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A refined quasi-3D hybrid-type higher order shear deformation theory for bending and Free vibration analysis of advanced composites beams

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Abstract. In this paper, a new displacement field based on quasi-3D hybrid-type higher order shear deformation theory is developed to analyze the static and dynamic response of exponential (E), power-law (P) and sigmoïd (S) functionally graded beams. Novelty of this theory is that involve just three unknowns with including stretching effect, as opposed to four or even greater numbers in other shear and normal deformation theories. It also accounts for a parabolic distribution of the transverse shear stresses across the thickness, and satisfies the zero traction boundary conditions at beams surfaces without introducing a shear correction factor. The beam governing equations and boundary conditions are determined by employing the Hamilton's principle. Navier-type analytical solutions of bending and free vibration analysis are provided for simply supported beams subjected to uniform distribution loads. The effect of the sigmoid, exponent and power-law volume fraction, the thickness stretching and the material length scale parameter on the deflection, stresses and natural frequencies are discussed in tabular and graphical forms. The obtained results are compared with previously published results to verify the performance of this theory. It was clearly shown that this theory is not only accurate and efficient but almost comparable to other higher order shear deformation theories that contain more number of unknowns.

Keywords: functionally graded beam; free vibration; bending; stress; shear deformation theory; stretching effect

1. Introduction

Functionally graded materials (FGMs) which are a new class of advanced composites, are made from a mixture of two different materials (metals and ceramics). The mechanical properties of FGM change with a smooth and continuous variation from layer to another through the thickness direction, and thus, eliminating the inter-laminar and residual stresses as founded in classical composites. Because of this feature, the FGMs are widely used in many engineering fields such as aircraft, aerospace, naval/marine, optical, civil, automotive, electronic, chemical, construction and mechanical engineering (Bousahla *et al*. 2014, Belkorissat *et al*. 2015, Taibi *et al.* 2015, Hamidi *et al.* 2015, Bellifa *et al*. 2016, Beldjelili *et al*. 2016, Bouderba *et al.* 2013, 2016. Bellifa *et al*. 2017a, Menasria *et al*. 2017, Mouffoki *et al*. 2017, Sekkal *et al*. 2017a, b, Zine *et al*. 2018, Bouhadra *et al.* 2018).

Due to the importance of these new materials, many theories have been developed by researchers to study the static and dynamic responses of functionally graded structures. Yang and Chen (2008) studied the free vibration and buckling of FGM beams with the presence of open cracks. Li *et al.* (2009) analyzed a small vibration of post-buckled FGM beams with surface-bonded piezoelectric layers in thermal environment by a numerical shooting method based the exact geometrically non-linear theory for axially extensible beams. Şimşek and Kocatürk (2009) analyzed the dynamic behavior of an FGM simply supported beam under a concentrated moving harmonic load, in which the effects of the material homogeneity, the velocity of the moving harmonic load, and the excitation frequency on the dynamic responses of the beam were discoursed.

In the framework of the first shear deformation theory or the Timoshenko beam theory, Li (2008) presented analytical solutions for the static bending and free vibration of FGM Timoshenko and Euler-Bernoulli beams. Huang and Li (2010) studied the free vibration of axially FGMs with non-uniform cross-sections by using the integration technique to transform the differential governing equations into the Fredholm integral equations. Bouremana *et al.* (2013) developed a new first shear deformation theory based on neutral surface position for FGM beams. Using Timoshenko beam theory, Arani and Kolahchi (2016) presented buckling analysis of embedded concrete columns armed with carbon nanotubes.

Based on higher order shear deformation theories, Aydogdu and Tashkin (2007) studied the free vibration behavior of a simply supported FGM beam based on the first, parabolic, and exponential shear deformation beam

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theories, respectively, in which natural frequencies were obtained by the Navier type solution method. Şimşek (2010) investigated the dynamic responses of functionally graded beams by different beam theories, in which a system of equations of motion was derived by Lagrange's equations. Mahi *et al.* (2010) analyzed the free vibration of FGM beams with the temperature dependent material properties. The formulation was derived based on a unified higher order shear deformation theory. The effects of the initial thermal stress on the natural frequencies were also discussed. The free vibration of thin and thick-walled FGM box beams has been studied by Ziane *et al*. (2013). Thai and Kim (2013), Thai and Choi (2014) and Belabed *et al*. (2014) presented a quasi-3D sinusoidal, polynomial and hyperbolic shear deformation theory, respectively, with only five unknowns for bending and free vibration analysis of E-, and P-FGM plates. Zidi *et al*. (2014) investigated the bending response of FG plates subjected to hygro-thermo-mechanical loading. Ahmed (2014) analyze the post-buckling of sandwich beams with functionally graded faces by employing a consistent HSDT. The influence of stretching effect and porosities on the free vibration behavior of thick FG beams was analyzed by Ait Atmane *et al.* (2015). A new simple and refined hyperbolic and sinusoïdal higher- order beam theory for bending and vibration analysis of FG beams with including the thickness stretching effect was performed by Bourada *et al.* (2015) and Meradjah *et al.* (2015), respectively. Based on physical neutral surface, Han *et al.* (2015) proposed a four variable refined theory to study a dynamic stability of S-FGM plates. Lee *et al.* (2015) developed a refined higher order shear and normal deformation theory with only five unknowns for predicting a bending response of E-, P-, and S-FGM plates on Pasternak elastic foundation. A static analysis of functionally graded (FG) single and sandwich beams by using a simple and efficient 4-unknown quasi-3D hybrid type theory, which includes both shear deformation and thickness stretching effects was presented by Mantari and Yarasca (2015). Akavci (2016) investigated mechanical response of FG sandwich plates on elastic foundation. Abdelaziz *et al.* (2017) presented an efficient hyperbolic shear deformation theory for bending, buckling and free vibration of FG sandwich plates with various boundary conditions. Kolahchi and Bidgoli (2016) presented a size-dependent sinusoidal beam model for dynamic instability of single-walled carbon nanotubes. Recent plate/beam theories can be consulted in references of Ait Amar Meziane *et al*. (2014), Fekrar *et al*. (2014), Al-Basyouni *et al.* (2015), Kolahchi *et al.* (2017a, b, c), Kolahchi *et al.* (2016a, b), Boukhari *et al.* (2016), Bounouara *et al.* (2016), Bilouei *et al.* (2016), Madani *et al.* (2016), Ahouel *et al*. (2016), Draiche *et al.* (2016), Bousahla *et al*. (2016), Kolahchi and Cheraghbak (2017), Zamanian *et al.* (2017), Zaoui *et al.* (2017a,b), Kolahchi (2017), El-Haina *et al.* (2017), Fahsi *et al.* (2017), Chikh *et al*. (2017), Hajmohammad *et al*. (2017), Zarei et al. (2017), Shokravi (2017a, b, c, d), Yazid *et al.* (2018), Youcef *et al.* (2018), Mokhtar *et al.* (2018), Fourn *et al.* (2018) and Bakhadda *et al.* (2018).

Recently, Yarasca *et al*. (2016) established an Hermite-Lagrangian finite element formulation to investigate the static behavior of functionally sandwich beams with including both shear deformation and stretching thickness effects. Simsek (2016) presented a two dimensional functionally graded materials (2D FGM) to investigate the buckling of beams with different boundary conditions. Benbakhti *et al.* (2016) presented a hyperbolic plate theory with 5-unknowns using stretching effects for thermo-mechanical bending of FG sandwich plates. Tounsi *et al.* (2016) presented a non-polynomial shear deformation theory with three unknowns for buckling and vibration analysis of FG sandwich plates. Tossapanon and Wattanasakulpong (2016) utilized Chebyshev collocation method to solve buckling and vibration problems of functionally graded (FG) sandwich beams resting on two-parameter elastic foundation including Winkler and shear layer springs. Kheroubi *et al.* (2016) proposed a simple and refined nonlocal hyperbolic higher-order beam theory for bending and vibration response of nanoscale beams. Meftah *et al.* (2017) analyzed the free vibration of thick FG plates resting on two parameter elastic foundation using a non-polynomial refined plate theory. The nonlinear thermal buckling of single-walled Borone Nitride nanotubes (SWBNNTs) using a new nonlocal first-order shear deformation beam theory was investigated by HadjElmerabet *et al.* (2017). Baseri *et al*. (2016) presented an analytical solution for buckling of embedded laminated plates based on higher order shear deformation plate theory. Recently, a new class of quasi-3D HSDTs is developed by several authors to study macro/nano-structures (Abualnour *et al*. 2018, Benchohra *et al*. 2018, Bennoun *et al*. 2016).

In this article, a new displacement field based on quasi-3D hybrid-type higher order shear deformation theory is developed to analyze the static and dynamic behaviors of exponential (E), power-law (P) and sigmoïd (S) functionally graded beams. Novelty of this theory is that contain just three unknowns with including stretching effect, as opposed to four or even greater numbers in other shear and normal deformation theories. It also does not require a shear correction factor because of the parabolic distribution of the transverse shear stresses across the thickness, and satisfies the free surfaces boundary conditions of transverse shear stresses. The Hamilton's principle is used to determine the beam governing equations. Navier-type analytical solutions of bending and free vibration analysis are provided for simply supported beams subjected to uniform distribution loads.

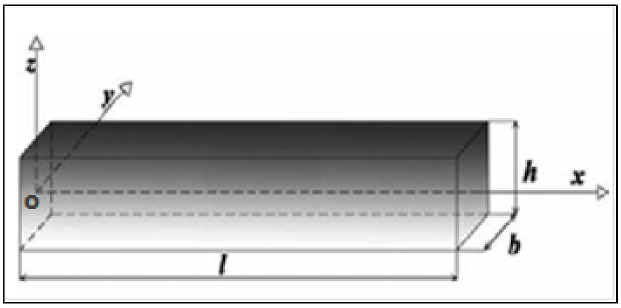


Fig. 1 Geometry and coordinates of a FG beam

The effect of the sigmoid, exponent and power-law volume fraction, the thickness stretching and the material length scale parameter on the deflection, stresses and natural frequencies are discussed. To evaluate the performance of this theory, the obtained results are compared with results available in literature.

2. Analytical modeling

2.1 Material properties

In this study, we consider a functionally graded beam of length L and rectangular cross-section with b the width and h the thickness, with the Cartesian coordinate system $O(x,y,z)$ and the origin at O as shown in Fig. 1. The material characteristics of the FGM are defined based on one of the following rules of mixture:

2.1.1 The power-law (P-FGM) variation

The volume fraction of the P-FGM beam given in the Fig. 2 is considered to change smoothly within the thickness of the beam in according to the power law form (Bao and Wang 1995, Tounsi *et al*. 2013, Hebali *et al.* 2014, Houari *et al.* 2016, Meksi *et al.* 2018) as follow

$$P(z) = P_m + (P_c - P_m)\left(\frac{1}{2} + \frac{z}{h}\right)^k \tag{1}$$

2.1.2 The exponential (E-FGM) variation

The volume fraction of the E-FGM beam that is plotted in the Fig. 3, is supposed to vary continuously in the thickness direction according to the exponential variation (Delale and Erdogan 1983) as given in Eq. (2)

$$P(z) = Ae^{k(z+h/2)},\ A = P_m,\ k = \frac{1}{h}\ln\left(\frac{P_c}{P_m}\right) \tag{2}$$

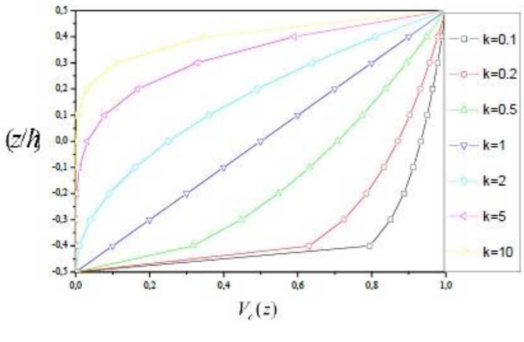


Fig. 2 Volume fraction profile of the material $V_c(z)$ through the thickness of P-FGM beam for different value of the parameter (k)

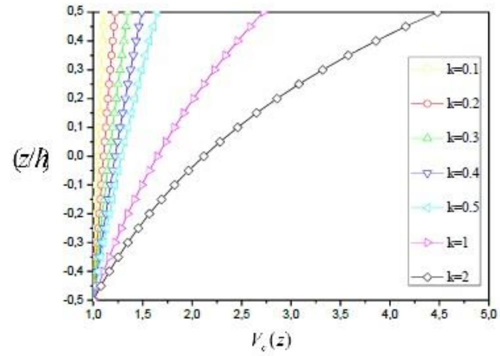


Fig. 3 Exponential function profile along the thickness of an EGM beam for different values of the parameter

2.1.3 The sigmoïd (S-FGM) variation

The volume fraction of the S-FGM beam as shown in the Fig. 4 is defined by using two power-low functions which ensure smooth distribution along the thickness direction (Lee *et al.* 2015) as follow

$$P_1(z) = \frac{1}{2}\left(\frac{h/2+z}{h/2}\right)^k P_c + \left(1 - \frac{1}{2}\left(\frac{h/2+z}{h/2}\right)^k\right) P_m \quad for \quad -\frac{h}{2} \le z \le 0 \tag{3a}$$

$$P_2(z) = \left(1 - \frac{1}{2}\left(\frac{h/2-z}{h/2}\right)^k\right) P_c + \frac{1}{2}\left(\frac{h/2-z}{h/2}\right)^k P_m \quad for \quad 0 \le z \le \frac{h}{2} \tag{3b}$$

Where P represents the effective material property, P_c and P_m denote the properties of metal and ceramic respectively, and k is the exponent that specifies the material distribution profile through the thickness. Poisson ratio ν is considered to be constant (Attia *et al*. 2015, 2018).

2.2 Kinematic and constitutive relations

According to the higher order shear deformation theory (HSDT), the displacement field of the proposed refined hybrid quasi-3D shear deformation theory can be expressed as

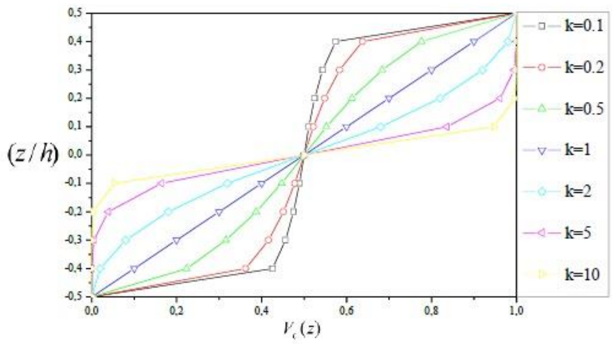


Fig. 4 Volume fraction profile of the material $V_c(z)$ through the thickness of S-FGM beam for different value of the parameter (k)

$$u(x,z,t)=u_0(x,t)-z\frac{\partial w_0}{\partial x}+f(z)\frac{\partial \varphi}{\partial x} \tag{4a}$$

$$w(x,z,t)=w_0(x,t)+g(z)\,\varphi(x,t) \tag{4b}$$

Where u_0, w_0, φ are in-plane displacement, transverse deflection and the shear rotation of transverse normal on the plane, respectively. $f(z)$ denotes a shape function determining the distribution of the transverse shear strains and stresses through the thickness.

In this study, an hybrid type shear strain shape functions are used

$$f(z)=\frac{h}{\pi}\sin\frac{\pi z}{h},\ and\ g(z)=\frac{1}{9}\left(1-4\frac{z^2}{h^2}\right) \tag{5}$$

The linear strain relations determined from the kinematic of Eqs. 4(a) and 4(b) can be constituted with Eqs. 6(a)-6(c)

$$\varepsilon_x=\varepsilon_x^0+z\,k_x^b+f(z)k_x^s \tag{6a}$$

$$\varepsilon_z=g'(z)\,\varepsilon_z^0 \tag{6b}$$

$$\gamma_{xz}=\left[g(z)+f'(z)\right]\gamma_{xz}^0 \tag{6c}$$

Where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\quad k_x^b=-\frac{\partial^2 w_0}{\partial x^2},\quad k_x^s=\frac{\partial^2\varphi}{\partial x^2}, \tag{7a}$$

$$\varepsilon_z^0=\varphi,\quad \gamma_{xz}^0=\frac{\partial\varphi}{\partial x},\quad g'(z)=\frac{\partial g(z)}{\partial z} \tag{7b}$$

For the FG beams, the linear elastic constitutive equations can be obtained by

$$\sigma_x=Q_{11}(z)\,\varepsilon_x+Q_{13}(z)\ \varepsilon_z \tag{8a}$$

$$\sigma_z=Q_{13}(z)\,\varepsilon_x+Q_{33}(z)\ \varepsilon_z \tag{8b}$$

$$\tau_{xz}=Q_{55}(z)\,\gamma_{xz} \tag{8c}$$

In which, ($\sigma_x, \sigma_z, \tau_{xz}$) and ($\varepsilon_x, \varepsilon_z, \gamma_{xz}$) are the stresses and the strain vectors with respect to the beam coordinate system. Q_{ij} are the elastic coefficients which they are defined in terms of engineering constants as below

$$Q_{11}(z)=Q_{33}(z)=\frac{E(z)}{1-\nu^2}, \tag{9a}$$

$$Q_{13}(z)=\nu Q_{11}(z), \tag{9b}$$

$$Q_{55}(z)=\frac{E(z)}{2(1+\nu)} \tag{9c}$$

2.3 Equations of motion

The equations of motion and the constitutive relations are derived by applying Hamilton's principle. The principle can be stated in analytical form as (Larbi Chaht *et al.* 2015, Zemri *et al.* 2015, Mahi *et al.* 2015, Bellifa *et al.* 2017b, Benadouda *et al.* 2017, Besseghier *et al.* 2017, Bouafia *et al.* 2017, Khetir *et al.* 2017, Klouche. *et al.* 2017, Zidi *et al.* 2017, Hachemi *et al.* 2017a, b, Belabed *et al.* 2018, Kaci *et al.* 2018)

$$\int_0^T(\delta U+\delta V-\delta K)dt=0 \tag{10}$$

where δU is the variation of strain energy; δV is the variation of the potential energy of external transverse load ; and δK is the variation of kinetic energy.

The variation of strain energy of the FG beam is given by

$$\begin{aligned}\delta U&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_x\,\delta\,\varepsilon_x+\sigma_z\,\delta\,\varepsilon_z+\tau_{xz}\,\delta\,\gamma_{xz}\right)d\,z\,d\,x\\&=\int_0^L\left(N_x\,\delta\,\varepsilon_x^0+N_z\,\delta\,\varepsilon_z^0+M_x^b\delta\,k_x^b+M_x^s\delta\,k_x^s+Q_{xz}^s\,\delta\,\gamma_{xz}^0\right)d\,x\end{aligned} \tag{11}$$

Where the stress resultants N_x, N_z, M_x^b, M_x^s and Q_{xz}^s are defined by

$$\begin{aligned}&\left(N_x, M_x^b, M_x^s\right)=\int_{-h/2}^{h/2}(1,z,f)\sigma_x dz,\ N_z=\int_{-h/2}^{h/2}g'(z)\ \sigma_z\ d\,z,\\&Q_{xz}^s=\int_{-h/2}^{h/2}\left(g(z)+f'(z)\right)\tau_{xz}\ d\,z\end{aligned} \tag{12}$$

The variation of the potential energy of external loadcan be expressed by

$$\delta V=-\int_0^L q\ \delta\ w_0\ dx \tag{13}$$

The variation of kinetic energy of the beam can be written as

$$\begin{aligned}\delta K&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\dot u\delta\,\dot u+\dot w\delta\,\dot w\right]\rho(z)\,dz\,dx\\&=\int_0^L\{I_0\left(\dot u_0\,\delta\,\dot u_0+\dot w_0\,\delta\,\dot w_0\right)+J_0(\dot w_0\,\delta\,\dot\varphi+\dot\varphi\,\delta\,\dot w_0)\\&+K_0\,\dot\varphi\,\delta\dot\varphi-I_1\left(\dot u_0\frac{\partial\delta\,\dot w_0}{\partial x}+\frac{\partial\,\dot w_0}{\partial x}\delta\,\dot u_0\right)\\&+I_2\left(\frac{\partial\,\dot w_0}{\partial x}\frac{\partial\delta\,\dot w_0}{\partial x}\right)+J_1\left(\dot u_0\frac{\partial\delta\,\dot\varphi}{\partial x}+\frac{\partial\,\dot\varphi}{\partial x}\delta\,\dot u_0\right)\\&-J_2\left(\frac{\partial\,\dot w_0}{\partial x}\frac{\partial\delta\,\dot\varphi}{\partial x}+\frac{\partial\,\dot\varphi}{\partial x}\frac{\partial\delta\,\dot w_0}{\partial x}\right)+K_2\frac{\partial\,\dot\varphi}{\partial x}\frac{\partial\delta\,\dot\varphi}{\partial x}\}dx\end{aligned} \tag{14}$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t ; $\rho(z)$

is the mass density; and (I_i, J_i, K_i) are mass inertias defined as

$$(I_0, I_1, I_2, J_1, J_2, J_0, K_0, K_2) = \int_{-h/2}^{h/2} (1, z, z^2, f, zf, g, g^2, f^2)\rho(z)\, dz \quad (15)$$

Substituting Eqs. (11), (13), and (14) into Eq. (10), integrating by parts, and collecting the coefficients of δu_0, δw_0 and $\delta \varphi$, the equations of motion are obtained in terms of efforts as follow

$$\delta u_0 : \frac{\partial N_x}{\partial x} = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} + J_1 \frac{\partial \ddot{\varphi}}{\partial x} \quad (16a)$$

$$\delta w_0 : \frac{\partial^2 M_x^b}{\partial x^2} + q = I_0 \ddot{w}_0 + J_0 \ddot{\varphi} + I_1 \frac{\partial \ddot{u}_0}{\partial x} - I_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} + J_2 \frac{\partial^2 \ddot{\varphi}}{\partial x^2} \quad (16b)$$

$$\delta \varphi : -\frac{\partial^2 M_x^s}{\partial x^2} + \frac{\partial Q_{xz}^s}{\partial x} - N_z = J_0 \ddot{w}_0 + K_0 \ddot{\varphi} - J_1 \frac{\partial \ddot{u}_0}{\partial x} + J_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} - K_2 \frac{\partial^2 \ddot{\varphi}}{\partial x^2} \quad (16c)$$

Substituting Eq. (6) into Eq. (8) and the subsequent results into Eq. (12), the stress resultants can be expressed in terms of generalized displacements (u_0, w_0, φ) as

$$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \\ N_z \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{11}^s & X_{13} \\ B_{11} & D_{11} & D_{11}^s & Y_{13} \\ B_{11}^s & D_{11}^s & H_{11}^s & Y_{13}^s \\ X_{13} & Y_{13} & Y_{13}^s & Z_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ -\frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial \varphi}{\partial x} \\ \varphi \end{Bmatrix} \quad (17)$$

$$Q_{xz}^s = A_{55}^s \frac{\partial \varphi}{\partial x}$$

Where A, B, D, B^s, D^s, etc… are the stiffnesses of the FG beam given by

$$(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s) = \int_{-h/2}^{h/2} Q_{11}(1, z, z^2, f, zf, f^2)dz$$

$$(X_{13}, Y_{13}, Y_{13}^s) = \int_{-h/2}^{h/2} Q_{13}(1, z, f) g' dz$$

$$Z_{33} = \int_{-h/2}^{h/2} Q_{33}(g'(z))^2 dz \quad (18)$$

$$A_{55}^s = \int_{-h/2}^{h/2} Q_{55}(g + f')^2 dz$$

Substituting Eqs. (17) into Eqs. (16), the equations of motion of the proposed quasi-3D hybrid-type HSDT can be expressed in terms of displacements (u_0, w_0, φ) as

$$\delta u_0 : A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_0}{\partial x^3} + B_{11}^s \frac{\partial^3 \varphi}{\partial x^3} + X_{13} \frac{\partial \varphi}{\partial x} = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} + J_1 \frac{\partial \ddot{\varphi}}{\partial x} \quad (19a)$$

$$\delta w_0 : B_{11} \frac{\partial^3 u_0}{\partial x^3} - D_{11} \frac{\partial^4 w_0}{\partial x^4} + D_{11}^s \frac{\partial^4 \varphi}{\partial x^4} + Y_{13} \frac{\partial^2 \varphi}{\partial x^2} + q = I_0 \ddot{w}_0 + J_0 \ddot{\varphi} + I_1 \frac{\partial \ddot{u}_0}{\partial x} - I_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} + J_2 \frac{\partial^2 \ddot{\varphi}}{\partial x^2} \quad (19b)$$

$$\delta \varphi : B_{11}^s \frac{\partial^3 u_0}{\partial x^3} + X_{13} \frac{\partial u_0}{\partial x} - Y_{13} \frac{\partial^2 w_0}{\partial x^2} - D_{11}^s \frac{\partial^4 w_0}{\partial x^4} + H_{11}^s \frac{\partial^4 \varphi}{\partial x^4} + (A_{55}^s - 2Y_{13}^S) \frac{\partial^2 \varphi}{\partial x^2} + Z_{33}\varphi = J_0 \ddot{w}_0 + K_0 \ddot{\varphi} - J_1 \frac{\partial \ddot{u}_0}{\partial x} + J_2 \frac{\partial^2 \ddot{w}_0}{\partial x^2} - K_2 \frac{\partial^2 \ddot{\varphi}}{\partial x^2} \quad (19c)$$

3. Closed-form solution for simply supported beam

Using Navier's procedure, the solution of the displacement variables satisfying the above boundary conditions can be expressed in the following Fourier series:

$$\begin{Bmatrix} u_0 \\ w_0 \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ W_m \sin(\lambda x) e^{i\omega t} \\ \Phi_m \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \quad (20)$$

Where (U_m, W_m, Φ_m)are unknown functions to be determined and ω is the natural frequency. λ isexpressed as

$$\lambda = m\pi / L \quad (21)$$

The transverse load q is also expanded in the double-Fourier sine series as

$$q(x) = \sum_{m=1}^{\infty} q_m \sin(\lambda x) \quad (22)$$

Where q_m is the intensity of the load calculated from the Eq. (23)

$$q_m = \frac{2}{L} \int_0^L q(x) \sin(\lambda x) dx \quad (23)$$

For sinusoidal distributed load

$$q_m = q_0, \quad (24)$$

For uniform distributed load

$$q_m = \frac{4q_0}{m\pi}, \quad (m = 1,3,5,.....) \quad (25)$$

Substituting Eqs. (20) and (22) into equations of motion (19),the closed-form solutions can be obtained from the following equations

Table 1 Materialspropertiesof metal and ceramic

Material	Properties		
	Young's modulus (GPa)	Poisson's ratio	Mass density (kg/m^3)
Aluminium (Al)	70	0.3	2702
Alumina (Al_2O_3)	380	0.3	3800

Table 2 Non-dimensional deflections and stresses of P-FGM beams under uniform load

k	Method	ε_z	$L/h=5$				$L/h=20$			
			$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Li *et al*. (2010)	$=0$	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	OuldLarbi *et al.* (2013)	$=0$	3.1253	0.9162	3.8091	0.7777	2.8908	0.2294	15.0194	1.0103
	Bouremana *et al*. (2013)	$=0$	3.1657	0.9209	3.7500	0.5990	2.8963	0.2303	15.0000	0.5993
	Meradjah *et al.* (2015)	$\neq 0$	3.1357	0.9261	3.8614	0.7438	2.8906	0.2300	15.2708	0.7656
	Present	$\neq 0$	3.1394	0.9285	3.8050	0.7354	2.8944	0.2302	15.0152	0.8283
0.5	Li *et al.* (2010)	$=0$	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	OuldLarbi *et al*. (2013)	$=0$	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Bouremana *et al.* (2013)	$=0$	4.8347	1.6331	4.9206	0.6270	4.4648	0.4083	19.6825	0.6266
	Meradjah *et al*. (2015)	$\neq 0$	4.7584	1.6124	5.0789	0.7604	4.4292	0.4010	20.0787	0.7824
	Present	$\neq 0$	4.7605	1.6120	4.9990	0.7692	4.4333	0.4010	19.7050	0.8940
2	Li *et al.* (2010)	$=0$	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	OuldLarbi *et al*. (2013)	$=0$	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Bouremana *et al.* (2013)	$=0$	8.0307	3.0741	6.7678	0.5101	7.4400	0.7686	27.0704	0.5102
	Meradjah *et al.* (2015)	$\neq 0$	7.8501	2.9703	6.9957	0.6838	7.2688	0.7390	27.5763	0.7044
	Present	$\neq 0$	7.8585	2.9904	6.8780	0.6293	6.7344	0.6994	27.0712	0.7102
5	Li *et al*. (2010)	$=0$	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	OuldLarbi *et al*. (2013)	$=0$	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Bouremana *et al.* (2013)	$=0$	9.6484	3.6496	7.9427	0.3926	8.8068	0.9120	31.7710	0.3927
	Meradjah *et al.* (2015)	$\neq 0$	9.6028	3.5488	8.2440	0.6079	8.6396	0.8798	32.3457	0.6271
	Present	$\neq 0$	9.5438	3.5939	8.0776	0.6310	8.6421	0.8814	31.8016	0.6721
10	Li *et al.* (2010)	$=0$	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	OuldLarbi *et al.* (2013)	$=0$	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Bouremana *et al.* (2013)	$=0$	10.7194	3.8098	9.5231	0.4288	9.6770	0.9524	38.0915	0.4292
	Meradjah *et al.* (2015)	$\neq 0$	10.7561	3.7501	9.8597	0.6625	9.5715	0.9278	38.7327	0.6835
	Present	$\neq 0$	10.6709	3.8019	9.6720	0.6291	9.5666	0.9288	38.1248	0.6704

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}\right) \begin{Bmatrix} U_m \\ W_m \\ \Phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_m \\ 0 \end{Bmatrix} \quad (26)$$

Where

$$\begin{aligned} & s_{11} = \lambda^2 A_{11}, \qquad s_{12} = -\lambda^3 B_{11}, \\ & s_{22} = \lambda^4 D_{11}, \qquad s_{13} = \lambda^3 B_{11}^s - \lambda X_{13}, \\ & s_{33} = \lambda^4 H_{11}^s + \lambda^2 (A_{55}^s - 2Y_{13}^s) + Z_{33} \\ & s_{23} = -\lambda^4 D_{11}^s + \lambda^2 Y_{13} \\ & m_{11} = I_0, \quad m_{12} = -\lambda I_1, \quad m_{13} = \lambda J_1, \\ & m_{22} = I_0 + \lambda^2 I_2, \quad m_{23} = J_0 - \lambda^2 J_2, m_{33} = K_0 + \lambda^2 K_2 \end{aligned} \quad (27)$$

4. Numerical results and discussions

In this section, a various numerical analyzes are presented to verify the accuracy of the present theory in predicting the bending and free vibration responses of three type of simply supported FG beam (P-FGM, E-FGM,S-FGM). Materials characteristics for metal and ceramic used in the FG beams are outlined in Table 1.

For convenience, the following non-dimensional parameters areused

$$\bar{w} = 100\frac{h^3 E_m}{L^4 q_0} w\left(\frac{L}{2}\right), \bar{u} = 100\frac{h^3 E_m}{L^4 q_0} u\left(0, -\frac{h}{2}\right), \quad (28)$$

$$\bar{\sigma}_x = \frac{h}{Lq_0}\sigma_x\left(\frac{L}{2},\frac{h}{2}\right),\ \bar{\tau}_{xz} = \frac{h}{Lq_0}\tau_{xz}(0,0)\ \bar{\varpi} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

4.1 Bending response

4.1.1 Case of P-FGM beam

To validate this study, a dimensionless displacements and stresses of thick and thin FG beam subjected to uniform distributed load using a power law variation of Young's modulus are presented in Table 2. The values obtained are compared with the 2D shear deformation theories given by Li *et al.* (2010), Ould Larbi *et al.* (2013), Bouremana *et al.* (2013) where the effect normal strain is neglected and the quasi-3D shear deformation theory reported by Meradjah *et al.* (2015) which taking into account for the effect of normal and transverse shear deformations $(\varepsilon_z \neq 0)$. Since the proposed and the quasi-3D theory of Meradjah *et al.* (2015) include the thickness-stretching effect, the results are close to each other. Meanwhile, 2D theories which the thickness stretching effect is omitted overestimate the results. This table gives also the effect of the volume fraction exponent ratio "k" and side-to-thickness ratio "L/h".

From these results, it can be noticed that the axial and transverse displacements "$\bar{u}$, $\bar{w}$" and in-plane stress"$\bar{\sigma}_x$"increase with the increasing value of power law index "k".The shear stress $\bar{\tau}_{xz}$ are also sensitive to the variation of "k".

In Figs. 5-8, the stress and displacement distribution s through the thickness of Al/Al_2O_3 FGM thick beam under uniform load, are presented. The results are plott ed as compared with those obtained by Meradjah *et al.* (2015) for several values of power law index "k".A very good accuracy between the solutions is observed, except a small difference between the results of the tr ansverse shear stress "$\bar{\tau}_{xz}$" is found (see Fig. 8). It is due to shear strain shape function.

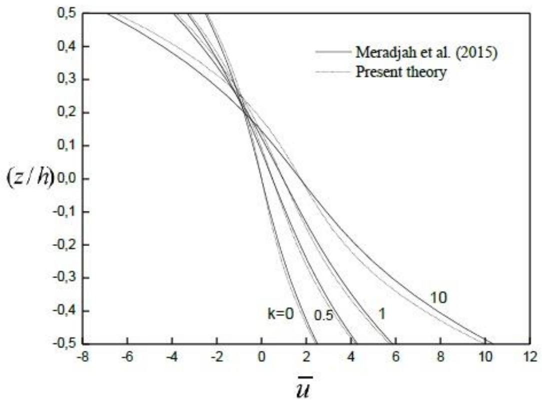


Fig. 5 The variation of the axial displacement $\bar{u}$ through-the-thickness of a P-FGM beam ($L = 2h$)

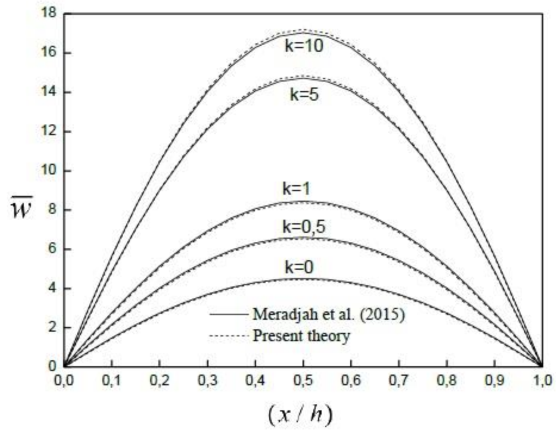


Fig. 6 The variation of the transversal displacement $\bar{w}$ through the length scale of a P-FGM beam ($L = 2h$)

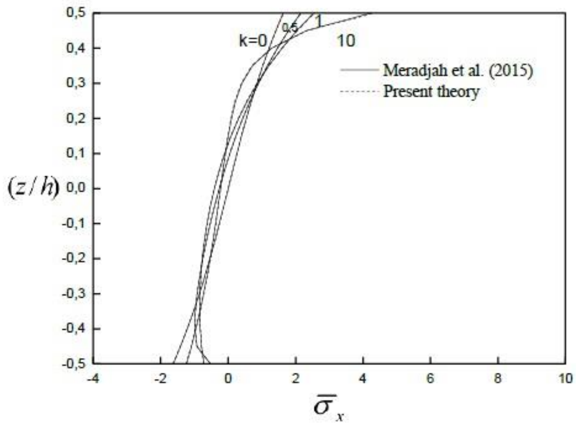


Fig. 7 The variation of the axial stress $\bar{\sigma}_x$ through the length scale of a P-FGM beam ($L = 2h$)

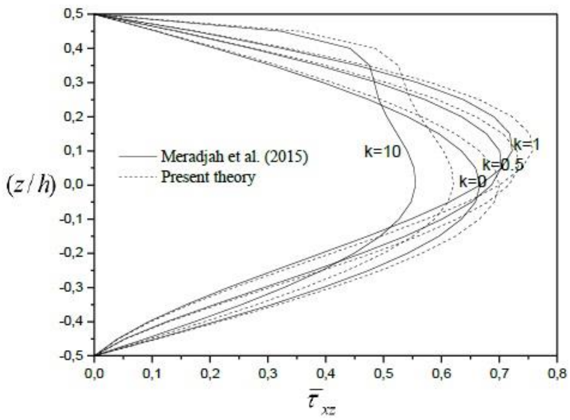


Fig. 8 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a P-FGM beam ($L = 2h$)

Table 3 Non-dimensional deflections and stresses of S-FGM beams under uniform load

k	Method	$L/h=5$				$L/h=20$			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Present $\varepsilon_z = 0$	4.9076	1.4491	3.8108	0.7539	4.4539	0.3544	15.0152	0.7689
	Present $\varepsilon_z \neq 0$	4.8164	1.3914	3.8296	0.8306	4.4446	0.3525	15.0242	1.0419
0.1	Present $\varepsilon_z = 0$	4.9364	1.5244	6.1814	0.7542	4.4824	0.3732	24.34367	0.7691
	Present $\varepsilon_z \neq 0$	4.8399	1.4530	6.2129	0.8307	4.4679	0.3678	24.35887	1.0419
0.2	Present $\varepsilon_z = 0$	5.0027	1.6006	6.0300	0.7545	4.5484	0.3922	23.7394	0.7694
	Present $\varepsilon_z \neq 0$	4.8942	1.5153	6.0615	0.8309	4.5218	0.3833	23.7545	1.0420
0.5	Present $\varepsilon_z = 0$	5.2806	1.8155	5.8638	0.7550	4.8256	0.4458	23.0732	0.7699
	Present $\varepsilon_z \neq 0$	5.1212	1.6908	5.8953	0.8312	4.7482	0.4271	3.08839	1.0421
1	Present (P-FGM) $\varepsilon_z =0$	5.7388	2.1012	5.8985	0.7539	5.2851	0.5174	23.20909	0.7689
	Present $\varepsilon_z = 0$	5.7388	2.1012	5.8985	0.7539	5.2851	0.5174	23.20909	0.7689
	Present (P-FGM) $\varepsilon_z \neq 0$	5.4950	1.9237	5.9303	0.8306	5.1232	0.4856	23.2244	1.0419
	Present $\varepsilon_z \neq 0$	5.4950	1.9237	5.9303	0.8306	5.1232	0.4856	23.2244	1.0419
2	Present $\varepsilon_z = 0$	6.3540	2.4508	6.1005	0.7494	5.9059	0.6055	24.0174	0.7642
	Present $\varepsilon_z \neq 0$	5.9957	2.2082	6.1328	0.8281	5.6297	0.5574	24.0330	1.0412
10	Present $\varepsilon_z = 0$	7.2178	2.9196	6.4914	0.7373	6.7848	0.7242	25.5903	0.7513
	Present $\varepsilon_z \neq 0$	6.6967	2.5884	6.5244	0.8214	6.3467	0.6542	25.6062	1.0392
15	Present $\varepsilon_z = 0$	7.2735	2.9493	6.5189	0.7362	6.8418	0.7318	25.7016	0.7502
	Present $\varepsilon_z \neq 0$	6.7418	2.6125	6.5519	0.8208	6.3932	0.6604	25.7175	1.0390
20	Present $\varepsilon_z = 0$	7.2955	2.9611	6.5298	0.7358	6.8643	0.7347	25.7457	0.7497
	Present $\varepsilon_z \neq 0$	6.7596	2.6219	6.5629	0.8206	6.4115	0.6628	25.7616	1.0390

4.1.2 Case of S-FGM beam

In this example, a simply supported S-FGM beam will be analyzed. A sigmoid function given in Eq. (3) is used to define the material properties of the beam.

The homogeneous $(k=0)$ and P-FGM $(k=1)$ beams are used herein for the verification. Table 3 shows the non-dimensional displacements, of simply supported beams, with various values of span-to-depth ratio "L/h" and power law index "k".

The calculated displacements and stresses of 2D and 3D present theory are compared. The present results are in good correlation, in the cases of homogeneous and functionally graded $(k=1)$ beams, with those obtained by Meradjah *et al*. (2015). This is due to the fact that the S-FGM material properties are the same with P-FGM, when the power law exponent is 1. The small difference between the present 2D and quasi-3D shear deformation results is due to omitting the thickness stretching effect. Also, it can be seen that the beam becomes stiffer when the effects of normal deformations is considered, and hence, leads to a decreasing of deflection and an increase in stresses.

In Fig. 9, variations of non-dimensional displacements and stresses according to the thickness of an S-FGM beam are plotted using the present theory with including the thick ness stretching effect. It can be shown that the fully ceramic beams produce the smallest displacements and stresses. As the volume fraction index increases, the deflection, axial displacement and stresses will increase.

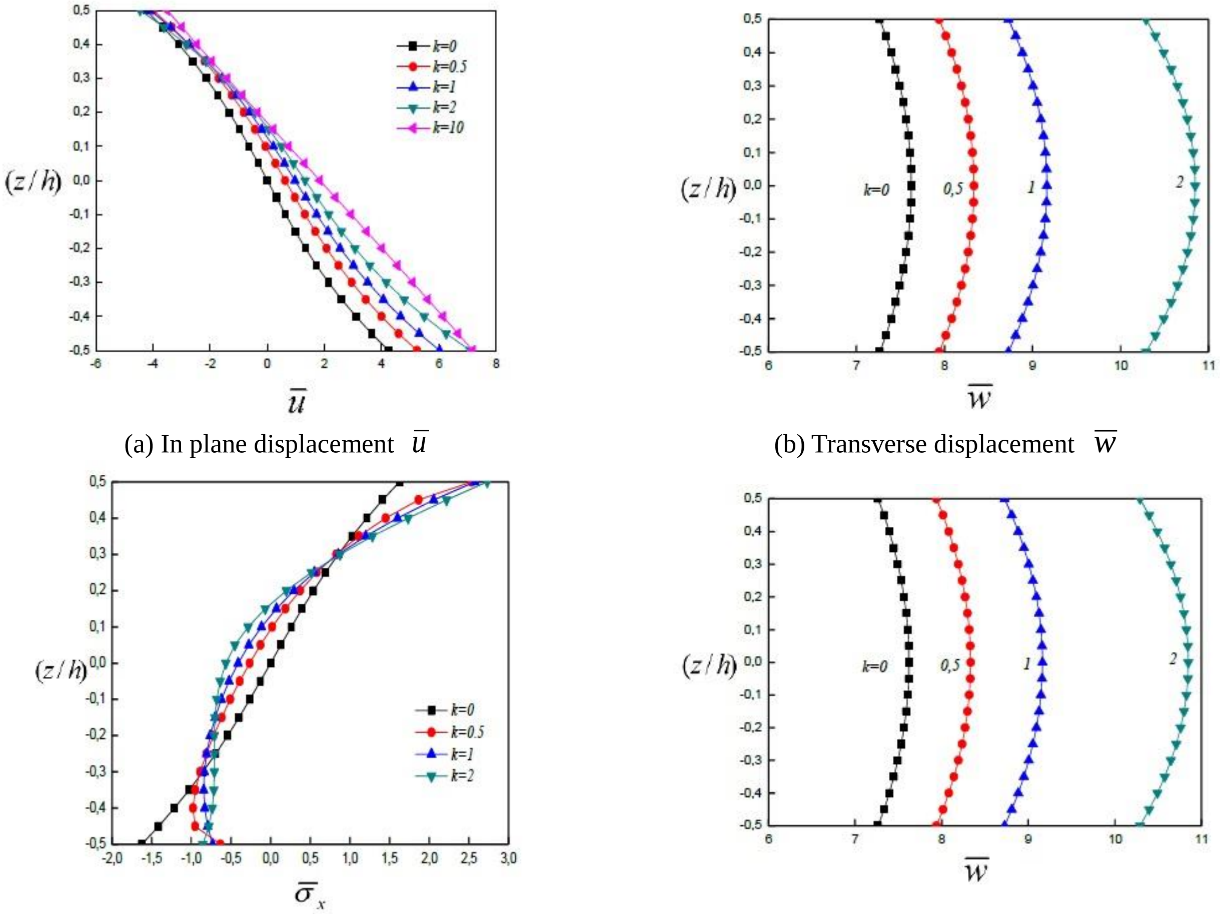
It should be noticed that Fig. 9(b) demonstrates the stretching thickness effect because it provides distributions of deflection within the thickness of beam. When this effect is ignored, these become constant through the thickness.

4.1.3 Parameter studies

In this party, parametric studies have been presented to evaluate the effect of power law index "k" and side-to-thickness ratio "L/h" on bending analysis of functionally graded beams with variable functions.

In Fig. 10, the influence of side-to-thickness ratio on non-dimensional deflection of simply supported beams with variable functions is plotted. It can be observed from this figure that, increasing of side-to-thickness ratio causes reducing of the magnitude of deflection, it means that the effect of shear deformation is significant when beams are thick $(L/h \leq 20)$, and negligible for thin beams.

Another comparative study for evaluating the dimensionless center deflections of P-FGM, E-FGM, S-FGM beams is carried out. From Fig.11, it is showed that the deflection at the center of P-FGM beam is larger than those of E-FGM and S-FGM beams. The S-FGM beam which has the smallest deflection is stiffer than the other FGM beams.



(a) In plane displacement $\bar{u}$ (b) Transverse displacement $\bar{w}$

(c) In plane stresses $\bar{\sigma}_x$ (d) Transverse shear stresses $\bar{\tau}_{xz}$

Fig. 9 Variation of displacements and stresses through the thickness of S-FGM beam (L = 2h)

4.2 free vibration response

To investigate the accurate of the present theory in predicting a free vibration response of functionally graded beams, dimensionless natural frequencies are calculated in Table 4 and compared with those available in literature. Table 4 reveals that the results of the present theory with only three unknowns exhibits an excellent agreement with those obtained by the sinusoidal shear deformation beam theory developed by Meradjah *et al.* (2015) with four unknowns $(\varepsilon_z \neq 0)$. This indicates that the same accuracy is realizable with the proposed theory using a few numbers of unknowns than other theories, and clearly highlights how the proposed theory can provide accurate solutions for free vibration problems.

Fig. 12(a) depicts the effect of power law index on natural frequencies of P-FGM beam with variable values of (L/h). It observed that the natural frequencies reduce with increasing of power law index and become almost constant with respect to the variation of power law index $(p \succ 5)$. Also, the maximum values of frequencies are showed in thick beams where the shear deformation is important. In Fig. 12(b), the variations of non-dimensional natural frequencies respect to side-to-thickness ratio for different values of power law index are displayed. The increasing value of side-to-thickness ratio leads to the increase of natural frequencies.

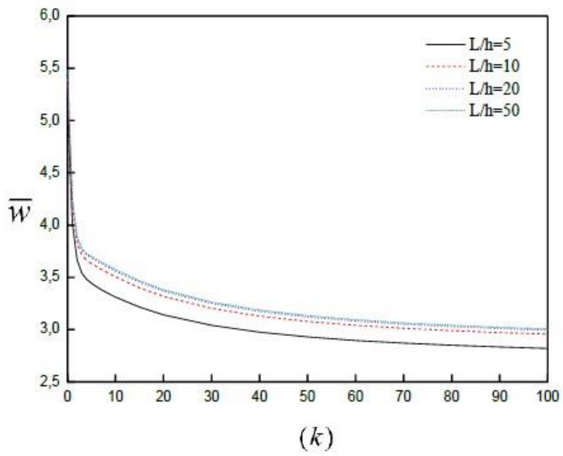


Fig. 10 Effect of side-to-thickness ratio (L/h) on non-dimensional deflection of FGM beam

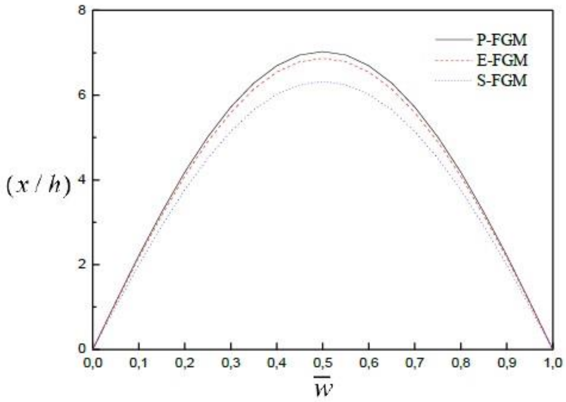
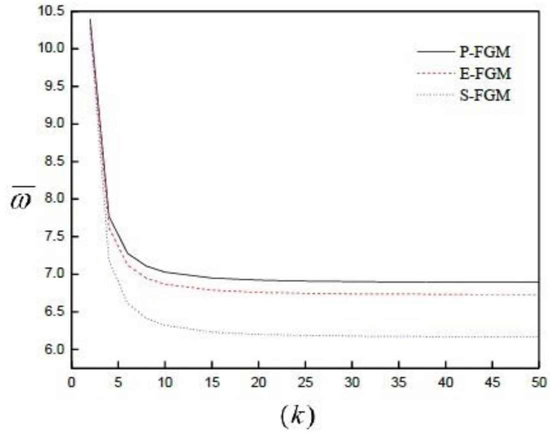
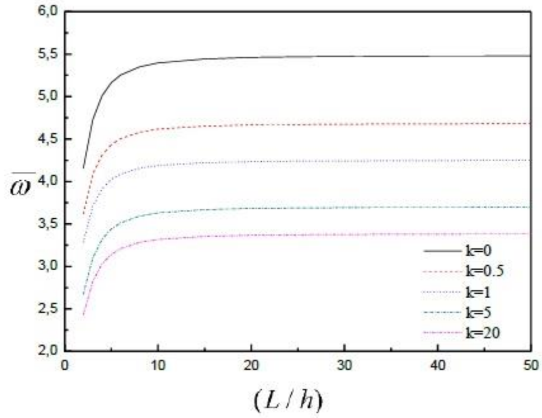


Fig. 11 The comparison of the deflection of E-FGM, P-FGM and S-FGM beams $(L/h=10)$



(a)



(b)

Fig. 12 Effect of the power-law index (k) and side-to-thickness ratio (L/h) on the natural frequency $\overline{\omega}$ of P-FGM beam

Figs. 13 and 14 plots the influence of side-to-thickness ratio and power law index on natural frequencies of S-FGM beams, respectively. It can be seen from Fig. 13, the diminishing effect of volume fraction exponent on natural frequencies. It is due to the fact that a higher value of "k" corresponds to lower value of volume fraction of the ceramic phase and thus leads to the decrease of the value of the elasticity modulus which makes the beams softer. It can be noticed from the Fig. 14 that for a given value of "k", as the thickness ratio increase, the natural frequency increase and the effect of side-to-thickness ratio becomes negligible for the values higher than 20.

5. Conclusions

The bending and free vibration responses of P, E, and S-FGM beams have been studied in this paper by using a new quasi-3D hybrid type HSDT. The theory is developed by making further simplifying assumptions to the existing HSDTs, with the incorporation of an undetermined integral term.

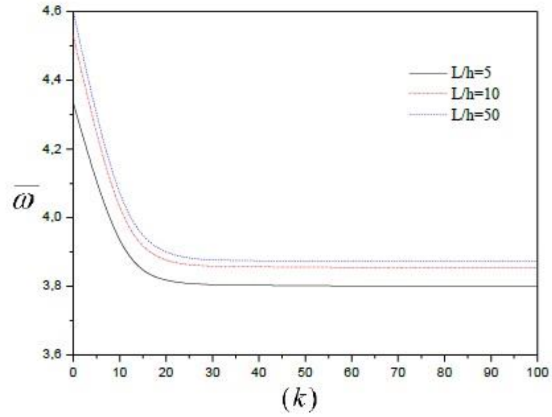


Fig. 13 Variation of fundamental natural frequency $\overline{\omega}$ versus power-law index (k) for different side-to-thickness ratio (L/h) of S-FGM beam

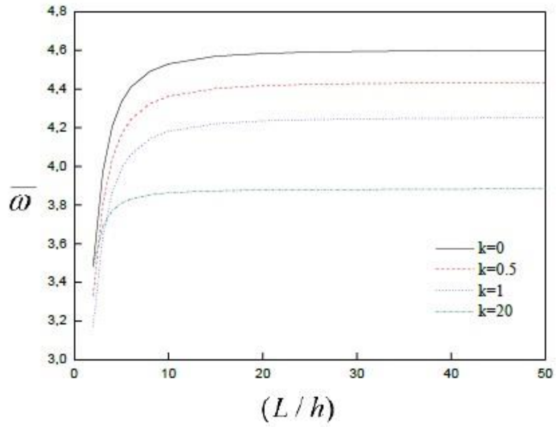


Fig. 14 Variation of fundamental natural frequency $\overline{\omega}$ versus side-to-thickness ratio (L/h) for different power-law index (k) of S-FGM beam

The number of variables and equations of motion of the proposed quasi-3D hybrid type HSDT are reduced by one, and hence, make this theory simple and efficient to use.
The equations of motion are obtained through the Hamilton's principle. These equations are analytically solved by utilizing Navier's procedure. Results demonstrate that the beam becomes stiffer when the thickness stretching effect is incorporated, and consequently, leads to a reduction of deflection and an increase of frequency. It was concluded that the present formulation provides a very accurate results compared to the other existing higher-order beam theories. So, it can be used as a reference for the prospective researchers to compare their results.

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