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Sensor and actuator design for displacement control of continuous systems

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Abstract. The present paper is concerned with the design of distributed sensors and actuators. Strain type sensors and actuators are considered with their intensity continuously distributed throughout a continuous structure. The sensors measure a weighted average of the strain tensor. As a starting point for their design we introduce the concept of collocated sensors and actuators as well as the so-called natural output. Then we utilize the principle of virtual work for an auxiliary quasi-static problem to assign a mechanical interpretation to the natural output of the sensors to be designed. Therefore, we take the virtual displacements in the principle of virtual work as that part of the displacement in the original problem, which characterizes the deviation from a desired one. We introduce different kinds of distributed sensors, each of them with a mechanical interpretation other than a weighted average of the strain tensor. Additionally, we assign a mechanical interpretation to the collocated actuators as well; for that purpose we use an extended body force analogy. The sensors and actuators are applied to solve the displacement tracking problem for continuous structures; i.e., the problem of enforcing a desired displacement field. We discuss feed forward and feed back control. In the case of feed back control we show that a PD controller can stabilize the continuous system. Finally, a numerical example is presented. A desired deflection of a clamped-clamped beam is tracked by means of feed forward control, feed back control and a combination of the two.

Keywords: continuous systems; distributed sensing/actuation; natural output; collocated control; dynamic displacement tracking.

1. Introduction

Smart structure technology has become a key technology in the design of modern, so-called intelligent, civil, mechanical and aerospace systems. Similar to human beings, these intelligent or smart systems are capable to react to disturbances exerted upon them by the environment they are operating in. For reviews see Crawley (1994) and Tani, *et al.* (1998) and for future challenges and opportunities see Liu, *et al.* (2005a). Practical applications of smart structures are e.g., in the fields of active structural vibration control (Alkhatib and Golnaraghi 2003) as well as active noise control (Gopinathan, *et al.* 2001 and Irschik, *et al.* 2003).

The design of the smart structure is a highly multi-disciplinary task, which involves the modeling of the structure, the interrogation and communication of the structure with a controller by means of

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suitable sensing and actuation, the integration of the smart system in the structure and the implementation of the system. One key aspect for a successful design is the communication between structure and controller, the so-called control-structure interaction (Gabbert and Tzou 2001). Sensors and actuators are responsible for the functioning of this communication. Sensors provide information about the state the structure is in; this information has to be interpreted and properly processed by the controller to provide the actu ator with information about what to do. For a discussion of strategic issues in the sensor design see Liu, *et al*. (2005b) and for frontiers in sensors/sensor systems see Glaser, *et al*. (2005). In typical continuous systems a crucial point is the spatial distribution of sensors to obtain proper information as well as to perform distributed control of continua (Gabbert and Tzou 2001). Finding these distributions and using them for structural control of continuous systems are the main topics of the present paper.

A common strategy for the modeling of continuous systems is based on the linearized theory of elasticity; within this paper we consider a three-dimensional linear elastic background body with sources of self-stress, which are also denoted as eigenstrains in the literature, see e.g., Reissner (1931), Nemenyi (1931), Mura (1991) and Irschik and Ziegler (1988). Self-stresses have many different physical sources, e.g. thermal stresses or piezoelectric stresses. In the present paper we consider only those types of eigenstrains that are produced in smart materials by physical mechanism, such as piezoelectricity, as most of the modern smart materials have both, actuation and sensing properties; often, piezoelectric sensors and actuators are integrated into the structure to achieve the goal of smartness. In the context of smart materials the actuation mechanism is typically denoted as a strain induced one, as is the sensing mechanism, see Tzou (1998).

The first part of this paper is concerned with the design of sensors. We use the power theorem of mechanics, see e.g., Chadwick (1999), to introduce the so-called collocated sensor output. As both, the actuation and the sensing mechanism are of the strain type, we are able to deduce a sensor output that is collocated to the actuation. Such an output represents a weighted average of the total strain tensor with respect to the volume of the body. For the definition of collocation see Preumont (2002). In a further step we introduce the natural sensor output, which we define as the weighted average of only a part of the strain tensor. This part stems from an additive decomposition of the displacement vector into two parts; one of them being a desired displacement vector to be tracked and the other one representing the deviation of the total displacement vector from the desired one. For the natural sensor output only the latter one is considered. We note that the sensor design targets on being applied to the problem of dynamic displacement tracking. For a first account of the dynamic displacement tracking problem by eigenstrains see Irschik, *et al*. (2005). The remainder of the first part of this paper is devoted to assign a mechanical interpretation to the natural sensor output. For that sake we use the principle of virtual work (e.g. Ziegler 1998 or Malvern 1969) for an auxiliary quasi-static problem. That part of the displacement that characterizes the deviation from a desired displacement is used as a virtual displacement such that we are able to relate the natural sensor output to the work done by the quasi-static auxiliary forces with respect to the latter deviation of the displacement in the original problem. For the case the collocated sensor output equals the natural sensor output results can be found in Krommer, *et al*. (2005). The latter case allows to track a zero displacement. As we are able to interpret the natural sensor output as a weighted average of the deviation of the displacement from a desired one, the choice of the auxiliary forces defines the actual mechanical meaning of the natural sensor output. In the context of our general sensor design method we are able to discuss well established types of distributed sensors, such as modal sensors (Lee and Moon 1990), nilpotent sensors (Irschik, *et al*. 1999a or Miu 1992) and displacement type sensors (Irschik, *et al*. 1999b) as well as new types, such as dilatational and rotational sensors.

As we have assumed the actuation to be collocated to the sensing, we need not design the actuators;

rather, their spatial distribution throughout the structure appears as a natural outcome of our design procedure. In order to make the actuator design useful also in non-collocated situations, we assign a mechanical interpretation to the actuators in the second part of the paper. For that sake we use an extension of the classical body force analogy, which was introduced by Duhamel (see Noda, *et al*. 2000) for static thermoelastic problems, with respect to dynamic eigenstrain problems, as it was presented in Irschik, *et al*. (2005). It is found that the collocated actuators produce a displacement in the body as it is induced by the quasi-static forces in the auxiliary problem, if we assign an arbitrary time variation to the latter ones. The stress due to the actuation equals the stress due the auxiliary forces (with arbitrary time variations) plus the source of self-stress itself, see Irschik, *et al*. (2005). In the following we will denote sources of self-stress as actuation stresses.

In the last part of the present paper the developed methodology for the sensor and actuator design is applied to solve the dynamic tracking problem. The latter problem is solved, if we consider the space wise distribution of the external forces acting on the three dimensional body as well as the space wise distributions that enter via the desired displacement to be tracked as the quasi-static forces in the auxiliary problem. Then, we immediately find the natural sensor output, which is the most suitable for the sake of dynamic displacement tracking. Moreover, we are able to proof that, if the time variation of the external forces are known, the displacement of the body equals the desired one, provided the time variations of the external forces as well as of the desired displacement are applied as the time variations of the actuators, see Irschik, *et al*. (2005) and Irschik and Krommer (2005). For the given external disturbances Haftka and Adelman (1985) introduced the special case of zero displacement tracking into the literature in context with disturbances slowly varying in time under the notion of "Shape Control Problem". For transient disturbances, Irschik and Pichler (2001) presented a solution within the context of three-dimensional thermoelasticity for a zero displacement to be tracked; a more general formulation was presented by Irschik and Pichler (2004). For a review on shape control see Irschik (2002) and for the dynamic shape control of subdomains of structures see Krommer and Varadan (2005) and Krommer (2005).

In contrast to the above references, in which a feed forward control strategy was used, the present paper also considers the case of external forces with unknown time variations. In such a case the natural sensor output is fed back to the corresponding collocated actuator. We use a simple PD controller, for which we are able to show that the closed loop system is likely to be stable in the sense of Liapunov; for the case of a circular plate with a zero displacement to be tracked see Gattringer, *et al.* (2003). Finally, we combine the two strategies, namely the feed forward one and the feed back one.

To validate the developed design methodology we study a simple clamped-clamped beam under the action of a transverse force loading, for which we specify a desired deflection to be tracked. The feed forward method is used for a given time variation of the transverse force loading; as the solution is exact, we are able to track the desired deflection identically. Then we compare the feed back strategy to the feed forward one, as well as to a combination of both. It turns out that the combination of the two methods results into a deflection that is very close to the desired one. We conclude this is due to a proper design of sensors and actuators for the purpose of dynamic displacement tracking.

2. Collocated sensing

We study a three-dimensional body in the small deformation regime; hence, we need not distinguish between the reference configuration and the current one. The volume of the body is V and it is bounded by ∂B. We consider the body to be constituted as one with sources of self – stress acting in a linear elastic background. The constitutive relation, which relates the deviation of the actuation stress tensor $\boldsymbol{\sigma}_A$ from the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}$ is

$$\boldsymbol{\sigma} - \boldsymbol{\sigma}_A = \mathbf{C}\boldsymbol{\varepsilon} \tag{1}$$

$\mathbf{C}$ is the elasticity tensor. In the small deformation regime the strain tensor is defined as the symmetric part of the gradient of the displacement vector $\mathbf{u}$, $\boldsymbol{\varepsilon} = sym(grad\mathbf{u})$. Within the volume V the motion of the body is governed by the Cauchy equations of motion (e.g. Chadwick 1999).

$$div\boldsymbol{\sigma} + \mathbf{b} = \rho\frac{d^2}{dt^2}\mathbf{u} \tag{2}$$

$\mathbf{b}$ is the body force vector and ρ denotes the mass density. We split the boundary into two parts; on ∂B_u we assume homogenous kinematical boundary conditions, $\mathbf{u} = \mathbf{0}$, and on ∂B_σ surface tractions $\mathbf{t}$ are applied, $\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}$. $\mathbf{n}$ is the unit normal vector of the boundary pointing outwards. The kinematical boundary conditions are such that they ensure no rigid body motion can occur. The initial conditions are taken homogenous throughout this paper; hence, $\mathbf{u} = \mathbf{0}$ and $\dot{\mathbf{u}} = \mathbf{0}$ for $t = 0$ within V; a superposed dot stands for the time derivative d/dt. Fig. 1 shows a sketch of the body under consideration. The actuation stress tensor $\boldsymbol{\sigma}_A$ represents any possible source of self-stress acting in the elastic background body; for early reports see the original papers by Reissner (1931) and Nemenyi (1931). Possible physical phenomena are e.g., thermal expansion strains or piezoelectric strains. A detailed discussion and review on shape control using piezoelectric eigenstrains as the actuation mechanism can be found in Irschik (2002).

To design sensors that are collocated to the actuation we utilize the power theorem of mechanics (e.g. Chadwick 1999), which can be written as

$$\frac{d}{dt}(T + W) = L_E + L_A \tag{3}$$

In Eq. (3) T is the kinetic energy, L_A characterizes the inelastic part of the stress power and is denoted as the power of the actuation in the following, the time derivative of W characterizes the remaining, purely elastic part of the stress power and L_E is the power of the external forces. These latter entities are

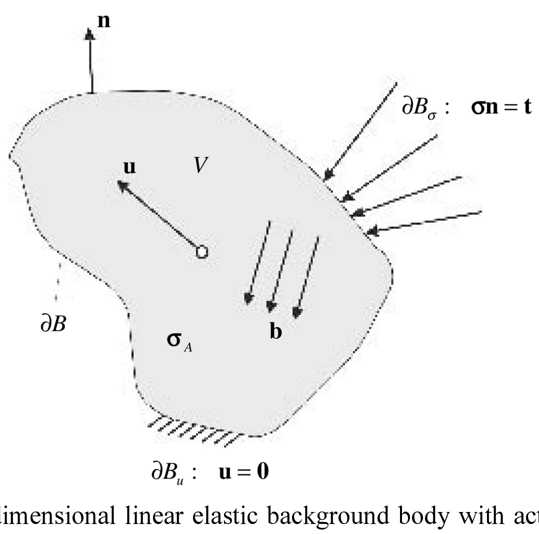

Fig. 1 Three–dimensional linear elastic background body with actuation stresses

defined as:

$$T = \int_V \rho \frac{1}{2} \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} dV,\ W = \int_V \frac{1}{2} \mathbf{C}\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} dV,\ L_E = \int_{\partial B_\sigma} \mathbf{t} \cdot \dot{\mathbf{u}} dS + \int_V \mathbf{b} \cdot \dot{\mathbf{u}} dV,\ L_A = -\int_V \boldsymbol{\sigma}_A : \dot{\boldsymbol{\varepsilon}} dV \tag{4}$$

Collocation between actuation and sensing means that the power of the actuation equals the product of the input signal and the time derivative of the output signal of a system, see Preumont (2002). To introduce this concept we assume the actuation be the sum of n individual actuations; each of them separable in space and time. Hence, the actuation has the form

$$\boldsymbol{\sigma}_A(\mathbf{x},t) = \sum_{i=1}^{n} u_i(t) \overset{s}{\boldsymbol{\sigma}}_{Ai}(\mathbf{x}) \equiv \vec{u}(t)^T \vec{\overset{s}{\boldsymbol{\sigma}}}_A(\mathbf{x}) \tag{5}$$

A matrix notation has been introduced in Eq. (5), in which an entity with an arrow stands for a column matrix with n components; the components can be scalars, vectors or tensors. The representation of the actuation in Eq. (5) is quite general, because in typical practical applications the individual actuations will often have a fixed distribution in space with an arbitrary time variation. In terms of control theory, $\vec{u}(t)$ is the input vector of the system. The output vector is denoted as $\vec{y}(t)$ and it has the same dimension as the input vector. We achieve collocation, if the condition

$$L_A = -\int_V (\vec{u}(t)^T \vec{\overset{s}{\boldsymbol{\sigma}}}_A(\mathbf{x})) : \dot{\boldsymbol{\varepsilon}} dV = -\sum_{i=1}^{n} u_i(t) \underbrace{\int_V \overset{s}{\boldsymbol{\sigma}}_{Ai}(\mathbf{x}) : \dot{\boldsymbol{\varepsilon}} dV}_{-\dot{y}_i(t)} \equiv \vec{u}^T(t) \dot{\vec{y}}(t) \tag{6}$$

is met; hence, from Eq. (6) we find the collocated output vector $\vec{y}(t)$ as

$$\vec{y}(t) = -\int_V \vec{\overset{s}{\boldsymbol{\sigma}}}_A(\mathbf{x}) : \boldsymbol{\varepsilon} dV \quad \text{with:} \quad y_i(t) = -\int_V \overset{s}{\boldsymbol{\sigma}}_{Ai}(\mathbf{x}) : \boldsymbol{\varepsilon} dV \tag{7}$$

Each output signal $y_i(t)$ measures a weighted average of the strain tensor, with a tensor valued weighting function $\overset{s}{\boldsymbol{\sigma}}_{Ai}(\mathbf{x})$ that is identical to the space wise distribution of the actuation. In the literature the sensor signals $y_i(t)$ are also denoted as the natural outputs of the system, see, Nijmeijer and van der Schaft (1991); in the present paper we prefer the notion collocated output vector for $\vec{y}(t)$. We note, as the weighting functions for the actuators and the sensors are identical, we are able to use an actuator as a sensor as well. Such an actuator is denoted as a self-sensing actuator; for the concept of self-sensing see e.g., Dosch and Inmann (1992), Tzou and Hollkamp (1994) and Irschik, *et al.* (2000).

In many practical applications one is not interesting in measuring the weighted average of the total strain, but only of a part of the strain. We introduce an additive decomposition of the displacement vector in the form $\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}$, for which we assume both parts to satisfy the homogenous kinematical boundary conditions as well as the homogenous initial conditions individually. Considering the displacement vector $\mathbf{u}_0$ to be the desired displacement vector of the three dimensional body, we are able to formulate an initial boundary value problem for the relative displacement vector $\Delta\mathbf{u}$, which represents the deviation from the desired displacement vector. This problem reads as

$$div\Delta\boldsymbol{\sigma} + \Delta\mathbf{b} = \rho \frac{d^2}{dt^2} \Delta\mathbf{u},\ \partial B_u{:}\Delta\mathbf{u} = \mathbf{0},\ \partial B_\sigma{:}\Delta\boldsymbol{\sigma}\mathbf{u} = \Delta\mathbf{t},\ t = \mathbf{0},\ \Delta\mathbf{u} = 0,\ \Delta\dot{\mathbf{u}} = \mathbf{0} \tag{8}$$

In Eq. (8) we have introduced some relative entities such as the relative stress tensor $\Delta\boldsymbol{\sigma}$, the relative body force vector $\Delta\mathbf{b}$ and the relative surface traction $\Delta\mathbf{t}$. The definition of these entities is:

$$\Delta\boldsymbol{\sigma} - \boldsymbol{\sigma}_A = \mathbf{C}\Delta\boldsymbol{\varepsilon}, \quad \Delta\mathbf{b} = \mathbf{b} - \rho\frac{d^2}{dt^2}\mathbf{u}_0 + div(\mathbf{C}\boldsymbol{\varepsilon}_0), \quad \Delta\mathbf{t} = \mathbf{t} - (\mathbf{C}\boldsymbol{\varepsilon}_0)\mathbf{n} \tag{9}$$

$\boldsymbol{\varepsilon}_0$ and $\Delta\boldsymbol{\varepsilon}$ denote the symmetric part of the gradient of the corresponding displacement vectors, $\boldsymbol{\varepsilon}_0 = sym(grad\mathbf{u}_0)$ and $\Delta\boldsymbol{\varepsilon} = sym(grad\Delta\mathbf{u})$. As for the original problem we can formulate a "relative power theorem" for the deviation problem as well.

$$\frac{d}{dt}(\Delta T + \Delta W) = \Delta L_E + \Delta L_A$$

$$\Delta T = \int_V \rho\frac{1}{2}\Delta\dot{\mathbf{u}} \cdot \Delta\dot{\mathbf{u}}dV, \quad \Delta W = \int_V \frac{1}{2}\mathbf{C}\Delta\boldsymbol{\varepsilon}:\Delta\boldsymbol{\varepsilon}dV$$

$$\Delta L_E = \int_{\partial B_\sigma} \Delta\mathbf{t} \cdot \Delta\dot{\mathbf{u}}dS + \int_V \Delta\mathbf{b} \cdot \Delta\dot{\mathbf{u}}dV, \; \Delta L_A = -\int_V \boldsymbol{\sigma}_A:\Delta\dot{\boldsymbol{\varepsilon}}dV \tag{10}$$

We note that this "relative power theorem can be derived from Eq. (8) by calculating the inner product with the relative displacement vector, integrating over the volume of the body and applying the Gauss integral theorem. Because the derivation of Eq. (10) is based on an additive decomposition of the displacement vector, our formulation of the "relative power theorem" is valid for linear problems only. Nonetheless, it is quite useful for the design of sensors in the linear case we are dealing with in the present paper. For that sake we introduce a concept, we would like to denote as relative collocation. This concept requires the product of the input signal and the time derivative of the output signal to equal the power of the actuation with respect to the relative displacement vector; hence, by virtue of Eq. (5) we have

$$\Delta L_A(t) = -\vec{u}^T(t)\int_V \vec{\boldsymbol{\sigma}}_A^s(\mathbf{x}):\Delta\dot{\boldsymbol{\varepsilon}}dV = \vec{u}^T(t)\Delta\dot{\vec{y}}(t) \tag{11}$$

finding an output vector $\Delta\vec{y}(t)$ as

$$\Delta\vec{y}(t) = -\int_V \vec{\boldsymbol{\sigma}}_A^s(\mathbf{x}):\Delta\boldsymbol{\varepsilon}dV \tag{12}$$

As we have already mentioned $\Delta\vec{y}(t)$ is of importance in applications, in which one is not interested in measuring the weighted average of the total strain, but only of parts of it; especially those parts that stem from a displacement vector representing the deviation of the total displacement vector from a desired one. Introducing the desired output vector as

$$\vec{y}_0(t) = -\int_V \vec{\boldsymbol{\sigma}}_A^s(\mathbf{x}):\boldsymbol{\varepsilon}_0 dV \tag{13}$$

we can relate the collocated output vector $\vec{y}(t)$ to the output vector $\Delta\vec{y}(t)$ as $\vec{y}(t) = \vec{y}_0(t) + \Delta\vec{y}(t)$. In the remainder of the paper we will refer to $\Delta\vec{y}(t)$ as the natural output vector, as it is the natural output vector of the problem given in Eq. (8).

The control problem, which we are dealing with, is the following. We prescribe a desired displacement vector $\mathbf{u}_0(\mathbf{x}, t)$ and we seek for an actuation in the form $\boldsymbol{\sigma}_A(\mathbf{x}, t) = \vec{u}(t)^T \vec{\boldsymbol{\sigma}}_A^s(\mathbf{x})$ that, if applied to the three dimensional body, results in a displacement vector coinciding with the desired one. The deviation of the displacement vector from the desired displacement vector, $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$, then is a direct measure for the deviation of the controlled displacement from the desired one. The problem is that we cannot measure $\Delta\mathbf{u}$ directly, but we are only able to measure a weighted average of the total strain tensor $\boldsymbol{\varepsilon}$ by virtue of the collocated output vector of Eq. (7). From the latter one we can calculate the natural output vector as $\Delta\vec{y}(t) = \vec{y}(t) - \vec{y}_0(t)$, in which the desired output vector $\vec{y}_0(t)$ is known, because $\mathbf{u}_0$ and hence $\boldsymbol{\varepsilon}_0$ is prescribed. How well $\Delta\vec{y}(t)$ measures the deviation of the controlled displacement from the desired one highly depends on the choice of the tensor valued weighting functions $\vec{\boldsymbol{\sigma}}^s_A(\mathbf{x})$.

3. Mechanical interpretation of natural output

In this section we discuss different types of sensors, which allow us to assign a different mechanical interpretation to the natural output vector other than a weighted average of the relative strain tensor. We derive these sensors by a proper choice of $\vec{\boldsymbol{\sigma}}^s_A(\mathbf{x})$. As we are only interested in the mechanical interpretation of the components of the natural output vector, we are only looking at a single natural output; hence, we ask for (we omit using an index)

$$\Delta y(t) = -\int_V \boldsymbol{\sigma}^s_A(\mathbf{x}):\Delta\boldsymbol{\varepsilon}\,dV \tag{14}$$

to have a desired mechanical meaning. As a starting point, we use the principle of virtual work in a formulation slightly extending the one that can be found in Malvern (1969). For that sake we study an auxiliary quasi-static problem. We apply static body forces $\mathbf{b}^{(aux)}$ to a body with the same dimension as the original one. At the boundary ∂B_σ an auxiliary traction $\mathbf{t}^{(aux)}$ is applied. The boundary ∂B_u is split into two parts, $\partial B_u = \partial B_{u1} + \partial B_{u2}$. At ∂B_{u2} we assume the auxiliary displacement vector to vanish, $\mathbf{u}^{(aux)} = 0$, whereas at ∂B_{u1} we allow auxiliary tractions $\mathbf{t}^{(aux)}$ to be present. The homogenous kinematical boundary conditions at ∂B_{u2} must ensure no rigid body motion can occur in the auxiliary problem. The principle of virtual work for the auxiliary problem states that the sum of the virtual work of the auxiliary body forces $\mathbf{b}^{(aux)}$ and tractions $\mathbf{t}^{(aux)}$ with respect to any kinematically admissible displacement vector $\delta\mathbf{u}$ and of the virtual work of any statically admissible stress tensor $\boldsymbol{\sigma}^{(aux)}$ with respect to the strain tensor $\delta\boldsymbol{\varepsilon} = sym(grad\,\delta\mathbf{u})$ associated with the kinematically admissible displacement vector $\delta\mathbf{u}$ vanishes. Hence,

$$\int_V \boldsymbol{\sigma}^{(aux)}:sym(grad\,\delta\mathbf{u})\,dV = \int_V \mathbf{b}^{(aux)}\cdot\delta\mathbf{u}\,dV + \int_{\partial B_\sigma} \mathbf{t}^{(aux)}\cdot\delta\mathbf{u}\,dS + \int_{\partial B_{u1}} \mathbf{t}^{(aux)}\cdot\delta\mathbf{u}\,dS \tag{15}$$

The kinematically admissible displacement vector $\delta\mathbf{u}$ has to satisfy the kinematical boundary conditions at ∂B_{u2} as well as some smoothness conditions we are not mentioning here. The statically admissible stress tensor has to satisfy the dynamical boundary conditions at ∂B_σ and at ∂B_{u1} as well as the equilibrium condition $div\boldsymbol{\sigma}^{(aux)} + \mathbf{b}^{(aux)} = \mathbf{0}$.

In order to relate the auxiliary problem to the original problem, we use the relative displacement vector $\Delta\mathbf{u}$ as the kinematically admissible displacement vector, $\delta\mathbf{u} = \Delta\mathbf{u}$. This is admissible, as $\Delta\mathbf{u}$ satisfies the homogenous kinematical boundary conditions at ∂B_{u2}. Moreover, $\Delta\mathbf{u}$ vanishes at ∂B_{u1} due to the boundary conditions in the original problem. Therefore, the last term in Eq. (15) is zero. We compare the principle of virtual work with the natural output of Eq. (14),

$$\int_V \boldsymbol{\sigma}^{(aux)}:\Delta\boldsymbol{\varepsilon}\,dV = \int_V \mathbf{b}^{(aux)}\cdot\delta\mathbf{u}\,dV + \int_{\partial B_\sigma} \mathbf{t}^{(aux)}\cdot\delta\mathbf{u}\,dS \leftrightarrow \Delta y(t) = -\int_V \boldsymbol{\sigma}^s_A(\mathbf{x}):\Delta\boldsymbol{\varepsilon}\,dV \tag{16}$$

and conclude that the natural output is identical to the negative work of an auxiliary static system of body forces $\mathbf{b}^{(aux)}$ and tractions $\mathbf{t}^{(aux)}$ with respect to the relative displacement, if the tensor valued distribution of the sensor is chosen to coincide with any statically admissible stress tensor for the latter static system of forces, $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}^{(aux)}$. Hence, such sensors measure a weighted average of the displacement vector.

We proceed by decomposing the relative displacement vector into two parts, $\Delta\mathbf{u} = \Delta\hat{\mathbf{u}} + \Delta\tilde{\mathbf{u}}$, each of them satisfying the homogenous kinematical boundary conditions at δB_u. Then, the natural output is also an additive decomposition, $\Delta y(t) = \Delta\hat{y}(t) + \Delta\tilde{y}(t)$, with the two parts defined as:

$$\Delta\hat{y}(t) = -\int_V \boldsymbol{\sigma}^{(aux)}:\Delta\hat{\boldsymbol{\varepsilon}}dV = -\int_{\partial B_\sigma} \mathbf{t}^{(aux)}\cdot\Delta\hat{\mathbf{u}}dS - \int_V \mathbf{b}^{(aux)}\cdot\Delta\hat{\mathbf{u}}dV$$

$$\Delta\tilde{y}(t) = -\int_V \boldsymbol{\sigma}^{(aux)}:\Delta\tilde{\boldsymbol{\varepsilon}}dV = -\int_{\partial B_\sigma} \mathbf{t}^{(aux)}\cdot\Delta\tilde{\mathbf{u}}dS - \int_V \mathbf{b}^{(aux)}\cdot\Delta\tilde{\mathbf{u}}dV \tag{17}$$

We now ask for a design of the natural output such that $\Delta y(t) = \Delta\hat{y}(t) + \Delta\tilde{y}(t) = \Delta\tilde{y}(t)$ holds. For that sake we need to find a solution to the equation

$$\Delta\hat{y}(t) = -\int_V \boldsymbol{\sigma}^{(aux)}:\Delta\hat{\boldsymbol{\varepsilon}}dV = -\int_{\partial B_\sigma} \mathbf{t}^{(aux)}\cdot\Delta\hat{\mathbf{u}}dS - \int_V \mathbf{b}^{(aux)}\cdot\Delta\hat{\mathbf{u}}dV = 0 \tag{18}$$

for which $\Delta\tilde{y}(t)$ has a desired mechanical interpretation. Eq. (18) is an orthogonality relation between a statically admissible stress tensor and a kinematically admissible strain tensor. In the following we discuss some practically relevant solutions to Eq. (18) and we discuss the mechanical meaning of the remaining natural output $\Delta\tilde{y}(t)$. The following discussion extends results presented in Krommer *et al.* (2005) for sensors measuring the weighted average of the total strain tensor to the present case concerned with the relative strain tensor.

3.1. Nilpotent sensors

The most obvious way to find a solution to Eq. (18) is choosing the auxiliary body force $\mathbf{b}^{(aux)}$ and the auxiliary traction $\mathbf{t}^{(aux)}$ at ∂B_σ to be zero. Then $\Delta\hat{y}(t) = 0$ is satisfied trivially. Moreover, $\Delta\tilde{y}(t)$ vanishes as well. Such sensors are called nilpotent sensors, as their output is trivial. In the context of beam theory a first exemplary result was presented by Miu (1992). Irschik, *et al.* (1999a) presented a general method to find sensor distributions with a vanishing output and introduced the notion of nilpotent sensors and actuators for thin beams in general. For a three-dimensional formulation see Irschik and Pichler (2005). The interesting aspect is that, although the nilpotent sensor measures a trivial signal, its space wise distribution characterized by $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(nilpotent)}$ is not trivial. It is a solution of the problem

$$div\boldsymbol{\sigma}_A^{(nilpotent)} = \mathbf{0},\ \partial B_\sigma:\boldsymbol{\sigma}_A^{(nilpotent)}\mathbf{n} = \mathbf{0},\ \ \partial B_{u1}:\boldsymbol{\sigma}_A^{(nilpotent)}\mathbf{n} = \mathbf{t}^{(aux)} \tag{19}$$

for which $\mathbf{t}^{(aux)}$ at ∂B_{u1} can be chosen arbitrary. Stress distributions satisfying Eq. (19) coincide with stress distributions in the original problem that are due to applied eigenstrains in the absence of any load stresses, such as body force and surface traction. Sensor distributions $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(nilpotent)}$ can be added to any other sensor distribution without changing the latter ones sensor signal; hence, this additional degree of freedom in the sensor design can be used, e.g., for the optimization of a sensor.

3.2. Modal sensors

A different solution to Eq. (18) is based on the orthogonality of eigenfunctions. We consider the auxiliary traction $\mathbf{t}^{(aux)}$ at ∂B_σ to be zero and we take the auxiliary body force vector $\mathbf{b}^{(aux)}$ to be proportional to the n-th vector valued eigenfunction; hence, $\mathbf{b}^{(aux)} = -K_n\mathbf{U}_n(\mathbf{x})$. By expanding the relative displacement into the eigenfunctions,

$$\Delta\mathbf{u} = \sum_{i=1}^{\infty}\mathbf{U}_i(\mathbf{x})\Delta A_i(t) = \underbrace{\sum_{(i\neq n)\ i=1}^{\infty}\mathbf{U}_i(\mathbf{x})\Delta A_i(t)}_{\Delta\hat{\mathbf{u}}} + \underbrace{\mathbf{U}_n(\mathbf{x})\ \Delta A_n(t)}_{\Delta\tilde{\mathbf{u}}} \tag{20}$$

in which we identify the series term $\mathbf{U}_n(\mathbf{x})\Delta A_n(t)$ as the part $\Delta\tilde{\mathbf{u}}$ of the relative displacement, we find the two parts of natural output in the form

$$\Delta\tilde{y}(t) = \int_V K_n\mathbf{U}_n(\mathbf{x})\cdot\Delta\tilde{\mathbf{u}}dV = \int_V K_n\mathbf{U}_n(\mathbf{x})\cdot\mathbf{U}_n(\mathbf{x})\Delta A_n(t)dV = K_n\Delta A_n(t)$$

$$\Delta\hat{y}(t) = \int_V K_n\mathbf{U}_n(\mathbf{x})\cdot\Delta\hat{\mathbf{u}}dV = \int_V K_n\mathbf{U}_n(\mathbf{x})\cdot\sum_{(i\neq n)\ i=1}^{\infty}\mathbf{U}_i(\mathbf{x})\Delta A_n(t)dV = 0 \tag{21}$$

Eq. (18) is satisfied and the remaining part of the natural output measures a signal proportional the n-th modal amplitude $\Delta A_n(t)$ of the relative displacement, from which the name modal sensor stems. Modal sensors and actuators were introduced in a fundamental contribution by Lee and Moon (1990). The sensor distribution for the modal sensor $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(modal)}$ is any solution of

$$div\boldsymbol{\sigma}_A^{(modal)} = -K_n\mathbf{U}_n,\ \partial B_\sigma{:}\boldsymbol{\sigma}_A^{(modal)}\mathbf{n} = \mathbf{0},\ \partial B_{u1}{:}\boldsymbol{\sigma}_A^{(modal)}\mathbf{n} = \mathbf{t}^{(aux)} \tag{22}$$

We find a simple solution to Eq. (22) by solving an eigenvalue problem of the form

$$div\underbrace{[\mathbf{C}sym\ (grad\mathbf{U}_n)]}_{\boldsymbol{\sigma}_A^{(modal)}} = \underbrace{-\rho\omega_n^2}_{K_n}\mathbf{U}_n,\ \ \partial B_\sigma{:}\underbrace{[\mathbf{C}sym\ (grad\mathbf{U}_n)]}_{\boldsymbol{\sigma}_A^{(modal)}}\mathbf{n}{=}\mathbf{0},\ \ \partial B_u{:}\mathbf{U}_n = \mathbf{0} \tag{23}$$

The sensor distribution of the modal sensor then is the superposition of the modal stress tensor and any nilpotent sensor distribution, $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \mathbf{C}sym(grad\mathbf{U}_n) + \boldsymbol{\sigma}_A^{(nilpotent)}$.

3.3. Dilatational and rotational sensors

By virtue of the Helmholtz decomposition theorem any vector can be decomposed into the gradient of a scalar valued function and the curl of vector valued function. Hence, we take the body force vector in the auxiliary problem as $\mathbf{b}^{(aux)} = grad\phi + curl\boldsymbol{\Phi}$, see Gurtin (1972). Accordingly, we consider the auxiliary traction vector at ∂B_σ in the form $\mathbf{t}^{(aux)} = -\phi\mathbf{n} - \boldsymbol{\Phi}\times\mathbf{n}$. Decomposing the relative displacement as well, $\Delta\mathbf{u} = grad\varphi + curl\boldsymbol{\Gamma}$, the natural output becomes

$$\Delta y(t) = \int_{\partial B_\sigma}(\phi\mathbf{n} + \boldsymbol{\Phi}\times\mathbf{n})\cdot(grad\varphi + curl\boldsymbol{\Gamma})dS - \int_V(grad\phi + curl\boldsymbol{\Phi})\cdot(grad\varphi + curl\boldsymbol{\Gamma})dV \tag{24}$$

Eq. (24) can be reformulated by virtue of the Gauss integral theorem, keeping in mind that the divergence of the curl of a vector as well as the curl of the gradient of a scalar vanish identically. Hence, Eq. (24) is easily decomposed into two parts:

$$\Delta y(t) = \int_V \phi\, div\, grad\, \varphi\, dV + \int_V \mathbf{\Phi} \cdot curl\, curl\, \mathbf{\Gamma}\, dV \tag{25}$$

We introduce a dilatational sensor as a sensor that measures the weighted average of the relative dilatation of the body, $\Delta e = div\Delta\mathbf{u} = div(grad\varphi + curl\mathbf{\Gamma} = divgrad\varphi)$. We find the sensor distribution of a dilatational sensor by taking the auxiliary body force vector as $\mathbf{b}^{(aux)} = grad\phi$ and the auxiliary traction vector at ∂B_σ as $\mathbf{t}^{(aux)} = -\phi\mathbf{n}$. Furthermore, we identify the first part of the natural output in Eq. (25) as $\Delta\tilde{y}(t)$ and the second one as $\Delta\hat{y}(t)$. Eq. (18) is satisfied and the remaining part of the natural output is the desired weighted average of the relative dilatation,

$$\Delta\tilde{y}(t) = \int_V \phi \Delta e\, dV \tag{26}$$

The distribution of the dilatational sensor $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(dilatational)} + \boldsymbol{\sigma}_A^{(nilpotent)}$ is the superposition of any nilpotent sensor distribution and a spherical stress tensor, because the equation

$$\begin{aligned} div\boldsymbol{\sigma}_A^{(dilatational)} + grad\phi &= div(\boldsymbol{\sigma}_A^{(dilatational)} + \mathbf{I}\phi) = \mathbf{0} \\ \partial B_\sigma : \boldsymbol{\sigma}_A^{(dilatational)}\mathbf{n} + \phi\mathbf{n} &= (\boldsymbol{\sigma}_A^{(dilatational)} + \mathbf{I}\phi)\mathbf{n} = \mathbf{0} \end{aligned} \tag{27}$$

has the solution $(\boldsymbol{\sigma}_A^{(dilatational)} = -\mathbf{I}\phi)$. If $\phi = 1$, the dilatational sensor measures the relative volume change of the body; for the measurement of the volume change of a closed cavity inside a three-dimensional body see Irschik and Pichler (2005).

The idea of a rotational sensor is similar to the one of a dilatational sensor, but the purpose is to measure a weighted average of the relative rotation vector of the body, $2\Delta\boldsymbol{\omega} = curl\Delta\mathbf{u} = curl(grad\varphi + curl\mathbf{\Gamma}) = curlcurl\mathbf{\Gamma}$. In this case the auxiliary body force vector is taken as $\mathbf{b}^{(aux)} = curl\mathbf{\Phi}$ with the boundary condition $\mathbf{t}^{(aux)} = -\mathbf{\Phi} \times \mathbf{n}$ at ∂B_σ. The second term of the natural output in Eq. (25) is identified as $\Delta\tilde{y}(t)$, whereas the first one is $\Delta\hat{y}(t)$. Eq. (18) is satisfied and the remaining part of the natural output is the desired weighted average of the relative rotation vector,

$$\Delta\tilde{y}(t) = 2\int_V \mathbf{\Phi} \cdot \Delta\boldsymbol{\omega}\, dV \tag{28}$$

The distribution of the rotational sensor is $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(rotational)} + \boldsymbol{\sigma}_A^{(nilpotent)}$, with $\boldsymbol{\sigma}_A^{(rotational)}$ skew symmetric, because it is a solution of

$$\begin{aligned} div\boldsymbol{\sigma}_A^{(rotational)} + curl\mathbf{\Phi} &= div(\boldsymbol{\sigma}_A^{(rotational)} + \mathbf{\Phi}^{skew}) = \mathbf{0} \\ \partial B_\sigma : \boldsymbol{\sigma}_A^{(rotational)}\mathbf{n} + \mathbf{\Phi} \times \mathbf{n} &= (\boldsymbol{\sigma}_A^{(rotational)} + \mathbf{\Phi}^{skew})\mathbf{n} = \mathbf{0} \end{aligned} \tag{29}$$

Hence, the solution is $\boldsymbol{\sigma}_A^{(rotational)} = -\mathbf{\Phi}^{skew}$, in which $\mathbf{\Phi}^{skew}$ is the skew symmetric tensor, to which the vector $\mathbf{\Phi}$ is the axial vector.

3.4. Displacement type sensors

Displacement type sensors focus directly on e.g., the measurement of the displacement at specific locations in a specified direction, the mean value of the displacement in a specified direction or higher order static moments of the displacement vector. Such sensors were extensively discussed in the literature for structural members, see e.g., Irschik, *et al.* (1999b).

In order to relate such sensors to our general methodology for the sensor design, see Eqs. (17) and (18), we first study a displacement type sensor that measures the displacement at an arbitrary location $\mathbf{x} = \bar{\mathbf{x}}$ in the direction $\mathbf{e}_n$. For that sake we introduce any displacement vector that satisfies the kinematical boundary conditions and whose component in the $\mathbf{e}_n$ - direction vanishes at $\mathbf{x} = \bar{\mathbf{x}}$; the latter displacement vector we denote as $\Delta\hat{\mathbf{u}}$. The relative displacement becomes $\Delta\mathbf{u} = \Delta\tilde{\mathbf{u}} + \Delta\hat{\mathbf{u}}$ and $\Delta\mathbf{u}(\bar{\mathbf{x}}, t) \cdot \mathbf{e}_n = \Delta\tilde{\mathbf{u}}(\bar{\mathbf{x}}, t) \cdot \mathbf{e}_n$ holds. We consider the auxiliary traction vector to vanish at ∂B_σ and we assume the auxiliary body force vector to be a single force applied at $\mathbf{x} = \bar{\mathbf{x}}$ in the direction $-\mathbf{e}_n$, $\mathbf{b}^{(aux)} = -1\delta(\mathbf{x} - \bar{\mathbf{x}})\mathbf{e}_n$. Eq. (18) is satisfied and the remaining part of the natural output is

$$\Delta\tilde{y}(t) = \int_V 1\delta(\mathbf{x} - \bar{\mathbf{x}})\mathbf{e}_n \cdot \Delta\tilde{\mathbf{u}}dV = 1\Delta\tilde{\mathbf{u}}(\bar{\mathbf{x}}) \cdot \mathbf{e}_n = 1\Delta u_n(\bar{\mathbf{x}}) \tag{30}$$

This sensor measures the relative displacement $\Delta u_n(\bar{\mathbf{x}})$ at $\mathbf{x} = \bar{\mathbf{x}}$ in the direction $\mathbf{e}_n$. We like to mention that we are able to design sensors that measure displacements directly by using sensors that are sensors that measure strain by their very nature. This is highly important, as we are able to use distributed sensors that can be integrated into the structure for that purpose. The result appears to be preferable to classical methods for the measurement of displacements.

A different sensor is obtained, if we consider $\Delta\hat{\mathbf{u}}$ be any displacement vector satisfying the kinematical boundary conditions, for which the mean value of its component in the $\mathbf{e}_n$ - direction vanishes. The mean value of the relative displacement vector in this direction then is the one of $\Delta\tilde{\mathbf{u}}$. To measure the latter one we consider no traction be applied at ∂B_σ and the body force vector be proportional to the unit vector $\mathbf{e}_n$, $\mathbf{b}^{(aux)} = -1/V\mathbf{e}_n$. Then we have the two parts of the natural output as,

$$\Delta\hat{y}(t) = \frac{1}{V}\int_V \mathbf{e}_n \cdot \Delta\hat{\mathbf{u}}dV = \mathbf{0},\ \Delta\tilde{y}(t) = \frac{1}{V}\int_V \mathbf{e}_n \cdot \Delta\tilde{\mathbf{u}}dV = 1\Delta\bar{u}_n \tag{31}$$

The non – vanishing part of the natural output measures the mean value $\Delta\bar{u}_n$ of the relative displacement vector in the $\mathbf{e}_n$ - direction.

In the above two examples for displacement type sensors we have studied auxiliary body forces that were either a single force or a constant force throughout the bodies volume, the latter one representing a parallel force field with constant intensity. It is near at hand to consider a central force field in the auxiliary problem as well. Hence we assume $\mathbf{b}^{(aux)} = 1\mathbf{x}$ and $\mathbf{t}^{(aux)} = \mathbf{0}$ at ∂B_σ, in which $\mathbf{x}$ is the position vector from an arbitrary reference point 0 to a point of the body and 1 ensures a correct dimension (as it did in the other examples). The natural output is

$$\Delta y(t) = \int_V \mathbf{x} \cdot 1\Delta\mathbf{u}dV = V_0 \tag{32}$$

the so-called Virial V_0 of the force field $1\Delta\mathbf{u}$ with respect to the reference point 0.

We conclude the discussion on displacement type sensor by noting that their distribution can be calculated from

$$div\boldsymbol{\sigma}^{(aux)} + \mathbf{b}^{(aux)} = \mathbf{0},\ \partial B_\sigma{:}\boldsymbol{\sigma}^{(aux)}\mathbf{n} = \mathbf{t}^{(aux)} \tag{33}$$

as the superposition of a nilpotent distribution and any statically admissible stress tensor, $\boldsymbol{\sigma}_A^s(\mathbf{x}) =$

$\boldsymbol{\sigma}_A^{(aux)} + \boldsymbol{\sigma}_A^{(nilpotent)}$. Depending on the choice of the auxiliary body force vector and of the auxiliary traction vector, a large amount of information about the deviation of the displacement vector $\Delta\mathbf{u}$ from a desired displacement vector $\mathbf{u}_0$ can be extracted by the above methodology. Furthermore, we note that superposing nilpotent sensor distributions may be used for minimizing the required amount of sensor intensity.

4. Mechanical interpretation of collocated actuator

In section 2 we have introduced the collocated output vector $\vec{y}(t)$. Each component of this vector (we omit using an index)

$$y(t) = -\int_V \boldsymbol{\sigma}_A^s(\mathbf{x}):\boldsymbol{\varepsilon}dV \tag{34}$$

is collocated to an actuator with the space wise distribution $\boldsymbol{\sigma}_A^s(\mathbf{x})$ and the time variation $u(t)$. We ask for a mechanical interpretation of this actuation. Therefore, we study the original problem with an applied actuation, but with no body force and no traction; we identify the latter purely actuation problem with the superscript A. Next, we consider the same body with no actuation applied, but with non vanishing body forces and tractions at ∂B_σ. These forces in general need not to be the original forces acting on the body. The force problem is characterized by a superscript f. For both problems the kinematical boundary conditions at ∂B_u are zero as are the initial conditions. The two problems are governed by the following equations:

$$div\boldsymbol{\sigma}^A = \rho\frac{d^2}{dt^2}\mathbf{u}^A, \quad \partial B_\sigma:\boldsymbol{\sigma}^A\mathbf{n} = 0, \quad \partial B_u:\mathbf{u}^A = 0, \quad \boldsymbol{\sigma}^A - \boldsymbol{\sigma}_A = \mathbf{C}\boldsymbol{\varepsilon}^A = \mathbf{C}sym(grad\mathbf{u}^A)$$

$$div\boldsymbol{\sigma}^f + \mathbf{b}^f = \rho\frac{d^2}{dt^2}\mathbf{u}^f, \quad \partial B_\sigma:\boldsymbol{\sigma}^A\mathbf{n} = \mathbf{t}^f, \quad \partial B_u:\mathbf{u}^f = 0, \quad \boldsymbol{\sigma}^f = \mathbf{C}\boldsymbol{\varepsilon}^f = \mathbf{C}sym(grad\mathbf{u}^f) \tag{35}$$

Subtracting the second set of equations from the first, an initial boundary value problem with homogenous initial conditions is obtained for the difference solution $\mathbf{u}^A - \mathbf{u}^f$,

$$div(\mathbf{C}sym(grad(\mathbf{u}^A - \mathbf{u}^f))) + (div\boldsymbol{\sigma}_A - \mathbf{b}^f) = \rho\frac{d^2}{dt^2}(\mathbf{u}^A - \mathbf{u}^f)$$

$$\partial B_\sigma:(\mathbf{C}sym(grad(\mathbf{u}^A - \mathbf{u}^f)))\mathbf{n} = -(\boldsymbol{\sigma}_A\mathbf{n} + \mathbf{t}^f), \; \partial B_u:(\mathbf{u}^A - \mathbf{u}^f) = 0 \tag{36}$$

As the initial boundary value problem of linear elasticity has a unique solution, the proof dating back to Kirchhoff and Neumann (see Chandrasekharaiah and Debnath 1994), we find that the difference solution $\mathbf{u}^A - \mathbf{u}^f$ vanishes, if

$$div\boldsymbol{\sigma}_A = \mathbf{b}^f \text{ and } \partial B_\sigma: \boldsymbol{\sigma}_A\mathbf{n} = -\mathbf{t}^f \tag{37}$$

Hence, we can formulate an extension of the classical body force analogy by Duhamel for static thermoelastic problems to the case of dynamic eigenstrain problems: The displacement vector $\mathbf{u}^A$ of a pure eigenstrain (actuation) dynamic problem with the actuation stress tensor $\boldsymbol{\sigma}_A(\mathbf{x}, t)$ is identical to the displacement vector of a pure force problem, if the divergence of the actuation stress tensor equals the body force vector and, if the negative traction vector at ∂B_σ equals the actuation stress vector $\boldsymbol{\sigma}_A\mathbf{n}$ at ∂B_σ. Furthermore, the stress tensor in the actuation problem $\boldsymbol{\sigma}^A$ equals the stress tensor in the force problem plus the actuation stress tensor in the actuation problem, $\boldsymbol{\sigma}^A = \boldsymbol{\sigma}^f + \boldsymbol{\sigma}_A$, which follows directly from the constitutive relations in the two cases by accounting for $\mathbf{u}^A = \mathbf{u}^f$. For further details on the classical analogy see Noda, *et al.* (2000) and for the presented extension see Irschik *et al.* (2005). With the aid of Eq. (37) we can easily find the mechanical interpretation for actuators, which are collocated to the sensors we introduced in section 3.

The distribution of the nilpotent sensor $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(nilpotent)}$, when used as the space wise distribution of an actuation $\boldsymbol{\sigma}_A(\mathbf{x}, t) = \boldsymbol{\sigma}_A^{(nilpotent)}(\mathbf{x})u(t)$, finds the following set of body forces and tractions from Eq. (37)

$$\mathbf{b}^{f(nilpotent)}(\mathbf{x}, t) = u(t)div(\boldsymbol{\sigma}_A^{(nilpotent)}(\mathbf{x})) = \mathbf{0} \text{ and } \partial B_\sigma{:}\mathbf{t}^{f(nilpotent)} = -\boldsymbol{\sigma}_A^{(nilpotent)}(\mathbf{x})\mathbf{n}u(t) = \mathbf{0} \tag{38}$$

see Eq. (19); hence, $\mathbf{u}^A = \mathbf{u}^f = \mathbf{0}$. A nilpotent actuator does not induce any displacement.

If we use the distribution of a modal sensor $\boldsymbol{\sigma}_A^s(\mathbf{x}) = \boldsymbol{\sigma}_A^{(modal)}$ for an actuator we find from Eqs. (22) and (37) that

$$\mathbf{b}^{f(modal)}(\mathbf{x}, t) = u(t)div(\boldsymbol{\sigma}_A^{(modal)}(\mathbf{x})) = -K_n\mathbf{U}_n u(t) \text{ and } \partial B_\sigma{:}\mathbf{t}^{f(modal)} = -\boldsymbol{\sigma}_A^{(modal)}(\mathbf{x})\mathbf{n}u(t) = \mathbf{0} \tag{39}$$

One can easily show that a body force field, whose space wise distribution is proportional to an arbitrary eigenfunction can only induce a displacement $\mathbf{u}^f = \mathbf{U}_n(x)f(t)$ proportional to that eigenfunction. Therefore, the modal actuator always induces a displacement field proportional to an eigenfunction.

For the dilatational sensor and the rotational sensor we find a body force vector and a traction vector resulting into a displacement vector that is also induced by the collocated actuator from Eqs. (27) and (29):

$$\mathbf{b}^{f(dilatational)}(\mathbf{x}, t) = -grad\phi(\mathbf{x})u(t) \text{ and } \partial B_\sigma{:}\mathbf{t}^{f(dilatational)} = \phi(\mathbf{x})\mathbf{n}u(t)$$

$$\mathbf{b}^{f(rotational)}(\mathbf{x}, t) = -curl\boldsymbol{\Phi}(\mathbf{x})u(t) \text{ and } \partial B_\sigma{:}\mathbf{t}^{f(rotational)} = (\boldsymbol{\Phi}(\mathbf{x}) \times \mathbf{n})u(t) \tag{40}$$

The dilatational actuator is an actuator that induces a displacement field, which coincides with a displacement field induced by an irrotational body force vector field in combination with normal forces applied at the boundary. In contrast, the rotational actuator induces a displacement field coinciding with one induced by a solenoidal body force vector field with only shear forces applied at the boundary.

The mechanical interpretation of actuators that are collocated to displacement type sensors is very simple. An actuator collocated to the sensor defined by Eq. (30) produces a displacement vector that is also produced by a single force applied at location $\mathbf{x} = \bar{\mathbf{x}}$ in the direction $\mathbf{e}_n$, one collocated to the sensor defined in Eq. (31) induces a displacement vector identical to the one of a parallel and homogenous body force field $\mathbf{b}^f = -1/V\mathbf{e}_n u(t)$ and an actuator that is collocated to the sensor of Eq. (32) results into a displacement vector due to a central body force field $\mathbf{b}^f = 1\mathbf{x}u(t)$.

5. Dynamic displacement tracking

In the dynamic displacement tracking problem we seek for the three dimensional body to have a desired displacement vector $\mathbf{u}_0$. The body under consideration is under the action of the body force vector $\mathbf{b}$ and the traction vector $\mathbf{t}$, which is applied at ∂B_σ. The goal is to apply an actuation stress $\boldsymbol{\sigma}_A$ such that the displacement vector of the body $\mathbf{u}$ is the desired one $\mathbf{u}_0$. To find a solution to this problem we start with the natural output vector as introduced in Eq. (12)

$$\Delta\vec{y}(t) = -\int_V \vec{\boldsymbol{\sigma}}_A^s(\mathbf{x}):\Delta\boldsymbol{\varepsilon}dV \tag{41}$$

$\Delta\boldsymbol{\varepsilon} = sym(grad\Delta\mathbf{u})$ is the relative strain tensor with $\Delta\mathbf{u}$ being the deviation of the displacement vector from the desired one. We assume the desired displacement vector, the body force vector and the traction vector be separable in space and time,

$$\mathbf{u}_0(\mathbf{x},t) = \vec{\mathbf{u}}_0^T(\mathbf{x})\vec{f}(t), \quad \mathbf{b}(\mathbf{x},t) = \vec{\mathbf{b}}^T(\mathbf{x})\vec{d}^b(t), \quad \mathbf{t}(\mathbf{x},t) = \vec{\mathbf{t}}^T(\mathbf{x})\vec{d}^t(t) \tag{42}$$

then the relative body force vector $\Delta\mathbf{b}$ and the relative surface traction $\Delta\mathbf{t}$, see Eq. (9) can be written in the general form

$$\Delta\mathbf{b}(\mathbf{x},t) = \Delta\vec{\mathbf{b}}^T(\mathbf{x})\vec{d}(t), \quad \partial B_\sigma : \Delta\mathbf{t}(\mathbf{x},t) = \Delta\vec{\mathbf{t}}^T(\mathbf{x})\vec{d}(t) \tag{43}$$

in which $\Delta\vec{\mathbf{b}}(\mathbf{x})$ and $\Delta\vec{\mathbf{t}}(\mathbf{x})$ are generalized space wise distributions of the disturbances and $\vec{d}(t)$ are their time variations. The definition follows from Eq. (9) as

$$\Delta\vec{\mathbf{b}}^T(\mathbf{x}) = [\Delta\vec{\mathbf{b}}_1^T \;\; \Delta\vec{\mathbf{b}}_2^T \;\; \Delta\vec{\mathbf{b}}_3^T \;\; \Delta\vec{\mathbf{b}}_4^T] = [\vec{\mathbf{b}}^T(\mathbf{x}) \;\; \vec{\mathbf{0}}^T \;\; div(\mathbf{C}sym(grad\vec{\mathbf{u}}_0^T(\mathbf{x}))) - \rho\vec{\mathbf{u}}_0^T(\mathbf{x})]$$

$$\Delta\vec{\mathbf{t}}^T(\mathbf{x}) = [\Delta\vec{\mathbf{t}}_1^T \;\; \Delta\vec{\mathbf{t}}_2^T \;\; \Delta\vec{\mathbf{t}}_3^T \;\; \Delta\vec{\mathbf{t}}_4^T] = [\vec{\mathbf{0}}^T \;\; \vec{\mathbf{t}}^T(\mathbf{x}) \;\; -(\mathbf{C}sym(grad\vec{\mathbf{u}}_0^T(\mathbf{x})))\mathbf{n} \;\; \vec{\mathbf{0}}^T]$$

$$\vec{d}^T(t) = [\vec{d}_1^T \;\; \vec{d}_2^T \;\; \vec{d}_3^T \;\; \vec{d}_4^T] = \left[\vec{d}^{bT}(t) \;\; \vec{d}^{tT}(t) \;\; \vec{f}^T(t) \;\; \ddot{\vec{f}}^T(t)\right] \tag{44}$$

Obviously, we have four sets of disturbances; the first is due to the original body force vector, the second one due to the original traction vector and the last two sets account for the desired displacement vector $\mathbf{u}_0$. We choose the auxiliary forces in Eq. (16) as the space wise distributions of the disturbances $\vec{\mathbf{b}}^{(aux)}(\mathbf{x}) = \Delta\vec{\mathbf{b}}(\mathbf{x})$ and $\vec{\mathbf{t}}^{(aux)}(\mathbf{x}) = \Delta\vec{\mathbf{t}}(\mathbf{x})$; hence, the natural output vector becomes

$$\Delta\vec{y}^T(t) = [\Delta\vec{y}_1^T \;\; \Delta\vec{y}_2^T \;\; \Delta\vec{y}_3^T \;\; \Delta\vec{y}_4^T],$$

$$\Delta\vec{y}_i(t) = -\int_V \vec{\boldsymbol{\sigma}}_{Ai}^s(\mathbf{x}):\Delta\boldsymbol{\varepsilon}dV = -\int_V \Delta\vec{\mathbf{b}}_i(\mathbf{x})\cdot\Delta\mathbf{u}dV - \int_{\partial B_\sigma} \Delta\vec{\mathbf{t}}_i(\mathbf{x})\cdot\Delta\mathbf{u}dS \tag{45}$$

The space wise distributions of the components of the natural output vector are calculated as statically admissible stresses according to Eq. (33):

$$div\vec{\boldsymbol{\sigma}}_{Ai}^s(\mathbf{x}) + \Delta\vec{\mathbf{b}}_i(\mathbf{x}) = \mathbf{0}, \quad \partial B_\sigma : \vec{\boldsymbol{\sigma}}_{Ai}^s(\mathbf{x})\mathbf{n} = \Delta\vec{\mathbf{t}}_i(\mathbf{x}), \quad \partial B_{u1} : \vec{\boldsymbol{\sigma}}_{Ai}^s(\mathbf{x})\mathbf{n} = \vec{\mathbf{t}}_i^{(arbitrary)}(\mathbf{x}) \tag{46}$$

$i = 1,2,3,4$. Due to the arbitrariness of $\vec{\mathbf{t}}_i^{(arbitrary)}$ at ∂B_{u1} these latter stresses are not unique. In Fig. 2 the control scheme for dynamic displacement tracking is shown. The three dimensional elastic background body is under the influence of the body force vector $\mathbf{b}(\mathbf{x},t) = \vec{\mathbf{b}}^T(\mathbf{x})\vec{d}^b(t)$ and the traction

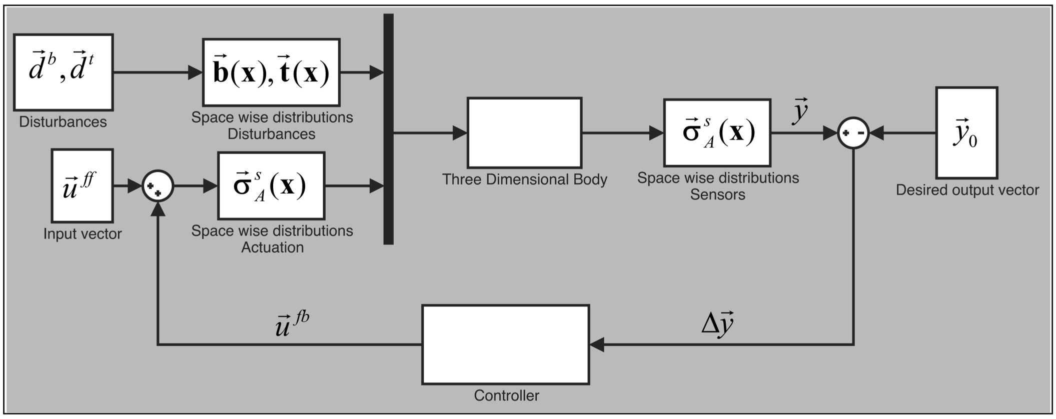

Fig. 2 Control scheme for displacement tracking

vector $\mathbf{t}(\mathbf{x}, t) = \vec{\mathbf{t}}^T(\mathbf{x})\vec{d}^b(t)$. The collocated output vector $\vec{y}$, whose distribution is characterized by $\vec{\boldsymbol{\sigma}}^s_A(\mathbf{x})$, is measured. A desired displacement vector $\mathbf{u}_0(\mathbf{x}, t)$ is prescribed and the desired output vector $\vec{y}_0$ is calculated and subtracted from the collocated output vector resulting into the natural output vector $\Delta\vec{y}$. The natural output vector is fed into a controller, which renders an input vector $\vec{u}^{fb}$ that is applied to the collocated actuators, which are again characterized by the distributions $\vec{\boldsymbol{\sigma}}^s_A(\mathbf{x})$. In addition to this latter feed back strategy a feed forward strategy is considered with the input vector $\vec{u}^{ff}$ applied to the collocated actuators as well. We note that a solution to Eq. (46) requires the space wise distributions of the disturbances be known. For the time being we consider this assumption be true. For the desired displacement vector this is satisfied in general, for the external forces it may not. Once we have calculated $\vec{\boldsymbol{\sigma}}^{sT}_A(\mathbf{x}) = [\vec{\boldsymbol{\sigma}}^{sT}_{A1}\ \vec{\boldsymbol{\sigma}}^{sT}_{A2}\ \vec{\boldsymbol{\sigma}}^{sT}_{A3}\ \vec{\boldsymbol{\sigma}}^{sT}_{A4}]$ we can use the collocated actuators, $\boldsymbol{\sigma}_{Ai}(\mathbf{x}, t) = (\vec{u}^{ffT}(t) + \vec{u}^{fb\,T}(t))\,\vec{\boldsymbol{\sigma}}^s_{Ai}(\mathbf{x})$, $i = 1,2,3,4$, to control the displacement of the body. From the relative power theorem, see Eq. (10) we find the relative power of the collocated actuation as well as of the external forces as

$$\Delta L_A = -(\vec{u}^{ffT}(t) + \vec{u}^{fb\,T}(t))\Big[\int_{\partial B_\sigma} \Delta\vec{\mathbf{t}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dS + \int_V \Delta\vec{\mathbf{b}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dV\Big]$$

$$\Delta L_E = \vec{d}^T(t)\Big[\int_{\partial B_\sigma} \Delta\vec{\mathbf{t}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dS + \int_V \Delta\vec{\mathbf{b}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dV\Big] \tag{47}$$

and the relative power theorem itself in the form

$$\frac{d}{dt}(\Delta T + \Delta W) = (\vec{d}^T - \vec{u}^{ffT}(t) - \vec{u}^{fb\,T}(t))\Big[\int_{\partial B_\sigma} \Delta\vec{\mathbf{t}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dS + \int_V \Delta\vec{\mathbf{b}}(\mathbf{x}) \cdot \Delta\dot{\mathbf{u}}\,dV\Big]$$

$$= (\vec{u}^{ffT}(t) + \vec{u}^{fb\,T}(t) - \vec{d}^T)\Delta\dot{\vec{y}}(t) \tag{48}$$

We consider the feed back part of the input vector to be provided by a PD controller, $\vec{u}^{fb}(t) = -\overleftrightarrow{P}\Delta\vec{y}(t) - \overleftrightarrow{D}\Delta\dot{\vec{y}}(t)$ where $\overleftrightarrow{P}$ and $\overleftrightarrow{D}$ are positive (semi-) definite matrices. Then Eq. (48) can be written as

$$\frac{d}{dt}\Big(\Delta T + \Delta W + \frac{1}{2}\Delta\vec{y}^T\,\overleftrightarrow{P}^T\Delta\vec{y}\Big) = -\Delta\dot{\vec{y}}^T\,\overleftrightarrow{D}^T\Delta\dot{\vec{y}} + (\vec{u}^{ffT}(t) - \vec{d}^T(t))\Delta\dot{\vec{y}}(t) \tag{49}$$

From Eq. (49) we conclude that, if $\vec{u}^{ff}(t) = \vec{d}(t)$ is satisfied, the relative displacement vector $\Delta\mathbf{u}$ vanishes in every point of the body and for all instances of time. The total displacement vector is identical to the desired one, $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}, t)$, and the goal of dynamic displacement tracking is reached exactly. This conclusion follows from the fact that the left hand side of Eq. (49) is the time derivative of a positive (semi-) definite function and it equals a negative (semi-) definite function. Hence, the bracketed quadratic form on the left hand side cannot be negative nor increase. As it is zero for the initial time due to the initial conditions for the relative displacement vector $\Delta\mathbf{u}$ it remains zero for all times and $\Delta\mathbf{u} = \mathbf{0}$ is satisfied. In this feed forward strategy sensors are not needed, other than for monitoring purposes.

Sensors come into the play, if $\vec{u}^{ff}(t) = \vec{d}^{t}$ cannot be satisfied. Then the error system needs to be stabilized. We find such a situation, whenever the time variation of the body force vector and the traction vector are not known, but their space wise distribution is. Then we will not use $\vec{u}_1^{ff}(t)$ and $\vec{u}_2^{ff}(t)$ as $\vec{d}_1(t)$ and $\vec{d}_2(t)$ are unknown. To proof stability of the free closed loop system, which is the system with $\vec{u}^{ff}(t) = \vec{d}(t) = \vec{y}_0(t) = \vec{0}$, we consider Eq. (49) again. The last term on the right hand is zero and, because $\vec{y}_0(t) = \vec{0}$, the total displacement vector $\mathbf{u}$ is identical to the relative displacement vector $\Delta\mathbf{u}$. Eq. (49) becomes

$$\frac{d}{dt}\left(T + W + \frac{1}{2}\vec{y}^{T}\overleftrightarrow{P}^{T}\vec{y}\right) = -\dot{\vec{y}}^{T}\overleftrightarrow{D}^{T}\dot{\vec{y}} \tag{50}$$

We note that the time derivative of the total energy of the free closed loop system, which is a positive (semi-) definite function, equals a negative (semi-) definite function. From this latter fact we conclude on the stability in the sense of Liapunov. For engineering applications we consider Eq. (50) to be sufficient for the stability; however, Eq. (50) does not proof the stability, as we are dealing with infinite – dimensional systems. For a discussion of stability of infinite – dimensional systems see Luo, *et al.* (1999) and Kugi (2001).

One problem remains to be discussed. So far we have assumed the space wise distribution of the body force vector and the traction vector to be known. This is necessary for a solution of Eq. (46). If we do not know these distributions, $\vec{\mathbf{b}}(\mathbf{x})$ and $\vec{\mathbf{t}}(\mathbf{x})$ represent test distributions, which we choose from either expectations we have about the real distributions or according to the different types of sensors and actuators we introduced in sections 3 and 4.

6. Example: Displacement tracking of a clamped – clamped beam

In this section we present some numerical results for a beam. We consider a clamped-clamped Bernoulli-Euler beam, which only performs a bending motion $w(x, t)$. The example problem we are using is fully non-dimensional with the following parameters

$$\hat{x} = \frac{x}{L},\quad \hat{t} = tL^2\sqrt{\frac{P}{D}},\quad \hat{w} = \frac{w}{h},\quad \hat{M} = \frac{ML^2}{Dh},\quad \hat{M}_A = \frac{M_A L^2}{Dh},\quad \hat{p}_z = \frac{p_z L^4}{Dh},\quad \hat{\omega} = \frac{\omega}{L^2}\sqrt{\frac{P}{D}} \tag{51}$$

For the sake of brevity we omit the characterization of non-dimensional entities by a hut in the following. The natural frequencies of the beam are

$$\omega_1 = 22.37,\ \omega_2 = 61.67,\ \omega_3 = 120.90 \tag{52}$$

The beam is under the action of a transverse force loading, which is a superposition of a linear, skew symmetric and a constant span-wise distribution, both with arbitrary time variations, $p_z(x, t) = (x-0.5)d_1(t) + 0.5d_2(t)$. The desired deflection we want to track is $w_0(x, t) = 1/2[1-\cos(2\pi x)][\cos t - 1] = w_0(x)f(t)$; hence, the relative transverse force loading becomes

$$\begin{aligned}\Delta p_z(x, t) &= p_z(x, t) - \ddot{w}_0(x, t) - w_0^{IV}(x, t) \\ &= (x-0.5)d_1(t) + 0.5d_2(t) + 8\pi^4[\cos(2\pi x)]\, f(t) - 0.5[1-\cos(2\pi x)]\, \ddot{f}(t) \\ &= \Delta p_{z1}(x)d_1(t) + \Delta p_{z2}(x)d_2(t) + \Delta p_{z3}(x)d_3(t) + \Delta p_{z4}(x)d_4(t)\end{aligned} \tag{53}$$

As we have no dynamical boundary conditions, we need not worry about any boundary conditions for the solution of $M''_{Ai}(x) = \Delta p_{zi}(x)$ to satisfy; hence, we choose the integration constants to be zero. In the present problem we find four span wise distributions for the sensors, which are

$$\begin{aligned}&M_{A1}(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2,\ M_{A2}(x) = \frac{1}{4}x^2 \\ &M_{A3}(x) = -2\pi^2[\cos(2\pi x)],\ M_{A4}(x) = -\frac{1}{2}\left[\frac{1}{2}x^2 + \frac{1}{4\pi^2}\cos(2\pi x)\right]\end{aligned} \tag{54}$$

In problems with dynamical boundary conditions we may have to add additional span wise distributions, see the definition of the generalized space wise distributions of the disturbances in Eq. (44) and the governing equations for the statically admissible stresses in Eq. (46). The integration constants are used to adjust the statically admissible bending moment to the dynamical boundary conditions.

With the aid of the four functions in Eq. (54) we find the collocated output vector, the desired output vector and the natural output vector in the form:

$$y_i(t) = -\int_0^1 M''_{Ai}(x)w(x, t)dx,\ y_{0i}(t) = -\int_0^1 M''_{Ai}(x)w_0(x, t)dx,\ \Delta y_i(t) = y_i(t) - y_{0i}(t) \tag{55}$$

The bending vibrations of the clamped-clamped beam are governed by the following initial boundary value problem

$$\begin{aligned}&w^{IV}(x, t) + \ddot{w}(x, t) = p_z(x, t) - \sum_{i=1}^{4} M_{Ai}(x)u_i(t) \\ &x = 0,\ L:\ w = 0,\ w' = 0;\ t = 0:\ w = 0,\ \dot{w} = 0\end{aligned} \tag{56}$$

For the simulation of the clamped-clamped beam we expand the deflection into the normalized eigenmodes, $w(x, t) = \sum_{i=1}^{N} W_i(x)A_i(t)$. A linear time invariant system of ordinary differential equations of first order can be easily derived as

$$\frac{d}{dt}\vec{x} = \overleftrightarrow{A}\vec{x} + \overleftrightarrow{B}\vec{u} + \overleftrightarrow{E}\vec{d},\quad \vec{y} = \overleftrightarrow{C}\vec{x},\quad \Delta\vec{y} = \vec{y} - \vec{y}_0 \tag{57}$$

In Eq. (57) $\vec{x}$ is the state vector with $2N$ components, N of them accounting for the modal amplitudes and N for the time derivative of the latter ones. $\vec{u} = [u_1\ u_2\ u_3\ u_4]^T$ is the input vector with four components, $\vec{d} = [d_1\ d_2]^T$ the disturbance vector with two components due the actual transverse force loading $p_z(x, t) = (x-0.5)d_1(t) + 0.5d_2(t)$ and $\vec{y} = [y_1\ y_2\ y_3\ y_4]^T$ is the collocated output vector. As we are simulating the original system of a clamped-clamped beam only two disturbances are present. The natural output vector can be directly computed as $\Delta\vec{y} = \vec{y} - \vec{y}_0$, because the desired one is known. The system matrix $\overleftrightarrow{A}(2N \times 2N)$, the input matrix $\overleftrightarrow{B}(2N\times4)$ and the output matrix $\overleftrightarrow{C}(4 \times 2N)$ have the following form

$$\overleftrightarrow{A} = \begin{bmatrix} \overleftrightarrow{0} & \overleftrightarrow{I} \\ -diag\{\lambda_i^4\} & \overleftrightarrow{0} \end{bmatrix}, \quad \overleftrightarrow{B} = \begin{bmatrix} \overleftrightarrow{0} \\ \overleftrightarrow{\bar{B}} \end{bmatrix}, \quad \overleftrightarrow{C} = \begin{bmatrix} \overleftrightarrow{\bar{B}}^T & \overleftrightarrow{0} \end{bmatrix} \tag{58}$$

The relation between $\overleftrightarrow{B}$ and $\overleftrightarrow{C}$ is due the collocated actuator and sensor design. The simple form of $\overleftrightarrow{A}$, in which λ_i are the eigenvalues of the corresponding eigenmode, follows from the non-dimensional formulation we are using. In the following examples we use five bending eigenmodes, $N = 5$, for the simulation within the Matlab/Simulink platform.

6.1. Feed-forward control

In the feed forward control strategy, in which the time variation of the disturbances $\vec{d}(t)$ is known, the input vector is chosen as

$$\vec{u}(t) = \vec{u}^{ff}(t) = \begin{bmatrix} u_1^{ff}(t) \\ u_2^{ff}(t) \\ u_3^{ff}(t) \\ u_4^{ff}(t) \end{bmatrix} = \begin{bmatrix} d_1(t) \\ d_2(t) \\ f(t) \\ \ddot{f}(t) \end{bmatrix} \tag{59}$$

see section 5. As we are not using a feed back strategy the sensors are only used for monitoring purposes. In the numerical simulation we use $d_1(t) = d_2(t) = 300H(t-1)$. Fig. 3 shows a comparison between the results for feed forward control and for the uncontrolled case. The desired outputs, the collocated outputs and the natural outputs are presented. In the controlled case the natural outputs are zero, because the collocated ones are identical the desired outputs. The goal of displacement tracking is reached exactly; not only are the natural outputs zero, but the deflection of the beam coincides exactly with the desired deflection $w_0 = 1/2[1-\cos(2\pi x)][\cos t-1]$. As we have already mentioned, the feed forward strategy requires the disturbances to be known.

6.2. Feed-back control

If the time variations $d_1(t)$ and $d_2(t)$ of the transverse force loading are unknown, but the span wise distribution is known, we use the feed back strategy with the input vector

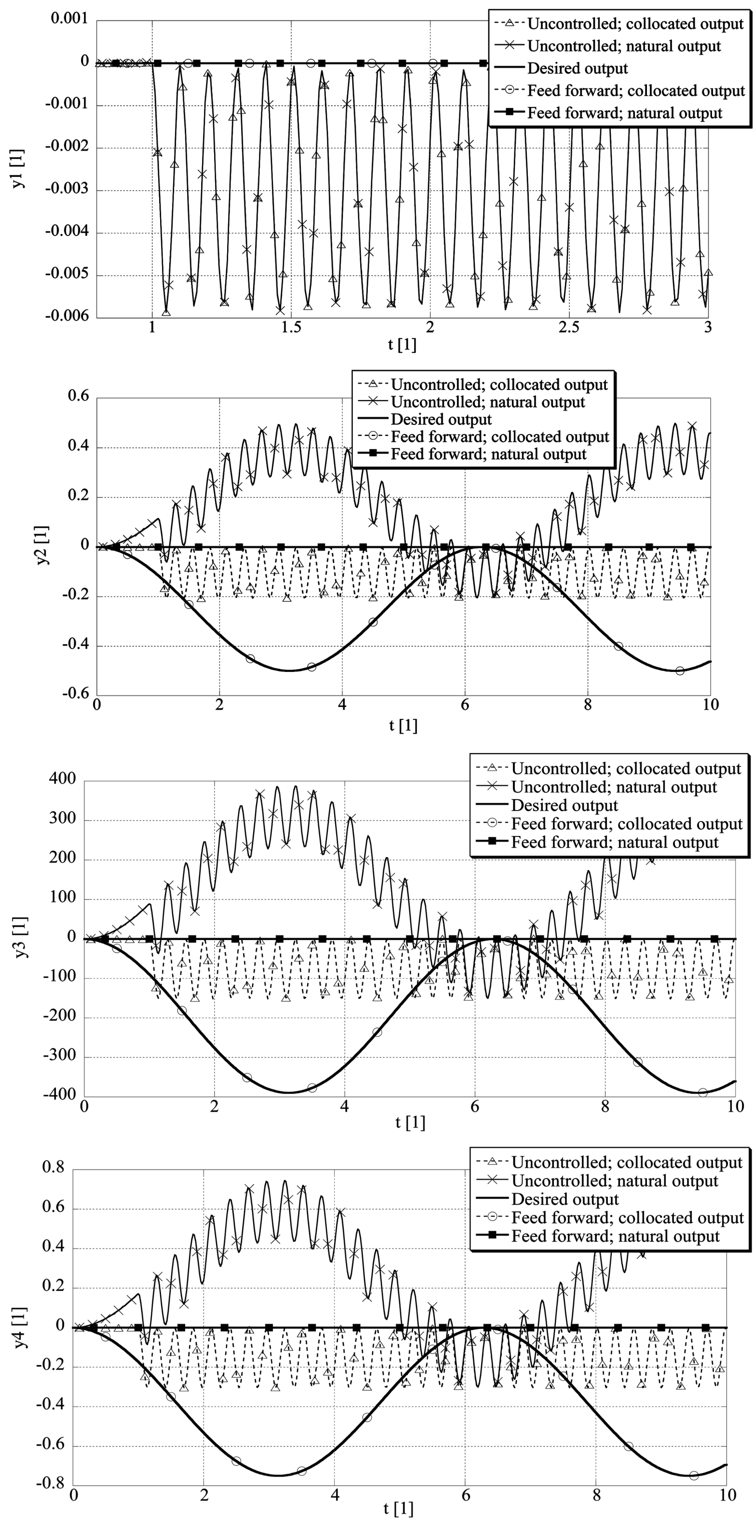

Fig. 3 Output signals for uncontrolled case and feed forward control

$$\vec{u}(t) = \vec{u}^{fb}(t) = \begin{bmatrix} u_1^{fb}(t) \\ u_2^{fb}(t) \\ u_3^{fb}(t) \\ u_4^{fb}(t) \end{bmatrix} = -\begin{bmatrix} P_1\Delta y_1(t) + D_1\Delta\dot{y}_1(t) \\ P_2\Delta y_2(t) + D_2\Delta\dot{y}_2(t) \\ P_3\Delta y_3(t) + D_3\Delta\dot{y}_3(t) \\ P_4\Delta y_4(t) + D_4\Delta\dot{y}_4(t) \end{bmatrix} \tag{60}$$

We chose the parameters of the PD controller as follows:

$$P_i = K_i^{-1}P, \quad D_i = 0.5P_i, \quad K_i = -\int_0^1 M''_{Ai}(x)w_0(x)dx, \quad i = 2, 3, 4 \tag{61}$$

with $P = 5$. For $i = 1$ Eq. (61) cannot be used, because $K_1 = 0$, which follows from the fact that $w_0(x)$ is symmetric and the sensor signal y_{01} only accounts for skew symmetric deformations. Therefore, we use $P_1 = 50K$, $D_1 = 0.1K$, with $K = 2000$; for these parameters we have found good results. We note that the present paper is dedicated to the presentation of a methodology to track displacements; in practical applications, the present controller design needs to be reconsidered. Simulation results for the output signals are shown in Fig. 4 for $d_1(t) = d_2(t) = 300H(t–1)$. For the simulation we have introduced a generalized output vector $\vec{z}$ such that the PD controller can be realized as a negative output feed back:

$$\vec{z} = \begin{bmatrix} \overleftrightarrow{B}^T & \overleftrightarrow{0} \\ \overleftrightarrow{0} & \overleftrightarrow{B}^T \end{bmatrix}\vec{x}, \quad \vec{u} = -\begin{bmatrix} \overleftrightarrow{P} & \overleftrightarrow{0} \\ \overleftrightarrow{0} & \overleftrightarrow{D} \end{bmatrix}\underbrace{(\vec{z} - \vec{z}_0)}_{\Delta\vec{z}} \tag{62}$$

Hence, we are avoiding the calculation of the time derivative of the natural output vector in the feed back control. The results shown in Fig. 4 are no longer as good as in the feed forward case, but we still achieve a significant reduction for the natural outputs, when compared to the uncontrolled case.

To improve the quality of the pure feed back strategy, more sophisticated methods of control theory may be applied. Nonetheless, we note that from a mechanical point of view the design of the actuators and sensors is optimal, as they can ensure an exact solution of displacement tracking for given disturbances. Instead of focusing on an improvement of the controller design, we combine the feed forward and the feed back strategy.

6.3. Combined feed forward and feed back control

For the combined method the input vector is taken in the form

$$\vec{u}(t) = \vec{u}^{ff}(t) + \vec{u}^{fb}(t) = \begin{bmatrix} 0 \\ 0 \\ f(t) \\ \ddot{f}(t) \end{bmatrix} - \begin{bmatrix} P_1\Delta y_1(t) + D_1\Delta\dot{y}_1(t) \\ P_2\Delta y_2(t) + D_2\Delta\dot{y}_2(t) \\ P_3\Delta y_3(t) + D_3\Delta\dot{y}_3(t) \\ P_4\Delta y_4(t) + D_4\Delta\dot{y}_4(t) \end{bmatrix} \tag{63}$$

As the disturbances are unknown, we only use that part of the feed forward input vector that accounts for the deflection to be tracked. With $d_1(t) = d_2(t) = 300H(t–1)$ and the same PD parameters as in the

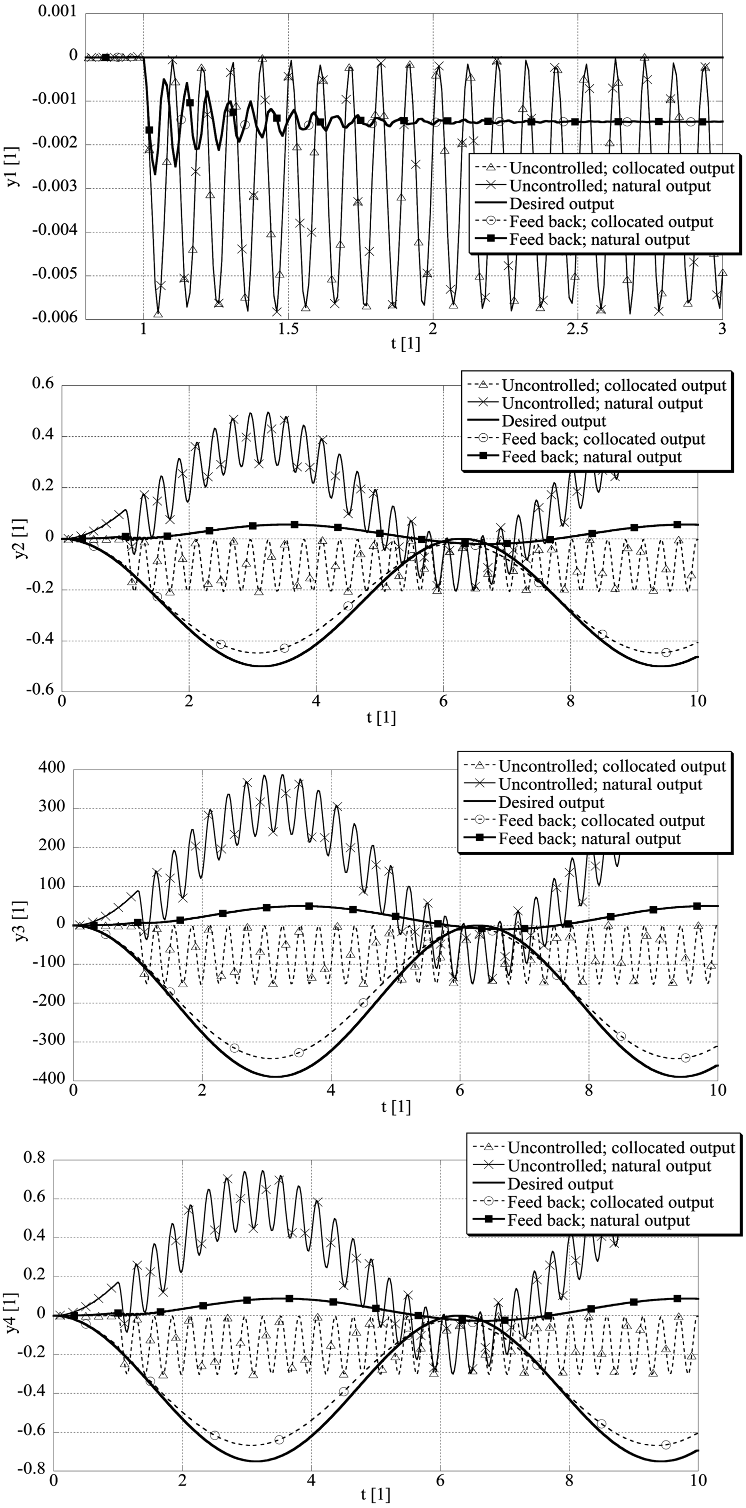

Fig. 4 Output signals for uncontrolled case and feed back control

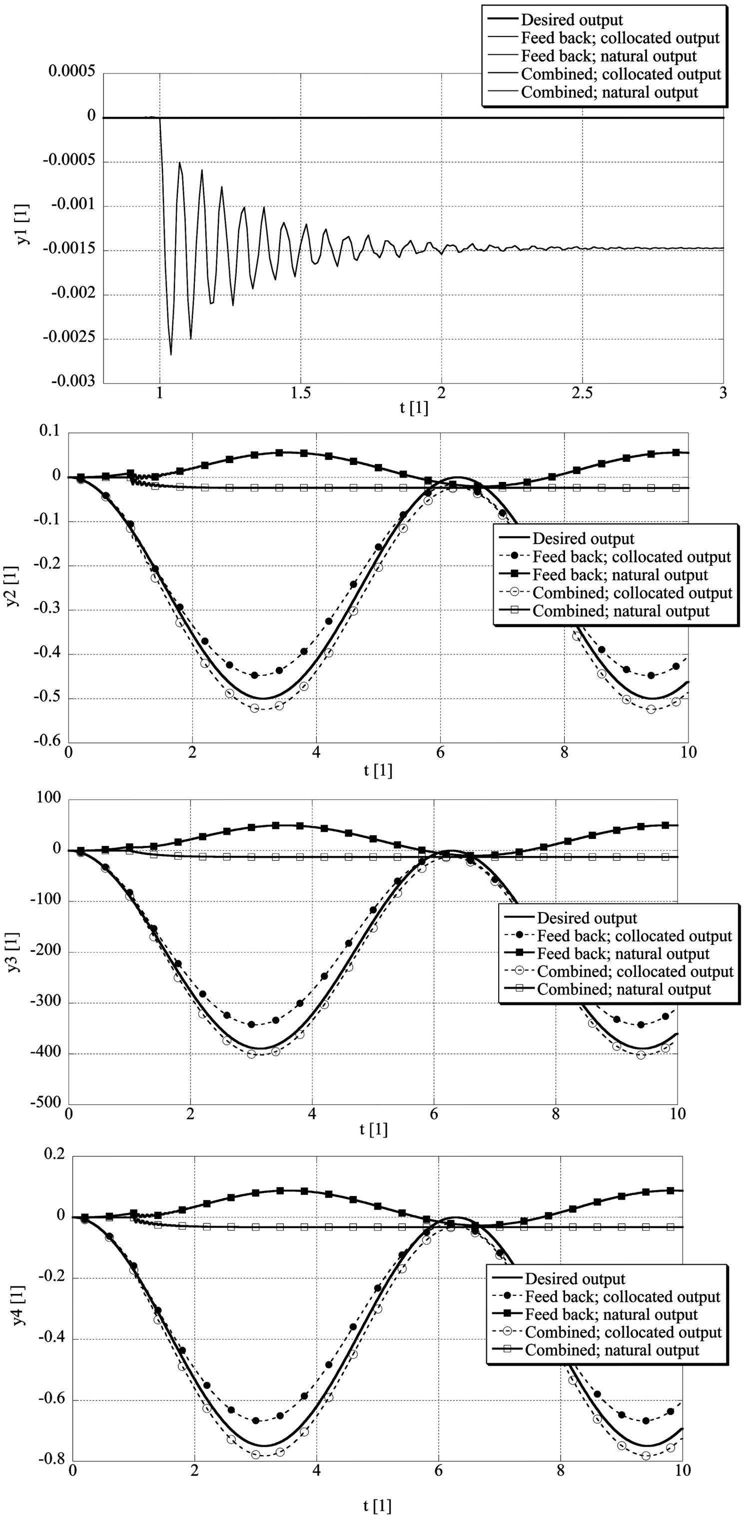

Fig. 5 Output signals for feed back control and combined control

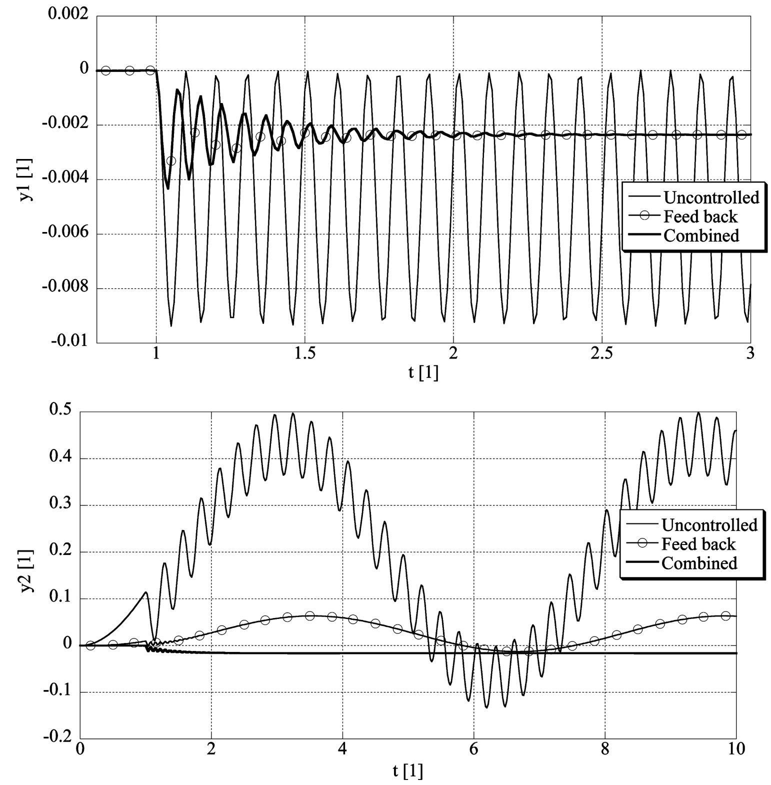

Fig. 6 Natural output for uncontrolled case, feed back control and combined control

previous example the results of Fig. 5 are obtained. In Fig. 5 we compare the combined strategy to the feed back one. We conclude that the combined strategy is preferable to the pure feed back one, as the natural output is significantly reduced without changing the parameters of the PD controllers. For $i = 1$ this is not true; the outputs are identical in both cases, because the feed forward part only induces vibrations in the symmetric modes and therefore has no influence on the skew symmetric vibration of the clamped-clamped beam. Hence, the first figure in Fig. 5 shows the desired output, which is zero, as well as four identical outputs in the two controlled cases.

As a final numerical example, we consider a case, in which the transverse force loading is unknown; with respect to its time variation and span wise distribution. We use

$$p_z(x, t) = 300H(x-0.5)[(x-0.5)+0.5]H(t-1) \tag{64}$$

Both, the actuators and sensors we have used in the previous examples are used in this one too. This is a reasonable design, as we are accounting for symmetric modes and skew symmetric modes.

Fig. 6 presents results for the first two natural outputs; the other ones are similar to the second one. We compare the uncontrolled case to both, the feed back case and the combined one. For the first sensor output no deviation between feed back and combined occurs, because the feed forward part has no influence on this natural output. In contrast, the second natural output shows that the combined strategy is preferable to the pure feed back strategy. In general, we note that, even for unknown disturbances, our sensor and actuator design gives very good results; this follows from the fact that we are accounting for all modes as well as for the desired deflection in the combined strategy.

7. Conclusions

In the present paper we presented a general method to design strain type sensors, the sensor output of which having a mechanical meaning, in extension of that of a weighted average of the strain tensor. Moreover, we were able to design these sensors such that they only account for a part of the total strain tensor, namely the part that accounts for the deviation of the displacement vector from a desired one. Such sensors can be advantageously used for the tracking of a desired displacement vector in a three-dimensional elastic background body. For the later sake we also discussed actuators, which are collocated to the sensors. First, we assigned a mechanical interpretation to these actuators and second we used the actuators to solve the dynamic displacement tracking problem.

The latter problem was targeted by means of both, a feed forward control strategy for given disturbances and a feed back control strategy in case the disturbances are unknown. We showed that the feed forward control strategy results in an exact solution of the dynamic displacement tracking problem. For the feed back control strategy we showed that the closed loop system is stable in the sense of Liapunov, as we used a PD controller in combination with collocated actuators and sensors. The controlled displacement was close to the desired one. Finally, we combined the feed forward and the feed back strategy for unknown disturbances resulting in a controlled displacement matching the desired one very well; at least much better than the feed back strategy alone. The reason our method works so well is the design of sensors and actuators, which was tailored for the dynamic displacement tracking problem.

One of the major problems of the method we presented is the need to put a possibly high number of spatially distributed sensors and actuators into practice. Future research must be targeted at replacing such sensors and actuators by dense networks of space wise constant sensors and actuators, for which the proper distribution is achieved by assigning weights to the individual members of the networks. By using such networks, one will be able to easily approximate a high number of spatial distributions in practical applications.

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