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Free vibration of deep and shallow curved FG nanobeam based on nonlocal elasticity

S.A.H. Hosseini*[1], O. Rahmani[2], V. Refaeinejad[2], H. Golmohammadi[2] and M. Montazeripour[1]

[1]Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
[2]Smart Structures and New Advanced Materials Laboratory, Department of Mechanical Engineering, University of Zanjan, Zanjan, Iran

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Abstract. In this paper, the effect of deepness on in-plane free vibration behavior of a curved functionally graded (FG) nanobeam based on nonlocal elasticity theory has been investigated. Differential equations and boundary conditions have been developed based on Hamilton's principle. In order to figure out the size effect, nonlocal theory has been adopted. Properties of material vary in radial direction. By using Navier solution technique, the amount of natural frequencies has been obtained. Also, to take into account the deepness effect on vibrations, thickness to radius ratio has been considered. Differences percentage between results of cases in which deepness effect is included and excluded are obtained and influences of power-law exponent, nonlocal parameter and arc angle on these differences percentage are studied. Results show that arc angle and power law exponent parameters have the most influences on the amount of the differences percentage due to deepness effect. It has been observed that the inclusion of geometrical deep term and material distribution results in an increase in sensitivity of dimensionless natural frequency about variation of aforementioned parameters and a change in variation range of natural frequency. Finally, several numerical results of deep and shallow curved functionally graded nanobeams with different geometry dimensions are presented, which may serve as benchmark solutions for the future research in this field.

Keywords: analytical solution; deep and shallow curved beam; deepness effect; free vibration; functionally graded nanobeam; nano structure

1. Introduction

Using nano material is a novel method in order to construct materials with perfect properties and manufacture tools with ultra-high stabilities. Development of nanotechnology occurred by out spreading of researchers' studies on nanostructures after the discovery of carbonic nanotubes (CNTs) by Iijima (1991). In continuance, in order to facilitate calculations and mathematical treatments and eliminate complicated simulation methods, continuum mechanics methods were employed. Among existing theories in continuum mechanics, according to Eringen's investigations (Eringen, Eringen, Eringen), it can be said that nonlocal elasticity theory is more capable than others to describe the behavior of various nanostructure. For the first time, Peddieson *et al.* (2003) presented a nonlocal Euler-Bernoulli beam model. The main difference between local and nonlocal

*Corresponding author, Professor, E-mail: hosseini@bzte.ac.ir

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models is, the way of defining stress function that, in local theories stress in a point is defined as a function of strain only in the same point, whereas in nonlocal theory stress field in a point is defined as a function of strain at all points in the near region. In the society of nanotechnology, great attention is paid to the application of nonlocal elasticity (Amara *et al.* 2010, Lim 2010, Ma *et al.* 2010, Zidour *et al.* 2012, Tounsi *et al.* 2013, Wang *et al.* 2013, Xu and Deng 2013, Aydogdu and Arda 2016, Rakrak *et al.* 2016, Barati 2017, Bouafia *et al.* 2017, Bensaid *et al.* 2018, Hosseini and Khosravi 2020), especially nonlocal Euler-Bernoulli beam theory (Seifoori and Liaghat 2013). In this regard, Ebrahimi *et al.* (2015) evaluated DTM method applications by investigating vibrations of FG nanobeams based on Euler-Bernoulli beam model. Also, they employed this method in order to analyze the vibration and buckling of FG nano beams by considering the physical neutral axis position (Eltaher *et al.*). Khosravi and Hosseini (2020) considered an attached disk as a mass nanoresonator to find the effects of nonlocal parameter, damping parameters, the mass nanoresonator on the dynamic responses of viscoelastic nanotube based on classical elasticity within the framework of exact solution and finite difference method. Alizadeh *et al.* (2020) carried out the vibration of a resonator based on nonlocal elasticity theory, Euler-Bernoulli beam model and Green-Naghdi approach in the presence of thermal and surface effects. Bastanfar *et al.* (2019) worked on the flexural behavior of imperfect Euler-Bernoulli nanobeam model, which contains a crack on it as well as rotational and axial springs at the defect location.

However, because of the size order of structures which was under investigation (being out of the micron and sub-micron scales) all of FGMs researches above were investigated and studied by classical theories which cannot involve size effect due to lacking inherent length scale parameter. After this, researches, scientists and engineers produced FG structures in the micron and submicron scales and wanted to investigate their characteristics. Therefore, more practical theories and models involving material length scale parameters capable of capturing the size effect were required in order to investigate the micron and sub-micron FG structural objects such as beams, plates etc. Khosravi and Hosseini (2020) employed the nonlocal elasticity theory and viscoelastic mass nanosensor to study the torsional behaviour of the model, the finite difference method was established to prove the accuracy of obtained results. Khosravi *et al.* (2020) established the nonlocal model along with the Rayleigh-Ritz theory to investigate the small scale torsional behaviour of the single-walled carbon nanotubes for free case and for the state in which model is subjected to the linear and harmonic torques. Khosravi *et al.* (2020) conducted torsional vibration of a single-walled carbon nanotube embedded in an elastic medium to evaluate the effect of the medium, excitation frequency, time constant, geometry and type of loads on the responses, also, the resonance behaviour was evaluated. Hosseini and Khosravi (2020)established the nonlocal theory to assess the free and forced torsional vibration of single-walled carbon nanotubes under both type of loadings.

Recently, FGMs have found a wide range of applications, and they are widely used in micro and nanostructures, and due to their importance, many various beam and plane models have been developed in buckling (Şimşek and Yurtcu 2012), bending (Eltaher *et al.* 2013, Şimşek and Yurtcu 2013, Ansari *et al.* 2015) and vibration analyses. Eltaher *et al.* (2012) investigated free vibration of size-dependent FG nanobeams using the finite element method based on the nonlocal continuum model. Simsek and Yurtcu (2012) studied the bending and buckling of FG nano beams in accordance with nonlocal Timoshenko and Euler-Bernoulli beam models. The application range of curved beam has been developed in mechanical, aerospace and civil engineering industries as the stabilizer of structures (Ai-min and Ming 2004, Zhu and Zhao 2008). Thus, static and dynamic characteristics of such beams and nanobeams have been turned into an interesting subject for researchers. It is noteworthy to mention that beam could be deep or shallow (Hajianmaleki and Qatu 2012, Kurtaran 2015, Kurtaran 2015, Ye *et al.* 2015).

In practice, all of the beams and nanobeams have some curvature, even with small angles. However, in most of the researches, owing to the simplification, straight beam theories have been adopted. Whereas, in comparison with straight beam theories, curved beam theories are more generalized, owing to the fact that if the radius of curved beam is set to infinity, the results of straight beam theory may be achieved. To the best of authors' knowledge about performed works and obtainable results, a gap exists in open literature that is no profound study has been conducted on free vibration characteristics of deep curved FG nanobeams based on nonlocal elasticity so far. The current paper aims to fill this gap.

In this paper, free vibration of a deep curved FG nanobeam has been studied based on nonlocal theory. Hamilton's Principle has been employed to derive governing equations of motion and related boundary conditions. Then Navier's approach has been utilized to solve the obtained differential equation. Influences of aspect ratio, nonlocal parameter, gradient index and interactive influences of such parameters on natural frequency have been investigated. For validation, the presented results have been compared with those of previous work. It is believed that the results of this paper will be a reference with which other researchers may compare their results on the future.

2. Governing equations

According to Euler-Bernoulli theory regarding curved beams, it should be assumed that after applying deformation to the beam, the plane which was already perpendicular to the plane of cross-sectional area before deformation, remains perpendicular, consequently. The radial displacement u and tangential displacement w are considered as follows

$$\begin{aligned} w(\theta,r,t) &= w_0(\theta,t) + \frac{z}{R}\left(w_0(\theta,t) + \frac{\partial u_0(\theta,t)}{\partial x}\right) \\ u(\theta,r,t) &= -u_0(\theta,t) \end{aligned} \tag{1}$$

where u_0, and w_0 are radial and tangential displacement of a mid-plane component, respectively. Also, R denotes radius of curvature. Parameters t, x, z mark time, the peripheral and radial directions of the nanobeam respectively.

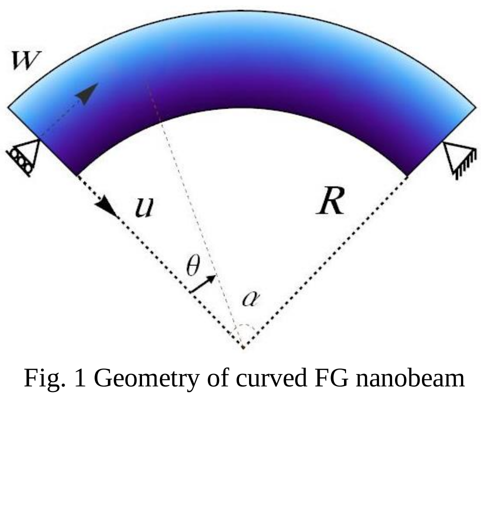


Fig. 1 Geometry of curved FG nanobeam

Fig. 1 illustrates a scheme of curved FG nanobeam. The strain of an element in the curved beam is defined as follows

$$\varepsilon_{xx} = \frac{1}{1+z/R}\left(\varepsilon_{xx}^{0} + zk_{x}^{0}\right) \tag{2}$$

where κ_x and ε_x are bending and tensional strains, respectively. These parameters are described as follows

$$\varepsilon_{xx}^{0} = \frac{1}{R}\left(-u_0(\theta,t) + \frac{\partial w_0(\theta,t)}{\partial\theta}\right) \tag{3}$$

$$k_{x}^{0} = \frac{1}{R^2}\left(\frac{\partial w_0(\theta,t)}{\partial\theta} + \frac{\partial^2 u_0(\theta,t)}{\partial\theta^2}\right) \tag{4}$$

The governing equations of motion and associated boundary conditions are developed based on Hamilton's principle

$$\int_0^t \delta(U - T + V)\,dt = 0 \tag{5}$$

where δU, δT and δV are taken into account for the virtual strain energy, the virtual kinetic energy, and the virtual potential of external loading, respectively. They can be calculated as follows

$$\delta U = \int_v \sigma_{ij}\delta\varepsilon_{ij}\,dV = \int_v (\sigma_{xx}\delta\varepsilon_{xx})\,dV = \int_0^\alpha (N(\delta\varepsilon_{xx}^{0}) + M(\delta k_x^{0}))R\,d\theta \tag{6}$$

where normal forces resultant and bending moment are given as follows

$$N = \int_A \sigma_{xx}\,dA, \quad M = \int_A \sigma_{xx} z\,dA \tag{7}$$

The first variation of kinetic energy is developed as follows

$$\begin{aligned}
T &= \frac{1}{2}\int_0^\alpha\int_A \rho(z,T)\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] R\,d\theta \\
&= \frac{1}{2}\int_0^\alpha\int_A \rho(z,T)\left[\left(-\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w_0}{\partial t} + \frac{z}{R}\left(\frac{\partial w_0}{\partial t} + \frac{\partial^2 u_0}{\partial t\,\partial\theta}\right)\right)^2\right] R\,d\theta \\
&= \frac{1}{2}\int_0^\alpha\int_A \rho(z,T)\left[\left(\frac{\partial u_0}{\partial t}\right)^2 + \left(\frac{\partial w_0}{\partial t}\right)^2 + \frac{z^2}{R^2}\left(\frac{\partial w_0}{\partial t} + \frac{\partial^2 u_0}{\partial t\,\partial\theta}\right)^2 + \frac{2z}{R}\frac{\partial w_0}{\partial t}\left(\frac{\partial w_0}{\partial t} + \frac{\partial^2 u_0}{\partial t\,\partial\theta}\right)\right] R\,d\theta
\end{aligned} \tag{8}$$

$$\begin{aligned}
\delta T = \int_0^\alpha \Bigg[& I_0\left(\frac{\partial u_0}{\partial t}\frac{\partial\delta u_0}{\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial\delta w_0}{\partial t}\right) + \frac{I_1}{R}\left(2\frac{\partial w_0}{\partial t}\frac{\partial\delta w_0}{\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial^2\delta u_0}{\partial t\,\partial\theta} + \frac{\partial^2 u_0}{\partial t\,\partial\theta}\frac{\partial\delta w_0}{\partial t}\right) + \\
& \frac{I_2}{R^2}\left(\frac{\partial w_0}{\partial t}\frac{\partial\delta w_0}{\partial t} + \frac{\partial w_0}{\partial t}\frac{\partial^2\delta u_0}{\partial t\,\partial\theta} + \frac{\partial^2 u_0}{\partial\theta\,\partial t}\frac{\partial\delta w_0}{\partial t} + \frac{\partial^2 u_0}{\partial t\,\partial\theta}\frac{\partial^2\delta u_0}{\partial t\,\partial\theta}\right)\Bigg] R\,d\theta
\end{aligned}$$

where I_0, I_1 and I_2 are the mass moments of inertia as follows

$$(I_0, I_1, I_2) = \int_A \rho(z)(1, z, z^2)dA \tag{9}$$

The first variation of performed work due to external forces resultant is defined as follows

$$\delta V = -b\int_0^{\alpha} \left(f\,\delta u + p\delta w\right) Rd\theta \tag{10}$$

where f and p denote radial and tangential distributed forces, respectively. b is the width of the nanobeam. By inserting Eqs. (6), (8) and (10) into Eqs. (5), it will be concluded as

$$-\frac{\partial N}{\partial \theta} - \frac{1}{R}\frac{\partial M}{\partial \theta} - p\,R = \left(-I_0\frac{\partial^2 w_0}{\partial t^2} - \frac{I_1}{R}\left(\frac{\partial^2 w_0}{\partial t^2} + \frac{\partial^3 u_0}{\partial \theta\,\partial t^2}\right) - \frac{I_2}{R^2}\left(\frac{\partial^2 w_0}{\partial t^2} + \frac{\partial^3 u_0}{\partial \theta\,\partial t^2}\right)\right)R \tag{11}$$

$$-N + \frac{\partial^2 M}{R\partial \theta^2} - f\,R = \left(-I_0\frac{\partial^2 u_0}{\partial t^2} + \frac{I_1}{R}\frac{\partial^3 w_0}{\partial \theta\,\partial t^2} + \frac{I_2}{R^2}\left(\frac{\partial^3 w_0}{\partial \theta\,\partial t^2} + \frac{\partial^4 u_0}{\partial \theta^2 \partial t^2}\right)\right)R \tag{12}$$

Also, associated boundary conditions are described as follows

$$N + \frac{M}{R} = 0 \text{ or } w_0 = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \alpha \tag{13a}$$

$$\frac{\partial M}{R\partial \theta} + I_1\frac{\partial^2 w_0}{\partial t^2} + \frac{I_2}{R}\left(\frac{\partial^2 w_0}{\partial t^2} + \frac{\partial^3 u_0}{\partial \theta\,\partial t^2}\right) = 0 \text{ or } u_0 = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \alpha \tag{13b}$$

$$M = 0 \text{ or } \frac{\partial u_0}{\partial \theta} = 0 \quad \text{at } \theta = 0 \text{ and } \theta = \alpha \tag{13c}$$

3. Functionally graded material properties

In this study, it is assumed that curved FG nanobeam is made of an aluminum and alumina composition. The properties of these two structural constituents have been presented in the result and discussion section. Based on the presented method the gradient variation may be chosen, randomly, although, in order to facilitate analytical solution, they are usually presented as exponential-type dependence. Furthermore, employing a power law gradient is significantly useful, especially from an experimental vantage point of view. Hence, material properties of curved FG nanobeam are defined as follows

$$E_f(z) = \left(E_c - E_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_m$$
$$\rho_f(z) = \left(\rho_c - \rho_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_m \tag{14}$$

k is a non-negative constant that adjusts the volume fraction of structural materials. c and m subscripts prescribe where a property is related to ceramic and metal of the curved FG nanobeam, respectively. For instance, if $z=-h/2$, E will be equal to $E = E_m$ and if $z=h/2$, E will be equal to $E = E_c$. For the sake of convenience, top and bottom surfaces of the curved FG nanobeam are assumed to be pure ceramic and pure aluminum, respectively. Although, the aforementioned power law equation is a theoretical relation it can be easily achieved and dominated experimentally. The properties of material vary in the thickness direction due to k. It should be pointed out that, if k=0, the material will be homogeneous.

3.1 Nonlocal elasticity theory

According to Eringen's nonlocal theory, stress field at point x does not only depend on the strain of the specified point, but also depends on the strain of all structural components. This issue has been proved by the atomic theory of lattice dynamic and phonon dispersion experimental observation. Stress tensor σ which is related to point x is calculated as follows

$$\bar{\sigma} = \int_{\Omega} K(|x' - x, \tau|)\sigma(x')\,dx' \tag{15}$$

σ is called the classical microscopic second Piola-kirchhoff stress at point x, Kernel function $k(|x' - x, \tau|)$ represents the nonlocal modulus, in which $(x' - x)$ is distance and τ depends on internal and external characteristic length. According to Hook's law, the macroscopic stress σ at point x in Hookean solid is considered for the strain at the same point which is defined as follows

$$\sigma(x) = C(x) : \varepsilon(x) \tag{16}$$

C represents the fourth order elasticity tensor which indicates the double-dot product. Eqs. (15) and (16) express the nonlocal constitutive behavior of Hookean solid. Eq. (15) represents the weighted average of strain field contributions owing to displacement of all points in the structure to the stress field at point x. For the sake of convenience, an equivalent model is utilized instead of integral constitutive relation. The model is evaluated as follows

$$(1-\mu\nabla^2)\bar{\sigma} = \sigma, \qquad \mu = \tau^2\ell^2 = e_0^2 a^2 \tag{17}$$

where τ is defined as $\tau = e_0 a/\ell$, in which e_0 is a constant value according to applied material, and a and ℓ denote internal (e.g., granular distance or lattice parameter) and external (e.g., wave or crack length) characteristic length, respectively. Also, the nonlocal parameter μ varies in accordance with different materials. For an elastic material in one-dimensional problem, the nonlocal relations can be simplified as follows

$$\sigma(x) - (e_0 a)^2 \frac{\partial^2 \sigma(x)}{\partial x^2} = E\varepsilon(x) \tag{18}$$

where E is Young's modulus. For a Euler-Bernoulli, nonlocal FG beam, Eq. (20) may be formulated as follows

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z)\varepsilon_{xx} \tag{19}$$

According to Eq. (19) resultants of normal forces and bending moment in the nonlocal Euler-Bernoulli theory can be obtained as follows

$$N - \mu \frac{\partial^2 N}{R^2 \partial\theta^2} = \frac{A}{R}\left(-u + \frac{\partial w}{\partial\theta}\right) - \frac{B}{R^2}\left(\frac{\partial w}{\partial\theta} + \frac{\partial^2 u}{\partial\theta^2}\right) \tag{20}$$

$$M-\mu\frac{\partial^2 M}{R^2\partial\theta^2}=\frac{B}{R^2}\left(-u+\frac{\partial w}{\partial\theta}\right)+\frac{D}{R^2}\left(\frac{\partial w}{\partial\theta}+\frac{\partial^2 u}{\partial\theta^2}\right)\tag{21}$$

where A, B, D for deep and shallow cases are obtained in the following. A, B, and D are extensional, coupling, and bending stiffness matrixes considering the deep curved FG beam, respectively, which are described as follows

$$(A,B,D)=\int_A\frac{E(z)}{1+z/R}(1,z,z^2)dA\tag{22}$$

Note that for the case in which (1+z/R) term is ignored in Eq. (2) (shallow case), A, B, and D values are obtained as follows

$$(A,B,D)=\int_A E(z)(1,z,z^2)dA\tag{23}$$

By performing some algebraic operations and inserting N and M from equations (20) and (21) in equations (11) and (12), the nonlocal equations of motion are obtained as follows

$$\begin{aligned}&\frac{A}{R}\left(-\frac{\partial u_0}{\partial\theta}+\frac{\partial^2 w_0}{\partial\theta^2}\right)+\frac{B}{R^2}\left(2\frac{\partial^2 w_0}{\partial\theta^2}+\frac{\partial^3 u_0}{\partial\theta^3}-\frac{\partial u}{\partial\theta}\right)+\frac{D}{R^3}\left(\frac{\partial^2 w_0}{\partial\theta^2}+\frac{\partial^3 u_0}{\partial\theta^3}\right)+\\&\frac{\mu}{R^2}\left(I_0R\frac{\partial^4 w_0}{\partial\theta^2\partial t^2}+I_1\left(2\frac{\partial^4 w_0}{\partial\theta^2\partial t^2}+\frac{\partial^5 u_0}{\partial t^2\partial\theta^3}\right)+\frac{I_2}{R}\left(\frac{\partial^4 w_0}{\partial\theta^2\partial t^2}+\frac{\partial^5 u_0}{\partial t^2\partial\theta^3}\right)\right)=\\&I_0R\frac{\partial^2 w_0}{\partial t^2}+I_1\left(2\frac{\partial^2 w_0}{\partial t^2}+\frac{\partial^3 u_0}{\partial t^2\partial\theta}\right)+\frac{I_2}{R}\left(\frac{\partial^2 w_0}{\partial t^2}+\frac{\partial^3 u_0}{\partial t^2\partial\theta}\right)\end{aligned}\tag{24}$$

$$\begin{aligned}&\frac{A}{R}\left(-u_0+\frac{\partial w_0}{\partial\theta}\right)+\frac{B}{R^2}\left(\frac{\partial w_0}{\partial\theta}+2\frac{\partial^2 u_0}{\partial\theta^2}-\frac{\partial^3 w_0}{\partial\theta^3}\right)-\frac{D}{R^3}\left(\frac{\partial^3 w_0}{\partial\theta^3}+\frac{\partial^4 u_0}{\partial\theta^4}\right)+Rf+\\&\frac{\mu}{R^2}\left(I_0R\frac{\partial^4 u_0}{\partial\theta^2\partial t^2}-I_1\frac{\partial^5 w_0}{\partial t^2\partial\theta^3}-\frac{I_2}{R}\left(\frac{\partial^5 w_0}{\partial t^2\partial\theta^3}+\frac{\partial^6 u_0}{\partial t^2\partial\theta^4}\right)+R\frac{\partial^2 f}{\partial\theta^2}\right)=\\&I_0R\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial t^2\partial\theta}-\frac{I_2}{R}\left(\frac{\partial^3 w_0}{\partial t^2\partial\theta}+\frac{\partial^4 u_0}{\partial t^2\partial\theta^2}\right)\end{aligned}\tag{25}$$

4. Analytical solution

In this part, the equations of motion for free vibration of a curved FG nano beam with associated simply supported boundary conditions is solved, analytically Navier's solution method is employed to perform the analytical approach known trigonometric displacement functions with unknown coefficients have been considered satisfying differential equations and associated boundary conditions. These displacement functions are described as following

$$u_0(\theta,t)=\sum_{n=1}^{\infty}U_n\sin\left(\frac{n\pi}{\alpha}\theta\right)e^{i\omega_n t}\tag{26}$$

$$w_0(\theta,t)=\sum_{n=1}^{\infty}W_n\cos(\frac{n\pi}{\alpha}\theta)e^{i\omega_n t} \tag{27}$$

where i is equal to $\sqrt{-1}$ and ω_n is natural frequency, U_n, and W_n denote Fourier coefficients which are unknown values.

In order to free vibration analysis of curved FG nanobeam, it is clear that total external forces should be equal to zero. By inserting Eqs. (26) and (27) into Eqs. (24) and (25), eigenvalues can be calculated as follows

$$\left(\begin{bmatrix} S_{11}^n & S_{12}^n \\ S_{12}^n & S_{22}^n \end{bmatrix}+\omega_n^2\begin{bmatrix} M_{11}^n & M_{12}^n \\ M_{12}^n & M_{22}^n \end{bmatrix}\right)\begin{Bmatrix} U_n \\ W_n \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{28}$$

where ${S^n}_{11}$, ${S^n}_{12}$, and ${S^n}_{22}$ are determined as follows

$$\begin{aligned} S_{11}^n &= -\frac{A}{R^2}-\frac{2B}{R^3}\left(\frac{n\pi}{\alpha}\right)^2-\frac{D}{R^4}\left(\frac{n\pi}{\alpha}\right)^4 \\ S_{12}^n &= -\frac{A}{R^2}\left(\frac{n\pi}{\alpha}\right)-\frac{B}{R^3}\left(\left(\frac{n\pi}{\alpha}\right)+\left(\frac{n\pi}{\alpha}\right)^3\right)-\frac{D}{R^4}\left(\frac{n\pi}{\alpha}\right)^4 \\ S_{21}^n &= S_{12}^n,\ S_{22}^n=-\frac{A}{R^2}\left(\frac{n\pi}{\alpha}\right)^2-\frac{2B}{R^3}\left(\frac{n\pi}{\alpha}\right)^2-\frac{D}{R^4}\left(\frac{n\pi}{\alpha}\right)^2 \end{aligned} \tag{29}$$

Also mass matrices coefficients are defined as follows

$$\begin{aligned} M_{11}^n &= I_0+\frac{I_2}{R^2}\left(\frac{n\pi}{\alpha}\right)^2+\frac{\mu}{R^2}\left(I_0\left(\frac{n\pi}{\alpha}\right)^2+\frac{I_2}{R^2}\left(\frac{n\pi}{\alpha}\right)^4\right) \\ M_{12}^n &= \frac{I_1}{R}\left(\frac{n\pi}{\alpha}\right)+\frac{I_2}{R^2}\left(\frac{n\pi}{\alpha}\right)+\frac{\mu}{R^2}\left(\frac{I_1}{R}\left(\frac{n\pi}{\alpha}\right)^3+\frac{I_2}{R^2}\left(\frac{n\pi}{\alpha}\right)^3\right),\ M_{21}^n=M_{12}^n \\ M_{22}^n &= I_0+2\frac{I_1}{R}+\frac{I_2}{R^2}+\frac{\mu}{R^2}\left(\left(I_0+2\frac{I_1}{R}+\frac{I_2}{R^2}\right)\left(\frac{n\pi}{\alpha}\right)^2\right) \end{aligned} \tag{30}$$

The dimensionless natural frequency of curved FG nanobeam has been obtained as follows

$$\bar{\omega}_n=\omega_n\alpha^2R^2\sqrt{\frac{\rho_c A}{E_c I}} \tag{31}$$

where A, and I stand for the cross section of curved homogeneous nanobeam, and its moment of inertia, respectively. E_c and ρ_c stand for Young's modulus and mass density both related to ceramic, respectively.

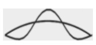
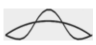
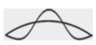
4. Results and discussion

By obtaining the amount of A, B, D for two cases of shallow and deep in curved FG nanobeam, the amount of frequencies for both aforementioned conditions are obtained. First of all, it should

Table 1 Comparison of the natural frequency for simply supported of straight and curved (α=π/360=0.5°) FG nanobeam with selected values of power-law exponent, aspect ratio and nonlocal parameters

L/h	μ	*k*=0		*k*=0.2		*k*=1		*k*=5	
		Present	Ref. (Eltaher *et al.* 2012)	Present	Ref. (Eltaher *et al.* 2012)	Present	Ref. (Eltaher *et al.* 2012)	Present	Ref. (Eltaher *et al.* 2012)
20	0	9.8594	9.8797	8.6858	8.7200	6.9885	7.0904	5.9370	6.0025
	1×10^{-12}	9.40622	9.4238	8.2865	8.3175	6.6672	6.7631	5.66411	5.7256
	2×10^{-12}	9.0102	9.0257	7.9376	7.9661	6.3865	6.4774	5.4256	5.4837
	3×10^{-12}	8.6603	8.6741	7.6294	7.6557	6.1385	6.2251	5.2449	5.2702
	4×10^{-12}	8.3483	8.3607	7.3545	7.3791	5.9174	6.0001	5.0271	5.0797
50	0	9.8679	9.8797	8.6937	8.7115	6.9951	7.0852	5.9421	5.9990
	1×10^{-12}	9.4143	9.4172	8.2940	8.3114	6.6735	6.7583	5.6689	5.7218
	2×10^{-12}	9.0180	9.0205	7.9448	7.9613	6.3925	6.4737	5.4302	5.4808
	3×10^{-12}	8.6678	8.6700	7.6363	7.5620	6.1437	6.2222	5.2194	5.2679
	4×10^{-12}	8.3555	8.3575	7.3612	7.3762	5.9229	5.9979	5.0314	5.0780
100	0	9.8692	9.8700	8.6948	8.7111	6.9960	7.0833	5.9428	5.9970
	1×10^{-12}	9.4154	9.4162	8.2951	8.3106	6.6744	6.7577	5.6696	5.7212
	2×10^{-12}	9.0191	9.0197	7.9459	7.9607	6.3934	6.4731	5.4309	5.4803
	3×10^{-12}	8.6689	8.6695	7.6373	7.6515	6.1452	6.2217	5.2201	5.2675
	4×10^{-12}	8.3565	8.3571	7.3622	7.3758	5.9237	5.9976	5.0320	5.0777

Table 2 Variation of dimensionless frequency of curved FG nanobeam for three first modes with *k*=0 and *L/h*=20

	μ	α=45°			α=60°			α=90°		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
Shallow curved beam	0	8.96683	38.4016	87.0973	8.31363	37.701	86.3883	6.61333	35.7521	84.3876
	1	8.55461	32.5159	63.3831	7.93144	31.9227	62.8671	6.3093	30.2725	61.4112
	2	8.19447	28.7061	52.2701	7.59753	28.1825	51.8446	6.04368	26.7256	50.6439
	3	7.87629	25.9829	45.4967	7.30253	25.5089	45.1263	5.80902	24.1903	44.0813
	4	7.59251	23.9118	40.8181	7.03943	23.4756	40.4858	5.59972	22.262	39.5482
Mode shape										
deep curved beam	0	8.96946	38.4132	87.1239	8.31781	37.7212	86.435	6.62012	35.794	84.489
	1	8.55712	32.5258	63.4024	7.93542	31.9398	62.9011	6.31579	30.308	61.4849
	2	8.19687	28.7149	52.2861	7.60135	28.1975	51.8726	6.0499	26.7569	50.7048
	3	7.8786	25.9908	45.5106	7.3062	25.5225	45.1507	5.81499	24.2186	44.1342
	4	7.59474	23.919	40.8306	7.04296	23.4881	40.5077	5.60548	22.2881	39.5957

be mentioned that properties of FG materials for this work on upper surface is pure Aluminum $\rho_m = 2702$ Kg/m^3 and $E_m = 70$ GPa and on the lower surface is the pure Aluminum oxide (Al_2O_3) $\rho_c = 3960$ Kg/m^3 and $E_c = 390$ GPa. Initially, in order to validation, obtained results for the case in which arc angle of curved FG nanobeam approaches to zero and consequently radius approaches to infinity, have been compared in Table 1 with results of a straight beam. An

Table 3 Variation of dimensionless frequency of curved FG nanobeam for three first modes with k=1 and L/h=20

	μ	α=45°			α=60°			α=90°		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
Shallow curved beam	0	6.82479	29.2137	66.2319	6.31452	28.6117	65.5306	5.00621	27.0107	63.7091
	1	6.51104	24.7362	48.1987	6.02423	24.2265	47.6884	4.77607	22.8709	46.3629
	2	6.23693	21.838	39.7481	5.77062	21.388	39.3272	4.575	20.1912	38.2341
	3	5.99476	19.7663	34.5973	5.54655	19.359	34.231	4.39736	18.2757	33.2795
	4	5.77877	18.1907	31.0396	5.34671	17.8159	30.7109	4.23893	16.8189	29.8573
deep curved beam	0	6.82725	29.225	66.2594	6.31816	28.6296	65.5742	5.01171	27.0442	63.7931
	1	6.51339	24.7458	48.2188	6.02771	24.2416	47.7201	4.78131	22.8993	46.424
	2	6.23918	21.8464	39.7646	5.77395	21.4013	39.3533	4.58002	20.2163	38.2845
	3	5.99693	19.7739	34.6117	5.54975	19.3711	34.2537	4.40219	18.2984	33.3234
	4	5.78086	18.1977	31.0525	5.3498	17.827	30.7313	4.24358	16.8398	29.8966

Table 4 Variation of dimensionless frequency of curved FG nanobeam for three first modes with k= 10 and L/h=20

	μ	α=45°			α=60°			α=90°		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
Shallow curved beam	0	5.72099	24.479	55.4667	5.29187	23.966	54.8582	4.19407	22.6098	53.293
	1	5.45799	20.7272	40.3646	5.0486	20.2928	39.9218	4.00127	19.1444	38.7827
	2	5.22821	18.2987	33.2875	4.83605	17.9152	32.9223	3.83282	16.9014	31.983
	3	5.02521	16.5627	28.9739	4.64828	16.2156	28.6561	3.68399	15.298	27.8385
	4	4.84415	15.2425	25.9944	4.4808	14.9231	25.7093	3.55126	14.0786	24.9757
deep curved beam	0	5.72553	24.4994	55.5157	5.2981	23.9959	54.9306	4.20268	22.6609	53.4201
	1	5.46232	20.7445	40.4003	5.05454	20.3181	39.9745	4.00948	19.1877	38.8753
	2	5.23236	18.3139	33.3169	4.84175	17.9376	32.9658	3.84068	16.9396	32.0593
	3	5.02919	16.5766	28.9995	4.65375	16.2359	28.6939	3.69156	15.3326	27.9049
	4	4.84799	15.2552	26.0174	4.48608	14.9417	25.7432	3.55855	14.1104	25.0353

excellent agreement is achieved for all amounts of the power-law exponent, the nonlocal parameter, etc.

Tables 2-4 report the amount of dimensionless frequency of deep and shallow cases for the first three modes of frequency, the various values of the nonlocal parameter, and the arc angle of curved FG nanobeam with respect to power-law exponents 0, 1, and 10 respectively. As it is clear in these tables, by increasing of arc angle, the amount of frequency decreases. Also, it can be observed that an increase in nonlocal parameter leads to a reduction in the amount of frequency, which means that when the amount of nonlocal parameter increases, atomic spacing increases which results in a reduction in material stiffness, therefore, the amount of natural frequency will be increased. By comparing the obtained amounts from two cases of shallow and deep, it can be concluded that the expected amounts of frequency fall. In this regard, it is also worth noting that this reduction has a larger rate initially, and then by an increase in the power-law exponent, reduction rate will decrease.

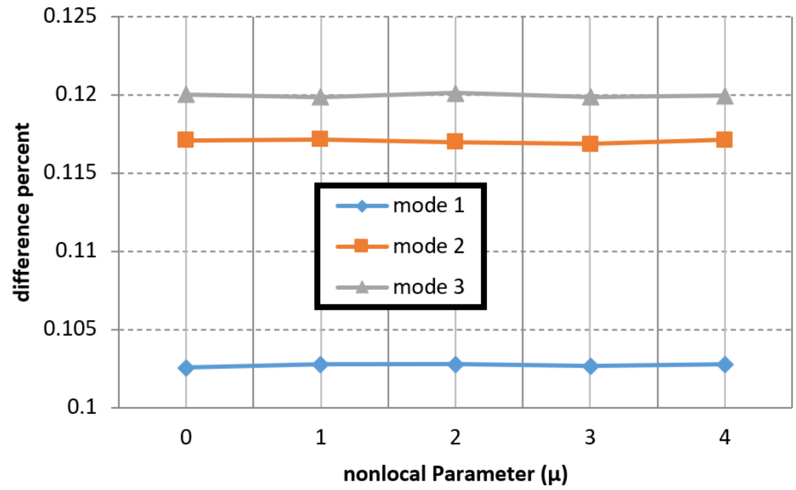


Fig. 2 Variation of the difference percentage versus nonlocal parameter for the first three modes with α=90° and k=0

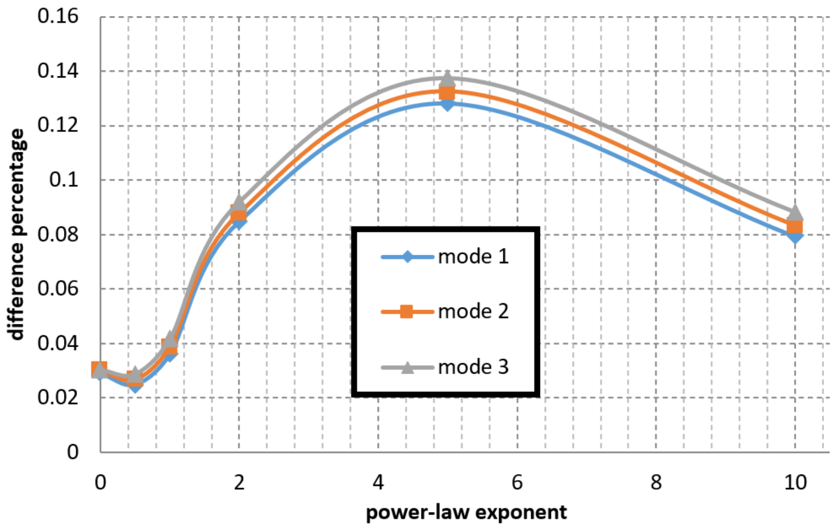


Fig. 3 Variation of percentage the difference versus power-law exponent for the first three modes number with α=45°

4.1 Effect of deepness term

Effects of deepness term and difference percentage between obtained amounts in shallow and deep cases and effective factors on these differences are studied in this section. For this purpose, the relation of differences between frequency amounts due to the inclusion of deepness term and exclusion of this term is expressed as following

$$\text{Difference percentage}=\frac{\bar{\omega}_{deep}-\bar{\omega}_{shallow}}{\bar{\omega}_{deep}}\times 100 \tag{32}$$

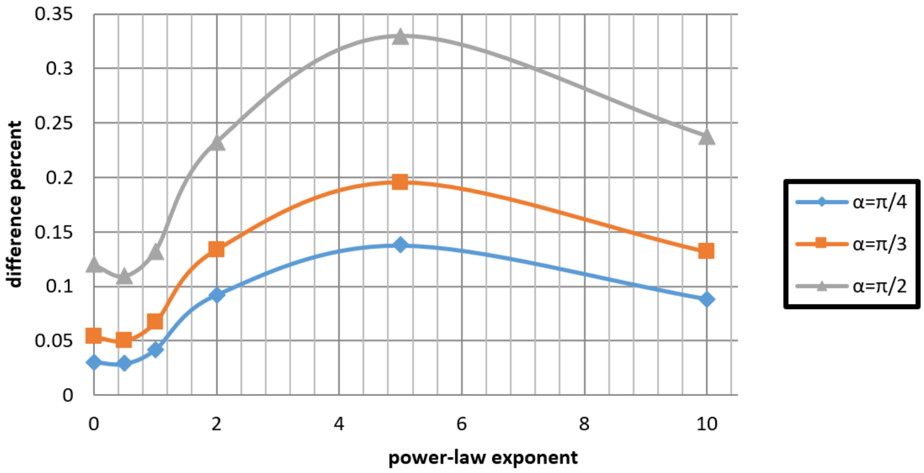


Fig. 4 Variation of the difference percentage versus power-law exponent for selected values of arc angle with third mode

Fig. 2 illustrates the changes' percentage of shallow and deep cases against nonlocal parameter for various values of mode number. As it is apparent from the figure provided, by increasing of nonlocal parameter, the changes' percentage remains almost constant. It is evident that the nonlocal parameter has no significant effect on deepness or shallowness of the curved FG beam. Moreover, it should be noticed that the third mode of vibration is by far the most significant difference percentage.

Fig. 3 is a diagram illustrating the changes' percentage against power-law exponent for various mode numbers. From the information supplied, it is interesting to note that the difference percentage see a drop initially due to a rise in the power-law exponent, then reaches its peak and finally experiences a gradual decline. Results show that for power-law exponent 0.1, the difference percentage between deep and shallow cases is less than the case of homogeneous material ($k = 0$). Furthermore, in $k = 5$ maximum difference exists between the cases in which deepness term is included, the point which should be investigated in analysis of curved FG beams. Also, evident is the fact that an increase in the amount of power-law exponent ($k \succ 5$) gives rise to a decline in the difference percentage. One particularly interesting fact highlighted by the figure is that the figure shows a negligible amount of the difference percentage variations in terms of each mode for the lower amount of power-law exponent, however, this variation will rise by increasing the power-low exponent parameter. Fig. 4 shows the difference percentage with respect to power-law exponent for various arc angles. It is evident that the amount of this parameter goes up from $k = 0.5$ to $k = 5$. In other words, in $k = 0.5$ distances between the difference percentage related to each of arc angles are minimum and in $k = 5$ they are maximum. Last but not least, the figure shows the highest amount of the difference percentage for $\alpha = \pi/2$, $\alpha = \pi/3$, $\alpha = \pi/4$ respectively, in all range of power-law exponent parameter.

5. Conclusions

In this paper, the effect of deepness on frequency responses of a curved FG nanobeam has been investigated. Differential equations and boundary conditions have been developed in accordance with Euler-Bernoulli curved beam model and Hamilton's principle. By using nonlocal theory, governing equations in the order of nano has been obtained. Natural frequency of the curved nano-beam with simply supported boundary conditions for both shallow and deep cases has been obtained analytically and influences of various parameters on natural frequency and differences between frequency of shallow and deep cases were investigated and the following results were obtained:

- Increasing of power-law-exponent, nonlocal parameter and opening angle lead to the reduction of natural frequency.
- Increasing of nonlocal parameter did not alter the difference percentage between shallow and deep cases. This observation acclaims the fact that, this matter is independent of the size effect.
- Rising of mode number results in an increase in the difference percentage.
- Opening angle significantly affects the difference percentage in such a way that increasing of opening angle causes the difference percentage and its sensitivity to rise.
- By growing of power-law-exponent, the difference percentage initially reduces, then goes up and finally declines again.

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