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Strong formulation finite element method for arbitrarily shaped laminated plates – Part I. Theoretical analysis

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Abstract. This paper provides a new technique for solving the static analysis of arbitrarily shaped composite plates by using Strong Formulation Finite Element Method (SFEM). Several papers in literature by the authors have presented the proposed technique as an extension of the classic Generalized Differential Quadrature (GDQ) procedure. The present methodology joins the high accuracy of the strong formulation with the versatility of the well-known Finite Element Method (FEM). The continuity conditions among the elements is carried out by the compatibility or continuity conditions. The mapping technique is used to transform both the governing differential equations and the compatibility conditions between two adjacent sub-domains into the regular master element in the computational space. The numerical implementation of the global algebraic system obtained by the technique at issue is easy and straightforward. The main novelty of this paper is the application of the stress and strain recovery once the displacement parameters are evaluated. Computer investigations concerning a large number of composite plates have been carried out. SFEM results are compared with those presented in literature and a perfect agreement is observed.

Keywords: static analysis; arbitrarily shaped plates; stress and strain recovery; generalized differential quadrature; strong formulation finite element method

1. Introduction

One of the most studied problem in mechanical, civil and aerospace engineering is related to laminated composite plates. In fact, this structural component has been applied to several practical applications (Timoshenko and Woinowsky-Krieger 1959, Leissa 1993, Liew *et al.* 1998 and Reddy 1999). Usually, the geometry under study is not regular. In other words it is not a rectangle or a circle, hence analytical solutions cannot be used for the development of such studies. Moreover, it is of common interest to investigate the stress and strain profiles through the thickness of a plate. Therefore, this work wants to show a methodology that can recovery stress and strain profiles through the thickness of arbitrarily shaped composite plates.

It is important to note that composite materials have achieved a considerable success since their high strength-to-weight ratio, high stiffness-to-weight ratio, and their capability to be tailored according to a given requirement. Hence, besides two dimensional structural components,

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higher-order one dimensional models have been introduced in literature for studying several composite structures (Carrera and Pagani 2013, 2014, Carrera *et al.* 2013 and Pagani *et al.* 2013, 2014).

The most common domain decomposition method is the Finite Element Method (FEM). The present technique divides the whole domain into several elements according to the problem geometry. Unlike FEM, which solves the weak formulation of the differential problem, the present method solves the strong formulation of the differential system at the master element level. For instance, a higher order numerical scheme such as Generalized Differential Quadrature (GDQ) method solves the equations of the problem in their strong form within their boundary conditions inside each element.

Together with FEM, the Spectral Element Method (SEM) represents another domain decomposition technique currently in use as in (Canuto *et al.* 2007 and Ostachowicz *et al.* 2011). SEM directly derived from the so-called Spectral Methods (SMs) (Boyd 2001, Canuto *et al.* 2006, Gottlieb and Orszag 1977). The key feature of SMs is the trial functions, which approximate the unknown parameters of the model. For the SMs, the approximating functions are linear combinations of suitable basis functions (Orszag 1969, 1980). Regarding GDQ, its starting point is the discretization of the derivative of unknown functions (Civan and Sliepcevich 1983 a,b, 1984, 1985). However, for some cases it can be demonstrated that GDQ and SMs are analogous (Bert and Malik 1996a, Quan and Chang 1989a, b).

As far as the domain decomposition is concerned, one of the most common extensions of SMs is the Spectral Element Method (SEM) which solves the weak formulation of the problem at the master element level (Patera 1984). A SEM combines the generality of the FEM with the accuracy of spectral techniques. In particular, the two approaches are very similar in some cases. Thus, it must be pointed out that FEM is a special case of SEM, where the trial functions are fixed a priori, whereas in SEM they depend on the number of grid points inside each element. For this reason these methods can be seen as particular weighted residual methods. The choice of the basis functions is one of the features which distinguishes spectral methods SMs from FEM. The basis functions for spectral methods are infinitely differentiable and global functions, whereas in the h-version of FEM, the domain is divided into small elements, and low-order trial functions are specified in each element.

It is noted that, Generalized Differential Quadrature can be seen as a particular spectral method, in which Lagrange polynomials are used as basis functions. With the advent of the GDQ method (Shu and Richards 1992a, b, Shu and Xue 1999 and Shu 2000) several applications have been hitherto presented in literature that it is impossible to cite them all (Bert *et al.* 1988, 1989, Tornabene and Viola 2007, 2008, 2009a, b, 2013, Ferreira *et al.* 2013, 2014, and Viola and Tornabene 2005, 2006, 2009). Among them several researchers tried to merge the mapping technique, used in FEM (Bert and Malik 1996b, Liu 1998, 1999, 2000, Liu and Liew 1999, Wang *et al.* 1998, Wang *et al.* 2004, Zhong *et al.* 2011, and Zong *et al.* 2005), with GDQ. In this way, it is possible to solve complex problems with mechanical and geometric discontinuities.

Summarizing, in this paper the static problem of multi-layered Reissner-Mindlin flat plates is investigated by using GDQ method when a domain decomposition technique is introduced. For the sake of generality this method is termed Strong Formulation Finite Element Method (SFEM), because the strong form of the partial differential system of equations is solved at the master element level. Moreover, GDQ is not the only strong formulation method to solve partial differential system of equations, because the derivative of a function can be generally solved using the mathematical developments of SMs.

Reliability and accuracy of the SFEM is investigated and the numerical results are compared with those found in literature and obtained through a finite element code, in the final section.

2. Moderately thick plate problem

Considering a two-dimensional plate theory, the structural shape is adequately defined by describing the geometry of its middle surface. The present linear theory is known also as Reissner-Mindlin theory, because the in-plane displacements are linear through-the-thickness, whereas the out-of-plane displacement is constant

$$
\begin{aligned}
U(x,y,z) &= u(x,y) + z\beta_x(x,y) \\
V(x,y,z) &= v(x,y) + z\beta_y(x,y) \\
W(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

The displacement parameters of the present model are u, v, w for the displacements along the three Cartesian directions and β_x, β_y for the two rotations about the y and x axes.

From the displacement field Eq. (1), using the definition of the three dimensional strain components for a two dimensional solid (Tornabene *et al.* 2013b) the strain characteristics can be found

$$
\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_x \\ \gamma_y \end{bmatrix}
=
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\
0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
0 & 0 & \frac{\partial}{\partial x} & 1 & 0 \\
0 & 0 & \frac{\partial}{\partial x} & 0 & 1
\end{bmatrix}
\begin{bmatrix} u \\ v \\ w \\ \beta_x \\ \beta_y \end{bmatrix}
\tag{2}
$$

Eq. (2) can be written, in compact matrix form, as

$$
\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} \tag{3}
$$

where $\mathbf{D}$ is termed the kinematic operator. The constitutive equations establish the relation between strain characteristics and stress resultants, which are integrated quantities of the stress components. In compact matrix form such relation can be expressed as

$$
\mathbf{S} = \mathbf{A}\boldsymbol{\varepsilon} \tag{4}
$$

where the vector $\mathbf{S} = \lfloor N_x \quad N_y \quad N_{xy} \quad M_x \quad M_y \quad M_{xy} \quad T_x \quad T_y \rfloor^T$ contains all the stress resultants and the constitutive matrix $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} & A_{16}^{(0)} & A_{11}^{(1)} & A_{12}^{(1)} & A_{16}^{(1)} & 0 & 0 \\ A_{12}^{(0)} & A_{22}^{(0)} & A_{26}^{(0)} & A_{12}^{(1)} & A_{22}^{(1)} & A_{26}^{(1)} & 0 & 0 \\ A_{16}^{(0)} & A_{26}^{(0)} & A_{66}^{(0)} & A_{16}^{(1)} & A_{26}^{(1)} & A_{66}^{(1)} & 0 & 0 \\ A_{11}^{(1)} & A_{12}^{(1)} & A_{16}^{(1)} & A_{11}^{(2)} & A_{12}^{(2)} & A_{16}^{(2)} & 0 & 0 \\ A_{12}^{(1)} & A_{22}^{(1)} & A_{26}^{(1)} & A_{12}^{(2)} & A_{22}^{(2)} & A_{26}^{(2)} & 0 & 0 \\ A_{16}^{(1)} & A_{26}^{(1)} & A_{66}^{(1)} & A_{16}^{(2)} & A_{26}^{(2)} & A_{66}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44}^{(0)} & A_{45}^{(0)} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45}^{(0)} & A_{55}^{(0)} \end{bmatrix} \tag{5}$$

The stiffness coefficients $A_{ij}^{(\tau)}$ depend on the elastic coefficients $\bar{Q}_{ij}^{(k)}$ (Tornabene 2009, 2011a, b, c, 2012, Tornabene and Fantuzzi 2014) and are defined as follows

$$\begin{aligned} A_{ij}^{(\tau)} &= \sum_{k=1}^{l} \int_{\zeta_k}^{\zeta_{k+1}} \bar{Q}_{ij}^{(k)} \zeta^{\tau} d\zeta \quad \text{for} \quad i, j = 1, 2, 6 \quad \text{and} \quad \tau = 0, 1, 2 \\ A_{ij}^{(\tau)} &= \sum_{k=1}^{l} \int_{\zeta_k}^{\zeta_{k+1}} \kappa \bar{Q}_{ij}^{(k)} \zeta^{\tau} d\zeta \quad \text{for} \quad i, j = 4, 5 \quad \text{and} \quad \tau = 0, 1, 2 \end{aligned} \tag{6}$$

where k identifies the current ply and $\kappa = 5/6$ is the shear correction factor. Finally, the indefinite equilibrium equations can be derived from the Hamilton principle (Tornabene 2009, 2011a, b, c, 2012, Tornabene and Fantuzzi 2014)

$$\mathbf{D}^{*}\mathbf{S} + \mathbf{q} = \mathbf{0} \tag{7}$$

where $\mathbf{D}^{*}$ is the equilibrium operator

$$\mathbf{D}^{*} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & -1 \end{bmatrix} \tag{8}$$

and $\mathbf{q} = \begin{bmatrix} q_x & q_y & q_z & m_x & m_y \end{bmatrix}^T$ represents the force vector. It is pointed out that, the loads have to be applied at the middle surface of the plate. Therefore, the actual loads are considered at the top and bottom surfaces of the plate and they are used for evaluating the middle surface loads as follows

$$
\begin{aligned}
q_x &= q_x^{(+)} + q_x^{(-)} \\
q_y &= q_y^{(+)} + q_y^{(-)} \\
q_z &= q_z^{(+)} + q_z^{(-)} \\
m_x &= q_x^{(+)} \frac{h}{2} - q_x^{(-)} \frac{h}{2} \\
m_y &= q_y^{(+)} \frac{h}{2} - q_y^{(-)} \frac{h}{2}
\end{aligned}
\tag{9}
$$

where $^{(+)}$, $^{(-)}$ refer to the top and bottom surfaces of the plate.

The equations of motion Eq. (7) can be expressed in terms of the displacement parameters using the constitutive Eq. (4) and kinematic Eq. (3) equations. As it is well-known, to solve a static problem the boundary conditions must be enforced. Since arbitrarily shaped plates are taken into account the external boundary conditions depend on the outward unit normal vector of each boundary. Moreover, the connectivity conditions between couples of adjoining elements have to be enforced in a domain decomposition application. It should be noted that for a classic GDQ implementation only external conditions are imposed (Tornabene *et al.* 2009, 2010, 2011, 2012a, b, c, 2013a, Tornabene and Ceruti 2013a, b). On the other hand, compatibility conditions between two adjacent edges have to be imposed. The implementation of these conditions is well presented in literature (Chen 2003a, b, 2004, 2006, Zhong and He 1998, 2003). In the previous works by the authors the term GDQFEM was proposed. However, GDQFEM can be derived from the SFEM because the former uses GDQ method, whereas the latter employs a general higher order differentiation scheme for discretizing derivatives. Hence, the two implementations follow the same guideline. Considering the equation of transformation from a Cartesian system x-y to the local reference system at the generic edge $n-s$ (where n is the normal component and s is the tangential one to the edge) the stress resultant vector $\mathbf{S_n} = \left[N_n \quad N_{ns} \quad T_n \quad M_n \quad M_{ns} \right]^T$ can be evaluated as follows

$$
\mathbf{S_n} = \mathbf{NS} \tag{10}
$$

where $\mathbf{N}$ is the transformation matrix that contains the directions cosines of the unit normal vector to the current edge

$$
\mathbf{N} = \begin{bmatrix}
n_x^2 & n_y^2 & 2n_x n_y & 0 & 0 & 0 & 0 & 0 \\
-n_x n_y & n_x n_y & n_x^2 - n_y^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & n_x & n_y \\
0 & 0 & 0 & n_x^2 & n_y^2 & 2n_x n_y & 0 & 0 \\
0 & 0 & 0 & -n_x n_y & n_x n_y & n_x^2 - n_y^2 & 0 & 0
\end{bmatrix} \tag{11}
$$

It is important to underline that the third relation of Eq. (10) has to be changed when inter-element compatibility conditions are treated, in order to avoid numerical instabilities. Thus matrix Eq. (11) becomes

$$\bar{\mathbf{N}} = \begin{bmatrix} n_x^2 & n_y^2 & 2n_x n_y & 0 & 0 & 0 & 0 & 0 \\ -n_x n_y & n_x n_y & n_x^2 - n_y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(n_x - n_y\right)^2 & \left(n_x + n_y\right)^2 \\ 0 & 0 & 0 & n_x^2 & n_y^2 & 2n_x n_y & 0 & 0 \\ 0 & 0 & 0 & -n_x n_y & n_x n_y & n_x^2 - n_y^2 & 0 & 0 \end{bmatrix} \tag{12}$$

3. Strong formulation finite element method

In the present paper, a domain decomposition technique that uses GDQ method for the static solution of a partial differential system of equations is carried out. Therefore, Lagrange polynomials are used for the approximation of the partial derivatives of the unknown functions u, v, w, β_x, β_y. Usually a complex domain is divided into several elements. Generally, these elements have an irregular shape that cannot be integrated directly within any integration or differentiation scheme. Thus, mapping technique must be introduced to transform an element described in Cartesian coordinates x-y, into a master element that is regular (square) and belongs to a computational domain $\xi - \eta$. Summarizing, the main steps of any numerical methodology, that has the scope to divide the whole domain into sub-domains, are:

1. Describe a physical problem that can have any complexity.
2. Domain discretization process which leads to a global mesh made of a collection of preselected finite elements.
3. Mapping technique that map an equation set from Cartesian coordinates to a computational domain.
4. Deduction of the element equations of the generated mesh:
   (a) Construct the strong formulation of the differential equations over the typical element.
   (b) Assume that a dependent variable can be approximated in the form

$$\bar{\mathbf{u}} \cong \sum_{i=1}^{N} \boldsymbol{\psi}_i \mathbf{u}_i$$

   Using a fixed degree of the interpolating polynomials, the number of the element degrees of freedom (dofs) is fixed. Therefore, once the degree of the polynomials is chosen, the computational cost is known a priori. Furthermore, substituting the interpolation into the previous step 4a the algebraic equations for a single element are obtained in the form

$$\mathbf{D}^*\mathbf{C}\mathbf{D}\mathbf{u} + \mathbf{q} = \mathbf{0}$$

   The main difference with standard FEM is that in the latter the element formulation is weak and the interpolation functions ψ_i can be either derived or taken from literature in order to compute the element matrices. On the contrary, for the present case the element stiffness matrices are evaluated through differential quadrature (strong form). It should be added that the element stiffness matrix is non-symmetric

$$\mathbf{K} = \mathbf{L} = \mathbf{D}^*\mathbf{C}\mathbf{D}$$

   (c) Evaluate the weighting coefficients of the selected collocation for the master element.
5. Assembly the element equations to obtain the equations for the whole problem:
   (a) Inter-element connectivity is based on the compatibility conditions. Compared to standard FEM, which a priori assume the element connectivity, that in general is a C^0 type, the connectivity for the present methodology is higher than C^0 and it is a posteriori enforced.
   (b) Enforce the external boundary conditions of the element that correspond to the external boundary of the physical problem.
   (c) Assemble element equations using the previous steps 5a and 5b.
6. Static solution of the assembled equations.
7. Post-processing phase: strain and stress recovery from the calculated displacement parameters.

These steps are based on the present formulation that considers the strong formulation of the problem. It is pointed out that the same approach is followed when weak or variational formulations are taken into account.

4. Mapping technique

It is well-known that coordinate transformation basically transforms an irregular element in the Cartesian x-y plane to a square computational domain in the natural coordinate $\xi-\eta$ through the following relationships

$$x = x(\xi,\eta) \quad y = y(\xi,\eta) \tag{13}$$

In literature equation Eq. (13) is also used in standard FEM to map any finite element. All the spatial derivatives of the configuration variables of the problem are mapped to the computational system $\xi-\eta$. The first and second order derivatives of an arbitrary function defined in the Cartesian x-y plane are given by:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} \xi_x & \eta_x & 0 & 0 & 0 \\ \xi_y & \eta_y & 0 & 0 & 0 \\ \xi_{xx} & \eta_{xx} & \xi_x^2 & \eta_x^2 & 2\xi_x\eta_x \\ \xi_{yy} & \eta_{yy} & \xi_y^2 & \eta_y^2 & 2\xi_y\eta_y \\ \xi_{xy} & \eta_{xy} & \xi_x\xi_y & \eta_x\eta_y & \xi_x\eta_y + \xi_y\eta_x \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial^2}{\partial \xi^2} \\ \frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial \xi \partial \eta} \end{bmatrix} \tag{14}$$

where ξ_x, η_x, ξ_y, η_y are the first order derivatives of ξ and η with respect to x and y, respectively. They can be demonstrated to be functions of the Jacobian matrix $\mathbf{J}$ of the transformation (Fantuzzi 2013 and Fantuzzi *et al.* 2014)

$$\xi_x = \frac{y_\eta}{\det \mathbf{J}}, \quad \xi_y = -\frac{x_\eta}{\det \mathbf{J}}$$
$$\eta_x = -\frac{y_\xi}{\det \mathbf{J}}, \quad \eta_y = \frac{x_\xi}{\det \mathbf{J}} \tag{15}$$

and the higher order derivatives are

$$\xi_{xx} = \frac{1}{\det \mathbf{J}^2}\left(y_\eta y_{\xi\eta} - \frac{y_\eta^2}{\det \mathbf{J}} \det \mathbf{J}_\xi - y_\xi y_{\eta\eta} + \frac{y_\xi y_\eta}{\det \mathbf{J}} \det \mathbf{J}_\eta \right)$$
$$\xi_{yy} = \frac{1}{\det \mathbf{J}^2}\left(x_\eta x_{\xi\eta} - \frac{x_\eta^2}{\det \mathbf{J}} \det \mathbf{J}_\xi - x_\xi x_{\eta\eta} + \frac{x_\xi x_\eta}{\det \mathbf{J}} \det \mathbf{J}_\eta \right)$$
$$\eta_{xx} = \frac{1}{\det \mathbf{J}^2}\left(-y_\eta y_{\xi\xi} + \frac{y_\eta y_\xi}{\det \mathbf{J}} \det \mathbf{J}_\xi + y_\xi y_{\xi\eta} - \frac{y_\xi^2}{\det \mathbf{J}} \det \mathbf{J}_\eta \right)$$
$$\eta_{yy} = \frac{1}{\det \mathbf{J}^2}\left(-x_\eta x_{\xi\xi} + \frac{x_\eta x_\xi}{\det \mathbf{J}} \det \mathbf{J}_\xi + x_\xi x_{\xi\eta} - \frac{x_\xi^2}{\det \mathbf{J}} \det \mathbf{J}_\eta \right) \tag{16}$$
$$\xi_{xy} = \frac{1}{\det \mathbf{J}^2}\left(-y_\eta x_{\xi\eta} + \frac{y_\eta x_\eta}{\det \mathbf{J}} \det \mathbf{J}_\xi + y_\xi x_{\eta\eta} - \frac{y_\xi x_\eta}{\det \mathbf{J}} \det \mathbf{J}_\eta \right)$$
$$\eta_{xy} = \frac{1}{\det \mathbf{J}^2}\left(-y_\xi x_{\xi\eta} + \frac{x_\eta x_\xi}{\det \mathbf{J}} \det \mathbf{J}_\xi + y_\eta x_{\xi\xi} - \frac{y_\xi x_\xi}{\det \mathbf{J}} \det \mathbf{J}_\eta \right)$$

In Eq. (16) $\det \mathbf{J}_\xi$, $\det \mathbf{J}_\eta$ denote the first order derivatives of the function $\det \mathbf{J}$ with respect to the natural coordinates ξ and η, respectively

$$\det \mathbf{J}_\xi = x_\xi y_{\xi\eta} - y_\xi x_{\xi\eta} + y_\eta x_{\xi\xi} - x_\eta y_{\xi\xi}$$
$$\det \mathbf{J}_\eta = -x_\eta y_{\xi\eta} + y_\eta x_{\xi\eta} - y_\xi x_{\eta\eta} + x_\xi y_{\eta\eta} \tag{17}$$

The above formulation is general, so various shape functions for coordinate transformation can be used. For the sake of conciseness the linear, quadratic and cubic functions are not illustrated, nevertheless they can be found in (Han and Liew 1997, Liew and Han 1997). In the present work, 8 node elements are used for the sake of generality. Thus, 8 node elements can map either curved or straight sided elements.

5. Boundary conditions implementation

As stated in the introduction the present section shows the implementation of the GDQ method using a domain decomposition technique that is termed SFEM. In the following, the main points of the boundary conditions implementation are reported. The compatibility conditions between two adjacent elements have a dominant influence on the numerical solution. In particular, the corner points are the key issue of SFEM implementation, in fact no theoretical background exists and each implemented formula has to be verified numerically. As far as the edge continuity conditions are concerned, they are expressed by

$$\begin{array}{ll} u^{(n)} = u^{(m)} & N_n^{(n)} = N_n^{(m)} \\ v^{(n)} = v^{(m)} & N_{ns}^{(n)} = N_{ns}^{(m)} \\ w^{(n)} = w^{(m)} \text{ on } B & T_n^{(n)} = T_n^{(m)} \text{ on } B \\ \beta_x^{(n)} = \beta_x^{(m)} & M_n^{(n)} = M_n^{(m)} \\ \beta_y^{(n)} = \beta_y^{(m)} & M_{ns}^{(n)} = M_{ns}^{(m)} \end{array} \tag{18}$$

where B indicates the generic boundary of an element and the two superscripts $^{(n)}$, $^{(m)}$ stand for the two elements involved in the connection. The kinematic conditions are only functions of the generalized displacements (u, v, w, β_x, β_y), whereas the internal stress resultants also depend on the outward local reference system $n-s$. The local reference system is described by the outward unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at a point on the interfaces of the two adjoining elements. The expressions of the normal vector components $\mathbf{n} = \left[n_x \quad n_y\right]^T$ on the four sides of a quadrilateral element assume the aspect:

$$\begin{aligned} \mathbf{n} &= \frac{\xi}{\sqrt{x_\eta^2 + y_\eta^2}} \left[y_\eta \quad -x_\eta \right]^T \quad \text{for} \quad \xi = \pm 1 \\ \mathbf{n} &= \frac{\eta}{\sqrt{x_\xi^2 + y_\xi^2}} \left[-y_\xi \quad x_\xi \right]^T \quad \text{for} \quad \eta = \pm 1 \end{aligned} \tag{19}$$

Eq. (19) is valid for the edges parallel to ξ axis and η axis, respectively. Once the mathematical expressions of the kinematic and static conditions Eq. (18) are developed, only the corner conditions have to be defined afterwards. Since the continuity condition means that kinematic and static relationships should be imposed at each connected boundary, the same must be done at the corners of the elements. However this rule is not always easy to follow especially when more than two elements concur at the same corner.

At first, a single external boundary corner is shown in Fig. 1(a). The corner under consideration can have two edges both clamped, both free or just one of them clamped. The edges are indicated as EB because they are external boundaries, whereas the internal boundaries will be indicated as IB. The represented element is named element 5. It is obvious that when at least one of the two edges of element 5 is clamped the corner is fixed too, so only kinematic conditions have to be imposed

$$U^{(n)} = 0 \tag{20}$$

where $U^{(n)}$ stands for the degrees of freedom of the plate element. It is recalled that in the present paper the degrees of freedom are u, v, w, β_x, β_y. However, a numerical issue arises when both the edges are set free, because it is not clearly defined the side at which the corner belongs to. Since it is physically a point of both sides, the natural boundary conditions of these edges have to be imposed, as follows

$$S_{\mathbf{n}_1}^{(n)} + S_{\mathbf{n}_2}^{(n)} = 0 \tag{21}$$

In Eq. (21) $S^{(n)}$ stands for one of the stress resultants of the model N_n, N_{ns}, T_n, M_n, M_{ns} which depend on the outward unit normal vector of the given edges $\mathbf{n}_1$, $\mathbf{n}_2$. In other words, it is important to insert both $S_{\mathbf{n}_1}^{(n)}$ and $S_{\mathbf{n}_2}^{(n)}$ in the static condition Eq. (21) when a corner is located between two free edges. A different configuration occurs when a corner point is next to a connected edge and a free edge as depicted in Fig. 1(b). The corners of the elements 3 and 5 have an external edge free EB = F and on the other edge the compatibility conditions should be set because the compatibility condition is physically stronger than a free edge condition. A generic corner configuration is shown in Fig. 2. This kind of configuration does not have a theoretical counterpart; hence the authors developed a particular procedure for the implementation of the continuity conditions. Firstly, an internal corner is studied in Fig. 2(a), where only internal boundaries (IBs) are present. Secondly, an external corner point, which leads to the external boundary conditions of the physical domain, is investigated in Fig. 2(b). For both corners either kinematic or static compatibility conditions must be prescribed. The present procedure sets a static condition between two corners and several kinematic ones among the others. It is pointed out that, the present methodology has to be repeated by the total number of degrees of freedom per node of the model. For instance, five elements concur at the displayed node. Once the concurring sequence of elements is defined, e.g. 1, 3, 5, 2, 4 four kinematic conditions $U^{(n)} = U^{(m)} \to U^{(n,m)}$ are written at the corner of the element *n* with respect to the corner of the element *m* and finally the static condition is enforced. From the mathematical point of view, the following equations are enforced by the code

$$U^{(1)} = U^{(3)}, \quad U^{(3)} = U^{(5)}, \quad U^{(5)} = U^{(2)}, \quad U^{(2)} = U^{(4)}, \quad S^{(4)} = S^{(1)} \tag{22}$$

In Eq. (22) $U^{(n)}$ stands for one of the displacement parameters u, v, w, β_x, β_y of the element $^{(n)}$ and $S^{(n)}$ represents the static external stress resultants N_n, N_{ns}, T_n, M_n, M_{ns} of the generic element $^{(n)}$. This procedure is repeated five times, hence if five elements concur at a point and five relations per corner have to be set, the present case represents twenty five algebraic conditions.

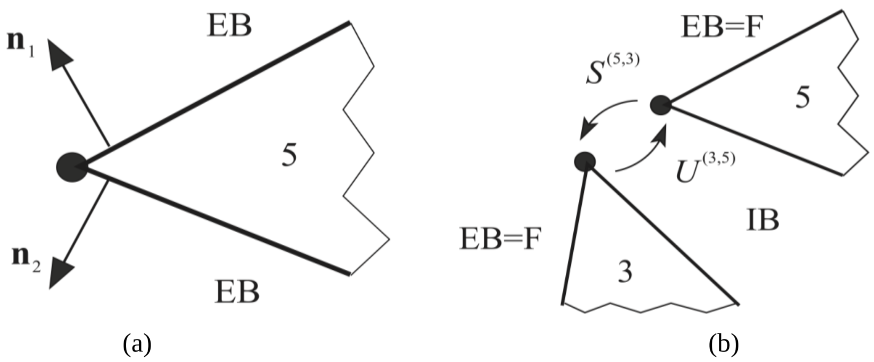

Fig. 1 Single corner boundary conditions schemes: (a) external corner of a single element; (b) external corner of two connected elements

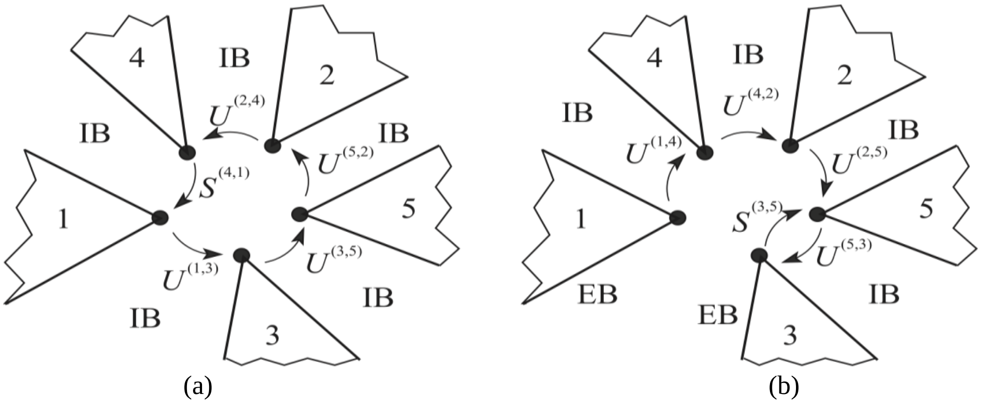

Fig. 2 Multiple corner boundary conditions schemes: (a) internal corner of five elements with IB conditions, (b) external corner of five elements with EB and IB conditions

In the second configuration illustrated in Fig. 2(b), two different boundary conditions are indicated. When a free boundary condition is imposed at the two external edges, the corner must satisfy the inter-element compatibility conditions. On the contrary, when a clamped edge is defined at one of two external edges, the corner conditions follow the clamped equations and they do not enforce any internal connectivity. When the corner is fixed, the kinematic conditions are written automatically by the code $U^{(1)}=0$ or $U^{(3)}=0$. When free boundary conditions are imposed at the two external edges, the previous rule, only valid for internal corners, must be used in a different manner. The corner conditions in a mathematical form, analogous to Eq. (22) take the form

$$U^{(1)}=U^{(4)},\quad U^{(4)}=U^{(2)},\quad U^{(2)}=U^{(5)},\quad U^{(5)}=U^{(3)},\quad S^{(3)}=S^{(5)} \tag{23}$$

In Eq. (23) $U^{(n)}$ and $S^{(n)}$ have the same meaning of the previous case. For the sake of conciseness, the kinematic and static relations are summarized in Fig. 2 using the compact form $U^{(n,m)}$ and $S^{(n,m)}$, where n and m are two generic elements. It must be pointed out that if the conditions $U^{(1,3)}$, $U^{(3,1)}$ or $S^{(1,3)}$, $S^{(3,1)}$ are set between external boundaries an inaccurate numerical solution is obtained. Therefore, the two corner configurations Eqs. (22) and (23) must be treated separately. In order to have a general view on the boundary conditions Fig. 3 is described below. This representation shows inter-element edges and external boundaries with solid blue and black lines, respectively. Moreover Fig. 4 shows the outward unit normal vectors nomenclature for the same mesh of Fig. 3. It can be noted that two groups of points occur: the points on the edges (E) and the ones on the corners (C). If a boundary is clamped a Dirichlet or kinematic condition types (E1) is imposed, as shown in Fig. 3 on the element $\Omega^{(1)}$. On the contrary the Neumann or static condition types (E2) are related to the external natural boundary conditions Eq. (11). In general a boundary type E2, referring to the edge $\underline{3}$ of element $\Omega^{(1)}$ can be indicated as $\mathbf{S}^{(1)}_{\mathbf{n}_{\underline{3}}}$. The subscript of the normal vector $\mathbf{n}$ indicates the edge, e.g. $\underline{3}$, and the superscript stands for the element in which the normal belong to, e.g. (1). Considering the corner conditions several configurations can occur. The two corners on the edge $\underline{4}$ of $\Omega^{(1)}$ are embedded into the Dirichlet condition (E1), because the clamped boundary type is stronger, from the physical

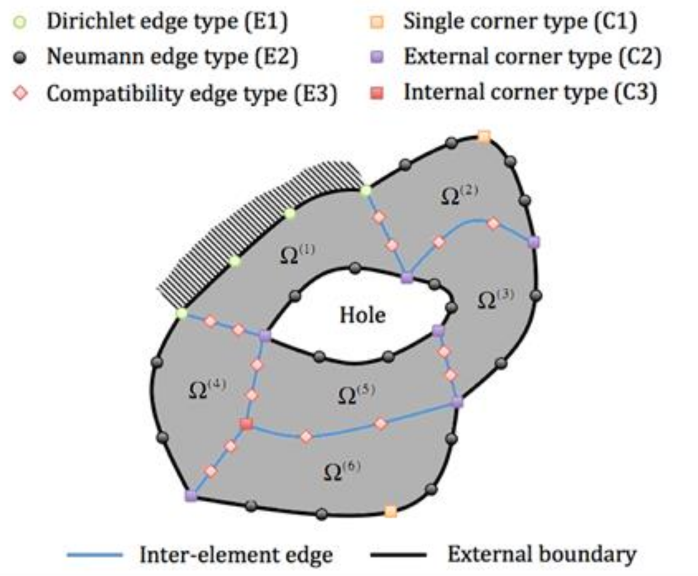

Fig. 3 Internal and external boundary conditions for element edges and corners

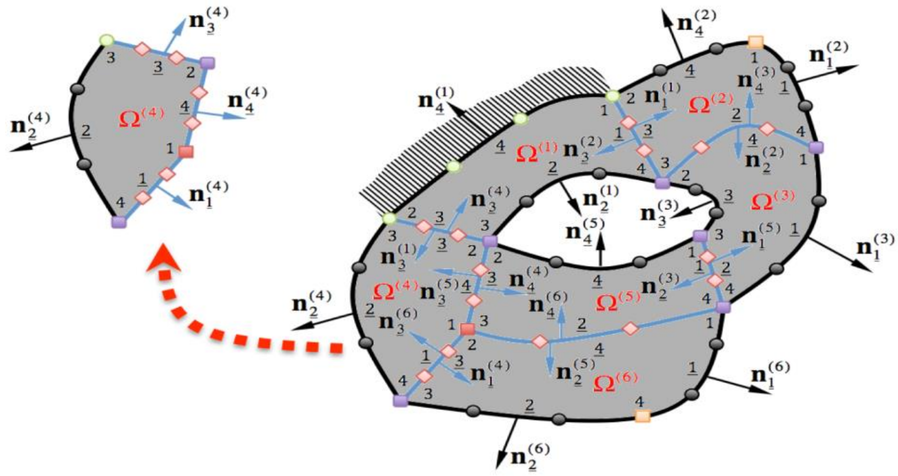

Fig. 4 Outward unit normal vectors definition for a generic sub-division

point of view, than a natural boundary type condition. The other corners indicated by C1 are different from the Neumann type conditions due to the coexistence of two natural conditions in a single point, as shown in Fig. 1(a) and previously reported. The compatibility conditions along the edges are indicated as E3 in Fig. 3. For instance, the grid points of edge $\underline{1}$ of element $\Omega^{(1)}$ are superimposed to the points of edge $\underline{3}$ of element $\Omega^{(2)}$. Only a group of points can be seen, nevertheless a double number of them are computationally considered for implementing the two conditions Eq. (18). Let the edge $\underline{1}$ of $\Omega^{(1)}$ and $\underline{3}$ of $\Omega^{(2)}$ be the two facing edges where the compatibility conditions have to be enforced (see Fig. 4). Mathematically speaking the continuity conditions on the linear interfaces are

$$
\begin{gathered}
u^{(1)}=u^{(2)}, \quad v^{(1)}=v^{(2)}, \quad w^{(1)}=w^{(2)}, \quad \beta_x^{(1)}=\beta_x^{(2)}, \quad \beta_y^{(1)}=\beta_y^{(2)} \\
N_{n(\mathbf{n}_{\underline{1}})}^{(1)}=N_{n(\mathbf{n}_{\underline{3}})}^{(2)}, \quad N_{ns(\mathbf{n}_{\underline{1}})}^{(1)}=N_{ns(\mathbf{n}_{\underline{3}})}^{(2)}, \quad T_{n(\mathbf{n}_{\underline{1}})}^{(1)}=T_{n(\mathbf{n}_{\underline{3}})}^{(2)}, \quad M_{n(\mathbf{n}_{\underline{1}})}^{(1)}=M_{n(\mathbf{n}_{\underline{3}})}^{(2)}, \quad M_{ns(\mathbf{n}_{\underline{1}})}^{(1)}=M_{ns(\mathbf{n}_{\underline{3}})}^{(2)}
\end{gathered}
\tag{24}
$$

In short Eq. (24) can be written in compact matrix form as

$$
\begin{aligned}
&\mathbf{u}^{(1)}=\mathbf{u}^{(2)} \quad \text{Kinematic conditions} \\
&\mathbf{S}_{\mathbf{n}_{\underline{1}}}^{(1)}=\mathbf{S}_{\mathbf{n}_{\underline{3}}}^{(2)} \quad \text{Static conditions}
\end{aligned}
\tag{25}
$$

The kinematic conditions are imposed on the points of one edge, for example $\underline{1}$ of $\Omega^{(1)}$, and the static conditions are enforced on the points of the other edge, for instance $\underline{3}$ of $\Omega^{(2)}$. Finally, the external and internal corner type conditions (C2) and (C3) are described considering Fig. 3 as a reference. Since the corner points are superimposed, as well as the points on the edges, the number of conditions that have to be imposed for each corner depend on the number of elements which concur at that node. For example, the two corners with (C2) conditions belong to two neighbor elements. In both cases the (C2) condition has the same form as the (E3) one, because only two elements concur at a corner. On the contrary, in the internal corner condition (C3) three conditions must be enforced. However, the continuity conditions Eq. (25) are made of two relationships only. The solution has been illustrated by Eq. (22). For the present case, depicted in Fig. 3, the kinematic conditions are written between $\Omega^{(4)}$ and $\Omega^{(6)}$ elements and between $\Omega^{(6)}$ and $\Omega^{(5)}$ elements, whereas the static condition is enforced between $\Omega^{(5)}$ and $\Omega^{(4)}$. Mathematically speaking the (C3) conditions can be written as

$$
\begin{aligned}
&\begin{cases}\mathbf{u}^{(4)}=\mathbf{u}^{(6)} \\ \mathbf{u}^{(6)}=\mathbf{u}^{(5)}\end{cases} \Rightarrow \mathbf{u}^{(5)}=\mathbf{u}^{(4)} \quad \text{First three corner conditions} \\
&\mathbf{S}_{\mathbf{n}_{\underline{3}}}^{(5)}=\mathbf{S}_{\mathbf{n}_{\underline{4}}}^{(4)} \quad \text{This is one of the conditions that can be chosen}
\end{aligned}
\tag{26}
$$

Following Eq. (26) the C^1 continuity conditions among the elements of a given mesh is satisfied. In other words a continuous and smooth stress distribution is guaranteed by Eq. (26).

6. The static problem

As any strong form based differential quadrature approach, a grid point distribution has to be set in order to proceed with the derivative approximation of the displacement field. The grid point location is determined in the master element that is a square domain. It has been shown by several papers in literature that sometimes it is convenient to use a different number of grid points when the two coordinate directions have different length $N \neq M$ (Tornabene and Reddy 2013 and Tornabene *et al.* 2013b, 2014a, b). Since the computational domain is usually regular, the same number of nodal points is used $N = M$. Only in particular occasions, the number is kept different $N \neq M$ due to highly stretched elements as in literature (Fantuzzi 2013 and Viola *et al.* 2013d, e, f). However, in the following applications it has been investigated only the case $N = M$.

In the following applications a Chebyshev-Gauss-Lobatto (C-G-L) grid along both ξ and η

directions is taken into account. It is possible to find other applications with different grid in literature (Marzani *et al.* 2008 and Viola *et al.* 2007, 2012, 2013a, b, c). For the computational domain $-1 \le \xi, \eta \le 1$, the C-G-L grid is defined as

$$\xi_i = \cos\left(\frac{i-1}{N-1}\pi\right), \quad i = 1, 2, \ldots, N$$
$$\eta_j = \cos\left(\frac{j-1}{N-1}\pi\right), \quad j = 1, 2, \ldots, N \tag{27}$$

where N is the total number of points along the two directions ξ and η. In general, domain decomposition technique yields to an algebraic system of equations that can be solved by Gaussian elimination technique, for the static case. As a result both strong and weak form element method have the same computational effort when the number of grid points per element and the type of mesh is set by the user.

Considering a mesh made of two elements the resulting global stiffness matrix is depicted in Fig. 5. The two neighboring elements are patched along an edge. Therefore, a kinematic and a static condition have to be enforced between the two elements. In addition the governing equations are discretized at all interior points of the two elements. Thus, as previously stated, the global system should have the form

$$\mathbf{K}_t = \begin{bmatrix} \bar{\mathbf{K}}_{bb} & \bar{\mathbf{K}}_{bd} \\ \bar{\mathbf{K}}_{db} & \bar{\mathbf{K}}_{dd} \end{bmatrix} \tag{28}$$

where each stiffness sub-matrix $\bar{\mathbf{K}}$ is functions of the two elements in different manners. Globally, the external and internal continuity equations are imposed in $\bar{\mathbf{K}}_{bb}$ and $\bar{\mathbf{K}}_{bd}$, whereas the fundamental equations are reported in $\bar{\mathbf{K}}_{db}$ and $\bar{\mathbf{K}}_{dd}$, for both elements. Fig. 5 shows the structure of the global stiffness matrix. Since the domain points of each element do not have any geometric relation among them, the fundamental equations are independent between the elements. In fact $\bar{\mathbf{K}}_{db}$ and $\bar{\mathbf{K}}_{dd}$ fill only the diagonal areas of the global stiffness matrix. On the contrary the bounded part contains not only the external boundary conditions $\mathbf{K}_b^{(n)}$ and $\mathbf{K}_b^{(m)}$ but also the

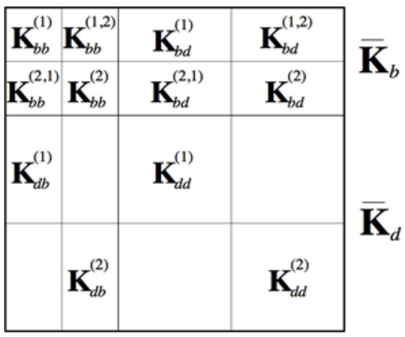

Fig. 5 Global stiffness matrix for a sample mesh made of two adjacent elements

continuity equations $\mathbf{K}_b^{(n,m)}$ and $\mathbf{K}_b^{(m,n)}$. In conclusion, firstly four stiffness matrices are computed for each element: $\mathbf{K}_{bb}^{(n)}$, $\mathbf{K}_{bd}^{(n)}$, $\mathbf{K}_{db}^{(n)}$ and $\mathbf{K}_{dd}^{(n)}$, for $n=1,2,...,n_e$. Secondly, the compatibility conditions are written in the coupling matrices $\mathbf{K}_{bb}^{(n,m)}$ and $\mathbf{K}_{bd}^{(n,m)}$, for $n, m=1,2,...,n_e$. Thirdly, by assembling the previous matrices and by following the connectivity of the mesh under consideration, a stiffness matrix similar to the one presented in Fig. 5 can be carried out. Finally, it is obvious that a classic linear static problem is obtained in the form

$$\begin{bmatrix} \bar{\mathbf{K}}_{bb} & \bar{\mathbf{K}}_{bd} \\ \bar{\mathbf{K}}_{db} & \bar{\mathbf{K}}_{dd} \end{bmatrix} \begin{bmatrix} \mathbf{U}_b \\ \mathbf{U}_d \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_b \\ \mathbf{Q}_d \end{bmatrix} \tag{29}$$

where $\mathbf{U}_b, \mathbf{U}_d$ are the domain and boundary displacement parameters u, v, w, β_x, β_y of the model at all the sampling points of the current domain. The symbols $\mathbf{Q}_b, \mathbf{Q}_d$ indicate the boundary and domain applied loads. The system Eq. (29) is usually solved by Gaussian elimination technique. In addition to reduce the computational effort of the algebraic system Eq. (29), the same problem can be condensed, using static condensation of the bounded degrees of freedom as

$$\left(\bar{\mathbf{K}}_{dd} - \bar{\mathbf{K}}_{db}\bar{\mathbf{K}}_{bb}^{-1}\bar{\mathbf{K}}_{bd}\right)\mathbf{U}_d = \mathbf{Q}_d - \bar{\mathbf{K}}_{db}\bar{\mathbf{K}}_{bb}^{-1}\mathbf{Q}_b \tag{30}$$

where the boundary displacements can be retrieved, once the vector $\mathbf{U}_d$ is computed by means of

$$\mathbf{U}_b = \bar{\mathbf{K}}_{bb}^{-1}\left(\mathbf{Q}_b - \bar{\mathbf{K}}_{bd}\mathbf{U}_d\right) \tag{31}$$

In this way the degrees of freedom of the system Eq. (29), $5N^2 \cdot n_e$, are reduced to $5(N-2)^2 \cdot n_e$ in Eq. (30).

7. Conclusions

Based on the previous analysis, it can be concluded that a new technique for studying the static behavior of arbitrarily shaped composite plates by using Strong Formulation Finite Element Method (SFEM) has been presented. Further improvements were introduced to the existing Generalized Differential Quadrature Finite Element Method (GDQFEM) presented by the authors in their previous papers. As it is well-known strong formulation based finite elements are cumbersome when they have to deal with inter-element conditions, nevertheless a simple procedure was illustrated for solving the present issue. Application examples and discussions will be presented in another paper where a numerical implementation will be shown to be easy and straightforward. The stress profiles of classical and square composite plates, found in literature, are compared. Finally, the arbitrarily shaped case is carried out, considering a more general example, where good agreement is observed.

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