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Exact solutions of vibration and postbuckling response of curved beam rested on nonlinear viscoelastic foundations

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Abstract. This paper presents the exact solutions and closed forms for of nonlinear stability and vibration behaviors of straight and curved beams with nonlinear viscoelastic boundary conditions, for the first time. The mathematical formulations of the beam are expressed based on Euler-Bernoulli beam theory with the von Kármán nonlinearity to include the mid-plane stretching. The classical boundary conditions are replaced by nonlinear viscoelastic boundary conditions on both sides, that are presented by three elements (i.e., linear spring, nonlinear spring, and nonlinear damper). The nonlinear integro-differential equation of buckling problem subjected to nonlinear nonhomogeneous boundary conditions is derived and exactly solved to compute nonlinear static response and critical buckling load. The vibration problem is converted to nonlinear eigenvalue problem and solved analytically to calculate the natural frequencies and to predict the corresponding mode shapes. Parametric studies are carried out to depict the effects of nonlinear boundary conditions and amplitude of initial curvature on nonlinear static response and vibration behaviors of curved beam. Numerical results show that the nonlinear boundary conditions have significant effects on the critical buckling load, nonlinear buckling response and natural frequencies of the curved beam. The proposed model can be exploited in analysis of macrosystem (airfoil, flappers and wings) and microsystem (MEMS, nanosensor and nanoactuators).

Keywords: analytical solutions; curved beam; nonlinear viscoelastic boundary conditions; static and dynamic stability

1. Introduction

Nowadays, beam structural member that is used in many applications in various disciplines (i.e., aerospace, marine, mechanical, architecture, and civil engineering) and in various scale (i.e., micro/nano-electromechanical systems (MEMS/NEMS) Benguediab *et al.* 2023), may suffer from geometric nonlinearities since it is not known a priori in which ranges of displacement magnitudes, (Mohamed *et al.* 2018). The buckling/postbuckling and vibration of structures are a highly researched area in the field of structural mechanics. Since 1997, Lacarbonara studied the nonlinear vibrations of a buckled beam around its first buckling mode shape by using Galerkin discretization and the direct reduction method. Ecsedi and Dluhi (2005) developed 1D linear

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mathematical model to investigate the static and dynamic responses of non-homogeneous curved beams and closed rings in cylindrical coordinate system. Çalım (2012) predicted the forced vibration responses of curved beams rested on elastic foundations by using Timoshenko beam theory and Laplace domain technique.

Sedighi *et al.* (2014) employed Homotopy perturbation method to examine dynamic pull-in instability of vibrating curved modified couple stress microbeams under electrostatic actuation. Lee and Jeong (2016) exploited Timoshenko beam and the Runge–Kutta method to study flexural and torsional vibrations of curved beams relying on Pasternak foundations. Mochida and Ilanko (2016) examined free vibration analysis of axially loaded slightly curved beams subject to partial axial restraints. Ghayesh *et al.* (2017) employed backward differentiation formula with the pseudo-arclength continuation method to investigate the vibration of three-layered modified couple stress curved microbeams. Eltaher *et al.* (2019) studied the influence of periodic and non-periodic imperfection modes on postbuckling stability and vibration of beams rested on nonlinear elastic foundations. Jin *et al.* (2020) used two-step perturbation with the modified Lindstedt-Poicaré method in static and dynamic to analyze snap-through of the imperfect postbuckled sandwich beams. Ding *et al.* (2022) exploited two-step perturbation method to determine buckling and resonances of functionally graded (FG) curved pipes under fluid moving. Mohamed *et al.* (2022a) developed a mathematical model to predict nonlinear bending, and buckling/postbuckling, responses of imperfect bioinspired composite beams by using DIQM. Yuan and Ding (2022, 2023) presented a novel finite element model based on the absolute nodal coordinate formula to study the dynamics curved pipe conveying fluid. She and Ding (2023) used Reddy's higher-order shear deformation shell theory and Galerkin method to study the nonlinear primary resonance of graphene platelet reinforced metal foams doubly curved shells under pre-stressing force. Yang *et al.* (2023) retrieved mode shapes of curved bridges from the contact responses of a single axle scanning vehicle using the variational mode decomposition and synchrosqueezed wavelet transform. Hosseini *et al.* (2023) studied free vibration of deep and shallow curved FG nanobeam based on nonlocal elasticity.

In general, mechanical analyses of elastic structures assume that the boundaries of the structure are supported classically by simply supported, fixed, or free. Only a few studies have considered the imperfections of the boundary, Ding *et al.* (2019). Based on nonlinear boundary conditions, Ma (2003) examined the existence of solutions of the nonlinear fourth order differential equation of beam with nonlinear boundary conditions. Ma and Da Silva (2004) developed iterative solution techniques for a beam equation with nonlinear boundary conditions of third order. Sedighi and Shirazi (2012) presented exact equivalent function for studying the cantilever beam vibration with a deadzone nonlinear boundary conditions. Ding *et al.* (2019) exploited three linear springs to construct a nonlinear isolation system with quasi-zero stiffness and investigate the transverse vibration of pre-pressure beams with nonlinear isolation effect. Ye *et al.* (2020) studied the nonlinear vibrations of slightly curved beam by harmonic balance method. It is shown that the nonlinear boundary conditions and initial deformation have a significant effect on the vibration behaviors of the beam. Geng *et al.* (2020) used impact damper at the free edge of cantilever beam to suppress multiple modal resonances. Zhao *et al.* (2022) predicted dynamic behavior of the beam structure with the nonlinear support and elastic boundary constraints. Tekin *et al.* (2023) used mixed-type finite element to investigate the axial vibration of bars with arbitrary boundary conditions. Wang *et al.* (2023) examined the vibration and resonance responses of non-uniform beam with randomly varying boundary conditions. Zhai *et al.* (2023) studied dynamic vibrational behavior of beam structures with nonlinear elastic foundations and boundaries by using finite element analysis. Alessi *et al.* (2023) developed finite element model to study the dynamic analysis of piezoelectric perforated cantilever bimorph energy harvester.

Analytical solutions were carried out for beams, plates, and shells over the years. Based on analytical solutions, Li and Qiao (2014) presented the exact bending curvature model for nonlinear free vibration of shear deformable anisotropic laminated beams. Chen and Li (2018) developed exact analytical solutions of buckling and postbuckling responses of the imperfect microbeams via Euler-Bernoulli beam and modified couple stress theory. Borjalilou *et al.* (2019) presented the exact solutions of bending, buckling and vibration of nonlocal FG nanobeams in frame of Timoshenko beam theory. Eltaher and Mohamed (2020) derived closed form solutions to examine the nonlinear static and dynamic stability of imperfect carbon nanotubes (CNTs) in prebuckling and postbuckling domains using Doublet mechanics theory. Juhász and Szekrényes (2020) estimated critical buckling loads and eigenfrequencies in closed form for delaminated composite spherical doubly curved shells. Rezaiee-Pajand and Kamali (2021) presented the exact solution for thermos-mechanical postbuckling of FG modified couple stress Timoshenko microbeams embedded on an elastic substrate. Adam *et al.* (2022) derived analytical expressions for free and forced vibrations of curved composite beams with symmetric layer arrangement and soft-hinged bearings. Almitani *et al.* (2022) developed in detail the exact solution of nonlinear postbuckling response of imperfect bioinspired composite beams resting on nonlinear foundations. Chang *et al.* (2022) presented closed solutions of static buckling and post buckling behaviors of functionally graded (FG) curved pipeline Euler-Bernoulli beam rested on Pasternak foundations. Mohamed *et al.* (2022b, 2023) developed an analytical closed form formula of the nonlinear load-deflection snap-through instability of helicoidal composite imperfect beams. Siam *et al.* (2023) exploited Navier analytical method to investigate the free vibration analysis of nonlocal viscoelastic nanobeam with holes and elastic foundations. Mohamed *et al.* (2024a) examined the nonlinear postbuckling and snap-through instability of movable simply supported BDFG porous plates rested on elastic foundations. Mohamed *et al.* (2024b) exploited the fractional differential quadrature method to study the nonlinear dynamics and forced vibrations of simply-supported fractional viscoelastic microbeams using.

The main objective of this paper is to investigate analytically the nonlinear static response in prebuckling and postbuckling domains, as well as linear vibration behaviors of straight and curved beam subjected to symmetric and asymmetric nonlinear boundary conditions, which are not considered elsewhere. The closed form solution for buckling, post buckling and vibration are derived in details. The proposed model is applicable only for a thin structure element.

2. Problem formulation

A curved beam with nonlinear elastic and damped boundary conditions is schematically shown in Fig. 1. The beam with length L in axial coordinate $\hat{x}$ and under external force $\hat{F}(\hat{x},\hat{t})$. The transverse deflection $\hat{w}$ in the direction of $\hat{z}$. The beam with initial imperfection $\hat{w}_0$ and rested on nonlinear viscoelastic boundaries with linear spring $\hat{K}$, nonlinear spring $\hat{K}_N$ and linear damper $\hat{C}$. The subscripts L and R indicate left and right boundaries.

Based on the Euler Bernoulli beam theory including the initial geometrical curvature, mid-plane stretching and nonlinear end viscoelastic boundary conditions (BCs), the equation of motion of beam can be portrayed as

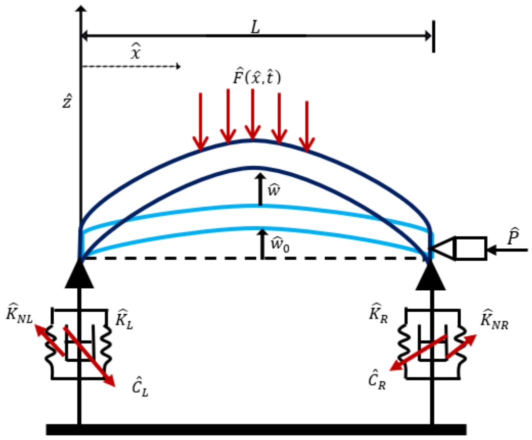

Fig. 1 Schematic of curved beam with nonlinear boundary conditions (Mohamed *et al.* 2024c)

$$m\frac{\partial^2 \hat{w}}{\partial \hat{t}^2} + EI\frac{\partial^4 \hat{w}}{\partial \hat{x}^4} + \hat{\mu}I\frac{\partial^5 \hat{w}}{\partial \hat{x}^4 \partial \hat{t}} + \frac{EA}{L}\left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{d^2 \hat{w}_0}{d\hat{x}^2}\right)\left[\frac{L}{EA}\hat{P} - \frac{1}{2}\int_0^L \left[2\frac{d\hat{w}_0}{d\hat{x}}\frac{d\hat{w}}{d\hat{x}} + \left(\frac{d\hat{w}}{d\hat{x}}\right)^2\right]d\hat{x}\right] + \frac{\hat{\mu}A}{L}\left[\int_0^L \left[\frac{\partial \hat{w}}{\partial \hat{x}}\frac{\partial^2 \hat{w}}{\partial \hat{x}\partial \hat{t}} + \frac{d\hat{w}_0}{d\hat{x}}\frac{\partial^2 \hat{w}}{\partial \hat{x}\partial \hat{t}}\right]d\hat{x}\right]\left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{d^2 \hat{w}_0}{d\hat{x}^2}\right) = \hat{F}(\hat{x},\hat{t}) \tag{1}$$

where the nonlinear end BCs are evaluated by

$$\frac{\partial^2 \hat{w}\,(0,t)}{\partial \hat{x}^2} = \frac{\partial^2 \hat{w}\,(L,t)}{\partial \hat{x}^2} = 0 \tag{2a}$$

$$\hat{K}_L\,\hat{w}(0,\hat{t}) + \hat{C}_L\frac{\partial \hat{w}(0,\hat{t})}{\partial \hat{t}} + \hat{K}_{NL}\hat{w}^3(0,\hat{t}) + EI\,\frac{\partial^3 \hat{w}\,(0,\hat{t})}{\partial \hat{x}^3} = 0 \tag{2b}$$

$$\hat{K}_R\,\hat{w}(L,\hat{t}) + \hat{C}_R\frac{\partial \hat{w}(L,\hat{t})}{\partial \hat{t}} + \hat{K}_{NR}\hat{w}^3(L,\hat{t}) - EI\,\frac{\partial^3 \hat{w}\,(L,\hat{t})}{\partial \hat{x}^3} = 0 \tag{2c}$$

The following non-dimensional parameters are used

$$x = \frac{\hat{x}}{L},\ w = \frac{\hat{w}}{r},\ w_0 = \frac{\hat{w}_0}{r},\ r = \sqrt{\frac{I}{A}}$$
$$, K_{NL} = \frac{\hat{K}_{NL}L^2}{EA}, K_{NR} = \frac{\hat{K}_{NR}L^2}{EA},\ P = \frac{\hat{P}L^2}{EI}$$
$$K_L = \frac{\hat{K}_L L^3}{EI}, K_R = \frac{\hat{K}_R L^3}{EI}, C_L = \hat{C}_L\sqrt{\frac{L^2}{EIm}},$$
$$C_R = \hat{C}_R\sqrt{\frac{L^2}{EIm}},\ \mu = \hat{\mu}\sqrt{\frac{I}{EmL^4}}, F = \frac{\hat{F}L^4}{rEI} \tag{3}$$

The governing equation and BCs in normalized form can be rewritten as

$$\ddot{w} + \mu\dot{w}'''' + w'''' + (w'' + w_0'')\left(P - \frac{1}{2}\int_0^1 (2w'w_0' + w'^2)dx\right) + \mu\int_0^1 (w'\dot{w}' + w_0'\dot{w}')dx = F(x,t) \tag{4}$$

with

$$w''(0,t) = w''(1,t) = 0 \tag{5a}$$

$$K_L w(0,t) + C_L \dot{w}(0,t) + K_{NL} w^3(0,t) + w'''(0,t) = 0 \tag{5b}$$

$$K_R w(1,t) + C_R \dot{w}(1,t) + K_{NR} w^3(1,t) + w'''(1,t) = 0 \tag{5c}$$

The field variable of governing Eq. (1) is expressed as

$$w(x,t) = w_s(x) + w_d(x,t) \tag{6}$$

meanwhile $w_s(x)$ is a static deflection and $w_d(x,t)$ is a small dynamic disturbance around static deflection. Substituting Eq. (6) into the governing Eq. (1) and the equations of boundary conditions (5). Then collecting the static parts, the result is time-independent equation, which represents the nonlinear buckling problem of curved beam

$$w_s'''' + (w_s'' + w_0'')\left(P - \frac{1}{2}\int_0^1 \left(2w_s' w_0' + w_s'^2\right)dx\right) = 0 \tag{7}$$

subjected to

$$w_s''(0) = w_s''(1) = 0 \tag{8a}$$

$$K_L w_s(0) + K_{NL} w_s^3(0) + w_s'''(0) = 0 \tag{8b}$$

$$K_R w_s(1) + K_{NR} w_s^3(1) - w_s'''(1) = 0 \tag{8c}$$

Assembling the time-dependent terms around the static deflection position. The result is the following nonlinear dynamic equation

$$\begin{gathered}\ddot{w}_d + \mu \dot{w}_d'''' + w_d'''' + w_d''\left(P - \frac{1}{2}\int_0^1 \left(2w_s' w_0' + w_s'^2\right)dx\right) - \frac{1}{2} w_d'' \int_0^1 w_d'^2 dx - \\ w_d'' \int_0^1 w_d'(w_s' + w_0')dx - \frac{1}{2}(w_s'' + w_0'')\int_0^1 w_d'^2 dx - (w_s'' + w_0'')\int_0^1 w_d'(w_s' + w_0')dx + \\ \mu \int_0^1 (w_s' + w_d' + w_0')\dot{w}_d' dx = F(x,t)\end{gathered} \tag{9}$$

With

$$w_d''(0,t) = w_d''(1,t) = 0 \tag{10a}$$

$$\begin{gathered}K_L w_d(0,t) + C_L \dot{w}_d(0,t) + K_{NL} w_d^3(0,t) + 3K_{NL} w_s(0) w_d^2(0,t) + 3K_{NL} w_s^2(0) w_d(0,t) + \\ w_d'''(0,t) = 0\end{gathered} \tag{10b}$$

$$\begin{gathered}K_R w_d(1,t) + C_R \dot{w}_d(1,t) + K_{NR} w_d^3(1,t) + 3K_{NR} w_s(1) w_d^2(1,t) + 3K_{NR} w_s^2(1) w_d(1,t) - \\ w_d'''(1,t) = 0\end{gathered} \tag{10c}$$

The total static deflection of the beam $\eta(x)$ can be evaluated by

$$\eta(x) = w_s(x) + w_0 \rightarrow w_s(x) = \eta(x) - w_0 \tag{11}$$

where w_0 is the initial shape of curvature, which can be assumed by

$$w_0 = g\sin(\pi x) \tag{12}$$

and g is the normalized amplitude.

Substituting Eqs. (11), (12) in the governing equations and BCs of static and dynamic problems. Eqs. (7), (8) become, respectively,

$$\eta'''' + \left(P - \frac{1}{2}\int_0^1 \left(\eta'^2 - w_0'^2\right)dx\right)\eta'' = w_0'''' \tag{13}$$

With

$$\eta''(0) = \eta''(1) = 0 \tag{14a}$$

$$K_L\eta(0) + K_{NL}\eta^3(0) + \eta'''(0) - w_0'''(0) = 0 \tag{14b}$$

$$K_R\eta(1) + K_{NR}\eta^3(1) - \eta'''(1) + w_0'''(1) = 0 \tag{14c}$$

Also, Eqs. (9) and (10) become

$$\ddot{w}_d + \mu\dot{w}_d'''' + w_d'''' + w_d''\left(P - \frac{1}{2}\int_0^1 \left(\eta'^2 - w_0'^2\right)dx\right) - \frac{1}{2}w_d''\int_0^1 w_d'^2 dx - w_d''\int_0^1 w_d'\eta' dx - \frac{1}{2}\eta''\int_0^1 w_d'^2 dx - \eta''\int_0^1 w_d'\eta' dx + \mu\int_0^1 (\eta' + w_d')\dot{w}_d' dx = F(x,t) \tag{15}$$

With

$$w_d''(0,t) = w_d''(1,t) = 0 \tag{16a}$$

$$K_L w_d(0,t) + C_L\dot{w}_d(0,t) + K_{NL}w_d^3(0,t) + 3K_{NL}\eta(0)w_d^2(0,t) + 3K_{NL}\eta^2(0)w_d(0,t) + w_d'''(0,t) = 0 \tag{16b}$$

$$K_R w_d(1,t) + C_R\dot{w}_d(1,t) + K_{NR}w_d^3(1,t) + 3K_{NR}\eta(1)w_d^2(1,t) + 3K_{NR}\eta^2(1)w_d(1,t) - w_d'''(1,t) = 0 \tag{16c}$$

3. Solution procedure

In this section, exact solutions of nonlinear buckling problem and linear vibration problem are developed.

3.1 Static buckling problem

The nonlinear buckling problem (13) can be written as

$$\eta'''' + \lambda^2\eta'' = w_0'''' \tag{17a}$$

$$\lambda^2 = P - \frac{1}{2}\int_0^1 \left(\eta'^2 - w_0'^2\right)dx \tag{17b}$$

Since Eq. (17a) is a fourth-order nonhomogeneous linear ordinary differential equation with constant coefficients, the general solution is

$$\eta(x) = \eta_h(x) + \eta_p(x) \tag{18}$$

whereas the homogeneous solution $\eta_h(x)$ can be obtained as

$$\eta_h(x) = c_1 + c_2x + c_3\sin(\lambda x) + c_4\cos(\lambda x)\,, \lambda \neq 0 \tag{19}$$

and the particular solution $\eta_p(x)$, which is dependent on initial shape of curvature w_0 (Eq. (12)), can be computed as

$$\eta_p(x) = -\frac{g\pi^2}{\lambda^2-\pi^2}\sin(\pi x)\,, \lambda \neq \pm\pi \tag{20}$$

(a) Straight beam ($w_0 = 0$)

The buckling problem of perfect beam is governed by Eqs. (17a&b) with $w_0 = 0$, that is

$$\eta'''' + \lambda^2\eta'' = 0 \tag{21a}$$

$$\lambda^2 = P - \frac{1}{2}\int_0^1 \eta'^2 dx \tag{21b}$$

Subjected to

$$\eta''(0) = \eta''(1) = 0 \tag{22a}$$

$$K_L\eta(0) + K_{NL}\eta^3(0) + \eta'''(0) = 0 \tag{22b}$$

$$K_R\eta(1) + K_{NR}\eta^3(1) - \eta'''(1) = 0 \tag{22c}$$

As a result, the general solution of Eq. (21a) is

$$\eta(x) = \eta_h(x) = c_1 + c_2x + c_3\sin(\lambda x) + c_4\cos(\lambda x)\,,\ \ \lambda \neq 0 \tag{23}$$

Substituting Eq. (22) into Eq. (21), results in the following nonlinear homogeneous algebraic equations

$$c_4\lambda^2 = 0 \tag{24a}$$

$$c_4\lambda^2 cos(\lambda) + c_3\lambda^2 sin(\lambda) = 0 \tag{24b}$$

$$k_{NL}(c_1 + c_4)^3 - c_3\lambda^3 + k_L(c_1 + c_4) = 0 \tag{24c}$$

$$k_R\big(c_1 + c_2 + c_3 sin(\lambda) + c_4 cos(\lambda)\big) + k_{NR}\big(c_1 + c_2 + c_3 sin(\lambda) + c_4 cos(\lambda)\big)^3 + c_3\lambda^3 cos(\lambda) - c_4\lambda^3 sin(\lambda) = 0 \tag{24d}$$

System (24) can be written in the form

$$\mathcal{N}(\lambda, \mathcal{C}) = 0, \mathcal{C} = [c_1, c_2, c_3, c_4]$$

which is nonlinear eigenvalue problem in both parameter λ and vector $\mathcal{C}$. For the system to have nontrivial solution and from Eq. (24a), (24b), we have

$$c_4 = 0 \ \ \rightarrow c_3\lambda^2 sin(\lambda) = 0 \ \ \ \rightarrow sin(\lambda) = 0 \ \ \rightarrow \lambda = n\pi, n = 1,2, \ldots \tag{25}$$

Substituting Eq. (25) into (24c), (24d), yields

$$k_{NL}{c_1}^3 - c_3\lambda^3 + k_L c_1 = 0 \rightarrow c_3 = \frac{c_1(k_{NL}{c_1}^2 + k_L)}{\lambda^3} \tag{26}$$

Substituting Eqs. (25), (26) into Eq. (23), get the total displacement as

$$\eta(x) = c_1\left(1 + \frac{k_{NL}{c_1}^2 + k_L}{\lambda^3}\sin(\lambda x)\right) + c_2 x \tag{27}$$

To determine the values of c_1 and c_2, substituting Eqs. (25), (26) into Eq. (24d) and substituting

Eq. (27) into Eq. (21b), the result is two nonlinear algebraic equations of polynomial type for constants c_1 and c_2

$$k_R(c_1 + c_2) + k_{NR}(c_1 + c_2)^3 + c_1(k_{NL}{c_1}^2 + k_L)cos(\lambda) = 0 \tag{28a}$$

$$c_1^2(k_{NL}{c_1}^2 + k_L)^2 + 2c_2^2\lambda^2 = 4(P - \lambda^2) \tag{28b}$$

Solving Eqs. (28a) and (28b), the values of c_1 and c_2 can be easily computed and the static response $\eta(x)$, Eq. (27), can be obtained. One has to notice that the left-hand side of Eq. (28b) is sum of squares and hence it is greater than or equal to zero irrespective of the values of the foundation constants. This implies that $P \geq \lambda^2$ and the case $P = \lambda^2$ corresponds to $c_1 = c_2 = 0 \to \eta(x) = 0$. For the first buckling mode, $\lambda = \pi$, and the critical buckling load $P_{cr} = \pi^2$ independent of the nonlinear foundation constants.

In the case of symmetric boundary conditions ($k_L = k_R$ and $k_{NL} = k_{NR}$), using Eq. (25) and Eq. (27), we have

$$\begin{cases} \eta(0) = c_1 \\ \eta(1) = c_1 + c_2 \end{cases} \to c_2 = 0$$

As a result

$$c_1(k_{NL}{c_1}^2 + k_L) = \pm 2\sqrt{P_0 - \lambda^2} \to \eta(x) = c_1 \pm 2\frac{\sqrt{P_0-\lambda^2}}{\lambda^3}\sin(\lambda x) \tag{29}$$

and hence the critical buckling loads in case of symmetric boundary conditions can be determined as

$$P_{cr} = \lambda^2$$

(b) Curved beam ($w_0 \neq 0$)

Since $\boldsymbol{w_0} \neq 0$, The solution of Eq. (17a) is

$$\eta(x) = c_1 + c_2 x + c_3\sin(\lambda x) + c_4\cos(\lambda x) - \frac{g\pi^2}{\lambda^2-\pi^2}\sin(\pi x)\,, \lambda \neq \{0, \pm\pi\} \tag{30}$$

Substituting Eq. (14) into Eq. (30), the result is the following nonlinear nonhomogeneous algebraic system

$$c_4\lambda^2 = 0 \tag{31a}$$

$$c_4\lambda^2 cos(\lambda) + c_3\lambda^2 sin(\lambda) = 0 \tag{31b}$$

$$k_{NL}(c_1 + c_4)^3 - c_3\lambda^3 + k_L(c_1 + c_4) = -g\pi^3 - \frac{g\pi^5}{\lambda^2-\pi^2} \tag{31c}$$

$$\begin{gathered} k_R\big(c_1 + c_2 + c_3 sin(\lambda) + c_4 cos(\lambda)\big) + k_{NR}\big(c_1 + c_2 + c_3 sin(\lambda) + c_4 cos(\lambda)\big)^3 + \\ c_3\lambda^3 cos(\lambda) - c_4\lambda^3 sin(\lambda) = -g\pi^3 - \frac{g\pi^5}{\lambda^2-\pi^2} \end{gathered} \tag{31d}$$

From Eqs. (31a), (31b), we have

$$c_4 = 0 \tag{32a}$$

$$c_3\lambda^2 sin(\lambda) = 0 \tag{32b}$$

It is noticed from Eq. (32b) that there are two cases

$$sin(\lambda) = 0 \to \lambda = n\pi, n = 2,3, ..n = \{0, 1\}$$

are excluded from Eq. (30) this case gives the higher buckling mode. Second (which gives the first buckling mode)

$$c_3 = 0 \tag{33}$$

Substituting Eq. (32a) and Eq. (33) into Eq. (30), the total displacement is

$$\eta(x) = c_1 + c_2 x - \frac{g\pi^2}{\lambda^2 - \pi^2}\sin(\pi x), \lambda \neq \{0, \pm\pi\} \tag{34}$$

Substituting Eq. (31a) and Eq. (33) into Eqs. (31c), (31d) and substituting Eq. (34) into Eq. (17b), the result is nonlinear nonhomogeneous algebraic system in constants c_1, c_2 and λ

$$k_{NL}c_1^3 + k_L c_1 = -g\pi^3 - \frac{g\pi^5}{\lambda^2 - \pi^2} \tag{35a}$$

$$k_R(c_1 + c_2) + k_{NR}(c_1 + c_2)^3 = -g\pi^3 - \frac{g\pi^5}{\lambda^2-\pi^2} \tag{35b}$$

$$(\lambda^2)^3 + \frac{1}{4}[2c_2^2 - 4P_0 - 8\pi^2 - g^2\pi^2](\lambda^2)^2 - \frac{1}{2}\pi^2[c_2^2 - g\pi^2 - \pi^2 - 4P_0]\lambda^2 = -\frac{1}{2}\pi^4[c_2^2 - 2P_0] \tag{35c}$$

Eqs. (35a), (35b), (35c) can be easily solved symbolically, using Matlab. As a result, the static response of the first buckling mode of curved beam can be easily computed.

In the case of symmetric boundary conditions ($k_L = k_R$ and $k_{NL} = k_{NR}$), using Eq. (33)

$$\begin{cases} \eta(0) = c_1 \\ \eta(1) = c_1 + c_2 \end{cases} \to c_2 = 0$$

As a result

$$(\lambda^2)^3 - \left[P_0 + 2\pi^2 + \frac{1}{4}g^2\pi^2\right](\lambda^2)^2 + 2\pi^2\left[P_0 + 2\pi^2 + \frac{1}{4}g\pi^2\right]\lambda^2 - \pi^4 P_0 = 0 \tag{36}$$

which is a cubic polynomial with respect to λ^2. It always has at least on real root, when the number of real roots changes, the buckling occurs. Therefore, the discriminant of Eq. (36) is set to zero. As a result, the first critical buckling load of curved beams with symmetric boundary conditions can be obtained as

$$P_{cr} = \pi^2\left(1 + \frac{3}{\sqrt[3]{16}}g^{\frac{2}{3}} - \frac{1}{4}g^2\right) \tag{37}$$

3.2 Linear vibration problem

By omitting the external excitation, damping and nonlinear terms in Eqs. (15) and (16), the governing equation and boundary conditions of the free vibration problem are

$$\ddot{w}_d + w_d'''' + w_d''\left(P - \frac{1}{2}\int_0^1(\eta'^2 - w_0'^2)dx\right) - \eta''\int_0^1 w_d'\eta' dx = 0 \tag{38}$$

With

$$w_d''(0,t) = w_d''(1,t) = 0 \tag{39a}$$

$$K_L w_d(0,t) + 3K_{NL}\eta^2(0)w_d(0,t) + w_d'''(0,t) = 0 \tag{39b}$$

$$K_R w_d(1,t) + 3K_{NR}\eta^2(1)w_d(1,t) - w_d'''(1,t) = 0 \tag{39c}$$

Assuming $w_d = \phi(x)e^{i\omega t}$ and using Eq. (17b), Eq. (38) can be simplified as follows

$$\phi^{iv} + \lambda^2\phi'' - \omega^2\phi = \eta'' \int_0^1 \eta'\phi' dx \tag{40}$$

in which ω denotes the natural frequency and $\phi(x)$ is the corresponding mode shape. The boundary conditions in terms of ϕ are given as

$$\phi''(0) = \phi''(1) = 0 \tag{41a}$$

$$K_L\phi(0) + 3K_{NL}\eta^2(0)\phi(0) + \phi'''(0) = 0 \tag{41b}$$

$$K_R\phi(1) + 3K_{NR}\eta^2(1)\phi(1) - \phi'''(1) = 0 \tag{41c}$$

Since Eq. (40) is fourth order nonhomogeneous linear ordinary differential equation with constant coefficients, the general solution of this equation can be represented as

$$\phi(x) = \phi_h(x) + \phi_p(x) \tag{42}$$

The homogeneous solution $\phi_h(x)$ is

$$\phi_h(x) = d_1 sin(r_1 x) + d_2 cos(r_1 x) + d_3 sinh(r_2 x) + d_4 cosh(r_2 x) \tag{43}$$

Whereas

$$r_{1,2} = \sqrt{\frac{1}{2}\sqrt{\lambda^4 + 4\omega^2} \pm \frac{1}{2}\lambda^2} \tag{44}$$

The particular solution takes the form

$$\phi_p(x) = d_5\eta''(x) \tag{45}$$

Substituting Eq. (42) into Eq. (40) and using Eq. (17), one obtains

$$d_5\left(w_0^{vi} - \left[\omega^2 + \int_0^1 \eta'\eta''' dx\right]\eta''\right) = \eta'' \int_0^1 \eta'\phi_h' dx \tag{46}$$

Substituting BCs Eq. (41) into Eq. (42) with Eq. (46) yields five homogeneous algebraic equations which can be written in the following matrix form

$$[\mathcal{K}]_{5\times5}[\boldsymbol{d}]_{5\times1} = [\boldsymbol{0}]_{5\times1}, \boldsymbol{d} = [d_1, d_2, d_3, d_4, d_5]^T \tag{47}$$

in which $\mathcal{K}$ is nonlinear stiffness matrix. Eq. (47) is nonlinear eigenvalue problem in parameter ω, in this case, for non-trivial solution, the determinant of the matrix must vanish. As a result, at a given axial load, the natural frequencies and the corresponding mode shapes around buckled position can be computed. The nonlinear stiffness matrix $\mathcal{K}$ for straight and curved beams are presented in Table 1.

Table 1 Nonlinear stiffness matrix of straight and curved beams in prebuckling and postbuckling states

	Nonlinear Stiffness matrix $\mathcal{K}$
	(a) Prebuckling State ($\eta(x) = 0 \rightarrow \lambda^2 = P$) $[\boldsymbol{\mathcal{K}}] = \begin{bmatrix} 0 & -r_1^2 & 0 & r_2^2 \\ -r_1^2 sin(r_1) & -r_1^2 cos(r_1) & r_2^2 sinh(r_2) & r_2^2 cosh(r_2) \\ -r_1^3 & k_L & r_2^3 & k_L \\ r_1^3 cos(r_1) + k_R sin(r_1) & k_R cos(r_1) - r_1^3 sin(r_1) & k_R sinh(r_2) - r_2^3 cosh(r_2) & k_R cosh(r_2) - r_2^3 sinh(r_2) \end{bmatrix}$
Straight beam	*(b) Postbuckling State ($\eta(x) = c_1\left(1 + \frac{k_{NL}{c_1}^2 + k_L}{\lambda^3}\sin(\lambda x)\right) + c_2 x \rightarrow \lambda^2 = n^2\pi^2$)* $[\boldsymbol{\mathcal{K}}] = \begin{bmatrix} 0 & -r_1^2 & 0 & r_2^2 & 0 \\ -r_1^2 sin(r_1) & -r_1^2 cos(r_1) & r_2^2 sinh(r_2) & r_2^2 cosh(r_2) & 0 \\ -r_1^3 & 3k_{NL}c_1^2 + k_L & r_2^3 & 3k_{NL}c_1^2 + k_L & \lambda^2 c_1(k_{NL}c_1^2 + k_L) \\ a_1 & a_2 & a_3 & a_4 & \lambda^2 \cos(\lambda)\, c_1(k_{NL}c_1^2 + k_L) \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix}$ $a_1 = r_1^3 cos(r_1) + k_R sin(r_1) + k_{NR}(c_1 + c_2)^2 \sin(r_1)$ $a_2 = k_R cos(r_1) - r_1^3 sin(r_1) + k_{NR}(c_1 + c_2)^2 \cos(r_1)$ $a_3 = k_R sinh(r_2) - r_2^3 cosh(r_2) + k_{NR}(c_1 + c_2)^2 sinh(r_2)$ $a_4 = k_R cosh(r_2) - r_2^3 sinh(r_2) + k_{NR}(c_1 + c_2)^2 cosh(r_2)$ $b_1 = -(\lambda^2 + r_2^2) sin(r_1)\left(c_2\lambda^2[r_1^2 - \lambda^2] + c_1 r_1^2[k_{NL}c_1^2 + k_L]cos(\lambda)\right)$ $b_2 = (\lambda^2 + r_2^2)(c_2\lambda^2[r_1^2 - \lambda^2][1 - \cos(r_1)] + c_1 r_1^2[k_{NL}c_1^2 + k_L][1 - cos(\lambda)\cos(r_1)])$ $b_3 = -(r_1^2 - \lambda^2) sinh(r_1)\left(c_2\lambda^2[\lambda^2 + r_2^2] + c_1 r_2^2[k_{NL}c_1^2 + kL]cos(\lambda)\right)$ $b_4 = (r_1^2 - \lambda^2)(c_2\lambda^2[\lambda^2 + r_2^2][1 - \cosh(r_2)] + c_1 r_2^2[k_{NL}c_1^2 + k_L][1 - cos(\lambda)\cosh(r_2)])$ $b_5 = (\lambda^2 + r_2^2)(r_1^2 - \lambda^2)\left(c_1^2 k_L\left[\frac{1}{2}c_1^4 k_{NL}^2 + c_1^2 k_{NL} + \frac{1}{2}k_L\right] - \omega^2\lambda^2\right)$
Curved beam	*Prebuckling and Postbuckling States ($\eta(x) = c_1 + c_2 x - \frac{g\pi^2}{\lambda^2 - \pi^2}sin(\pi x)$, $\lambda \neq \{0, \pm\pi\}$)* $[\boldsymbol{\mathcal{K}}] = \begin{bmatrix} 0 & -r_1^2 & 0 & r_2^2 & 0 \\ -r_1^2 sin(r_1) & -r_1^2 cos(r_1) & r_2^2 sinh(r_2) & r_2^2 cosh(r_2) & 0 \\ -r_1^3 & 3k_{NL}c_1^2 + k_L & r_2^3 & 3k_{NL}c_1^2 + k_L & -\frac{g\pi^7}{\lambda^2 - \pi^2} \\ a_1 & a_2 & a_3 & a_4 & -\frac{g\pi^7}{\lambda^2 - \pi^2} \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix}$ $e_1 = (\pi^2 + r_2^2)(\lambda^2 - \pi^2)\, sin(r_1)\,(c_2[\lambda^2 - \pi^2][r_1^2 - \pi^2] + g\pi^3 r_1^2)$ $e_2 = -(\pi^2 + r_2^2)(\lambda^2 - \pi^2)(c_2[\lambda^2 - \pi^2][r_1^2 - \pi^2][1 - \cos(r_1)] - g\pi^3 r_1^2[1 + \cos(r_1)])$ $e_3 = (r_1^2 - \pi^2)(\lambda^2 - \pi^2) sinh(r_2)(c_2[\lambda^2 - \pi^2][r_2^2 + \pi^2] + g\pi^3 r_2^2)$ $e_4 = -(r_1^2 - \pi^2)(\lambda^2 - \pi^2)(c_2[\lambda^2 - \pi^2][r_2^2 + \pi^2][1 - \cosh(r_2)] - g\pi^3 r_2^2[1 + \cosh(r_2)])$ $e_5 = -(\pi^2 + r_2^2)(r_1^2 - \pi^2)\left(-\pi^2[(\lambda^2 - \pi^2)^3] + \frac{1}{2}g\pi^8 - \omega^2[(\lambda^2 - \pi^2)^2]\right)$

4. Numerical results

In this section, the effects of symmetric and asymmetric nonlinear BC on static and linear vibration behaviors of straight and curved beam are investigated.

4.1 Static results

The influences of nonlinear BCs on critical buckling load of straight and curved beams for symmetric and asymmetric cases are shown in Tables 2, 3. It is noted that the critical buckling

Table 2 Effect of symmetric BCs on nondimensional first critical buckling load of straight and curved beams

Imperfection amplitude	$k_{NR} = k_{NL} = 10$			$k_{NR} = k_{NL} = 30$		
	$k_R = k_L = 5$	20	40	$k_R = k_L = 5$	20	40
$g = 0$	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
$g = 1$	19.1525	19.1525	19.1525	19.1525	19.1525	19.1525
$g = 2$	18.6524	18.6524	18.6524	18.6524	18.6524	18.6524
$g = 3$	12.1045	12.1045	12.1045	12.1045	12.1045	12.1045

Table 3 Effect of asymmetric BCs on nondimensional first critical buckling load of straight and curved beams

	k_{NL}	$k_{NR} = 5$				$k_{NR} = 20$			
		$k_R = 5$		$k_R = 15$		$k_R = 5$		$k_R = 15$	
		$k_L = 10$	20	$k_L = 10$	20	$k_L = 10$	20	$k_L = 10$	20
$g = 0$	10	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
	30	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696	9.8696
$g = 1$	10	19.2945	19.3997	19.1865	19.2454	19.1905	19.1581	19.2266	19.1755
	30	19.6816	19.7704	19.4388	19.5049	19.1768	19.1986	19.1587	19.1714
$g = 2$	10	18.8389	18.9400	18.7230	18.7893	18.7166	18.6746	18.7546	18.6988
	30	19.3934	19.4829	19.1349	19.2076	18.6833	18.7036	18.6644	18.6780
$g = 3$	10	12.3276	12.4263	12.2077	12.2777	12.1908	12.1441	12.2298	12.1716
	30	13.0191	13.1089	12.7530	12.8289	12.1409	12.1604	12.1215	12.1356

load of straight beam has no change as the value of the linear and nonlinear foundation constant changes in two cases symmetric and asymmetric. This is expected as presented in Eq. (2), the forces of foundation are functions of transverse deflection (w). So, for straight beam under axial compressive load, the only deformation occurs in the axial direction U, until reach to the first buckling load, then cause the transverse deflection. Therefore, the first linear buckling load does not affected by the foundations. If we add axial elastic spring on the boundaries, we can get influence on first buckling load for straight beam.

For curved beam, in case of symmetric BCs, the value of foundation constants has no effect on the critical buckling load (see Eq. (37)). However, in asymmetric case, as the value of foundation constants changes, the critical buckling load changes.

Figs. 2 and 3 show the nonlinear responses in prebuckling and postbuckling states of straight beam for different values of linear and nonlinear elastic foundation constants, respectively. The solid lines display stable solutions and dotted line displays the unstable one. It is noted that the foundation constants has no effect on the response of straight beam in prebuckling state (the response is zero). While, in postbuckling state, foundation parameters have a great influence on the nonlinear response and this effect becomes more tangible when the value of axial load increases.

The influences of linear and nonlinear elastic foundation constants on the nonlinear response of curved beam at a given imperfection amplitude $g = 1$ are plotted in Figs. 4 and 5, respectively. It is observed that the foundation constants have great effect on the nonlinear response of curved beam in prebuckling and postbuckling domains. Furthermore, one can note that the nonlinear foundation parameters have the most effect on the nonlinear response.

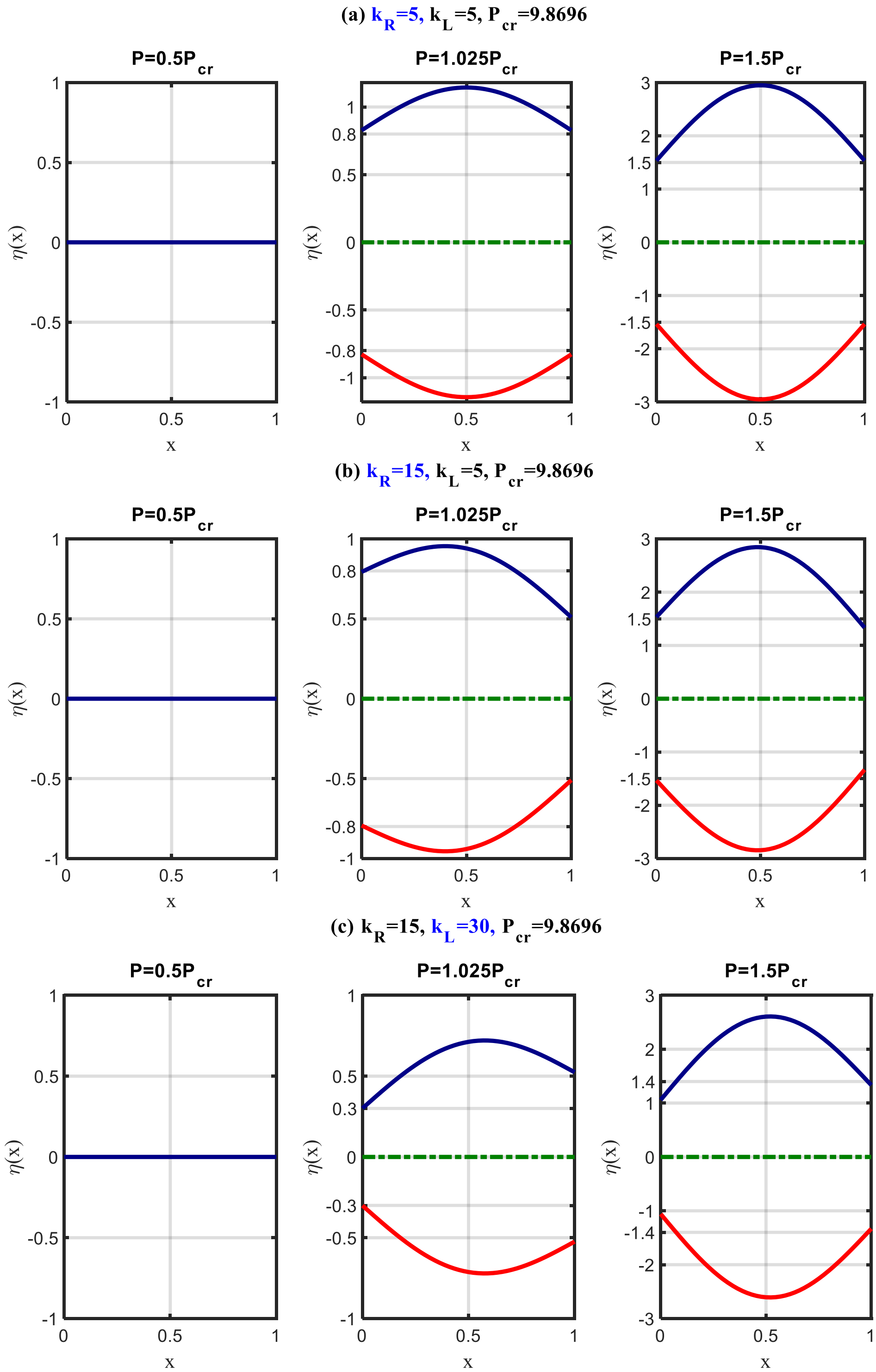

Fig. 2 Effect of k_R and k_L on nonlinear static response of straight beam in prebuckling and postbuckling states ($k_{NL} = k_{NR} = 10$)

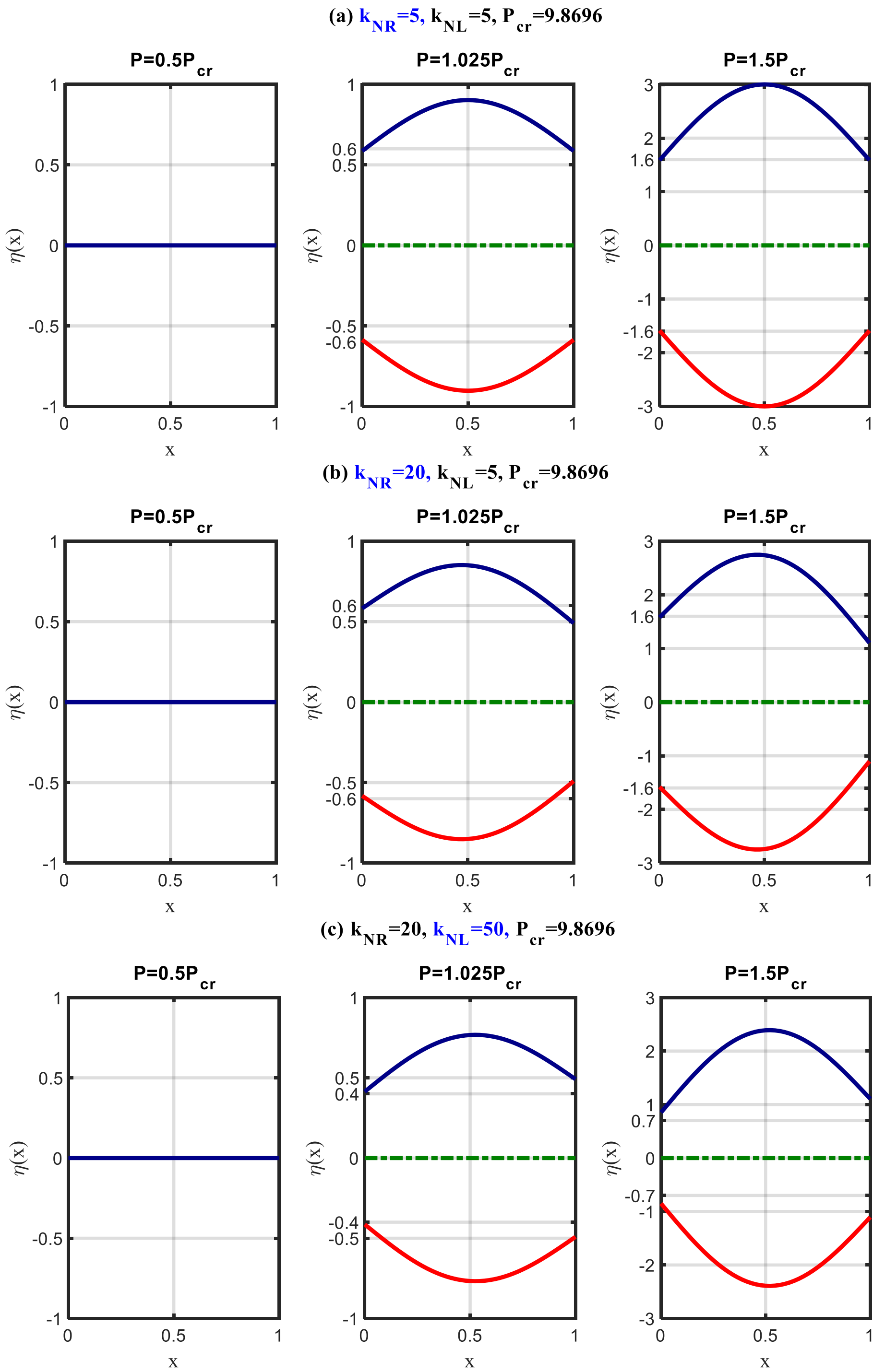

Fig. 3 Effect of k_{NR} and k_{NL} on nonlinear static response of straight beam in prebuckling and postbuckling state ($k_L = k_R = 15$)

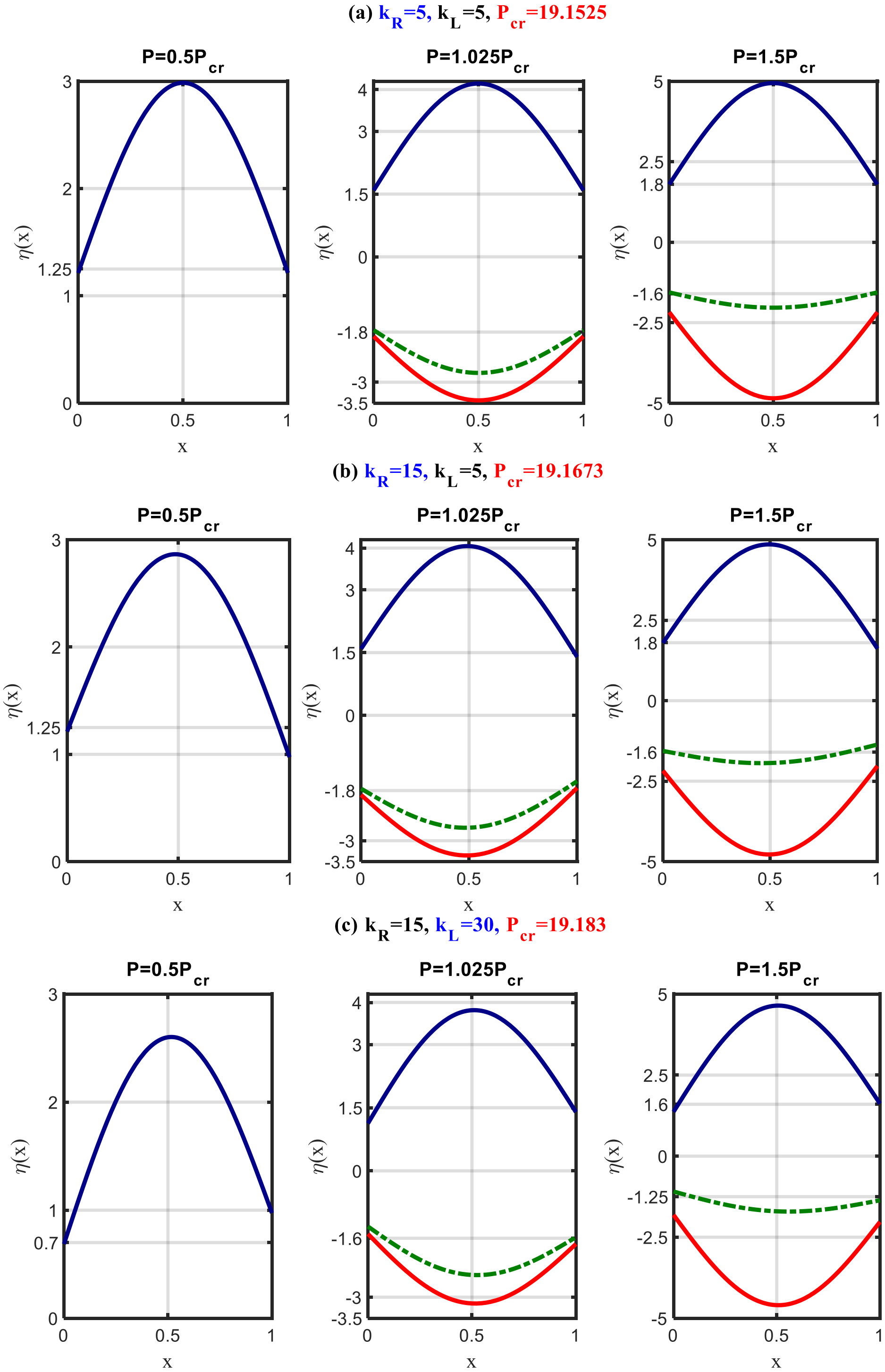

Fig. 4 Effect of k_R and k_L on nonlinear static response of curved beam in prebuckling and postbuckling states ($g = 1, k_{NL} = k_{NR} = 10$)

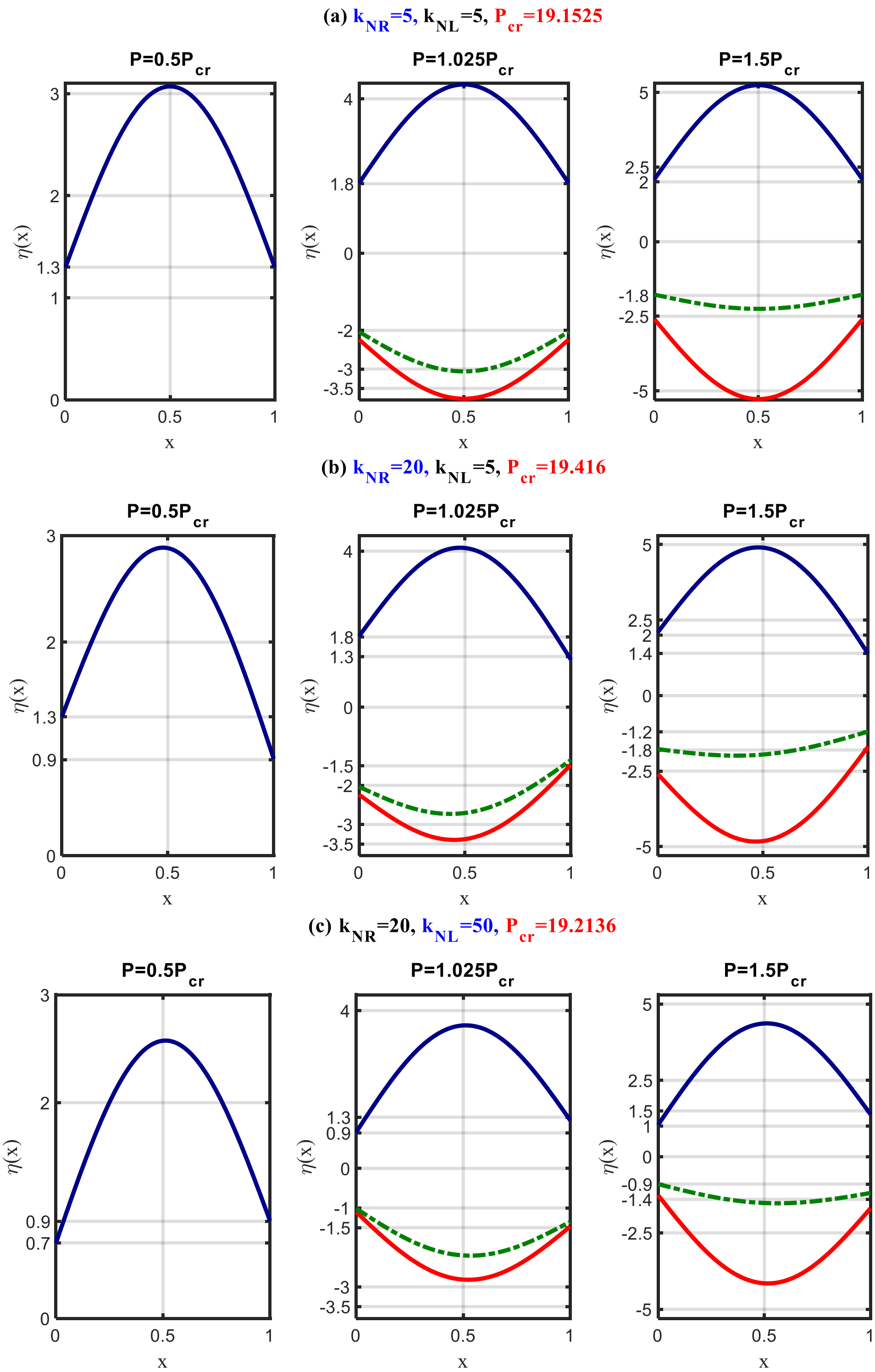

Fig. 5 Effect of k_{NR} and k_{NL} on nonlinear static response of curved beam in prebuckling and postbuckling state ($g = 1, k_L = k_R = 10$)

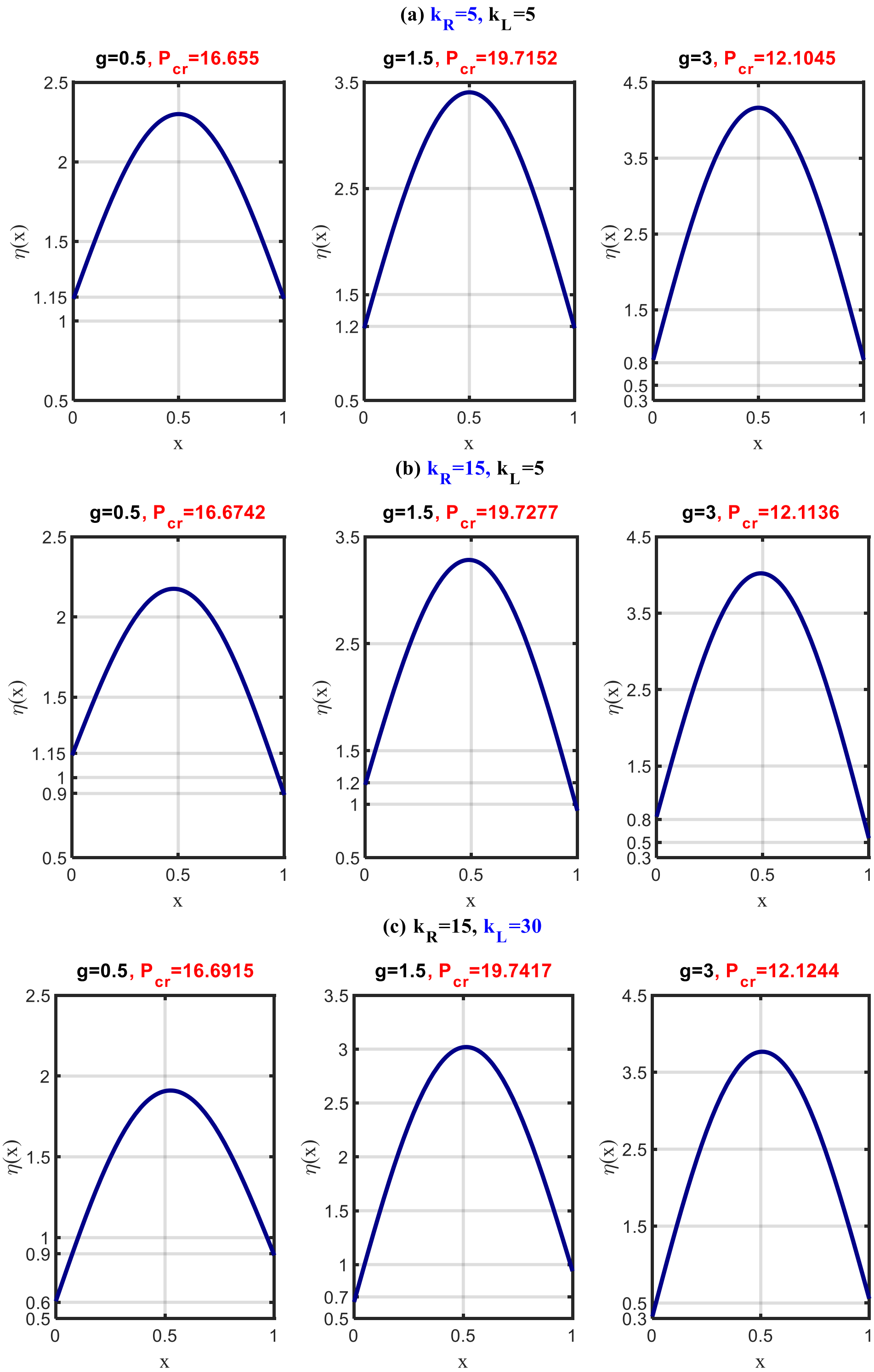

Fig. 6 Effect of k_R and k_L on static response of curved beam in prebuckling state with various values of g ($P = 0.5P_{cr}, k_{NL} = k_{NR} = 10$)

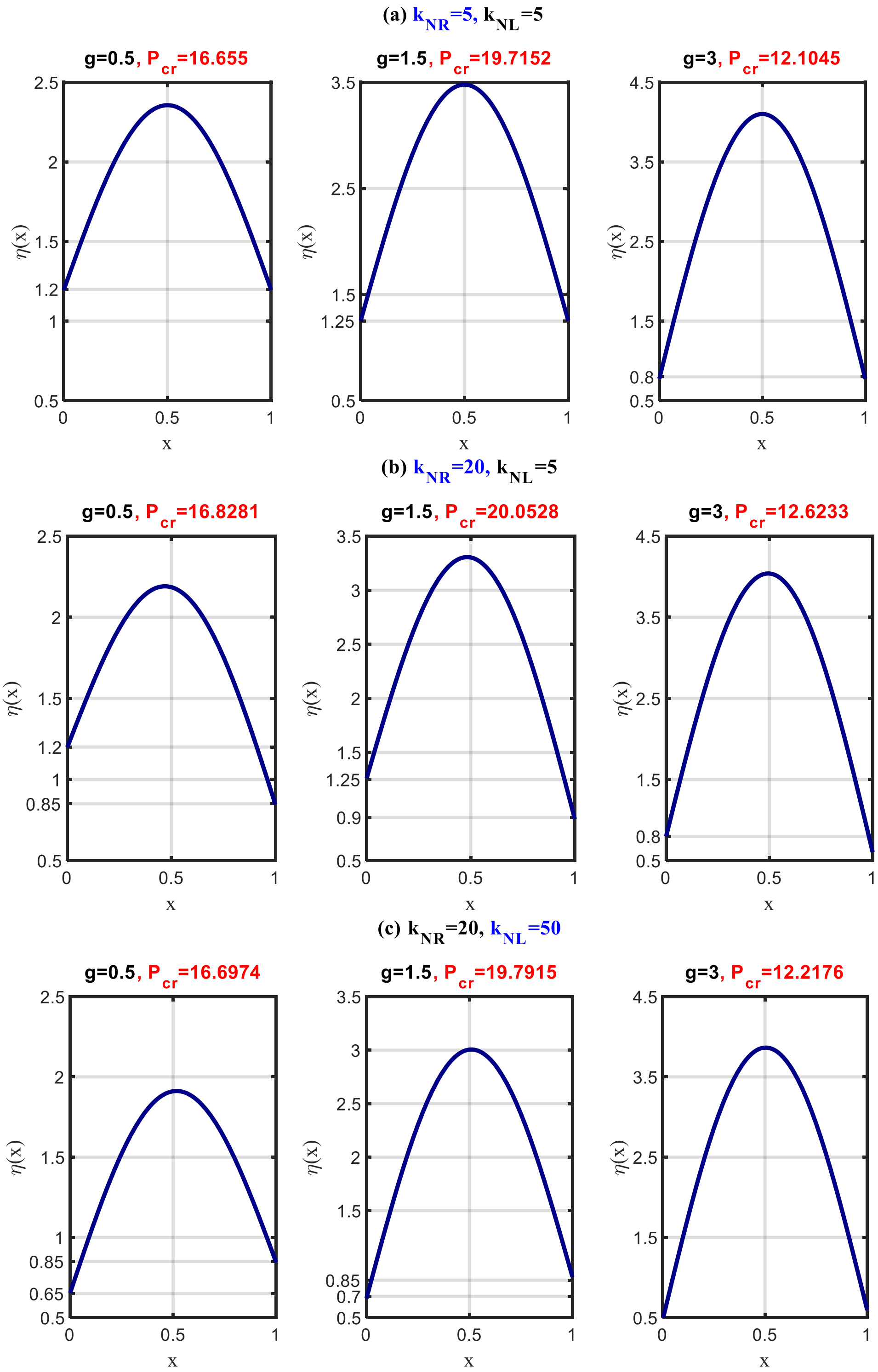

Fig. 7 Effect of k_{NR} and k_{NL} on static response of curved beam in prebuckling state with various values of g ($P = 0.5P_{cr}, k_L = k_R = 10$)

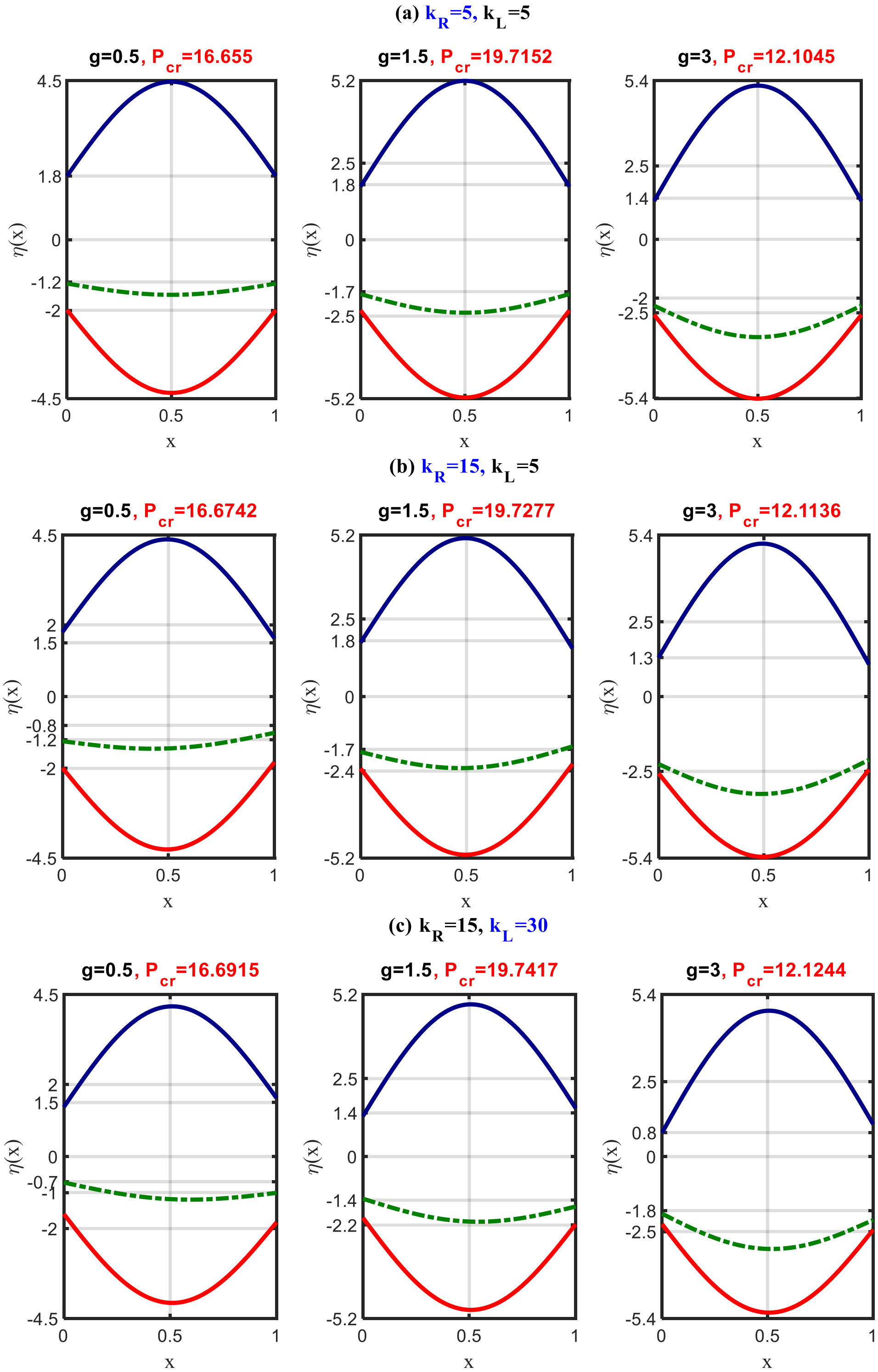

Fig. 8 Effect of k_R and k_L on static response of curved beam in postbuckling state with various values of g ($P = 1.5P_{cr}, k_{NL} = k_{NR} = 10$)

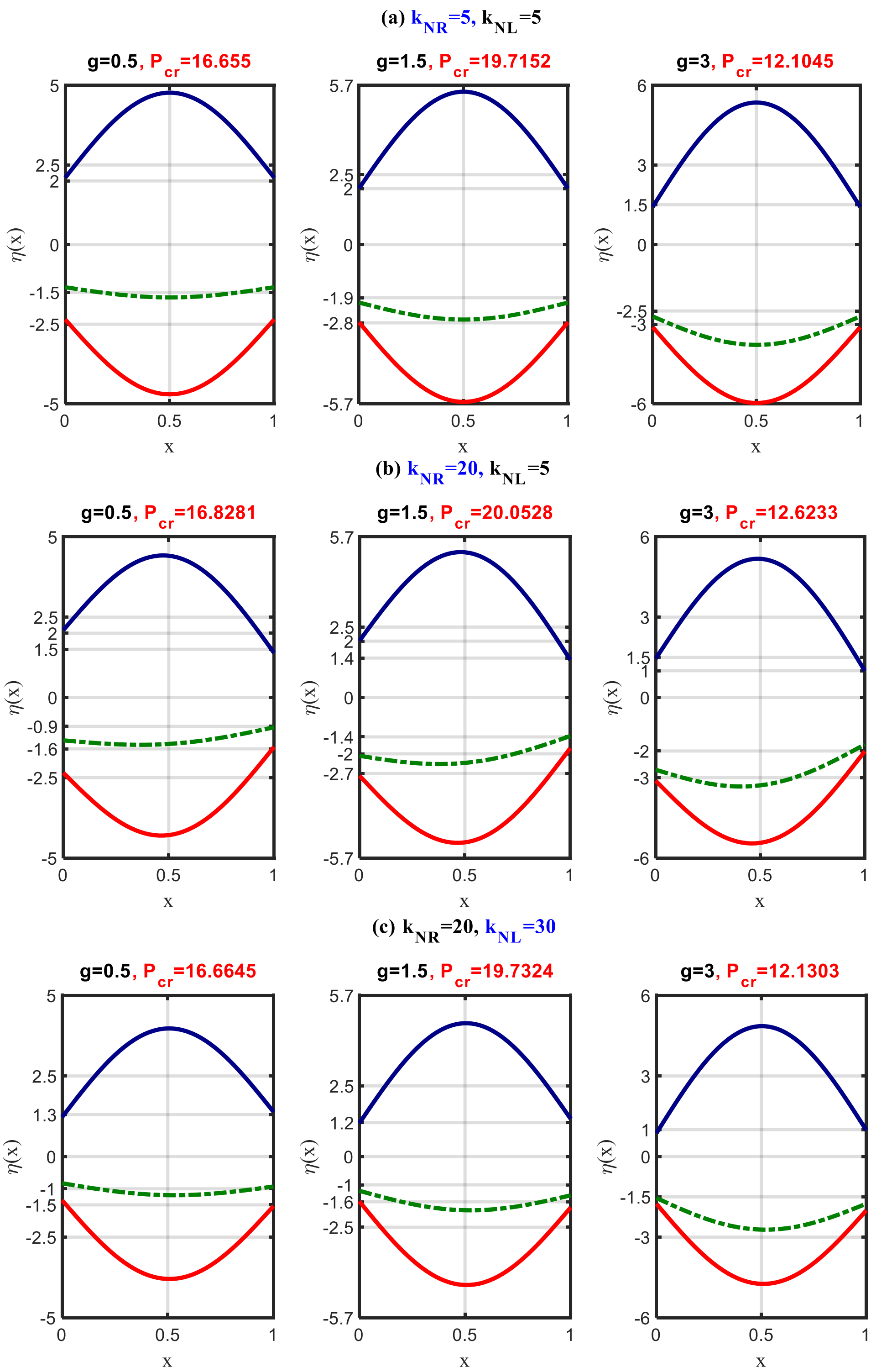

Fig. 9 Effect of k_{NR} and k_{NL} on static response of curved beam in postbuckling state with various values of g $(P = 1.5P_{cr}, k_L = k_R = 10)$

Table 4 Effect of symmetric boundary conditions on nondimensional first three natural frequencies of straight and curved beam in prebuckling state at axial load $P = 0.5P_{cr}$, (P_{cr} is reported in Table 2)

		$k_{NR} = k_{NL} = 10$			$k_{NR} = k_{NL} = 30$		
		$k_R = k_L = 5$	20	40	$k_R = k_L = 5$	20	40
$g = 0$ $P_{cr} = 9.8696$	ω_1	3.0361	5.4364	6.7968	3.0361	5.4364	6.7968
	ω_2	5.4447	10.6959	14.7683	5.4447	10.6959	14.7683
	ω_3	23.2631	25.8498	29.0382	23.2631	25.8498	29.0382
$g = 1$ $P_{cr} = 19.1525$	ω_1	10.1650	9.7529	10.1702	11.2111	10.8608	10.8346
	ω_2	15.5193	14.5731	15.5318	18.3239	17.3074	17.2348
	ω_3	30.8649	30.0289	30.8764	33.7490	32.6317	32.5551
$g = 2$ $P_{cr} = 18.6524$	ω_1	12.7205	12.1432	13.1751	14.4575	13.8368	14.0216
	ω_2	14.7246	13.8961	15.4051	17.4995	16.4495	16.7543
	ω_3	32.1241	31.6174	32.5767	34.1916	33.3387	33.5771

The mutual effects of foundation parameters with amplitude of initial curvature on the nonlinear response of curved beam in the prebuckling domain are displayed in Figs. 6 and 7.

Figs. 8 and 9 show the influence of initial curvature amplitude with linear and nonlinear foundation constants on the nonlinear response of curved beam in postbuckling domain. It is observed that, as the value of the amplitude of initial curvature increases, leads to increase in the absolute values of the amplitude of the stable and unstable responses.

4.2 Dynamic results

In Table 4, the first three natural frequencies of straight and curved beam in prebuckling state in case of symmetric BCs for different values of foundation constants are tabulated. It is evident from Table 4 that with increasing the variation of linear foundation constant the natural frequencies of straight and curved beam change. For the case of straight beam, the nonlinear foundation constant has no rule on the value of natural frequencies in prebuckling state. The reason for this can be seen from boundary conditions Eqs. (39b), (39c), which indicate that the nonlinear foundation constant depends on the static response $\eta(x)$ at ends. As seen from Figs. 2, 3, the static response of straight beam in prebuckling state is zero, so the nonlinear foundation constant plays no rule in prebuckling domain. On the contrary, for curved beam, the nonlinear foundation constant increases the value of natural frequencies.

Figs. 10 and 11 shows the variations of the first four mode shapes of the straight beam in postbuckling domain when $P = 1.5P_{cr}$ with different values of foundation constants. The mode shapes are normalized such that $\phi(0) = 1$.

The effect of the amplitude of initial curvature on the first two natural frequencies is shown in Fig. 12. It is noted that an increase of initial curvature decreases the second natural frequency of transverse vibration of the curved beam. Furthermore, as can be seen from Fig. 12, as the amplitude of initial curvature increases, the one to one internal resonance is activated $(\omega_1 = \omega_2)$, Fig. 12(c).

It is observed from Fig. 12(c) that the one to one internal resonance between the first and second mode occurs for curved beam in prebuckling state when $g = 3$ at $P \cong 0.3647P_{cr}(\omega_1 = \omega_2 = 10.8199)$. Fig. 13 shows the first and second vibration mode shapes of curved beam when

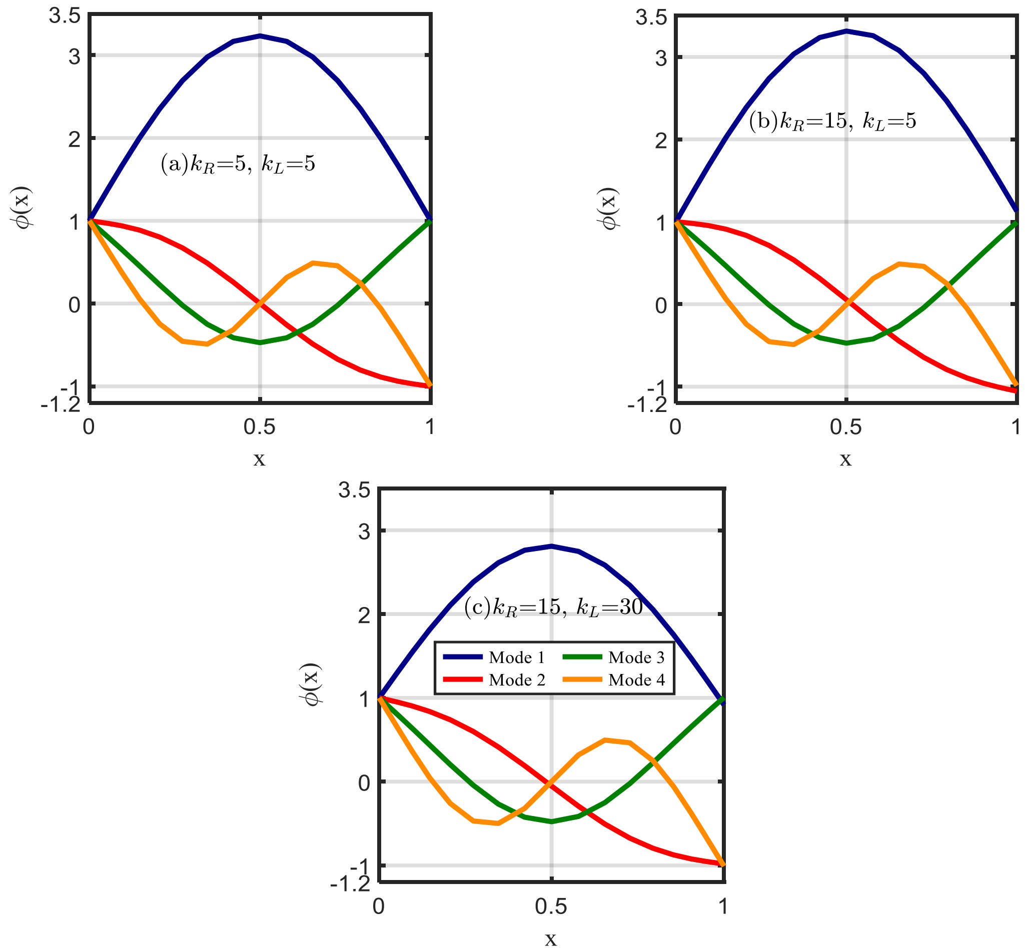

Fig. 10 Effect of k_R and k_L on first four mode shapes of straight beam in postbuckling state ($P = 1.5P_{cr}, k_{NL} = k_{NR} = 10$)

$g = 3$ in the prebuckling domain (a) before internal resonance, $P = 0.3P_{cr}$, and (b) after internal resonance, $P = 0.4P_{cr}$.

5. Conclusions

This study analyzed post buckling stability and dynamic responses of straight and curved beams subjected to symmetric and asymmetric nonlinear boundary conditions analytically. For the first time, exact solutions of nonlinear static response of straight and curved beams were derived. Closed form formulas for the critical buckling loads of straight and curved beams under symmetric boundary conditions were deduced. Besides, the natural frequencies and corresponding mode shapes around buckled position were analytically computed. The proposed formula can be exploited in analysis of airfoil, flappers and wings by engineers and designers. The main results of the current analysis can be summarized as:

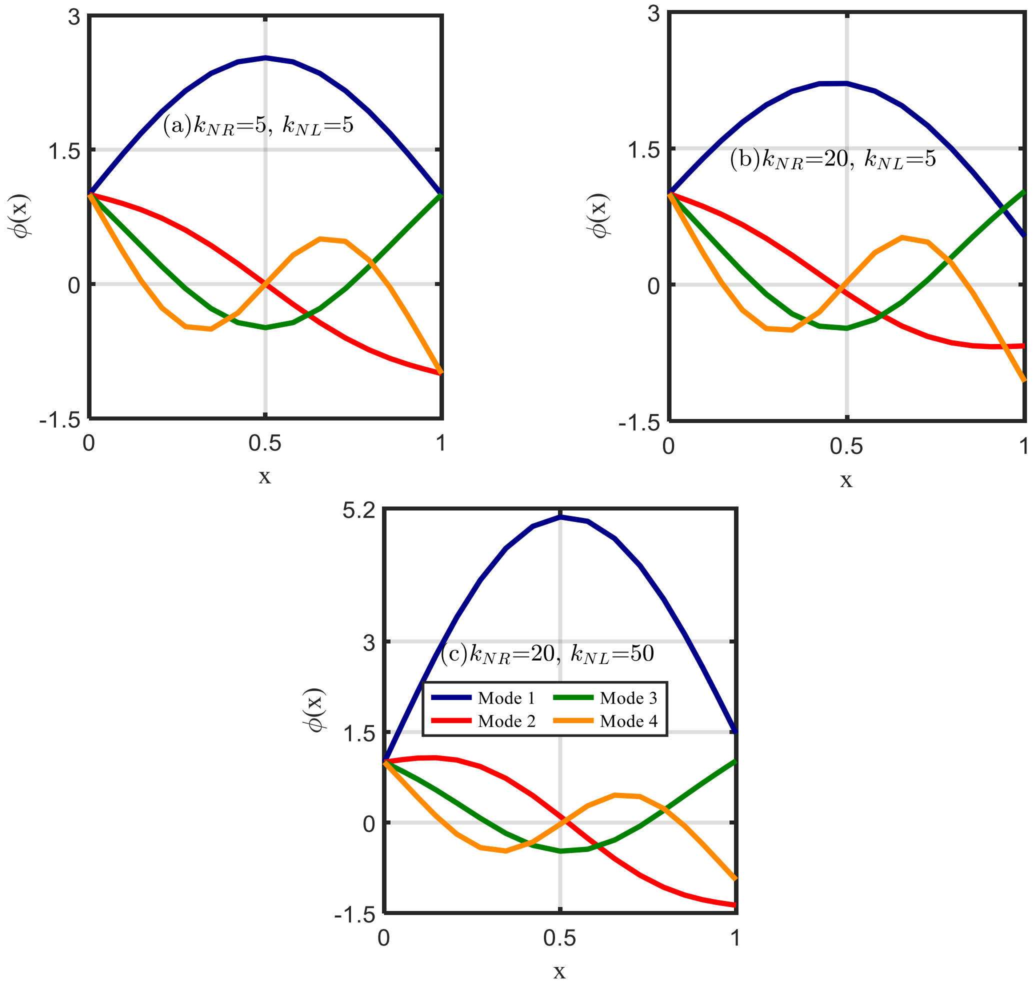

Fig. 11 Effect of k_{NR} and k_{NL} on first four mode shapes of straight beam in postbuckling state ($P = 1.5P_{cr}, k_L = k_R = 15$)

• For straight beam, the foundation constants have no effect on the critical buckling load in both symmetric and asymmetric nonlinear boundary conditions.
• For curved beam, the foundation constants play no rule on the critical buckling load in case of symmetric boundary conditions. Whereas, in the case of asymmetric boundary conditions, they have great influence on the critical buckling load.
• In both symmetric and asymmetric boundary conditions, the static response of straight beam has no change with the variation of foundation constants (the static response is zero).
• The variation of foundation constants have a great influence on nonlinear static response of curved beam in prebuckling and postbuckling domains.
• The foundation constants and the amplitude of initial curvature have significant effects on the natural frequencies of the curved beam.
• As the amplitude of initial curvature increases, one to one internal resonance can be activated.

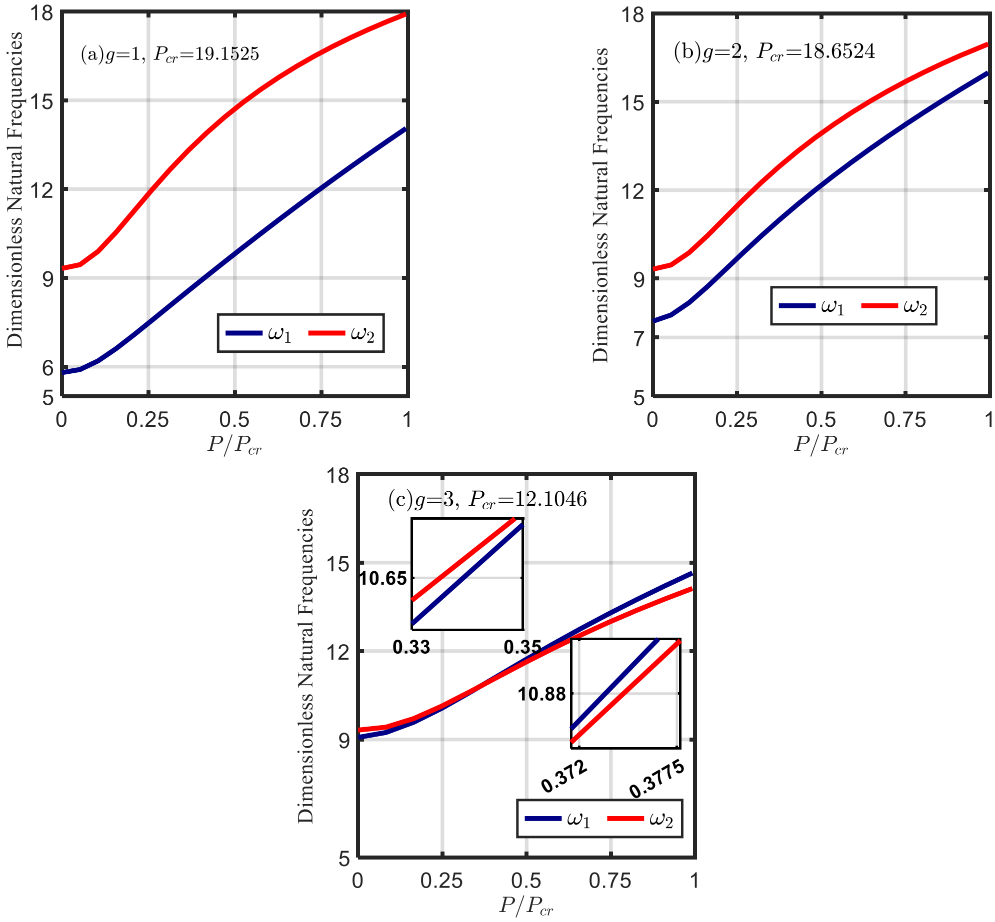

Fig. 12 Effect of imperfection amplitude g on first two natural frequencies of curved beam in prebuckling state ($k_L = k_R = 15, k_{NL} = k_{NR} = 10$)

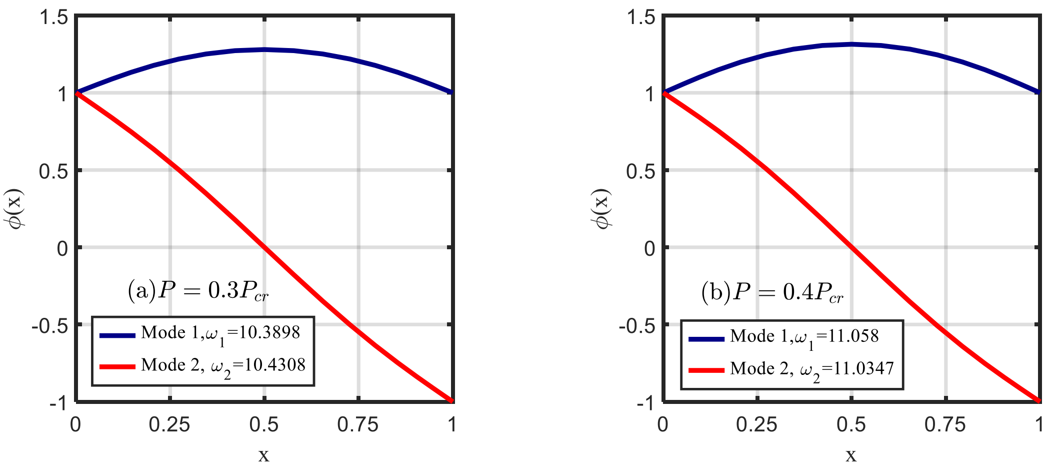

Fig. 13 First two vibration mode shapes of curved beam in prebuckling state at different values of axial load ($g = 3, k_L = k_R = 15, k_{NL} = k_{NR} = 10, P_{cr} = 12.1045$)

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