a, b, and Tornabene and Fantuzzi 2014). In fact, they are applied element by element. When the fundamental system is solved, applying the GDQ method at the master element level, to the two-dimensional equations of moderately thick composite plates, the numerical values of the displacement components are evaluated in all the points of the middle surface (i, j) of the plate

$$u_{(ij)}, v_{(ij)}, w_{(ij)}, \beta_{x(ij)}, \beta_{y(ij)} \quad \text{with} \quad i = 1,...,N, \quad j = 1,...,M \tag{1}$$

Once the displacements and rotations are known the displacement field can be calculated at any point of the three-dimensional solid using the kinematic model of paper part I (Fantuzzi and Tornabene 2014), as

$$\begin{aligned} U_{(ijm)} &= u_{(ij)} + z_m \beta_{x(ij)} \\ V_{(ijm)} &= v_{(ij)} + z_m \beta_{y(ij)} \\ W_{(ijm)} &= w_{(ij)} \end{aligned} \tag{2}$$

with $i = 1,...,N$, $j = 1,...,M$, $m = 1,...,T$, where m is the index that varies along the plate thickness. The kinematic equations are functions of both the displacement components Eq. (2) and their derivatives, hence the strain characteristics can be evaluated directly using the GDQ method as follows

$$\varepsilon^0_{x(ij)} \cong \sum_{k=1}^{N} \varsigma^{x(1)}_{ik} u_{(kj)} \qquad \chi_{x(ij)} \cong \sum_{k=1}^{N} \varsigma^{x(1)}_{ik} \beta_{x(kj)}$$

$$\varepsilon^0_{y(ij)} \cong \sum_{k=1}^{M} \varsigma^{y(1)}_{jk} v_{(ik)} \qquad \chi_{y(ij)} \cong \sum_{k=1}^{M} \varsigma^{y(1)}_{jk} \beta_{y(ik)}$$

$$\gamma^0_{xy(ij)} \cong \sum_{k=1}^{N} \varsigma_{ik}^{x(1)} v_{(kj)} + \sum_{k=1}^{M} \varsigma_{jk}^{y(1)} u_{(ik)} \qquad \chi_{xy(ij)} \cong \sum_{k=1}^{N} \varsigma_{ik}^{x(1)} \beta_{y(kj)} + \sum_{k=1}^{M} \varsigma_{jk}^{y(1)} \beta_{x(ik)} \tag{3}$$

The symbol ς_{ij} indicates the GDQ weighting coefficient that are evaluated as in literature (Tornabene 2009, 2011a, b, c, and Tornabene *et al.* 2009, 2010, 2011, 2013a). The in-plane deformation components, in discrete form, are carried out from the strain characteristics as

$$\begin{aligned} \varepsilon_{x(ijm)} &= \varepsilon^0_{x(ij)} + z_m \chi_{x(ij)} \\ \varepsilon_{y(ijm)} &= \varepsilon^0_{y(ij)} + z_m \chi_{y(ij)} \\ \gamma_{xy(ijm)} &= \gamma^0_{xy(ij)} + z_m \chi_{xy(ij)} \end{aligned} \tag{4}$$

It is pointed out that, Eq. (3) is defined upon the middle surface, whereas expressions Eq. (4) are also functions of the normal abscissa z. As it is well-known, the shear strains γ_{xz}, γ_{yz} can be calculated only approximately with FSDT. It is assumed that the shear strain and stresses are constant through the thickness. Recalling the hypothesis of small thickness, the transversal strain ε_z is also negligible. Nevertheless, it will be shown how to recover these strains using the constitutive equations of the three dimensional solid.

Considering the stress definitions $\sigma_x, \sigma_y, \tau_{xy}$ (Reddy 2003 and Tornabene 2012), the in-plane stress components can be obtained as functions of the strain components Eq. (4) as

$$\begin{aligned} \sigma_{x(ijm)} &= \bar{Q}_{11}^{(m)} \varepsilon_{x(ijm)} + \bar{Q}_{12}^{(m)} \varepsilon_{y(ijm)} + \bar{Q}_{16}^{(m)} \gamma_{xy(ijm)} \\ \sigma_{y(ijm)} &= \bar{Q}_{12}^{(m)} \varepsilon_{x(ijm)} + \bar{Q}_{22}^{(m)} \varepsilon_{y(ijm)} + \bar{Q}_{26}^{(m)} \gamma_{xy(ijm)} \\ \tau_{xy(ijm)} &= \bar{Q}_{16}^{(m)} \varepsilon_{x(ijm)} + \bar{Q}_{26}^{(m)} \varepsilon_{y(ijm)} + \bar{Q}_{66}^{(m)} \gamma_{xy(ijm)} \end{aligned} \tag{5}$$

Once the stress components are known, the GDQ method can be used to approximately retrieve their derivatives $\sigma_{x,x}, \sigma_{y,y}, \tau_{xy,x}, \tau_{xy,y}$ as

$$\begin{aligned} \left.\frac{\partial \sigma_x}{\partial x}\right|_{(ijm)} &\cong \sum_{k=1}^{N} \varsigma_{ik}^{x(1)} \sigma_{x(kjm)} \qquad & \left.\frac{\partial \tau_{xy}}{\partial x}\right|_{(ijm)} &\cong \sum_{k=1}^{N} \varsigma_{ik}^{x(1)} \tau_{xy(kjm)} \\ \left.\frac{\partial \sigma_y}{\partial y}\right|_{(ijm)} &\cong \sum_{k=1}^{M} \varsigma_{jk}^{y(1)} \sigma_{y(ikm)} \qquad & \left.\frac{\partial \tau_{xy}}{\partial y}\right|_{(ijm)} &\cong \sum_{k=1}^{M} \varsigma_{jk}^{y(1)} \tau_{xy(ikm)} \end{aligned} \tag{6}$$

It is important to calculate Eq. (6) because they can be used in the two indefinite equilibrium equations of the three-dimensional elasticity (Reddy 2003 and Tornabene 2012) to recover the out-of-plane shear stresses τ_{xz}, τ_{yz}. These equations are reported in the following:

$$\begin{aligned} \frac{\partial \tau_{xz}}{\partial z} &= -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} \\ \frac{\partial \tau_{yz}}{\partial z} &= -\frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_y}{\partial y} \end{aligned} \tag{7}$$

Since the quantities $\sigma_{x,x}, \sigma_{y,y}, \tau_{xy,x}, \tau_{xy,y}$ were calculated using Eq. (6) all the terms on the right hand of Eq. (7) are known. It is noted that, the two equilibrium equations Eq. (7) are independent, so they can be solved separately. The discrete form of Eq. (7) as a function of the index $m = 2,...,T$ is

$$\sum_{k=1}^{T} \varsigma_{mk}^{z(1)} \overline{\tau}_{xz(ijk)} = -\left.\frac{\partial \sigma_x}{\partial x}\right|_{(ijm)} - \left.\frac{\partial \tau_{xy}}{\partial y}\right|_{(ijm)} \tag{8}$$

$$\sum_{k=1}^{T} \varsigma_{mk}^{z(1)} \overline{\tau}_{yz(ijk)} = -\left.\frac{\partial \tau_{xy}}{\partial x}\right|_{(ijm)} - \left.\frac{\partial \sigma_y}{\partial y}\right|_{(ijm)} \tag{9}$$

In order to solve the two algebraic systems of Eqs. (8) and (9), the boundary conditions must be introduced. As already stated in literature (Tornabene 2012, Tornabene *et al.* 2012b, c and Viola *et al.* 2012, 2013b, c) these systems are of the first order but have two distinct boundary conditions to satisfy

$$\tau_{xz}\left(z = -h/2\right) = q_x^{(-)} \quad \text{or} \quad \tau_{xz}\left(z = h/2\right) = q_x^{(+)} \tag{10}$$

$$\tau_{yz}\left(z = -h/2\right) = q_y^{(-)} \quad \text{or} \quad \tau_{yz}\left(z = h/2\right) = q_y^{(+)} \tag{11}$$

The bottom boundary conditions are used to solve Eqs. (8) and (9) and the second one is used within a linear correction of the obtained profile. The first boundary conditions at $z = -h/2$ in discrete form are

$$\overline{\tau}_{xz(ij1)} = q_{x(ij)}^{(-)} \qquad \overline{\tau}_{yz(ij1)} = q_{y(ij)}^{(-)} \tag{12}$$

whereas, the second conditions

$$\overline{\tau}_{xz(ijT)} = q_{x(ij)}^{(+)} \qquad \overline{\tau}_{yz(ijT)} = q_{y(ij)}^{(+)} \tag{13}$$

are satisfied with the following linear correction

$$\begin{aligned} \tau_{xz(ijm)} &= \overline{\tau}_{xz(ijm)} + \frac{q_{x(ij)}^{(+)} - \overline{\tau}_{xz(ijT)}}{h}\left(z_m + \frac{h}{2}\right) \\ \tau_{yz(ijm)} &= \overline{\tau}_{yz(ijm)} + \frac{q_{y(ij)}^{(+)} - \overline{\tau}_{yz(ijT)}}{h}\left(z_m + \frac{h}{2}\right) \end{aligned} \qquad \text{for } m = 1,...,T \tag{14}$$

Once the shear stresses τ_{xz} and τ_{yz} are recovered, the normal stress σ_n can be derived from the third indefinite equilibrium equation of the three-dimensional elasticity. The present equations is

$$\frac{\partial \sigma_z}{\partial z} = -\frac{\partial \tau_{xz}}{\partial x} - \frac{\partial \tau_{yz}}{\partial y} \tag{15}$$

The derivatives $\tau_{xz,x}$ and $\tau_{yz,y}$ are calculated as the previous ones (see Eq. (6))

$$\left.\frac{\partial \tau_{xz}}{\partial x}\right|_{(ijm)} \cong \sum_{k=1}^{N} \varsigma_{ik}^{x(1)} \tau_{xz(kjm)} \qquad \left.\frac{\partial \tau_{yz}}{\partial y}\right|_{(ijm)} \cong \sum_{k=1}^{M} \varsigma_{jk}^{y(1)} \tau_{yz(ikm)} \tag{16}$$

so that all the terms on the right hand of Eq. (15) are known. Thus, the third 3D equilibrium equation in discrete form becomes

$$\sum_{k=1}^{T} \varsigma_{mk}^{z(1)} \bar{\sigma}_{z(ijk)} = -\left.\frac{\partial \tau_{xz}}{\partial x}\right|_{(ijm)} - \left.\frac{\partial \tau_{yz}}{\partial y}\right|_{(ijm)} \tag{17}$$

with the index $m = 2,...,T$ variable along the thickness of the plate. It should be pointed out that, also in this case, two boundary conditions occur for a first order differential problem. The two boundary conditions are

$$\sigma_z\left(z = -h/2\right) = q_z^{(-)} \quad \text{or} \quad \sigma_z\left(z = h/2\right) = q_z^{(+)} \tag{18}$$

where $q_z^{(-)}$ and $q_z^{(+)}$ are the normal loads applied at the bottom and top surface of the structure, respectively. In discrete form, the first boundary condition at the bottom surface of the plate is taken into account for solving equation Eq. (17)

$$\bar{\sigma}_{z(ij1)} = q_{z(ij)}^{(-)} \tag{19}$$

Afterwards, the normal stress profile has to be corrected using the second boundary condition

$$\bar{\sigma}_{z(ijT)} = q_{z(ij)}^{(+)} \tag{20}$$

with a relationship similar to the ones proposed above (see Eq.(14))

$$\sigma_{z(ijm)} = \bar{\sigma}_{z(ijm)} + \frac{q_{z(ij)}^{(+)} - \bar{\sigma}_{z(ijT)}}{h}\left(z_m + \frac{h}{2}\right) \qquad \text{for } m = 1,...,T \tag{21}$$

In the introductory part of the present section it was recalled that, the shear strains were supposed to be constant and the normal strain was negligible according to the main hypotheses of FSDT. However, they can be recovered like the shear and normal stresses considering the constitutive equations of a laminated composite plate. For a generically oriented lamina the shear stresses can be calculated as

$$\begin{bmatrix} \tau_{xz}^{(k)} \\ \tau_{yz}^{(k)} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{44}^{(k)} & \bar{Q}_{45}^{(k)} \\ \bar{Q}_{45}^{(k)} & \bar{Q}_{55}^{(k)} \end{bmatrix} \begin{bmatrix} \gamma_{xz}^{(k)} \\ \gamma_{yz}^{(k)} \end{bmatrix} \tag{22}$$

Since the shear stresses τ_{xz} and τ_{yz} have been recovered, the shear strains γ_{xz} and γ_{yz} can be obtained

$$\begin{bmatrix} \gamma_{xz}^{(k)} \\ \gamma_{yz}^{(k)} \end{bmatrix} = \frac{1}{\bar{Q}_{55}^{(k)}\bar{Q}_{44}^{(k)} - \left(\bar{Q}_{45}^{(k)}\right)^2} \begin{bmatrix} \bar{Q}_{55}^{(k)} & -\bar{Q}_{45}^{(k)} \\ -\bar{Q}_{45}^{(k)} & \bar{Q}_{44}^{(k)} \end{bmatrix} \begin{bmatrix} \tau_{xz}^{(k)} \\ \tau_{yz}^{(k)} \end{bmatrix} \tag{23}$$

Eq. (23) leads to the following discrete form

$$\begin{gathered} \gamma_{xz(ijm)} = \frac{\bar{Q}_{55}^{(m)}\tau_{xz(ijm)} - \bar{Q}_{45}^{(m)}\tau_{yz(ijm)}}{\bar{Q}_{55}^{(m)}\bar{Q}_{44}^{(m)} - \left(\bar{Q}_{45}^{(m)}\right)^2} \\ \gamma_{yz(ijm)} = \frac{\bar{Q}_{44}^{(m)}\tau_{yz(ijm)} - \bar{Q}_{45}^{(m)}\tau_{xz(ijm)}}{\bar{Q}_{55}^{(m)}\bar{Q}_{44}^{(m)} - \left(\bar{Q}_{45}^{(m)}\right)^2} \end{gathered} \tag{24}$$

Now, considering the unreduced elastic constants from the constitutive equations

$$\boldsymbol{\sigma} = \bar{\boldsymbol{C}}\boldsymbol{\varepsilon} \tag{25}$$

where $\bar{C}_{ij}^{(k)}$ are the elastic coefficients obtained considering the problem reference system and not the material reference system, the normal strain ε_z can be investigated. As shown in literature (Reddy 2003 and Tornabene 2012) the stiffness matrix $\bar{\boldsymbol{C}}$ is defined as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{bmatrix} \tag{26}$$

By inverting the third of Eq. (26) the normal strain ε_z can be worked out

$$\varepsilon_z^{(k)} = \frac{\sigma_z^{(k)} - \bar{C}_{13}^{(k)}\varepsilon_x^{(k)} - \bar{C}_{23}^{(k)}\varepsilon_y^{(k)} - \bar{C}_{36}^{(k)}\gamma_{xy}^{(k)}}{\bar{C}_{33}^{(k)}} \tag{27}$$

The generic $\bar{C}_{ij}^{(k)}$ constants for an orthotropic material can be evaluated using the $\bar{C}_{ij}^{(k)}$ constants, where $C_{36} = 0$, as

$$\begin{aligned} \bar{C}_{13} &= C_{13}\cos^2\theta + C_{23}\sin^2\theta \\ \bar{C}_{23} &= C_{23}\cos^2\theta + C_{13}\sin^2\theta \\ \bar{C}_{33} &= C_{33} \\ \bar{C}_{36} &= \left(C_{13} - C_{23}\right)\cos\theta\sin\theta \end{aligned} \tag{28}$$

For the sake of completeness the $\bar{C}_{ij}^{(k)}$ stiffness constants are reported as a function of the mechanical properties of a generic ply

$$
\begin{aligned}
&C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} && C_{23} = \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_3\Delta} \\
&C_{12} = \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2E_3\Delta} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta} && C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} \\
&C_{13} = \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta} && C_{44} = G_{13} \\
&C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} && C_{55} = G_{23} \\
& && C_{66} = G_{12} \\
&\text{with} \quad \Delta = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}
\end{aligned}
\tag{29}
$$

Considering Eq. (29) and substituting into Eq. (28), the values of the coefficients $\overline{C}_{ij}^{(k)}$ are known and the normal strain ε_z is calculated as a special case of an orthotropic material. Thus the discrete form of Eq. (27) can be written as

$$
\varepsilon_{z(ijm)} = \frac{\sigma_{z(ijm)} - \overline{C}_{13}^{(m)}\,\varepsilon_{x(ijm)} - \overline{C}_{23}^{(m)}\varepsilon_{y(ijm)} - \overline{C}_{36}^{(m)}\gamma_{xy(ijm)}}{\overline{C}_{33}^{(m)}}
\tag{30}
$$

Since an approximated value of the shear and normal strains is found, the normal stresses Eq. (5), calculated at the beginning of the procedure, can be corrected using Eqs. (24) and (30). In fact, instead of using the constitutive equations with the reduced constants $\overline{Q}_{ij}^{(k)}$ the general Eq. (26) can be used instead for evaluating the first three equations. In the following numerical applications the in-plane stresses are implicitly calculated through this procedure and they are not the ones from Eq. (5).

3. Results of the strain and stress recovery

In the present section, the post processing technique of stress recovery is applied to the static analysis of arbitrarily shaped laminated composite plates. In order to show the accuracy of the present formulation a square composite plate is firstly studied. For the present case the shear and normal stress profiles through the thickness of the plate are shown for several lamination schemes compared to literature and to numerical results carried out by using a commercial finite element code. The plate geometry is depicted in Fig. 1 and it spotlights the points where the stresses have been recovered and presented in Figs. 2-4. The plate has two equal sides $a = b = 1\,\text{m}$ and constant thickness $h = 0.1\,\text{m}$. Each composite lamina is made of Graphite-Epoxy: $E_1 = 137.9\,\text{GPa}$, $E_2 = E_3 = 8.96\,\text{GPa}$, $G_{12} = G_{13} = 7.1\,\text{GPa}$, $G_{23} = 6.21\,\text{GPa}$, $\nu_{12} = \nu_{13} = 0.3$, $\nu_{23} = 0.49$ and $\rho = 1450\ \text{kg/m}^3$. The mesh is made of four square elements $n_e = 4$, with a C-G-L grid using $N = M = 21$ points on the middle surface and $T = 51$ points along the thickness of each single element. Finally, it

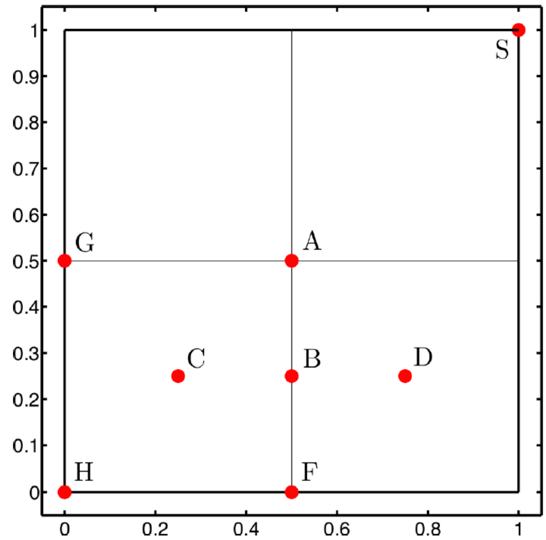


Fig. 1 Identification of the points used for the stress profiles and square plate geometry ($a=b=1\,\mathrm{m}$)

must be underlined that in all the following figures the acronym GDQFEM is indicated. A particular version of SFEM when GDQ is used for approximating the derivative can be also called GDQFEM as in the work (Viola *et al.* 2013a and Fantuzzi *et al.* 2014).

The first comparison, in terms of shear stresses τ_{xz} and τ_{yz} , is shown in Fig. 2 with respect to different cross-ply lamination schemes ($0/90/0/\ldots$). The reference solution is a semi-analytical solution given by (Reddy 2003) for a simply supported square plate subjected to a uniformly distributed load at the top surface of the plate $q_z^{(+)}=-8\,\mathrm{kPa}$. Perfect agreement is observed in Fig. 2 for all the shown cases.

The second example (Fig. 3) reports the stress tensor at the point $C=(0.25a,0.25a)$, for a ($90/0$) simply supported square plate subjected to $q_z^{(+)}=-10\,\mathrm{kPa}$ uniformly distributed top load.

It should be pointed out that the in-plane stresses have been corrected using the recovered shear and normal strains Eqs. (24) and (30), in fact there is a very good agreement between the present solution and the exact solution given by Reddy (2003).

For the following solution a three layered ($0/90/0$) simply supported plate subjected to a uniform load $q_z^{(+)}=-10\,\mathrm{kPa}$ is considered. The comparison shown in Fig. 4 is performed for several points of the plate, some of them are taken inside the plate geometry and some others at the plate boundaries. It is underlined that for all the reported cases very good agreement is observed.

In the following, other comparisons have been done with respect to a three dimensional finite element solution. The meshes used in the computations are depicted in Fig. 5. In detail Figs. 5(a)-5(c) represent three finite element meshes with 20 nodes brick elements with 3 dofs per node. Fig. 5(d) shows the SFEM mesh, where the four elements are presented with 8 nodes. Inside each element a C-G-L $N=M=21$ grid is considered. It must be noted the different computational effort between the 3D models with 319300 dofs and the 2D one with 8820 dofs. Thus, the present two dimensional approach brings a great advantage compared to 3D FEM because it leads to the same results with a lot less degrees of freedom.

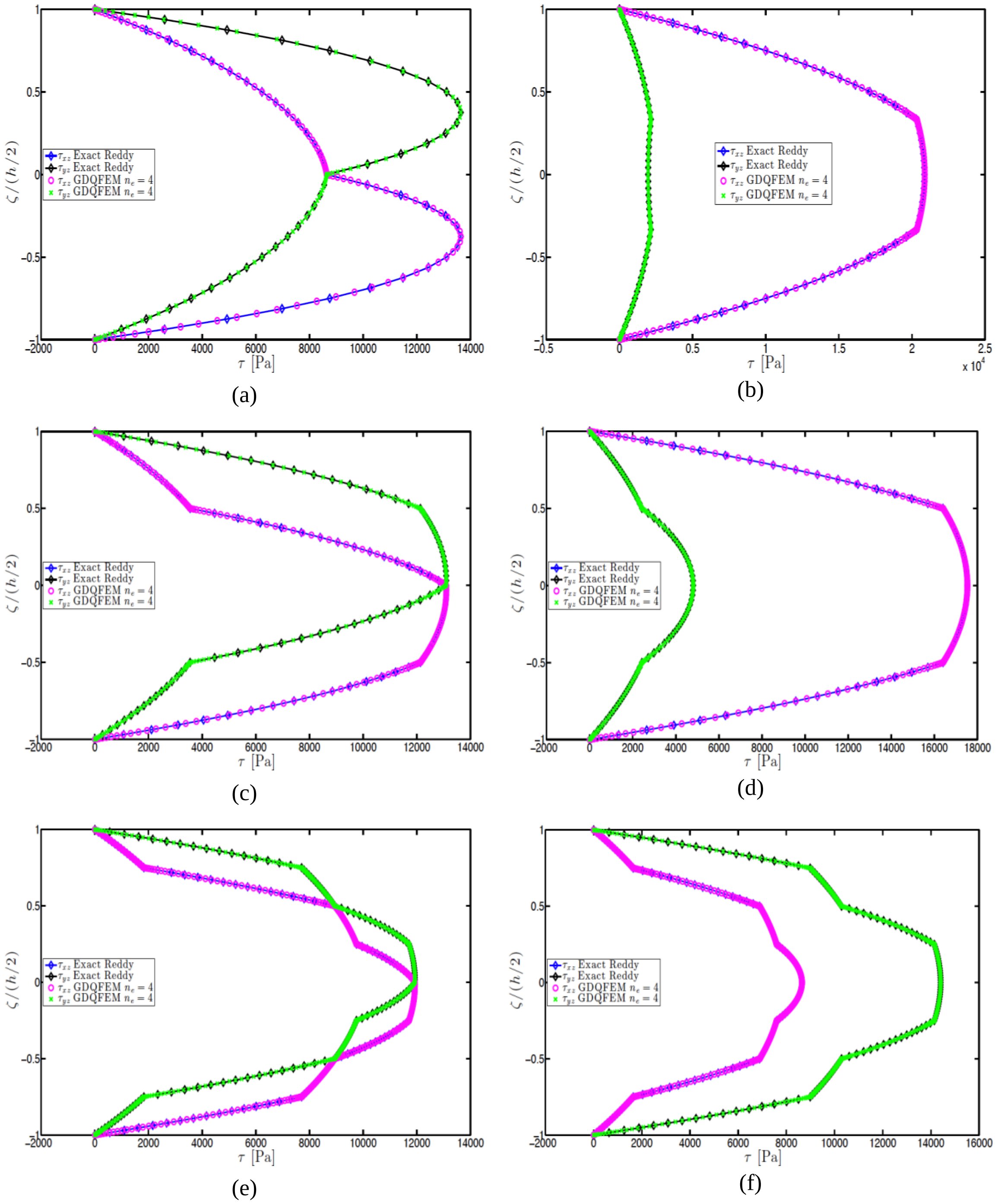


Fig. 2 Through the thickness variation of the shear stresses τ_{xz}, τ_{yz} [Pa] at the point $C = (0.25a, 0.25a)$ for a S-S-S-S square plate subjected to a uniformly distributed load at the top surface $q_n^{(+)} = -8\,\text{kPa}$ for different lamination schemes: (a) $(0/90)$, (b) $(0/90/0)$, (c) $(0/90/0/90)$, (d) $(0/90/90/0)$, (e) $(0/90/0/90/0/90/0/90)$, (f) $(90/0/90/0/0/90/0/90)$

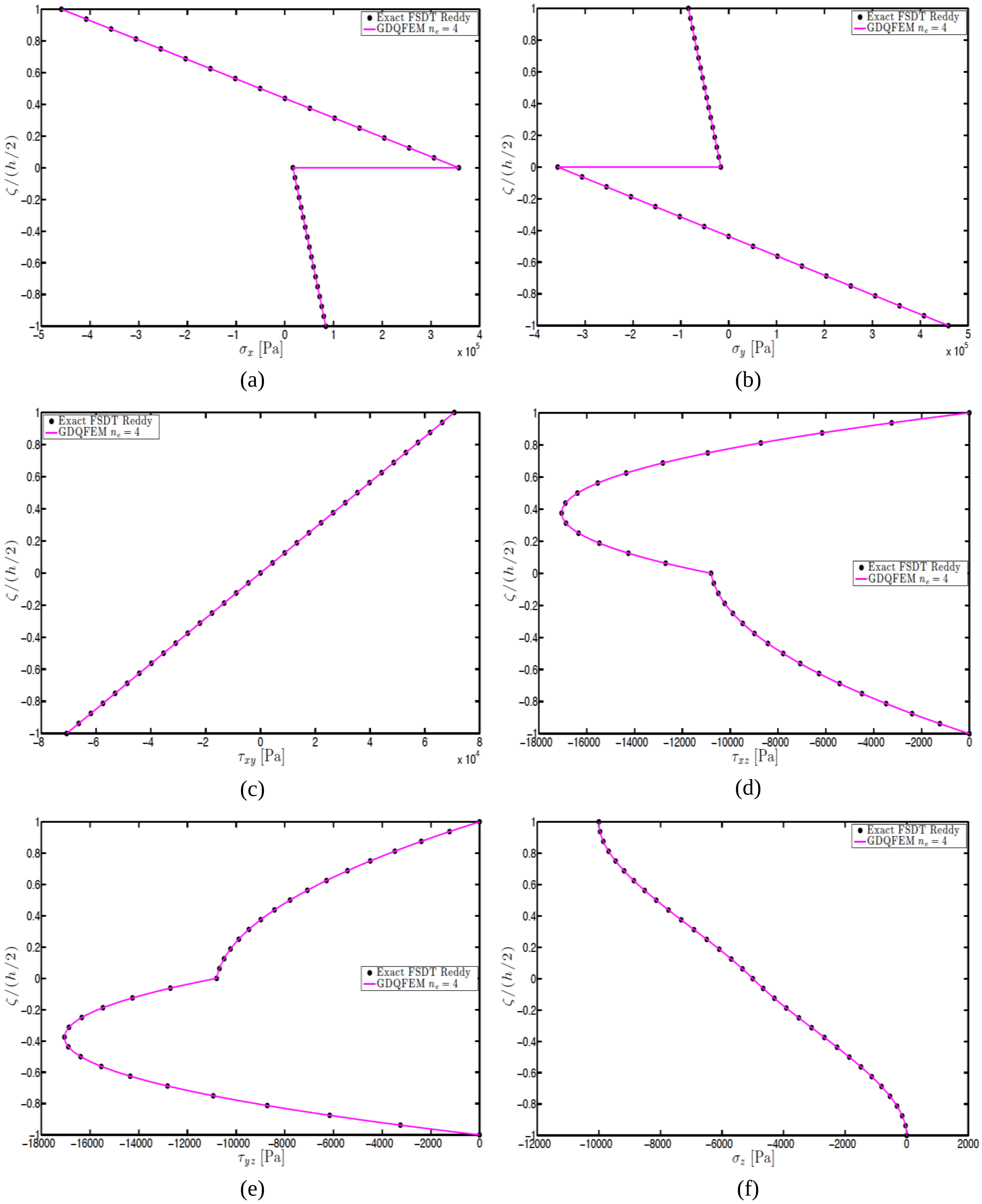


Fig. 3 Through the thickness variation of the stress components [Pa] at the point $C = (0.25a, 0.25a)$ for a $(90/0)$ S-S-S-S square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$

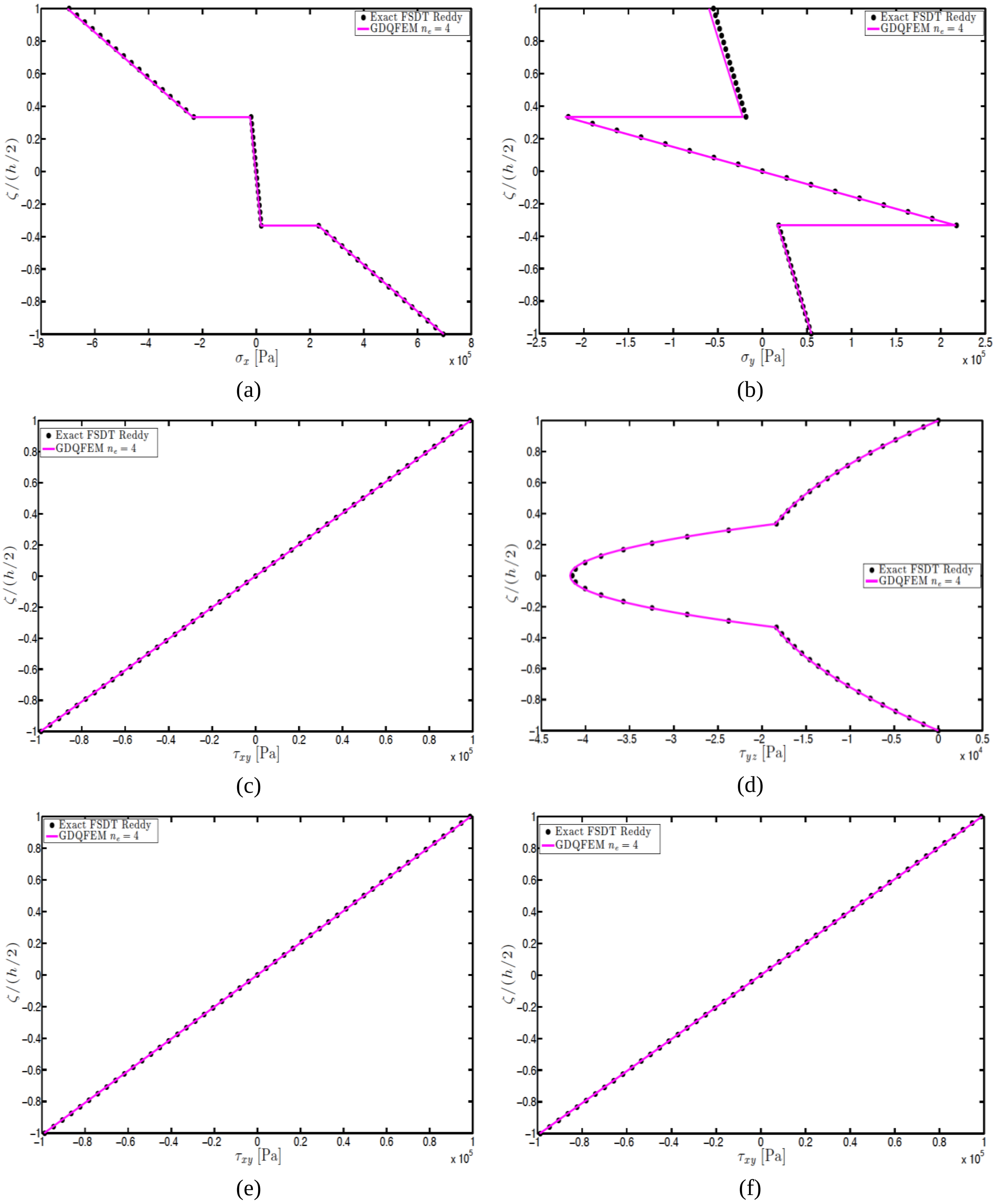


Fig. 4 Through the thickness variation of the stress components [Pa] for a $(0/90/0)$ S-S-S-S square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$ at different points: (a) $C = (0.25a, 0.25a)$, (b) $C = (0.25a, 0.25a)$, (c) $G = (0, 0.5a)$, (d) $F = (0.5a, 0)$, (e) $H = (0,0)$, (f) $S = (a, a)$

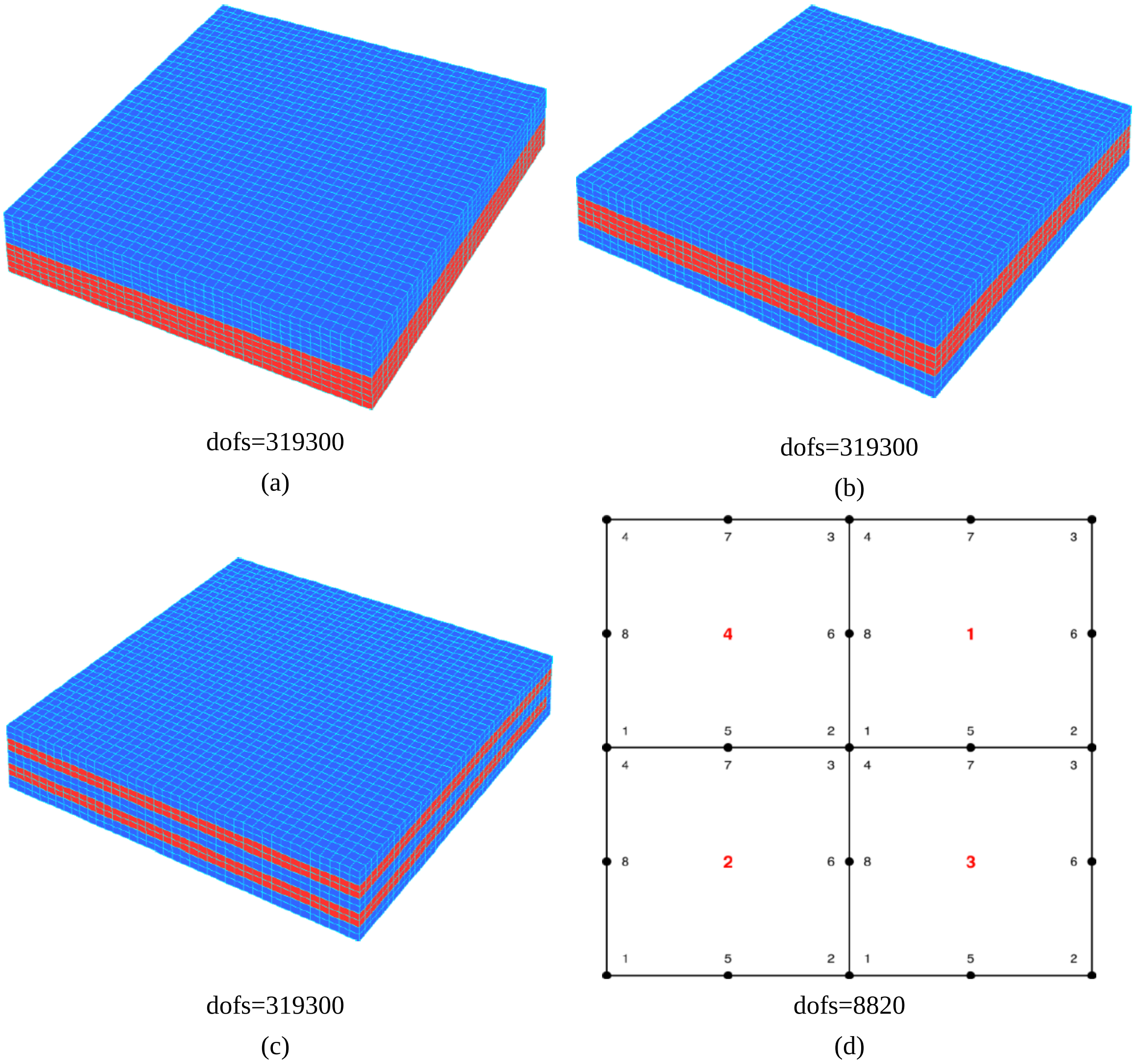


Fig. 5 Model comparison among three 3D finite element models with dofs = 319300: (a) a two plies composite plate, (b) a three plies composite plate and (c) a five plies composite plate, (d) a SFEM mesh with $n_e = 4$ and dofs = 8820

In Figs. 6 and 7 a clamped square plate with a $(90/0)$ lamination scheme subjected to a uniform top load $q_z^{(+)} = -10\,\text{kPa}$ is investigated. Two different points are shown $C = (0.25a, 0.25a)$ and $B = (0.5a, 0.25a)$. It is noted that due to the particular symmetry of the load and the geometry in *B* two shear stresses are null, whereas in *C* all the stress components are different from zero through the thickness. On the contrary if the same plate is made of $(30/60)$ lamination scheme, that yields to an anisotropic plate behavior, the stress components are all different from zero both in *B* and *C*, as illustrated in Figs. 8 and 9 where very good agreement with FEM is observed.

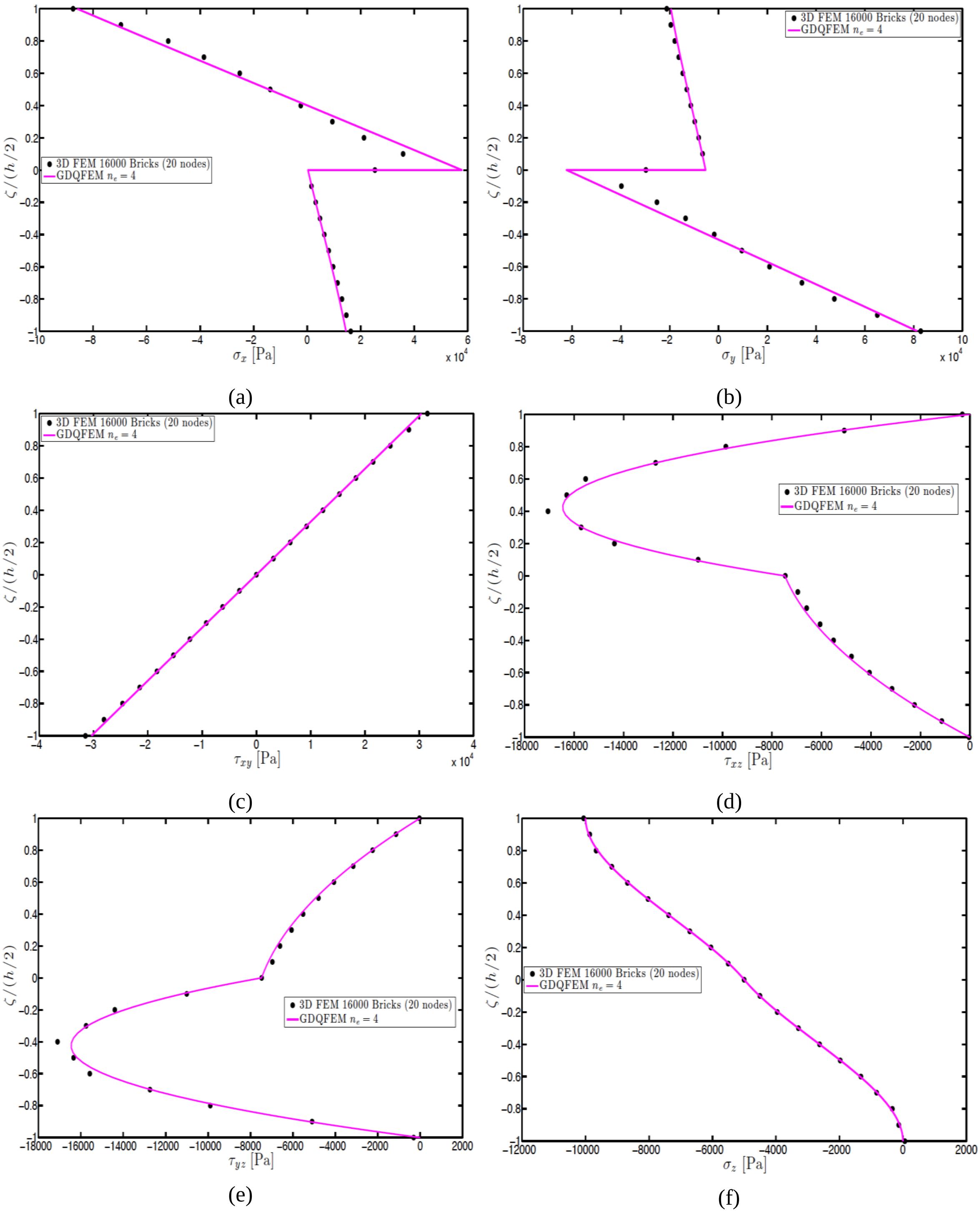
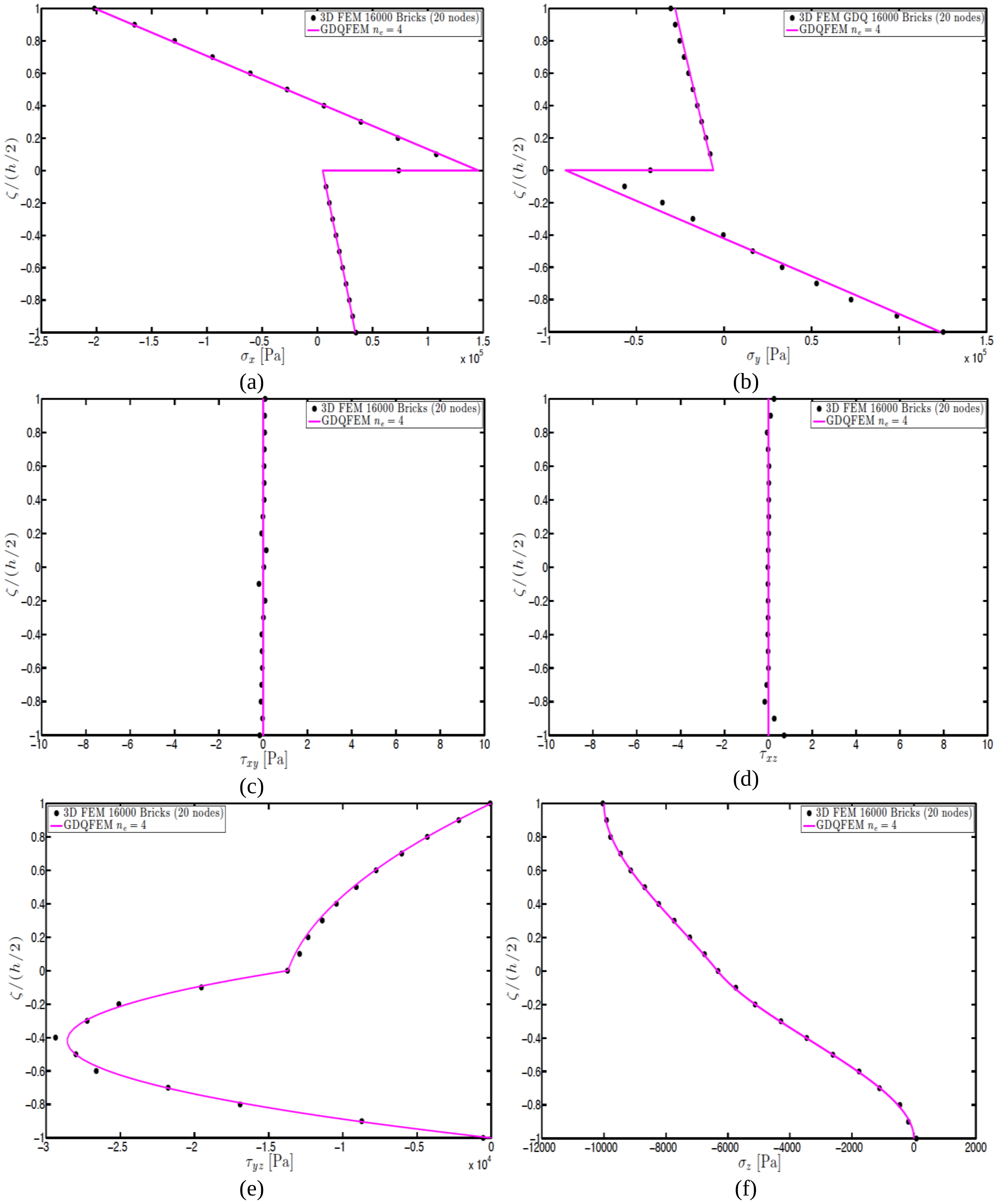


Fig. 6 Through the thickness variation of the stress components [Pa] at the point $C = (0.25a, 0.25a)$ for a $(90/0)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\,\text{m},\ h_1 = h_2 = 0.05\,\text{m}$



(a) (b) (c) (d) (e) (f)

Fig. 7 Through the thickness variation of the stress components [Pa] at the point $B = (0.5a, 0.25a)$ for a $(90/0)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\,\text{m}$, $h_1 = h_2 = 0.05\,\text{m}$

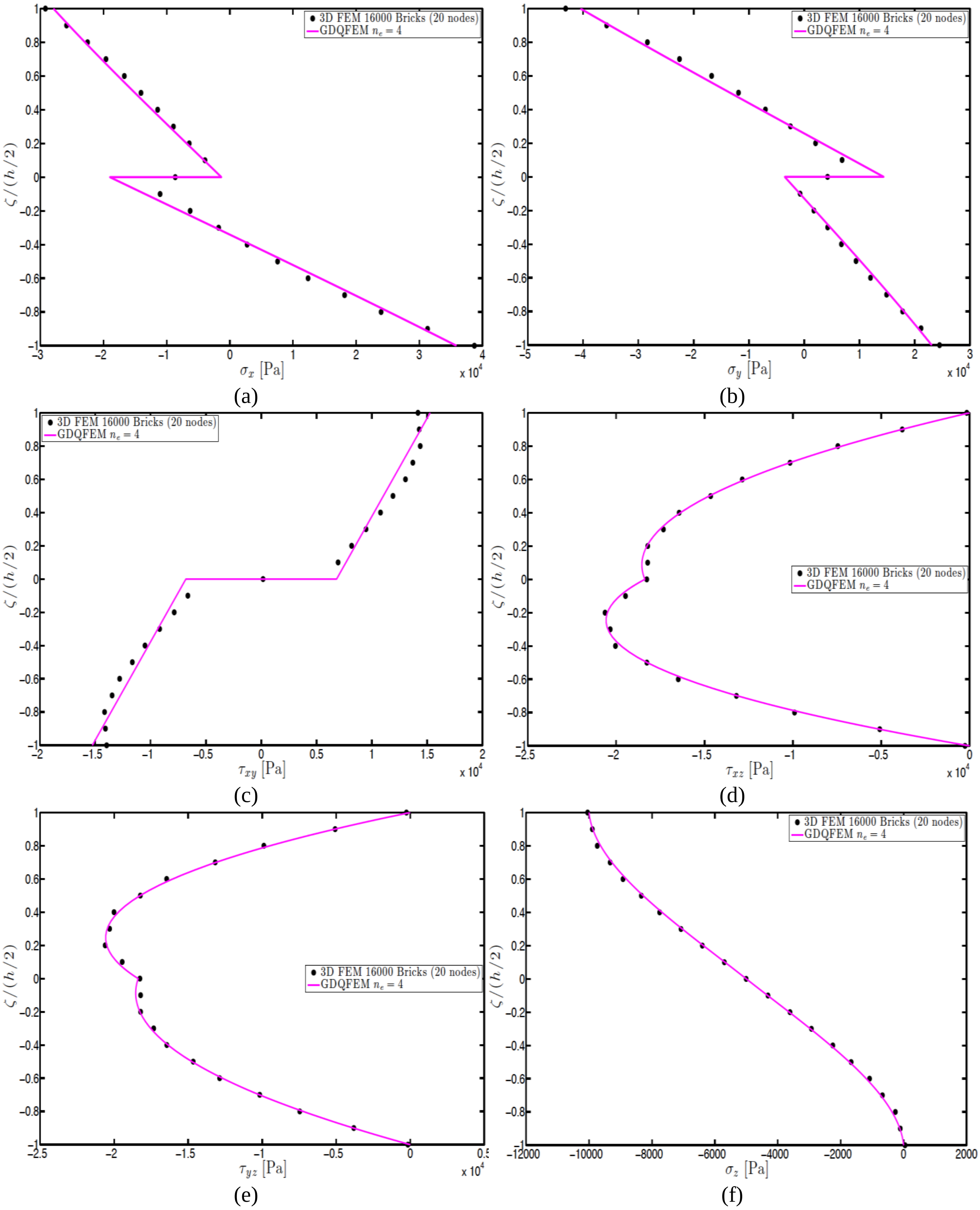


Fig. 8 Through the thickness variation of the stress components [Pa] at the point $C = (0.25a, 0.25a)$ for a $(30/60)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\,\text{m},\ h_1 = h_2 = 0.05\,\text{m}$

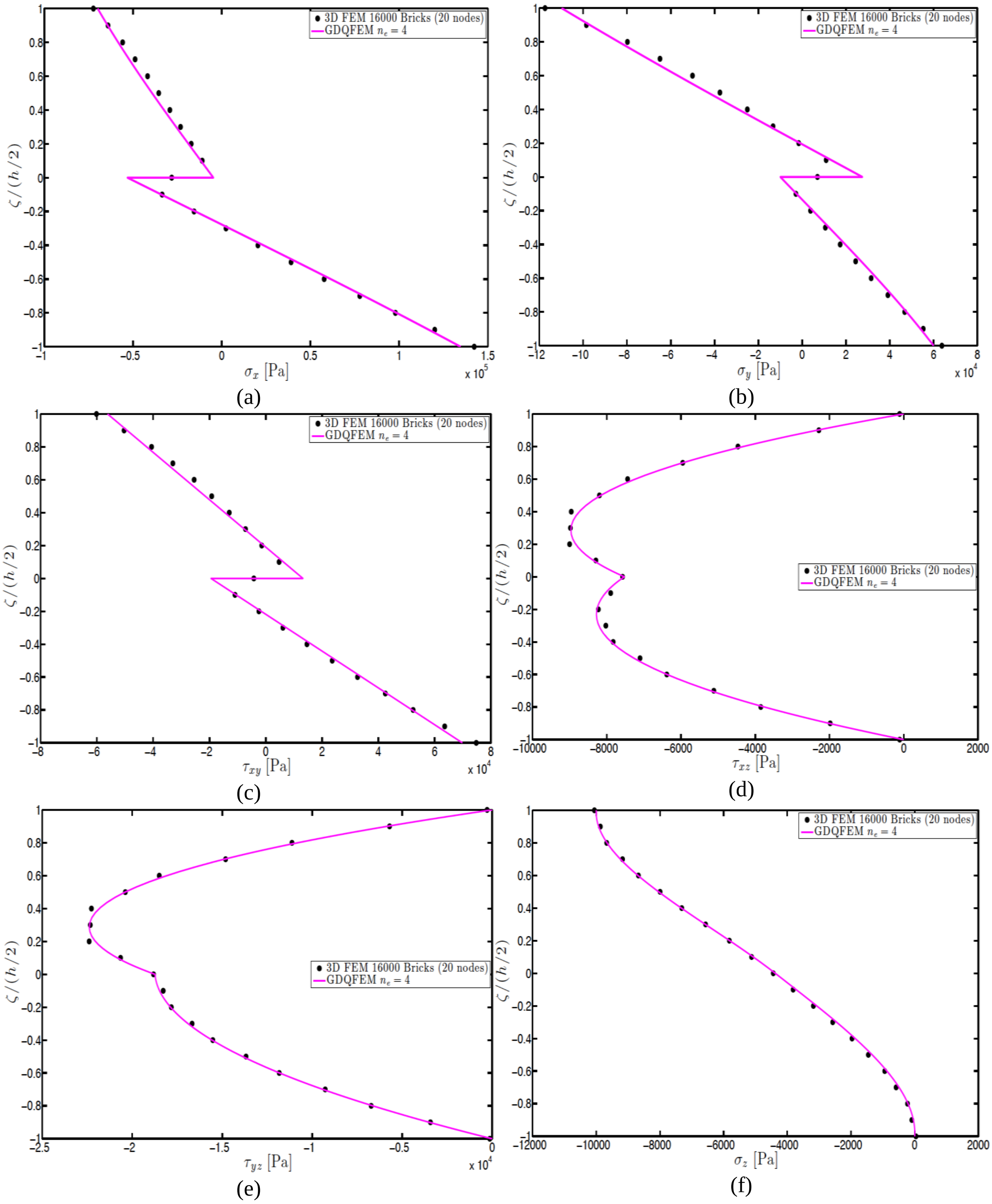


Fig. 9 Through the thickness variation of the stress components [Pa] at the point $B = (0.5a, 0.25a)$ for a (30 / 60) C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\,\text{m}$, $h_1 = h_2 = 0.05\,\text{m}$

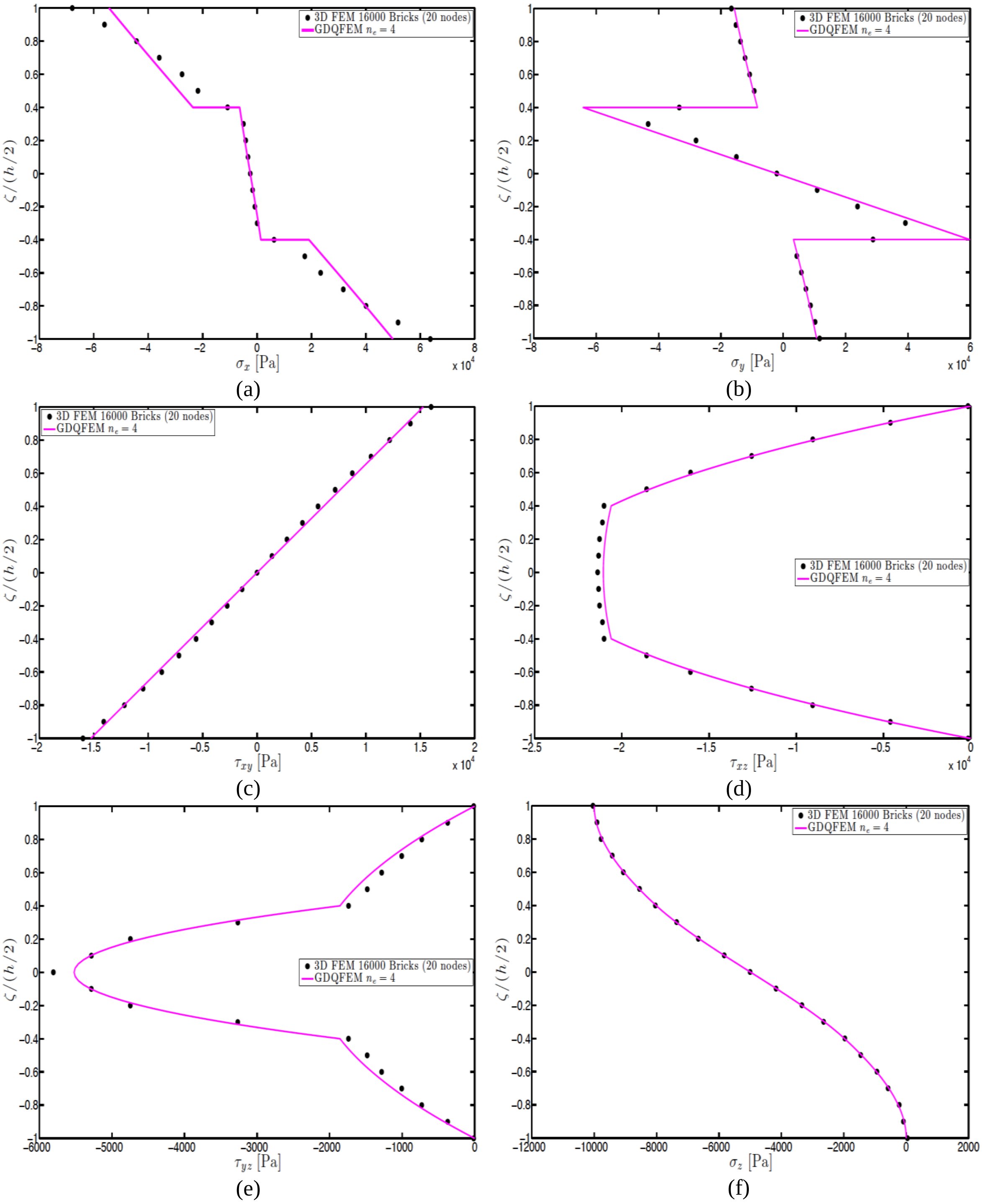


Fig. 10 Through the thickness variation of the stress components [Pa] at the point $C=(0.25a, 0.25a)$ for a $(0/90/0)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)}=-10\,\text{kPa}$. Geometric properties: $a=b=1\,\text{m}$, $h_1=h_3=0.03\,\text{m}$, $h_2=0.04\,\text{m}$

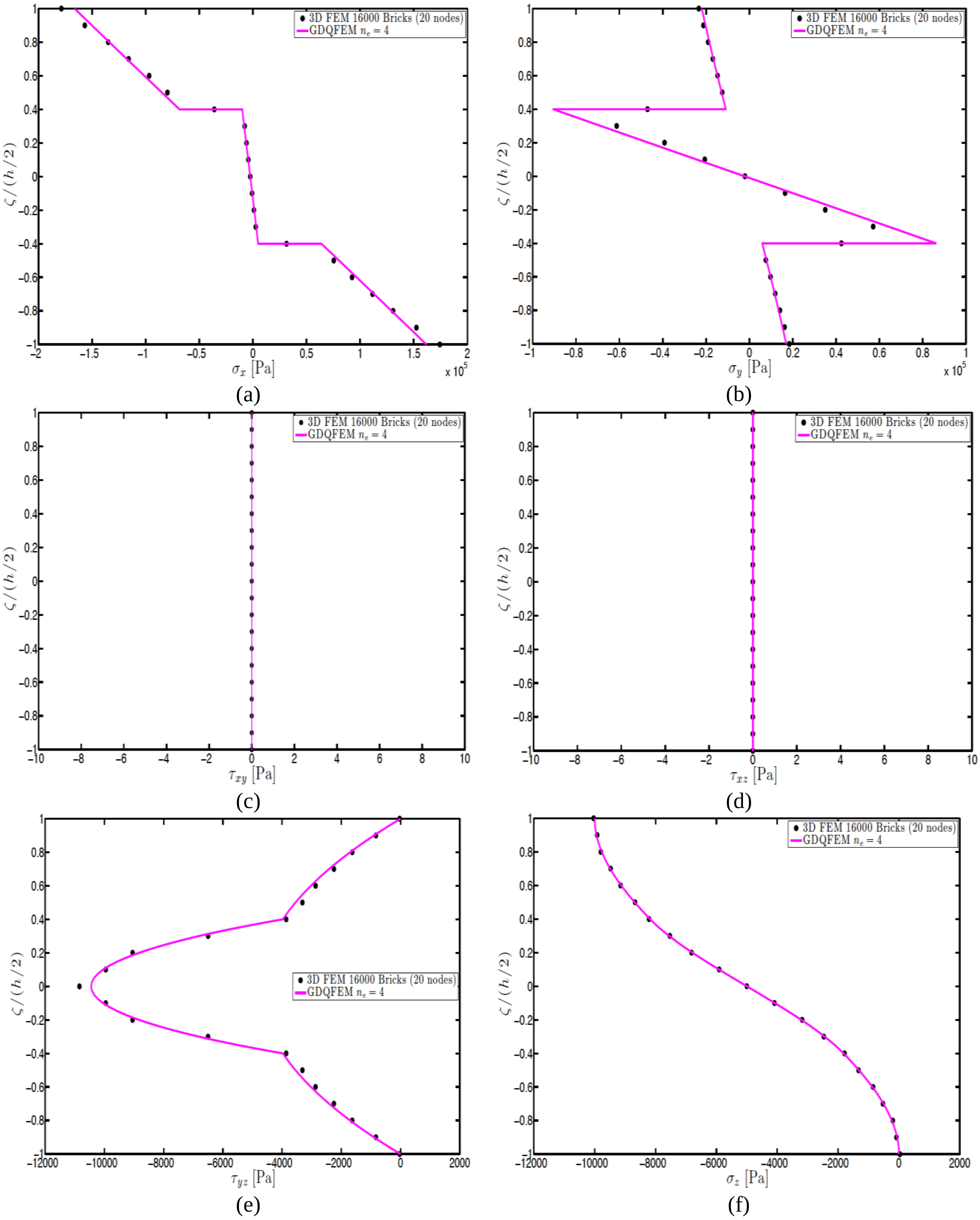


Fig. 11 Through the thickness variation of the stress components [Pa] at the point $B = (0.5a, 0.25a)$ for a $(0/90/0)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}, h_1 = h_3 = 0.03\text{m}, h_2 = 0.04\text{m}$

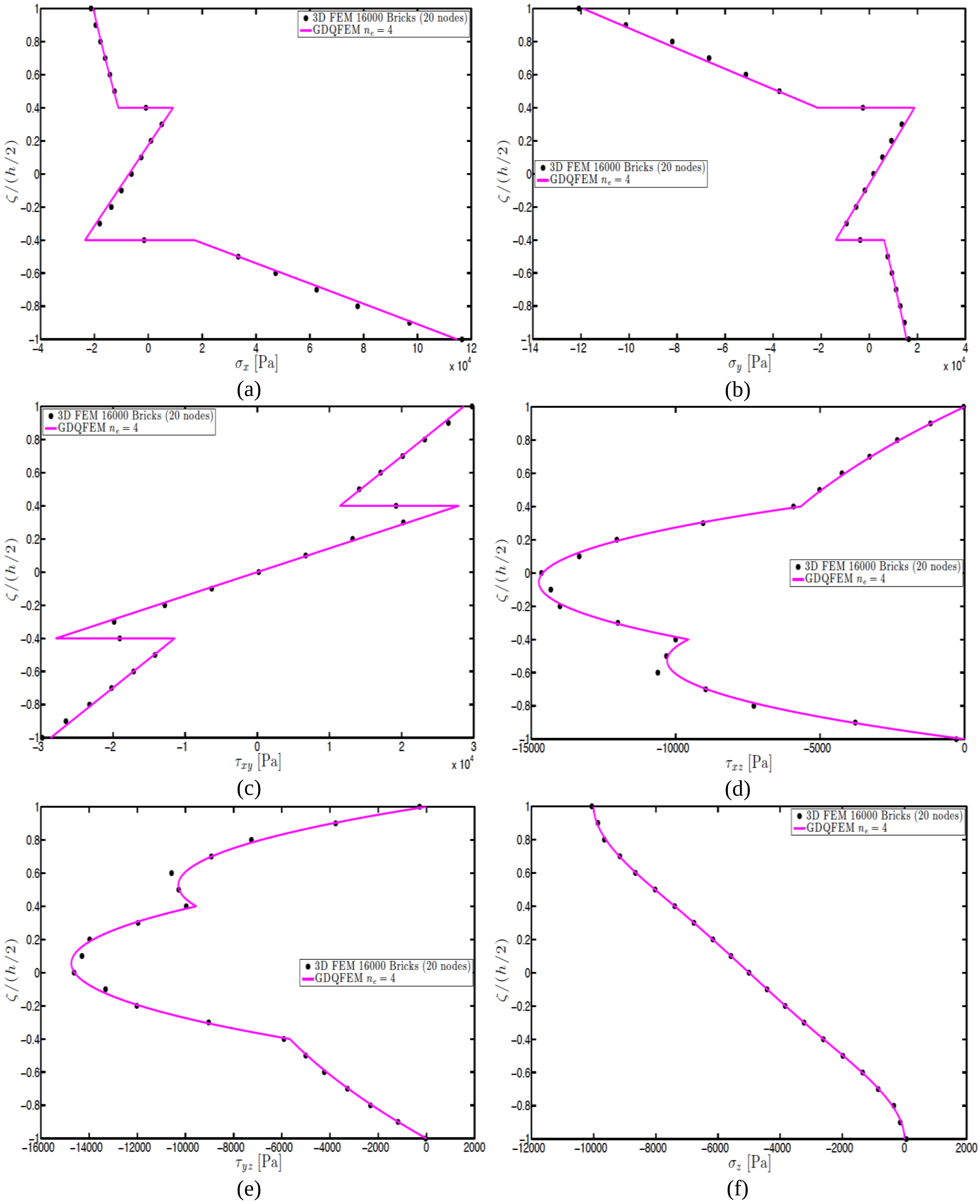


Fig. 12 Through the thickness variation of the stress components [Pa] at the point $C = (0.25a, 0.25a)$ for a $(0/45/90)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}, h_1 = h_3 = 0.03\text{m}, h_2 = 0.04\text{m}$

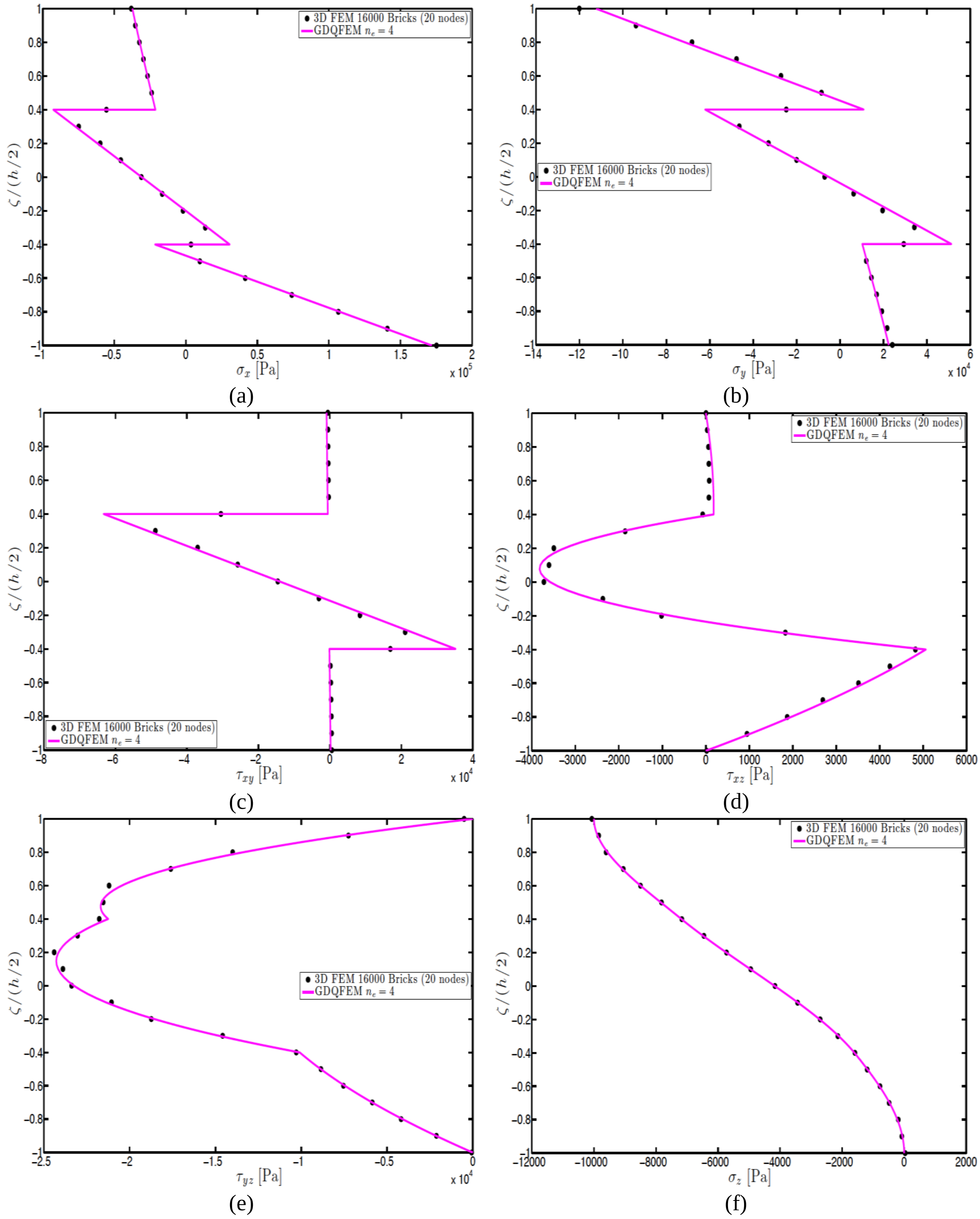


Fig. 13 Through the thickness variation of the stress components [Pa] at the point $B = (0.5a, 0.25a)$ for a $(0/45/90)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}$, $h_1 = h_3 = 0.03\text{m}$, $h_2 = 0.04\text{m}$

Analogously, in Figs. 10-13 a similar approach is shown for a three layer laminated plate. The plate is made of a laminate with the top and bottom layer $h_1 = h_3 = 0.03\text{m}$ and the central one $h_2 = 0.04\text{m}$. The first case (Figs. 10 and 11) is related to a symmetric lamination scheme $(0/90/0)$ that has two negligible stress components in B, as expected, whereas in the second case (Figs. 12 and 13) the lamination scheme is $(10/45/90)$ where all the stress components are different from zero either in B and C.

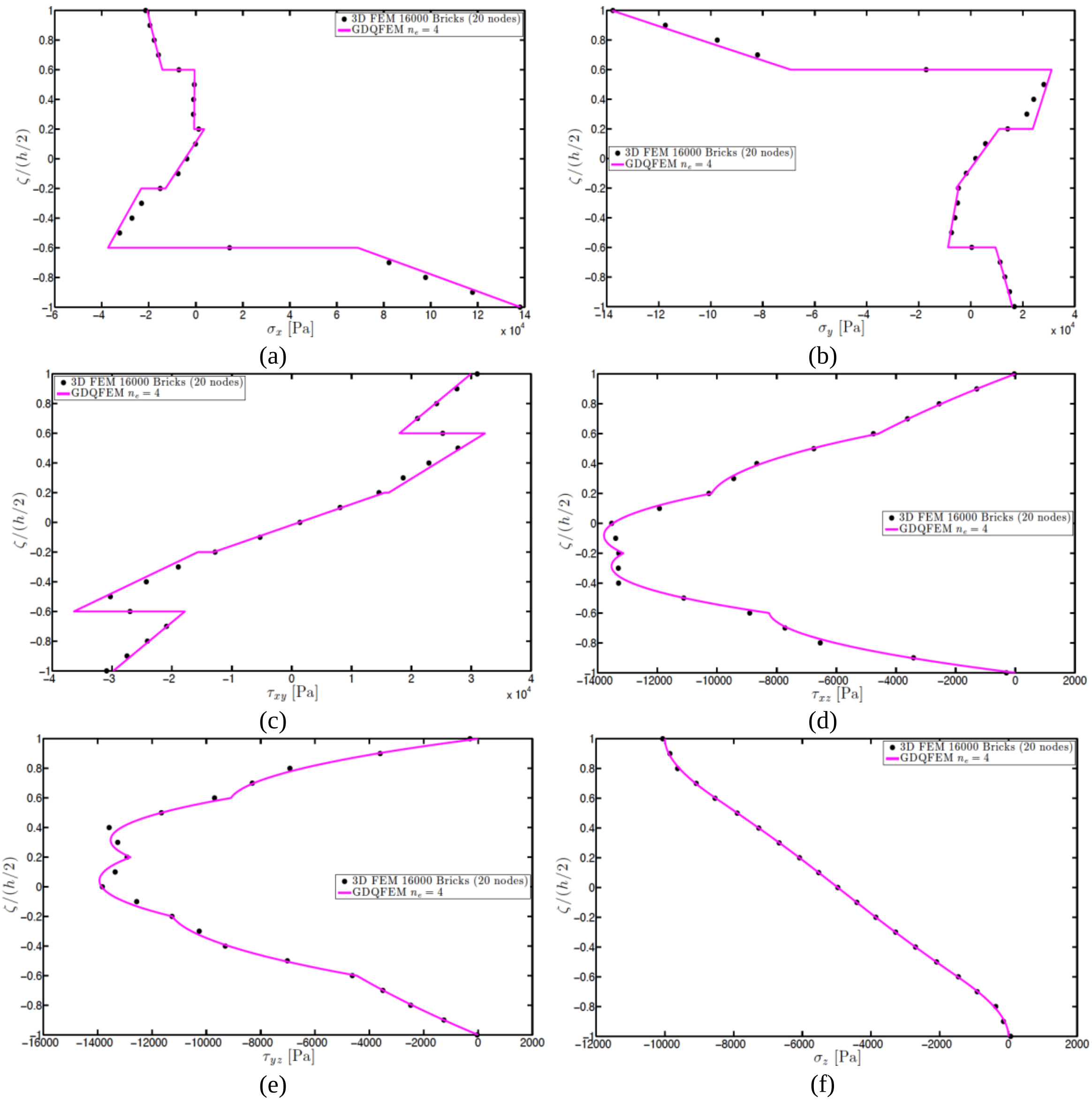


Fig. 14 Through the thickness variation of the stress components [Pa] at the point $C = (0.25a, 0.25a)$ for a $(0/30/45/65/90)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}, h_1 = h_2 = h_3 = h_4 = h_5 = 0.02\text{m}$

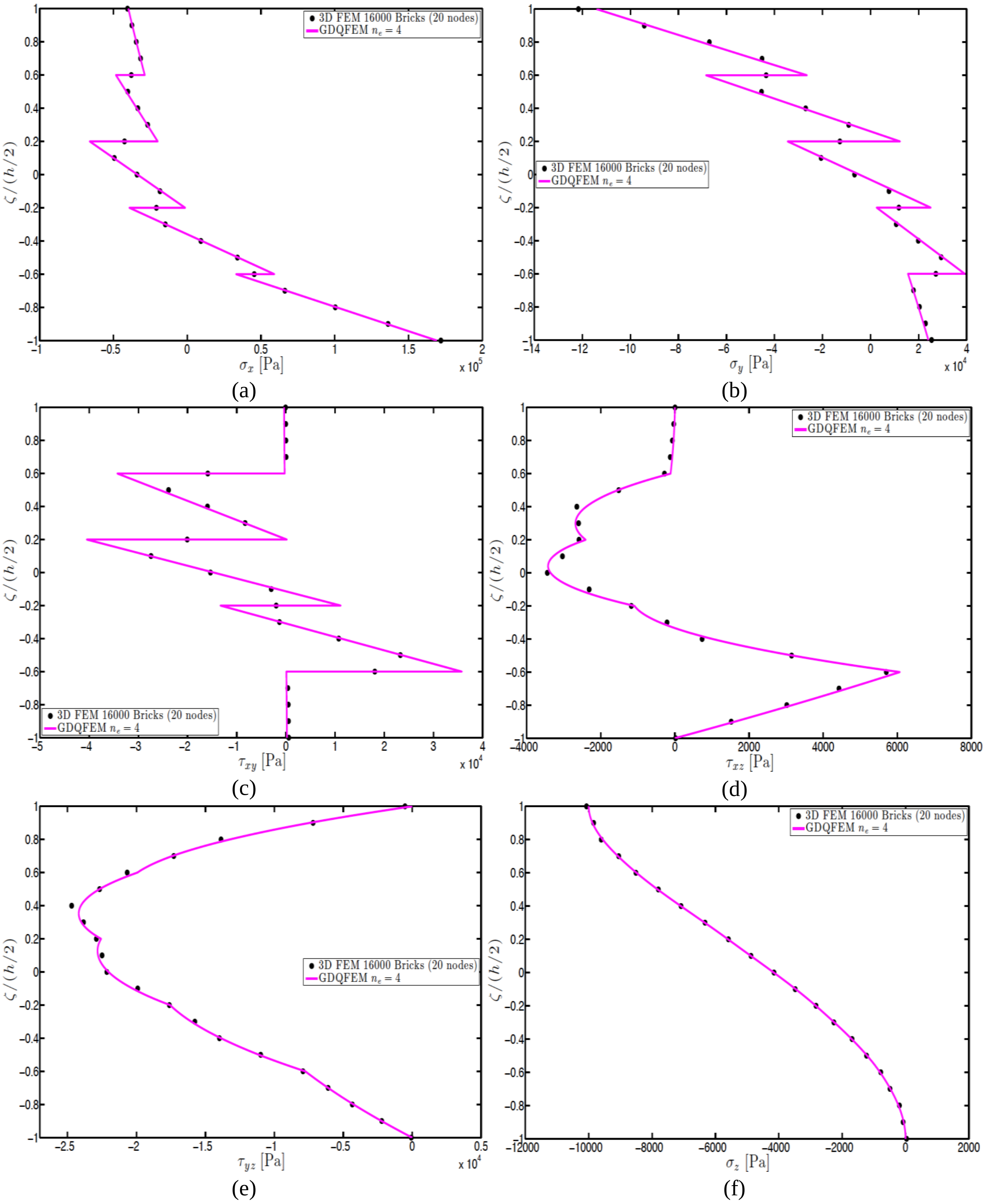


Fig. 15 Through the thickness variation of the stress components [Pa] at the point $B = (0.5a, 0.25a)$ for a $(0/30/45/65/90)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}$, $h_1 = h_2 = h_3 = h_4 = h_5 = 0.02\text{m}$

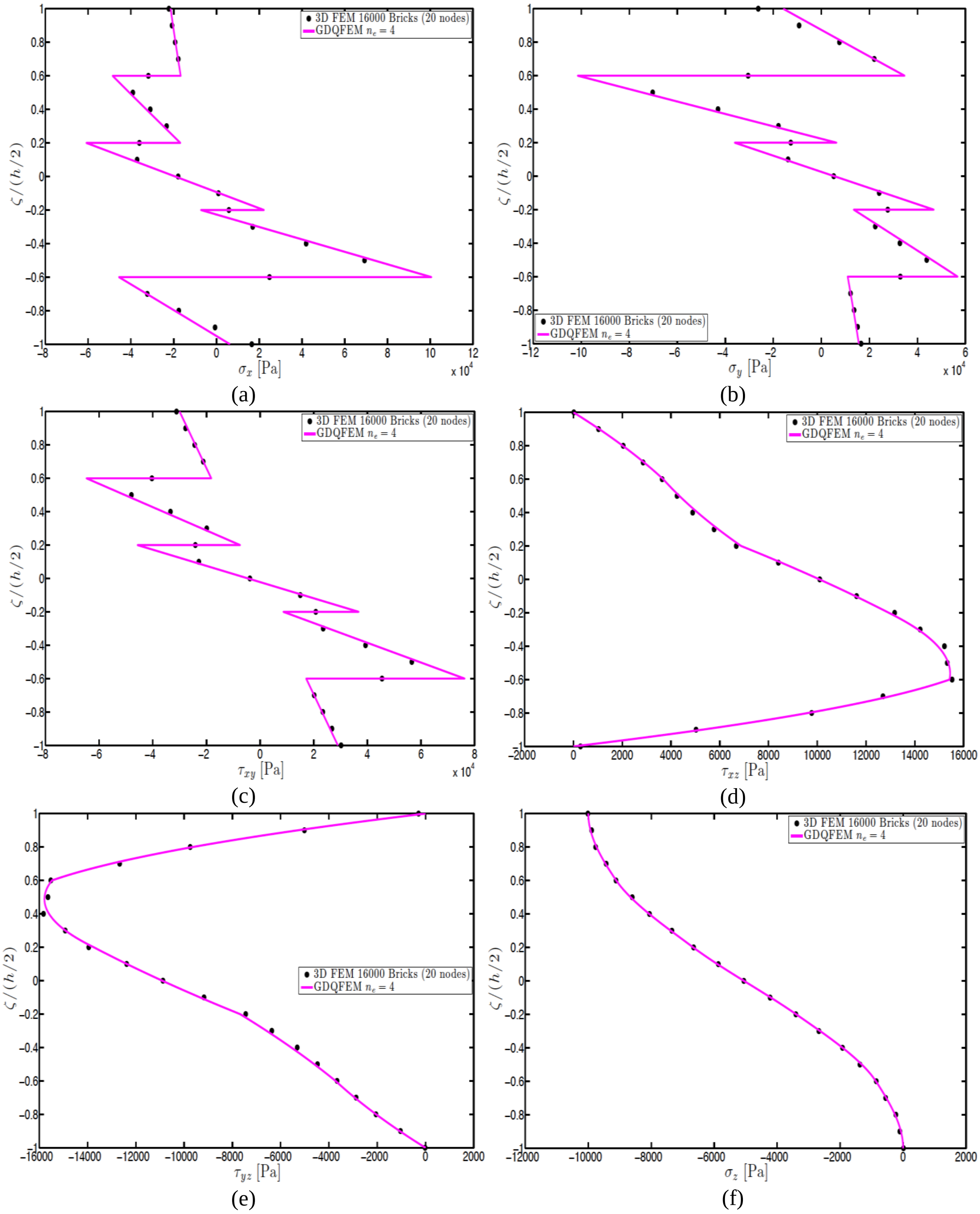
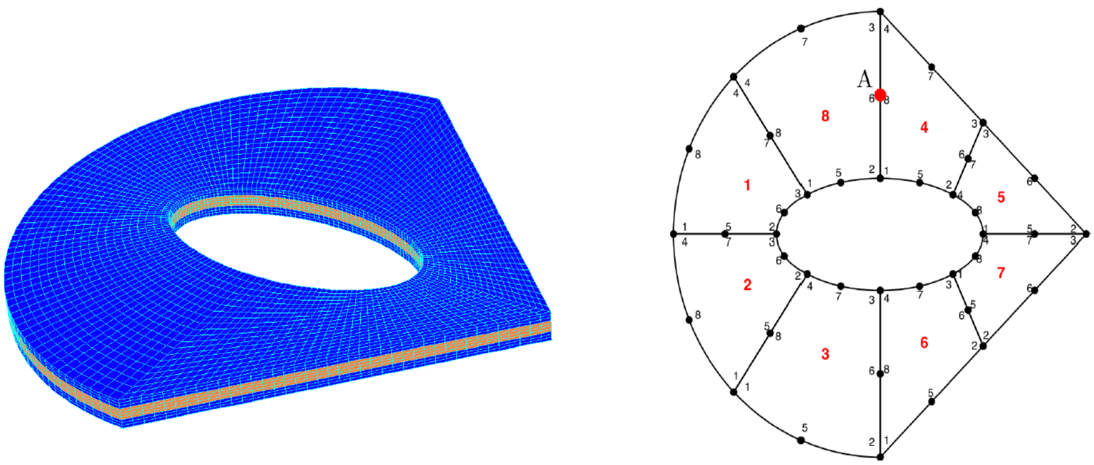


Fig. 16 Through the thickness variation of the stress components [Pa] at the point $D = (0.75a, 0.25a)$ for a $(0/30/45/65/90)$ C-C-C-C square plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $a = b = 1\text{m}, h_1 = h_2 = h_3 = h_4 = h_5 = 0.02\,\text{m}$

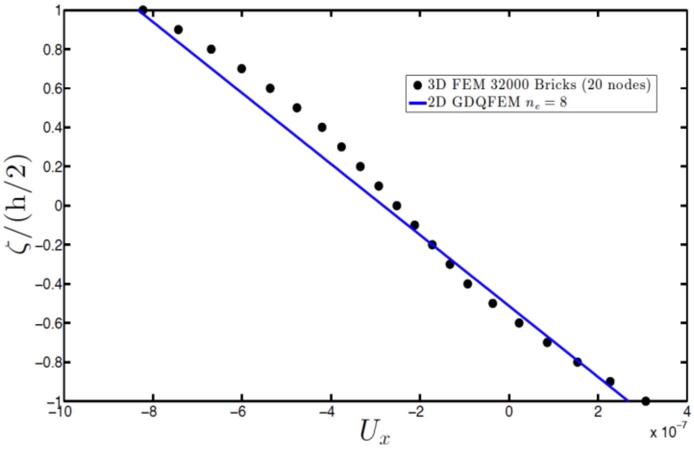
The final square plate example is referred to a five layer composite plate ($0/30/45/65/90$) with $h_1 = h_2 = h_3 = h_4 = h_5 = 0.02\,\text{m}$ and a uniform top load $q_z^{(+)} = -10\ \text{kPa}$. The plate is square and clamped and the laminae proprieties are the same as the previous examples: Graphite-Epoxy. The stress components evaluated by using the procedure illustrated in the previous section are compared with 3D FEM in three distinct points $C = (0.25a, 0.25a)$ (Fig. 14), $B = (0.5a, 0.25a)$ (Fig. 15) and $D = (0.75a, 0.25a)$ (Fig. 16). Since the lamination scheme is neither symmetric nor antisymmetric all the profiles in B and D are completely different. It is noted that a perfect agreement can be seen between the present solution and the 3D FEM one.

The latest static application is related to an arbitrarily shaped composite plate and it is represented in Fig. 17, where (on the left) a 3D finite element model made of three plies and (on the right) a 2D SFEM $n_e = 8$ mesh are shown. The geometry description of this plate can be found in Viola *et al*. (2013a). It must be noted that there is a large difference with respect to the dofs. In fact the former has 412800 dofs and the latter has 17640 dofs. The plate has all its external edges fixed and the internal ones, where an elliptic hole is drawn, are free. The lamination scheme is ($30/65/45$) with the top and bottom layers made of Graphite-Epoxy and the middle layer made of Glass-Epoxy: $E_1 = 53.78\ \text{GPa}$, $E_2 = E_3 = 13.93\ \text{GPa}$, $G_{12} = G_{13} = 8.96\ \text{GPa}$, $G_{23} = 3.45\ \text{GPa}$, $\nu_{12} = \nu_{13} = 0.25$, $\nu_{23} = 0.34$ and $\rho = 1900\ \text{kg/m}^3$. Moreover, the top and bottom sheets have a thickness $h_1 = h_3 = 0.03\,\text{m}$ and the middle layer has $h_2 = 0.04\,\text{m}$. The load $q_z^{(+)} = -10\ \text{kPa}$ is applied at the top surface. For the present case, the displacements, strain and stress components are depicted at the point A as represented in Fig. 17. From Figs. 18-20 good agreement is reported between the two theories, nevertheless it is clear that a FSDT theory is not enough to capture the nonlinear behavior that is registered in some layers of the stacking sequence. Thus, a higher order shear deformation theory (HSDT) should be considered. For this reason, one of the future aims of the authors is to develop a HSDT for free vibrations and static analysis of arbitrarily shaped laminated composite plates using the Carrera Unified Formulation (CUF) as already done by the authors in the papers (Tornabene *et al.* 2013b and Tornabene *et al.* 2014a).

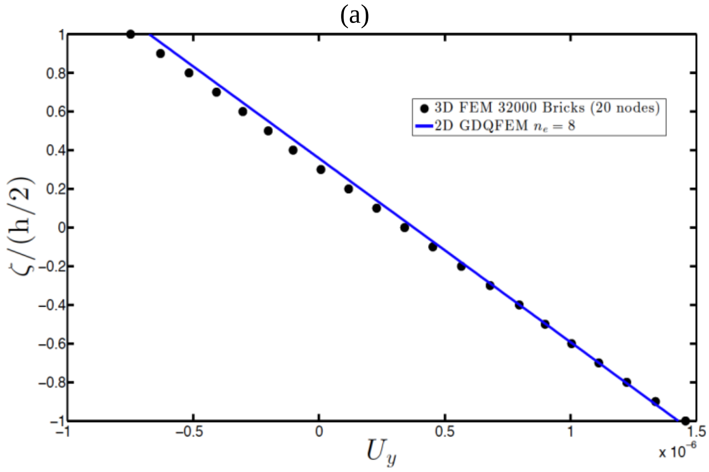


(a) dofs= 412800 (b) dofs= 17640

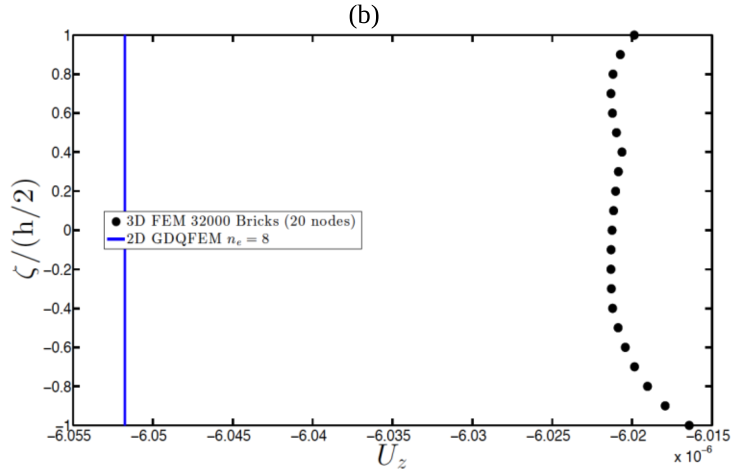
Fig. 17 Comparison between a 3D finite element model and a 2D SFEM model for an arbitrarily shaped composite plate made of three plies



(a)



(b)



(c)

Fig. 18 Through the thickness variation of the displacement components [m] at the point A for a $(30/65/45)$ arbitrarily shaped plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\mathrm{kPa}$. Geometric properties: $h_1 = h_3 = 0.03\mathrm{m}$, $h_2 = 0.04\mathrm{m}$

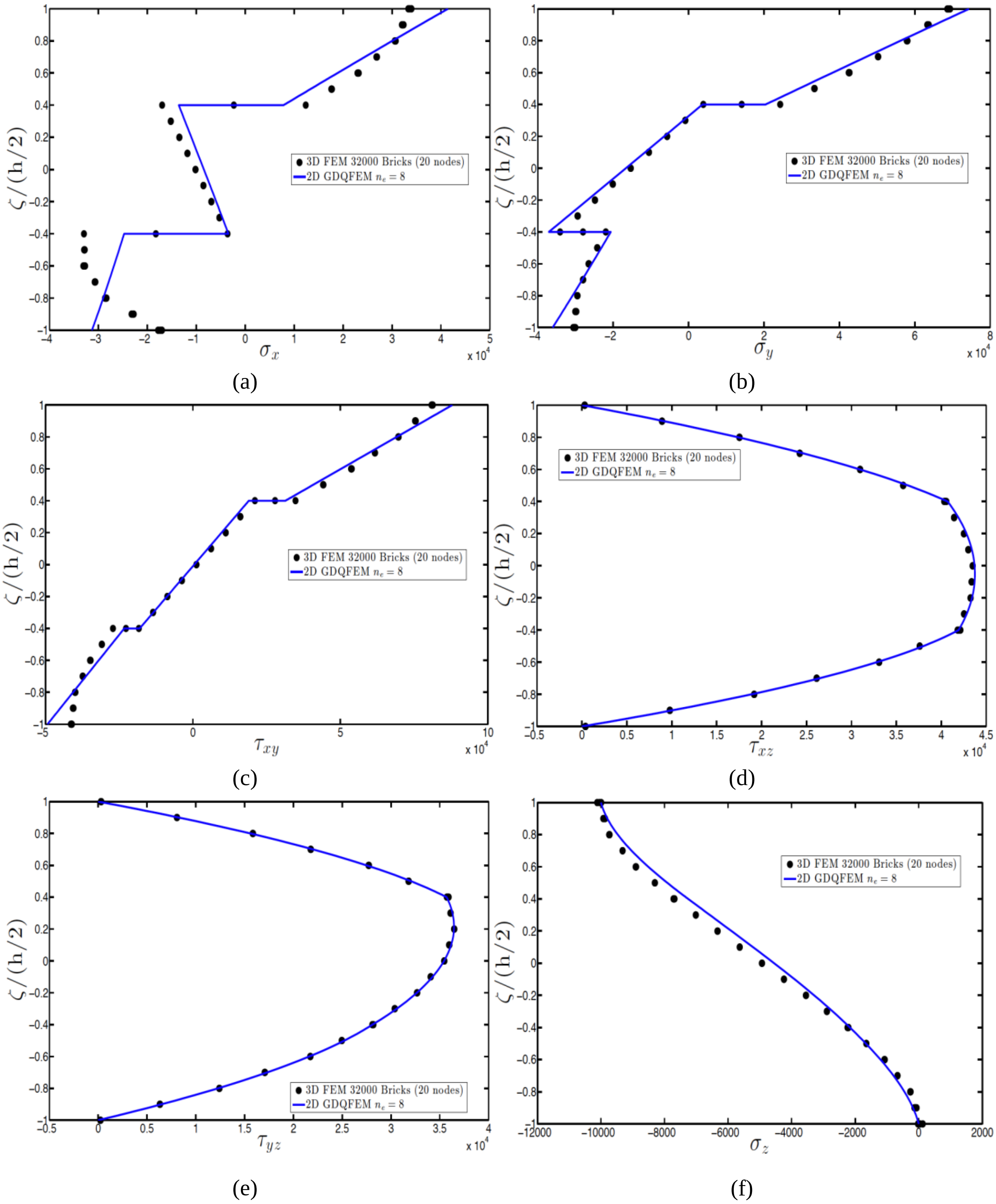


Fig. 19 Through the thickness variation of the stress components [Pa] at the point A for a $(30/65/45)$ arbitrarily shaped plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\mathrm{kPa}$. Geometric properties: $h_1 = h_3 = 0.03\,\mathrm{m}$, $h_2 = 0.04\,\mathrm{m}$

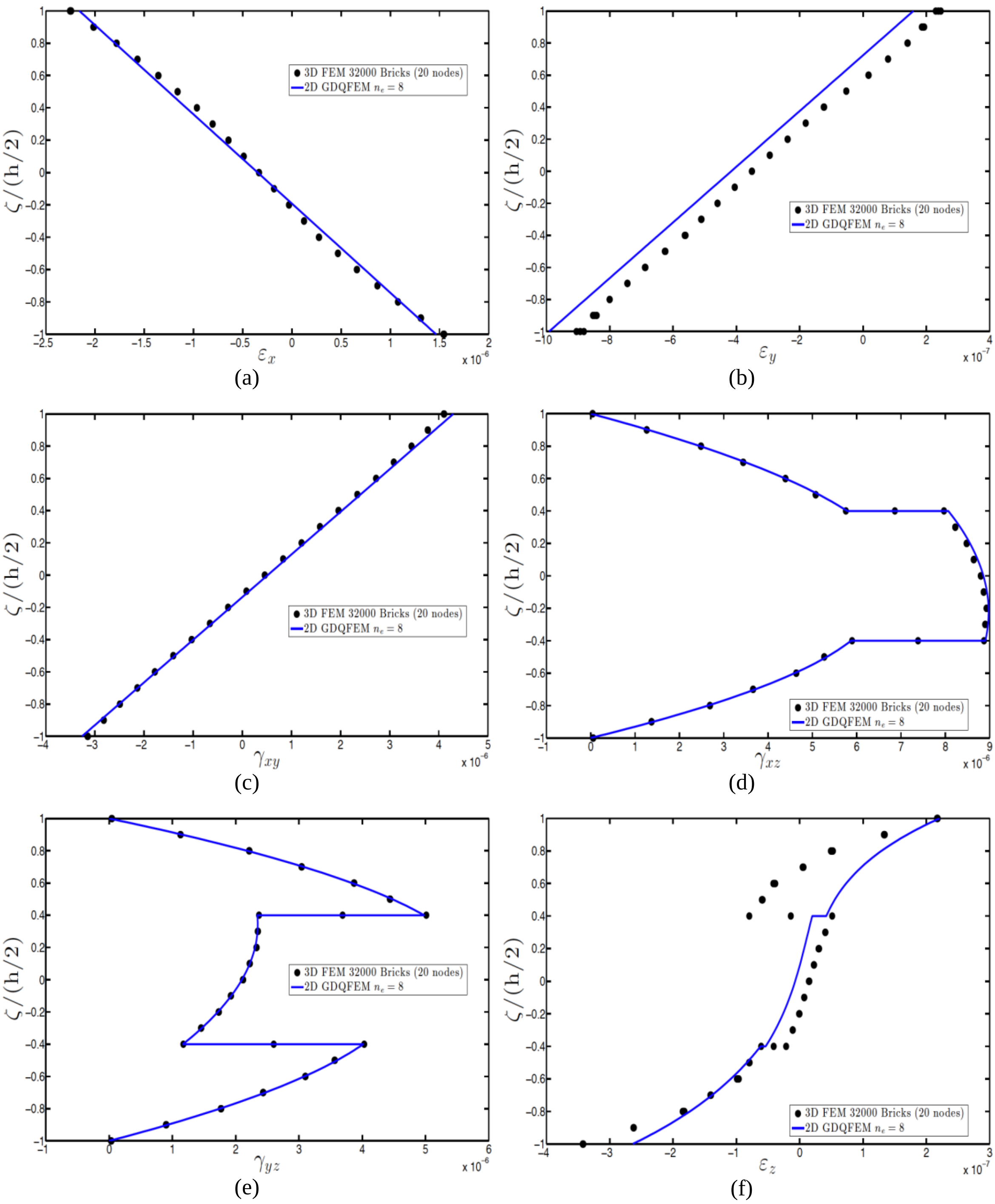


Fig. 20 Through the thickness variation of the strain components at the point A for a $(30/65/45)$ arbitrarily shaped plate subjected to a uniformly distributed load at the top surface $q_z^{(+)} = -10\,\text{kPa}$. Geometric properties: $h_1 = h_3 = 0.03\,\text{m}$, $h_2 = 0.04\,\text{m}$

4. Conclusions

A number of important conclusions concerning the SFEM can be drawn from the results of the numerical experiments presented in this paper. The approximation of the partial derivatives of an unknown field performed within GDQ method is accurate not only when applied to single domain, as already presented in literature by the authors, but also using a domain decomposition technique. As a result, the decomposition and the following enforcement of the inter-element compatibility conditions do not affect the accuracy and stability of the strong formulation at hand. The present technique appear to fit the semi-analytical results of cross-ply plates well-known from literature and analogous cross-ply plates of composite materials studied with a FEM code. In addition laminated composite plates with a generic orientation of the laminae are also studied and it was observed that the SFEM yields to very accurate results also due to the correction of the in-plane stresses. Finally, an arbitrarily shaped sandwich plate with a soft core was considered. It can be noted that the only limit on the accuracy is due to the implemented FSDT. The results become inaccurate when soft-core laminated plates are taken into account, in fact for these cases the FSDT is no longer valuable for the analysis. Thus, this aspect is not related to the employed numerical analysis, but on the approximation model under use. For further reading on the subject the reader can refer to some recent papers of the authors where Higher-order Shear Deformation Theories are employed for studying the soft core behavior of some doubly-curved shell structures using a classic GDQ implementation.

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