Smart Structures and Systems, *Vol. 22, No. 5 (2018) 527-546*
DOI: https://doi.org/10.12989/sss.2018.22.5.527

Size-dependent forced vibration response of embedded micro cylindrical shells reinforced with agglomerated CNTs using strain gradient theory

H. Tohidi[*1], S.H. Hosseini-Hashemi[2] and A. Maghsoudpour[1]

[1]*Department of Mechanical and Aerospace Engineering, Science and Research Branch, Islamic Azad University, Tehran, Iran, 14515-775*
[2]*School of mechanical Engineering Iran university of Science and Technology, Narmak, 16842-13114 Tehran, Iran*

(Received July 13, 2018, Revised October 5, 2018, Accepted October 8, 2018)

Abstract. This article presents an analysis into the nonlinear forced vibration of a micro cylindrical shell reinforced by carbon nanotubes (CNTs) with considering agglomeration effects. The structure is subjected to magnetic field and transverse harmonic mechanical load. Mindlin theory is employed to model the structure and the strain gradient theory (SGT) is also used to capture the size effect. Mori-Tanaka approach is used to estimate the equivalent material properties of the nanocomposite cylindrical shell and consider the CNTs agglomeration effect. The motion equations are derived using Hamilton's principle and the differential quadrature method (DQM) is employed to solve them for obtaining nonlinear frequency response of the cylindrical shells. The effect of different parameters including magnetic field, CNTs volume percent and agglomeration effect, boundary conditions, size effect and length to thickness ratio on the nonlinear forced vibrational characteristic of the of the system is studied. Numerical results indicate that by enhancing the CNTs volume percent, the amplitude of system decreases while considering the CNTs agglomeration effect has an inverse effect.

Keywords: forced vibration; micro cylindrical shell; Mindlin theory; size-dependent model; agglomeration effect

1. Introduction

Recently the subject of CNTs has attracted attentions of researchers because of their excellent physical and chemical properties such as high tensile strengths, high stiffness, high aspect ratio and low density (Yakobson *et al*. 1996, Saito *et al.* 1998, Qian *et al.* 2002, Yu *et al.* 2000). Numerous investigations are carried out to study different aspect of behavior of CNTs and the results of these studies shows that CNTs have extraordinary mechanical, electronic, electromechanical and thermal properties (Esawi and Farag, 2007, Liew *et al*. 2015, Shi and Feng 2004). Xiaodong *et al.* (2006) investigated the non-linear forced vibration of axially moving viscoelastic beams. By referring to the quasi-static stretch assumption, the partial-differential non-linearity is reduced to an integro-partial-differential one. The method of multiple scales is directly applied to the governing equations with the two types of non-linearity, respectively. The amplitude of near- and exact-resonant steady state is analyzed by use of the solvability condition of eliminating secular terms. Rougui *et al*. (2007) studied the geometrically non-linear free and forced vibrations of simply supported circular cylindrical shells. The non-linear dynamic variational problem obtained by applying Lagrange's equations was then transformed into a static case by adopting the harmonic balance method. Dynamic analysis of an embedded single-walled carbon nanotube (SWCNT) traversed by a moving nanoparticle, which was modeled by Simsek (2011) as a moving load, was investigated in this study based on the nonlocal Timoshenko beam theory, including transverse shear deformation and rotary inertia. Multiple time scale solutions were presented by Shooshtari and Rafiee (2011) to study the nonlinear forced vibration of a beam made of symmetric functionally graded (FG) materials based on Euler–Bernoulli beam theory and von Kármán geometric nonlinearity. The effects of material property distribution and end supports on the nonlinear dynamic behavior of FG beams were discussed. He *et al.* (2012) provided the analysis of nonlinear forced vibration of multi-layered graphene sheets. Based on the vdW explicit formulation, a nonlinear continuum model is developed for the vibrations of MLGSs subjected to out of plane harmonic excitation in spectral neighborhood of lower resonances. Ghayesh *et al.* (2013a) presented the nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory. Hamilton's principle is used to derive the nonlinear partial differential equation governing the motion of the system which is then discretized into a set of second- order nonlinear ordinary differential equations (ODEs) by means of the Galerkin technique. A change of variables is then introduced to this set of second-order ODEs, and a new set of ODEs is obtained consisting of first-order nonlinear ordinary differential equations. This new set is solved numerically employing the pseudo-arclength continuation technique. The geometrically nonlinear size-dependent behaviour of a Timoshenko microbeam was examined numerically by Ghayesh *et al*. (2013b). Ghayesh *et al*. (2013b) studied the

∗Corresponding author, Ph.D.
E-mail: Havjin.tohidi@gmail.com

http://www.techno-press.com/journals/sss&subpage=7
ISSN: 1738-1584 (Print), 1738-1991 (Online)

nonlinear resonant dynamics of a microscale beam. Assadi *et al.* (2013) analyzed the size dependent forced vibration of nanoplates with consideration of surface effects. The effects of surface properties including surface elasticity, surface residual stresses and surface mass density are considered which are bases for size dependent behaviors due to increase in surface to volume ratios at smaller scales. By using the superposition principle, closed form solution is derived for time response of nanoplates under general harmonic loads. Ghayesh *et al*. (2013d) investigated the nonlinear size-dependent behaviour of an electrically actuated MEMS resonator based on the modified couple stress theory. Rafiee *et al.* (2013) provided the nonlinear free and forced thermo-electro-aero-elastic vibration and dynamic response of piezoelectric functionally graded laminated composite shells. The governing equations are derived using improved Donnell shell theory ignoring the shallowness of cylindrical shells and kinematic nonlinearity and the physical neutral surface concept are taken into consideration. The nonlinear forced vibrations of a microbeam were investigated by Ghayesh *et al.* (2013e), employing the strain gradient elasticity theory. Farokhi et al. (2013f) nvestigated the nonlinear dynamics of a geometrically imperfect microbeam numerically on the basis of the modified couple stress theory. Alijani *et al.* (2014) implemented the non-linear static bending and forced vibrations of rectangular plates retaining nonlinearities in rotations and thickness deformation. The boundary conditions of the plate are assumed to be simply supported immovable and the equations of motion are derived by using a Lagrangian approach. The numerical solutions are obtained by using pseudo arc-length continuation and collocation scheme. The three-dimensional nonlinear size-dependent motion characteristics of a microbeam were investigated by Ghayesh *et al.* (2014) numerically, with special consideration to one-to-one internal resonances between the in-plane and out-of-plane transverse modes. Three-dimensional elasticity solution was extended by Javanbakht (2012) to investigate a functionally graded piezoelectric material (FGPM) finite length, simply supported shell panel under dynamic pressure excitation. Based on the high-order theory (HOT) of sandwich structures, the response of sandwich cylindrical shells with flexible core and any sort of boundary conditions under a general distributed static loading was investigated by Shokrollahi *et al.* (2015). A trigonometric refined beam theory for the bending, buckling and free vibration analysis of carbon nanotube-reinforced composite (CNTRC) beams resting on elastic foundation was developed by Tagrara *et al.* (2015). Gholipour *et al.* (2015) the in-plane and out-of-plane nonlinear size-dependent dynamics of a microplate resting on an elastic foundation, constrained by distributed rotational springs at boundaries. Tadi Beni *et al.* (2015) considered the nonlinear analysis of forced vibration of nonlocal third-order shear deformable beam model of magneto-electro-thermo elastic nanobeams. The equations are discretized using the GDQ method. Thereafter, using a Galerkin-based numerical technique, the set of nonlinear governing equations is reduced into a time varying set of ordinary differential equations of Duffing type. Ghayesh and Farokhi (2015a) studied the nonlinear dynamics of a microplate based on the modified couple stress theory. Farokhi *et al.* (2015) investigated the three-dimensional motion characteristics of perfect and imperfect Timoshenko microbeams under mechanical and thermal forces. The nonlinear dynamical behaviour of a geometrically imperfect microplate was examined by Farokhi and Ghayesh (2015) based on the modified couple stress theory. Ghayesh and Farokhi (2015b) investigated the complex sub and supercritical global dynamics of a parametrically excited microbeam. Jump and bifurcation phenomena for geometrical nonlinear cantilever beam were investigated by Motallebi *et al.* (2016) regarding forced vibration. The size-dependent dynamical performance of a microgyroscope was investigated by Ghayesh *et al*. (2016) via use of the modified couple stress theory. The nonlinear ODE of system is obtained by using Galerkin method. Dey et al. (2017) presented the non-linear vibration analysis of laminated composite circular cylindrical shells. Donnell's shell theory incorporating first order shear deformation, in-plane and rotary inertia is used to model the cylindrical shell. Galerkin's method is used to reduce the governing partial differential equations to a set of non-linear ordinary differential equations. These equations are solved using Incremental Harmonic Balance (IHB) method to obtain frequency-amplitude responses for free and forced vibration. Dai *et al.* (2016) studied the Surface effect on the nonlinear forced vibration of cantilevered nanobeams. The nonlinear partial differential equation (PDE) is discretized into a set of nonlinear ODEs by means of the Galerkin's technique. Sofiyev (2016) implemented the nonlinear free vibration of shear deformable orthotropic functionally graded cylindrical shells. The equations of motion of the FG orthotropic cylindrical shells are derived from the Donnell's non-linear shell theory, and then the superposition and Galerkin methods are adopted to convert the equation of motion into a non-linear ordinary differential equation. Fernandes *et al.* (2016) formulated a nonlinear finite strain and velocity gradient framework for the Euler–Bernoulli beam theory. Forced vibration analysis of a simple supported viscoelastic nanobeam was studied by Akbas (2016) based on modified couple stress theory (MCST). This formulation includes finite strain and the strain gradient within the strain energy generalization as well as velocity and its gradient within the kinetic energy generalization. Shokravi and jalili (2017) investigated nonlocal temperature-dependent dynamic buckling analysis of embedded sandwich micro plates reinforced by functionally graded carbon nanotubes (FG-CNTs). Şimşek et al. (2017) presented the size-dependent forced vibration of an imperfect FGM microplate with porosities subjected to a moving load using the modified couple stress theory. The equations of motion of FG microplate are solved in time domain by means of Newmark's method. Pasha Zanoosi *et al*. (2017) studied free and forced vibration of flexible polyurethane foam using multiple time scales method. The governing equation of motion was an integro-differential equation. The coupled nonlinear mechanical behaviour of extensible functionally graded microbeams, when both viscoelasticity and imperfection are present, was investigated by Ghayesh (2018a). Ghayesh (2018b) investigated the nonlinear vibration characteristics of axially functionally graded (AFG) shear deformable tapered beams subjected to external harmonic excitations. Ghayesh (2018c) presented a size-dependent continuum-based model for the coupled nonlinear dynamics of extensible functionally graded (FG) microbeams with viscoelastic properties.

According to the author's knowledge, to date no research for the nonlinear forced vibration of micro nanocomposite cylindrical shell has been found in the literature. However, size-dependent nonlinear forced vibration analysis of embedded micro Mindlin cylindrical shell subjected to magnetic field and harmonic mechanical load was studied in this research for the first time. The structure is reinforced with agglomerated CNTs. Based on energy method and Hamilton's principal, the motion equations are derived considering size effects using SGT. DQM is applied for obtaining the nonlinear frequency response and the effects of magnetic field, CNTs volume percent and agglomeration effect, boundary conditions, size effect and length to thickness ratio are discussed in detail.

2. Mathematical formulation of problem

Fig. 1 shows the geometry of the embedded micro cylindrical shell with radius, R, length, L, and thickness h. The structure is reinforced with agglomerated CNTs and is subjected to an axial magnetic field and harmonic mechanical load.

2.1 Displacement field and kinematic relations

There are many new theories for modeling of different structures. Some of the new theories have been used by Tounsi and co-authors (Bessaim 2013, Bouderba 2013, Belabed 2014, Ait Amar Meziane 2014, Zidi 2014, Hamidi, 2015, Bourada 2015, Bousahla *et al.* 2016a, b, Beldjelili 2016, Boukhari 2016, Draiche 2016, Bellifa 2015, Attia 2015, Mahi 2015, Ait Yahia 2015, Bennoun 2016, El-Haina 2017, Menasria 2017, Chikh 2017, Zemri 2015, Larbi Chaht 2015, Belkorissat 2015, Ahouel 2016, Bounouara 2016, Bouafia 2017, Besseghier 2017, Bellifa 2017, Mouffoki 2017, Khetir 2017).

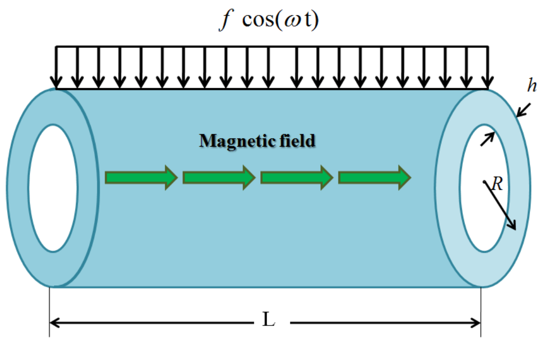

Fig. 1 A micro cylindrical shell reinforced with agglomerated CNTs under transverse uniform harmonic load subjected to magnetic field

Mindlin theory is applied for the micro cylindrical shell. Therefore, the displacement filed of the structure can be considered as (Tadi Beni *et al.* 2015)

$$U\left(x,\theta,z,t\right)=u_0\left(x,\theta,t\right)+z\,\phi_x\left(x,\theta,t\right), \tag{1}$$

$$V\left(x,\theta,z,t\right)=v_0\left(x,\theta,t\right)+z\,\phi_\theta\left(x,\theta,t\right), \tag{2}$$

$$W\left(x,\theta,z,t\right)=w_0\left(x,\theta,t\right), \tag{3}$$

where (u_0, v_0, w_0) are the displacement components of the mid-plane of the shell along the axial, circumferential and transverse directions, respectively. Also, ϕ_x and ϕ_θ indicate the rotations of the cylindrical shell cross section about x- and θ- directions, respectively. So, the nonlinear kinematic relations of the cylindrical shell may be obtained using Eqs. (1)-(3), as follows

$$\varepsilon_{xx}=\frac{\partial u_0}{\partial x}+z\frac{\partial \phi_x}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2, \tag{4a}$$

$$\varepsilon_{\theta\theta}=\frac{1}{r}\left(\frac{\partial v_0}{\partial \theta}+z\frac{\partial \phi_\theta}{\partial \theta}+w_0\right)+\frac{1}{2r^2}\left(\frac{\partial w_0}{\partial \theta}\right)^2, \tag{4b}$$

$$\varepsilon_{x\theta}=\frac{1}{2}\left(\frac{\partial v_0}{\partial x}+z\frac{\partial \phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial u_0}{\partial \theta}+\frac{z}{r}\frac{\partial \phi_x}{\partial \theta}+\frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial \theta}\right), \tag{4c}$$

$$\varepsilon_{xz}=\frac{1}{2}\left(\frac{\partial w_0}{\partial x}+\phi_x\right), \tag{4d}$$

$$\varepsilon_{\theta z}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}+\phi_\theta-\frac{v_0}{r}\right). \tag{4e}$$

2.2 Derivation of motion equations

In this section, the governing equations of the structure are derived using energy method and based on the Hamilton's principle which may be described as follows

$$\int_0^t \delta\left(U-K-W\right)dt=0, \tag{5}$$

where T is the kinetic energy; U and W are the total potential strain energy and the external works, respectively. Also δ denote the variation operator.

Based on the SGT, the potential strain energy of structure can be expressed as follows (Assadi 2013)

$$U = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + P_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^{\ s}\chi_{ij}^{\ s} \right) dV \,, \tag{6}$$

where ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ and χ_{ij} denote the strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor, respectively and can be can be written as (Assadi 2013)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} \right), \tag{7}$$

$$\gamma_i = \varepsilon_{mm,i}\,, \tag{8}$$

$$\begin{aligned} \eta_{ijk} &= \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k} \right) - \frac{1}{15}\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m} \right) \\ &- \frac{1}{15}\left[\delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m} \right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m} \right) \right], \end{aligned} \tag{9}$$

$$\chi_{ij}^{\ s} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i} \right), \tag{10}$$

in which "," refer to the partial derivative; u_i and δ_{ij} indicate the displacement vector and the Kronecker delta, respectively. Also, the rotation vector (θ_i) can be defined as (Assadi 2013)

$$\theta_i = \left(\frac{1}{2} curl\left(\mathbf{u} \right) \right)_i . \tag{11}$$

where $\mathbf{u}$ is the displacement vector. Substituting Eqs. (1)-(4) into Eqs. (7)-(11), the non-zero components of the dilatation gradient vector, the deviatoric stretch gradient tensor and the symmetric rotation gradient tensor may be written by

$$\begin{aligned} \gamma_x &= \frac{\partial^2 u_0}{\partial x^2} + z\frac{\partial^2 \phi_x}{\partial x^2} + \frac{1}{r}\frac{\partial^2 v_0}{\partial x\,\partial\theta} + \frac{z}{r}\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} \\ &+ \frac{1}{r}\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial x\,\partial\theta}\frac{\partial w_0}{\partial\theta}, \end{aligned} \tag{12a}$$

$$\begin{aligned} \gamma_\theta &= \frac{1}{r}\frac{\partial^2 u_0}{\partial x\,\partial\theta} + \frac{z}{r}\frac{\partial^2 \phi_x}{\partial x\,\partial\theta} + \frac{1}{r^2}\frac{\partial^2 v_0}{\partial\theta^2} + \frac{z}{r^2}\frac{\partial^2 \phi_\theta}{\partial\theta^2} \\ &+ \frac{1}{r^2}\frac{\partial w_0}{\partial\theta} + \frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial\theta} + \frac{1}{r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}, \end{aligned} \tag{12b}$$

$$\gamma_z = \frac{\partial \phi_x}{\partial x} + \frac{1}{r}\frac{\partial \phi_\theta}{\partial\theta}. \tag{12c}$$

$$\begin{aligned} \eta_{xxx} &= \frac{2}{5}\frac{\partial^2 u_0}{\partial x^2} + \frac{2}{5}z\frac{\partial^2 \phi_x}{\partial x^2} - \frac{1}{5r}\frac{\partial^2 v_0}{\partial x\,\partial\theta} - \frac{z}{5r}\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} \\ &- \frac{1}{5r}\frac{\partial w_0}{\partial x} - \frac{1}{5}\frac{\partial^2 v_0}{\partial x\,\partial\theta} - \frac{z}{5}\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} - \frac{1}{5r}\frac{\partial^2 u_0}{\partial\theta^2} - \frac{z}{5r}\frac{\partial^2 \phi_x}{\partial\theta^2} \\ &+ \frac{2}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} - \frac{2}{5r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial x\,\partial\theta} - \frac{1}{5r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2}, \end{aligned} \tag{13a}$$

$$\begin{aligned} \eta_{\theta\theta\theta} &= \frac{2}{5r}\frac{\partial^2 v_0}{\partial\theta^2} + \frac{2z}{5r}\frac{\partial^2 \phi_\theta}{\partial\theta^2} + \frac{2}{5r}\frac{\partial w_0}{\partial\theta} \\ &- \frac{1}{5}\frac{\partial^2 u_0}{\partial x\,\partial\theta} - \frac{z}{5}\frac{\partial^2 \phi_x}{\partial x\,\partial\theta} - \frac{1}{5}\frac{\partial^2 v_0}{\partial x\,\partial\theta} - \frac{z}{5}\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} \\ &- \frac{1}{5r}\frac{\partial^2 u_0}{\partial\theta^2} - \frac{z}{5r}\frac{\partial^2 \phi_x}{\partial\theta^2} + \frac{2}{5r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2} \\ &- \frac{2}{5r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial\theta} - \frac{1}{5r}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial\theta}, \end{aligned} \tag{13b}$$

$$\eta_{zzz} = -\frac{1}{5}\frac{\partial^2 w_0}{\partial x^2} - \frac{1}{5}\frac{\partial \phi_x}{\partial x} - \frac{1}{5r}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{1}{5}\frac{\partial \phi_\theta}{\partial\theta}, \tag{13c}$$

$$\begin{aligned} \eta_{xx\theta} = \eta_{x\theta x} = \eta_{\theta xx} &= \frac{4}{15}\frac{\partial^2 v_0}{\partial x^2} + \frac{4z}{15}\frac{\partial^2 \phi_\theta}{\partial x^2} \\ &+ \frac{4}{15r}\frac{\partial^2 u_0}{\partial x\,\partial\theta} + \frac{4z}{15r}\frac{\partial^2 \phi_x}{\partial x\,\partial\theta} + \frac{4}{15}\frac{\partial^2 u_0}{\partial x\,\partial\theta} \\ &+ \frac{4z}{15}\frac{\partial^2 \phi_x}{\partial x\,\partial\theta} - \frac{1}{5r}\frac{\partial^2 v_0}{\partial\theta^2} - \frac{z}{5r}\frac{\partial^2 \phi_\theta}{\partial\theta^2} - \frac{1}{5r}\frac{\partial w_0}{\partial\theta} \\ &+ \frac{4}{15r}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial\theta} + \frac{8}{15r}\frac{\partial^2 w_0}{\partial x\,\partial\theta}\frac{\partial w_0}{\partial x} - \frac{1}{5r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}, \end{aligned} \tag{13d}$$

$$\begin{aligned} \eta_{xxz} = \eta_{xzx} = \eta_{zxx} &= \frac{4}{15}\frac{\partial w_0}{\partial x} + \frac{4}{15}\frac{\partial \phi_x}{\partial x} \\ &- \frac{1}{15r}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{1}{15}\frac{\partial \phi_\theta}{\partial\theta}, \end{aligned} \tag{13e}$$

$$\begin{aligned} \eta_{\theta\theta x} = \eta_{\theta x\theta} = \eta_{x\theta\theta} &= \frac{4}{15}\frac{\partial^2 v_0}{\partial x\,\partial\theta} + \frac{4}{15}z\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} \\ &+ \frac{4}{15r}\frac{\partial^2 u_0}{\partial\theta^2} + \frac{4z}{15r}\frac{\partial^2 \phi_x}{\partial\theta^2} + \frac{4}{15r}\frac{\partial^2 v_0}{\partial x\,\partial\theta} \\ &+ \frac{4z}{15r}\frac{\partial^2 \phi_\theta}{\partial x\,\partial\theta} + \frac{4}{15r^2}\frac{\partial w_0}{\partial x} - \frac{2}{15}\frac{\partial^2 u_0}{\partial x^2} \\ &- \frac{2z}{15}\frac{\partial^2 \phi_x}{\partial x^2} + \frac{8}{15r^2}\frac{\partial^2 w_0}{\partial x\,\partial\theta}\frac{\partial w_0}{\partial\theta} \\ &+ \frac{4}{15r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{1}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}, \end{aligned} \tag{13f}$$

$$\eta_{\theta\theta z}=\eta_{z\theta z}=\eta_{z\theta\theta}=\frac{4}{15r}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{4}{15}\frac{\partial\phi_\theta}{\partial\theta}-\frac{1}{15}\frac{\partial^2 w_0}{\partial x^2}-\frac{1}{15}\frac{\partial\phi_x}{\partial x}, \tag{13g}$$

$$\begin{aligned}\eta_{zzx}=\eta_{zxz}=\eta_{xzz}=&-\frac{1}{5}\frac{\partial^2 u_0}{\partial x^2}-\frac{z}{5}\frac{\partial^2\phi_x}{\partial x^2}\\&-\frac{1}{15r}\frac{\partial^2 v_0}{\partial x\partial\theta}-\frac{z}{15r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial w_0}{\partial x}\\&-\frac{1}{15}\frac{\partial^2 v_0}{\partial x\partial\theta}-\frac{z}{15}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial^2 u_0}{\partial\theta^2}-\frac{z}{15r}\frac{\partial^2\phi_x}{\partial\theta^2}\\&-\frac{1}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}-\frac{2}{15r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{1}{15r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2},\end{aligned} \tag{13h}$$

$$\begin{aligned}\eta_{zz\theta}=\eta_{z\theta z}=\eta_{\theta zz}=&-\frac{1}{5r}\frac{\partial^2 v_0}{\partial\theta^2}-\frac{z}{5r}\frac{\partial^2\phi_\theta}{\partial\theta^2}\\&-\frac{1}{5r}\frac{\partial w_0}{\partial\theta}-\frac{1}{15}\frac{\partial^2 u_0}{\partial x\partial\theta}-\frac{z}{15}\frac{\partial^2\phi_x}{\partial x\partial\theta}-\frac{1}{15}\frac{\partial^2 v_0}{\partial x^2}\\&-\frac{z}{15}\frac{\partial^2\phi_\theta}{\partial x^2}-\frac{1}{15r}\frac{\partial^2 u_0}{\partial x\partial\theta}-\frac{z}{15r}\frac{\partial^2\phi_x}{\partial x\partial\theta}-\frac{1}{5r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}\\&-\frac{2}{15r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial\theta},\end{aligned} \tag{13i}$$

$$\eta_{x\theta z}=\eta_{\theta zx}=\eta_{zx\theta}=\frac{1}{6r}\frac{\partial^2 w_0}{\partial x\partial\theta}+\frac{1}{6}\frac{\partial\phi_\theta}{\partial x}+\frac{1}{6}\frac{\partial^2 w_0}{\partial x\partial\theta}+\frac{1}{6}\frac{\partial\phi_x}{\partial\theta}, \tag{13j}$$

$$\chi_{xx}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial v_0}{\partial x}+\frac{z}{r}\frac{\partial\phi_\theta}{\partial x}+\frac{\partial\phi_\theta}{\partial x}-\frac{1}{r}\frac{\partial^2 w_0}{\partial x\partial\theta}\right], \tag{14a}$$

$$\chi_{\theta\theta}=\frac{1}{2r}\begin{bmatrix}\dfrac{\partial^2 w_0}{\partial x\partial\theta}-\dfrac{\partial\phi_x}{\partial\theta}-\dfrac{\partial v_0}{\partial x}-z\dfrac{\partial\phi_\theta}{\partial x}\\+\dfrac{1}{r}\dfrac{\partial u_0}{\partial\theta}+\dfrac{z}{r}\dfrac{\partial\phi_x}{\partial\theta}\end{bmatrix}, \tag{14b}$$

$$\chi_{x\theta}=\frac{1}{4}\begin{bmatrix}\dfrac{\partial^2 w_0}{\partial x^2}-\dfrac{\partial\phi_x}{\partial x}+\dfrac{1}{r^2}\dfrac{\partial v_0}{\partial\theta}-\dfrac{1}{r^2}\dfrac{\partial^2 w_0}{\partial\theta^2}\\+\dfrac{z}{r^2}\dfrac{\partial\phi_\theta}{\partial\theta}+\dfrac{1}{r}\dfrac{\partial\phi_\theta}{\partial\theta}\end{bmatrix}, \tag{14c}$$

$$\chi_{xz}=\frac{1}{4}\begin{bmatrix}-\dfrac{\partial^2 v_0}{\partial x^2}-z\dfrac{\partial^2\phi_\theta}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial^2 u_0}{\partial x\partial\theta}\\+\dfrac{z}{r}\dfrac{\partial^2\phi_x}{\partial x\partial\theta}+\dfrac{1}{r}\phi_\theta\end{bmatrix}, \tag{14d}$$

$$\chi_{\theta z}=\frac{1}{4}\begin{bmatrix}-\dfrac{1}{r}\dfrac{\partial w_0}{\partial x}+\dfrac{1}{r}\phi_x-\dfrac{1}{r}\dfrac{\partial^2 v_0}{\partial x\partial\theta}\\-\dfrac{z}{r}\dfrac{\partial^2\phi_\theta}{\partial x\partial\theta}+\dfrac{1}{r^2}\dfrac{\partial^2 u_0}{\partial\theta^2}+\dfrac{z}{r^2}\dfrac{\partial^2\phi_x}{\partial\theta^2}\end{bmatrix}, \tag{14e}$$

$$\chi_{zz}=\frac{1}{2}\left[-\frac{\partial\phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial\phi_x}{\partial\theta}\right]. \tag{14f}$$

Furthermore, the higher-order stresses including p_i, $\tau_{ijk}^{(1)}$ and m_{ij}^s can be given by the following relations (Assadi 2013)

$$\sigma_{ij}=\lambda tr\,\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}, \tag{15}$$

$$p_i=2\mu l_0^2\gamma_i, \tag{16}$$

$$\tau^1_{ijk}=2\mu l_1^2\eta^1_{ijk}, \tag{17}$$

$$m_{ij}^s=2\mu l_2^2\chi_{ij}^s, \tag{18}$$

in which σ_{ij} is the classical stress tensor; p_i, $\tau_{ijk}^{(1)}$ and m_{ij} are the higher order stresses; λ and μ indicate the bulk and shear modulus and (l_0,l_1,l_2) are the material length scale parameters. Substituting Eqs. (12(a))-(12(c)), (13(a))-(13(j)) and (14(a))-(14(f)) into Eqs. (15)-(18), the stresses tensor components can be expressed as following relations

$$\begin{aligned}\sigma_{xx}=&\frac{E}{1-\nu^2}\left(\varepsilon_{xx}+\nu\varepsilon_{\theta\theta}\right)=\frac{E}{1-\nu^2}\Bigg[\\&\frac{\partial u_0}{\partial x}+z\frac{\partial\phi_x}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\\&+\frac{\nu}{r}\left(\frac{\partial v_0}{\partial\theta}+z\frac{\partial\phi_\theta}{\partial\theta}+w_0+\frac{1}{2r}\left(\frac{\partial w_0}{\partial\theta}\right)^2\right)\Bigg],\end{aligned} \tag{19a}$$

$$\begin{aligned}\sigma_{\theta\theta} &= \frac{E}{1-\nu^2}\left(\varepsilon_{\theta\theta}+\nu\varepsilon_{xx}\right)=\frac{E}{1-\nu^2}\Bigg[\\ &\frac{1}{r}\left(\frac{\partial v_0}{\partial\theta}+z\,\frac{\partial\phi_\theta}{\partial\theta}+w_0+\frac{1}{2r}\left(\frac{\partial w_0}{\partial\theta}\right)^2\right)\\ &+\nu\left(\frac{\partial u_0}{\partial x}+z\,\frac{\partial\phi_x}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right)\Bigg],\end{aligned}\tag{19b}$$

$$\tau_{x\theta}=2\mu\varepsilon_{x\theta}=\mu\left[\begin{aligned}&\frac{\partial v_0}{\partial x}+z\,\frac{\partial\phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial u_0}{\partial\theta}\\ &+\frac{z}{r}\frac{\partial\phi_x}{\partial\theta}+\frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial\theta}\end{aligned}\right],\tag{19c}$$

$$\tau_{xz}=2\mu\varepsilon_{xz}=\mu\left[\frac{\partial w_0}{\partial x}+\phi_x\right],\tag{19d}$$

$$\tau_{\theta z}=2\mu\varepsilon_{\theta z}=\mu\left[\frac{1}{r}\frac{\partial w_0}{\partial\theta}+\phi_\theta-\frac{v_0}{r}\right],\tag{19e}$$

$$\begin{aligned}p_x &= 2\mu l_0^2\gamma_x=2\mu l_0^2\Bigg(\frac{\partial^2u_0}{\partial x^2}+z\,\frac{\partial^2\phi_x}{\partial x^2}\\ &+\frac{1}{r}\frac{\partial^2v_0}{\partial x\partial\theta}+\frac{z}{r}\frac{\partial^2\phi_\theta}{\partial\theta\partial x}+\frac{1}{r}\frac{\partial w_0}{\partial x}\\ &+\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x^2}+\frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial x\partial\theta}\Bigg),\end{aligned}\tag{20a}$$

$$\begin{aligned}p_\theta &= 2\mu l_0^2\gamma_\theta=2\mu l_0^2\Bigg(\frac{1}{r}\frac{\partial^2u_0}{\partial x\partial\theta}+\frac{z}{r}\frac{\partial^2\phi_x}{\partial x\partial\theta}\\ &+\frac{1}{r^2}\frac{\partial^2v_0}{\partial\theta^2}+\frac{z}{r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\\ &+\frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x\partial\theta}+\frac{1}{r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial\theta^2}\Bigg),\end{aligned}\tag{20b}$$

$$p_z=2\mu l_0^2\gamma_z=2\mu l_0^2\left(\frac{\partial\phi_x}{\partial x}+\frac{1}{r}\frac{\partial\phi_\theta}{\partial\theta}\right),\tag{20c}$$

$$\begin{aligned}\tau_{xxx} &= 2\mu l_1^2\eta_{xxx}=2\mu l_1^2\Bigg(\frac{2}{5}\frac{\partial^2u_0}{\partial x^2}+\frac{2}{5}z\,\frac{\partial^2\phi_x}{\partial x^2}\\ &-\frac{1}{5r}\frac{\partial^2v_0}{\partial x\partial\theta}-\frac{z}{5r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{5r}\frac{\partial w_0}{\partial x}-\frac{1}{5r}\frac{\partial^2v_0}{\partial x\partial\theta}\\ &-\frac{z}{5r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{5r^2}\frac{\partial^2u_0}{\partial\theta^2}-\frac{z}{5r^2}\frac{\partial^2\phi_x}{\partial\theta^2}+\frac{2}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x^2}\\ &-\frac{2}{5r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial x\partial\theta}-\frac{1}{5r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial\theta^2}\Bigg),\end{aligned}\tag{21a}$$

$$\begin{aligned}\tau_{\theta\theta\theta} &= 2\mu l_1^2\eta_{\theta\theta\theta}=2\mu l_1^2\Bigg(\frac{2}{5r^2}\frac{\partial^2v_0}{\partial\theta^2}+\frac{2z}{5r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}\\ &+\frac{2}{5r^2}\frac{\partial w_0}{\partial\theta}-\frac{1}{5r}\frac{\partial^2u_0}{\partial x\partial\theta}-\frac{z}{5r}\frac{\partial^2\phi_x}{\partial x\partial\theta}-\frac{1}{5}\frac{\partial^2v_0}{\partial x^2}\\ &-\frac{z}{5}\frac{\partial^2\phi_\theta}{\partial x^2}-\frac{1}{5r}\frac{\partial^2u_0}{\partial x\partial\theta}-\frac{z}{5r}\frac{\partial^2\phi_x}{\partial x\partial\theta}+\frac{2}{5r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial\theta^2}\\ &-\frac{2}{5r}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x\partial\theta}-\frac{1}{5r}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial x^2}\Bigg),\end{aligned}\tag{21b}$$

$$\begin{aligned}\tau_{zzz} &= 2\mu l_1^2\eta_{zzz}=2\mu l_1^2=\\ &\left(\begin{aligned}&-\frac{1}{5}\frac{\partial^2w_0}{\partial x^2}-\frac{2}{5}\frac{\partial\phi_x}{\partial x}-\frac{1}{5r^2}\frac{\partial^2w_0}{\partial\theta^2}\\ &-\frac{2}{5r}\frac{\partial\phi_\theta}{\partial\theta}+\frac{1}{5r^2}\frac{\partial v_0}{\partial\theta}\end{aligned}\right),\end{aligned}\tag{21c}$$

$$\begin{aligned}\tau_{x\theta z} &= 2\mu l_1^2\eta_{x\theta z}=2\mu l_1^2\\ &\left(\begin{aligned}&\frac{1}{6r}\frac{\partial^2w_0}{\partial x\partial\theta}+\frac{1}{3}\frac{\partial\phi_\theta}{\partial x}+\frac{1}{6r}\frac{\partial^2w_0}{\partial\theta\partial x}\\ &+\frac{1}{3r}\frac{\partial\phi_x}{\partial\theta}-\frac{1}{6r}\frac{\partial v_0}{\partial x}\end{aligned}\right),\end{aligned}\tag{21d}$$

$$\begin{aligned}\tau_{xx\theta} &= 2\mu l_1^2\eta_{xx\theta}=2\mu l_1^2\Bigg(\frac{4}{15}\frac{\partial^2v_0}{\partial x^2}+\frac{4z}{15}\frac{\partial^2\phi_\theta}{\partial x^2}\\ &+\frac{4}{15r}\frac{\partial^2u_0}{\partial x\partial\theta}+\frac{4z}{15r}\frac{\partial^2\phi_x}{\partial x\partial\theta}+\frac{4}{15r}\frac{\partial^2u_0}{\partial x\partial\theta}+\frac{4z}{15r}\frac{\partial^2\phi_x}{\partial x\partial\theta}\\ &-\frac{1}{5r^2}\frac{\partial^2v_0}{\partial\theta^2}-\frac{z}{5r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}-\frac{1}{5r^2}\frac{\partial w_0}{\partial\theta}+\frac{4}{15r}\frac{\partial^2w_0}{\partial x^2}\frac{\partial w_0}{\partial\theta}\\ &+\frac{8}{15r}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x\partial\theta}-\frac{1}{5r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial\theta^2}\Bigg),\end{aligned}\tag{21e}$$

$$\begin{aligned}\tau_{xxz} &= 2\mu l_1^2\eta_{xxz}=2\mu l_1^2\\ &\left(\begin{aligned}&\frac{4}{15}\frac{\partial^2w_0}{\partial x^2}+\frac{8}{15}\frac{\partial\phi_x}{\partial x}-\frac{1}{15r^2}\frac{\partial^2w_0}{\partial\theta^2}\\ &-\frac{2}{15r}\frac{\partial\phi_\theta}{\partial\theta}+\frac{1}{15r^2}\frac{\partial v_0}{\partial\theta}\end{aligned}\right),\end{aligned}\tag{21f}$$

$$\begin{aligned}\tau_{\theta\theta x} &= 2\mu l_1^2\eta_{\theta\theta x}=2\mu l_1^2\Bigg(\frac{4}{15r}\frac{\partial^2v_0}{\partial x\partial\theta}+\frac{4}{15r}z\,\frac{\partial^2\phi_\theta}{\partial x\partial\theta}\\ &+\frac{4}{15r^2}\frac{\partial^2u_0}{\partial\theta^2}+\frac{4z}{15r^2}\frac{\partial^2\phi_x}{\partial\theta^2}+\frac{4}{15r}\frac{\partial^2v_0}{\partial x\partial\theta}+\frac{4z}{15r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}\\ &+\frac{4}{15r^2}\frac{\partial w_0}{\partial x}-\frac{2}{15}\frac{\partial^2u_0}{\partial x^2}-\frac{2z}{15}\frac{\partial^2\phi_x}{\partial x^2}+\frac{8}{15r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2w_0}{\partial x\partial\theta}\\ &+\frac{4}{15r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial\theta^2}-\frac{1}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2w_0}{\partial x^2}\Bigg),\end{aligned}\tag{21g}$$

$$\tau_{\theta\theta z}=2\mu l_1^2\eta_{\theta\theta z}=2\mu l_1^2 \left(\begin{array}{l}\frac{4}{15r^2}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{8}{15r}\frac{\partial\phi_\theta}{\partial\theta}-\frac{1}{15}\frac{\partial^2 w_0}{\partial x^2}\\ -\frac{2}{15}\frac{\partial\phi_x}{\partial x}-\frac{4}{15r^2}\frac{\partial v_0}{\partial\theta}\end{array}\right), \tag{21h}$$

$$\begin{aligned}\tau_{zzx}=2\mu l_1^2\eta_{zzx}=2\mu l_1^2\Bigg(&-\frac{1}{5}\frac{\partial^2 u_0}{\partial x^2}-\frac{z}{5}\frac{\partial^2\phi_x}{\partial x^2}\\ &-\frac{1}{15r}\frac{\partial^2 v_0}{\partial x\partial\theta}-\frac{z}{15r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial w_0}{\partial x}-\frac{1}{15}\frac{\partial^2 v_0}{\partial x\partial\theta}\\ &-\frac{z}{15}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial^2 u_0}{\partial\theta^2}-\frac{z}{15r}\frac{\partial^2\phi_x}{\partial\theta^2}-\frac{1}{5}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\\ &-\frac{2}{15r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{1}{15r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2}\Bigg),\end{aligned} \tag{21i}$$

$$\begin{aligned}\tau_{zz\theta}=2\mu l_1^2\eta_{zz\theta}=2\mu l_1^2\Bigg(&-\frac{1}{5r^2}\frac{\partial^2 v_0}{\partial\theta^2}-\frac{z}{5r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}\\ &-\frac{1}{5r^2}\frac{\partial w_0}{\partial\theta}-\frac{1}{15r}\frac{\partial^2 u_0}{\partial x\partial\theta}-\frac{z}{15r}\frac{\partial^2\phi_x}{\partial x\partial\theta}-\frac{1}{15}\frac{\partial^2 v_0}{\partial x^2}\\ &-\frac{z}{15}\frac{\partial^2\phi_\theta}{\partial x^2}-\frac{1}{15r}\frac{\partial^2 u_0}{\partial x\partial\theta}-\frac{z}{15r}\frac{\partial^2\phi_x}{\partial x\partial\theta}-\frac{1}{5r^2}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}\\ &-\frac{2}{15r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{1}{15r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta}\Bigg),\end{aligned} \tag{21j}$$

$$m_{xx}=2\mu l_2^2\chi_{xx}=\mu l_2^2\left[\begin{array}{l}\frac{1}{r}\frac{\partial v_0}{\partial x}+\frac{z}{r}\frac{\partial\phi_\theta}{\partial x}\\ +\frac{\partial\phi_\theta}{\partial x}-\frac{1}{r}\frac{\partial^2 w_0}{\partial x\partial\theta}\end{array}\right], \tag{22a}$$

$$m_{\theta\theta}=2\mu l_2^2\chi_{\theta\theta}=\frac{\mu l_2^2}{r}\left[\begin{array}{l}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{\partial\phi_x}{\partial\theta}-\frac{\partial v_0}{\partial x}\\ -z\frac{\partial\phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial u_0}{\partial\theta}+\frac{z}{r}\frac{\partial\phi_x}{\partial\theta}\end{array}\right], \tag{22b}$$

$$m_{zz}=2\mu l_2^2\chi_{zz}=\mu l_2^2\left[-\frac{\partial\phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial\phi_x}{\partial\theta}\right], \tag{22c}$$

$$m_{x\theta}=2\mu l_2^2\chi_{x\theta}=\frac{\mu l_2^2}{2}\left[\begin{array}{l}\frac{\partial^2 w_0}{\partial x^2}-\frac{\partial\phi_x}{\partial x}+\frac{1}{r^2}\frac{\partial v_0}{\partial\theta}-\\ \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{z}{r^2}\frac{\partial\phi_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial\phi_\theta}{\partial\theta}\end{array}\right], \tag{22d}$$

$$m_{xz}=2\mu l_2^2\chi_{xz}=\frac{\mu l_2^2}{2}\left[\begin{array}{l}-\frac{\partial^2 v_0}{\partial x^2}-z\frac{\partial^2\phi_\theta}{\partial x^2}+\frac{1}{r}\frac{\partial^2 u_0}{\partial x\partial\theta}\\ +\frac{z}{r}\frac{\partial^2\phi_x}{\partial x\partial\theta}+\frac{1}{r}\phi_\theta\end{array}\right], \tag{22e}$$

$$m_{\theta z}=2\mu l_2^2\chi_{\theta z}=\frac{\mu l_2^2}{2}\left[\begin{array}{l}-\frac{1}{r}\frac{\partial w_0}{\partial x}+\frac{1}{r}\phi_x-\frac{1}{r}\frac{\partial^2 v_0}{\partial x\partial\theta}\\ -\frac{z}{r}\frac{\partial^2\phi_\theta}{\partial x\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_0}{\partial\theta^2}+\frac{z}{r^2}\frac{\partial^2\phi_x}{\partial\theta^2}\end{array}\right]. \tag{22f}$$

The kinetic energy of the structure can be calculated using the following equation

$$K=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^{2\pi}\int_0^{l}\rho\left[\left(\frac{\partial U}{\partial t}\right)^2+\left(\frac{\partial V}{\partial t}\right)^2+\left(\frac{\partial W}{\partial t}\right)^2\right]dx\,d\theta\,dz \tag{23}$$

where ρ denotes the density of structure.

The external work due to the axial magnetic field and uniform transverse load can be given as (Kolahchi *et al.* 2016, Pasha Zanoosi *et al.* 2017)

$$W_m=\int\left(\eta\,\mathrm{h}\,H_x^2\frac{\partial^2 w}{\partial x^2}\right)w\,dA, \tag{24a}$$

$$W_t=-\int\left(f\ \cos(\omega \mathrm{t})\right)w\,dA, \tag{24b}$$

where η is the magnetic permeability of the CNTs and H_x is related to the magnetic field; f and ω are amplitude and excitation frequency, respectively.

2.3 Material properties of the structure

In this section, the Mori-Tanaka model is used since in this theory, the agglomeration of CNTs can be assumed. The effective material properties of the CNTs reinforced cylindrical shell are obtained based on the Mori-Tanaka approach which is able to consider the agglomeration effect of CNTs. The experimental results indicate that the assumption of uniform distribution for CNTs in the matrix material is not accurate and the most of CNTs are agglomerated in some regions of the matrix material. These regions are in spherical shapes, and are named as ‘‘inclusions’’ with different elastic properties from the other regions. Therefore, the Mori-Tanaka approach is used which is simple and accurate even at high volume fractions of the inclusions. The total volume V_r of CNTs can be divided into two parts described as follows (Shi and Feng 2004)

$$V_r = V_r^{inclusion} + V_r^{m}, \qquad (25)$$

in which $V_r^{inclusion}$ and V_r^{m} are the volumes of CNTs dist ributed in the spherical inclusions and in the matrix m aterial, respectively. To define the agglomeration of CN Ts, two parameters ξ and ζ are presented as below

$$\xi = \frac{V_{inclusion}}{V}, \qquad (26)$$

$$\zeta = \frac{V_r^{inclusion}}{V_r}. \qquad (27)$$

Also, the average volume fraction C_r of CNTs in the equivalent composite are considered as

$$C_r = \frac{V_r}{V}. \qquad (28)$$

Assuming the randomly orientation of the CNTs in the matrix material, the effective bulk modulus (K) and effective shear modulus (G) may be calculated by the following relations (Shi and Feng 2004)

$$K = K_{out}\left[1 + \frac{\xi\left(\frac{K_{in}}{K_{out}} - 1\right)}{1 + \alpha(1-\xi)\left(\frac{K_{in}}{K_{out}} - 1\right)}\right], \qquad (29)$$

$$G = G_{out}\left[1 + \frac{\xi\left(\frac{G_{in}}{G_{out}} - 1\right)}{1 + \beta(1-\xi)\left(\frac{G_{in}}{G_{out}} - 1\right)}\right], \qquad (30)$$

in which

$$K_{in} = K_m + \frac{(\delta_r - 3K_m\chi_r)C_r\zeta}{3(\xi - C_r\zeta + C_r\zeta\chi_r)}, \qquad (31)$$

$$K_{out} = K_m + \frac{C_r(\delta_r - 3K_m\chi_r)(1-\zeta)}{3\left[1 - \xi - C_r(1-\zeta) + C_r\chi_r(1-\zeta)\right]}, \qquad (32)$$

$$G_{in} = G_m + \frac{(\eta_r - 3G_m\beta_r)C_r\zeta}{2(\xi - C_r\zeta + C_r\zeta\beta_r)}, \qquad (33)$$

$$G_{out} = G_m + \frac{C_r(\eta_r - 3G_m\beta_r)(1-\zeta)}{2\left[1 - \xi - C_r(1-\zeta) + C_r\beta_r(1-\zeta)\right]}, \qquad (34)$$

where $\chi_r, \beta_r, \delta_r, \eta_r$ are assumed as

$$\chi_r = \frac{3(K_m + G_m) + k_r - l_r}{3(k_r + G_m)}, \qquad (35)$$

$$\beta_r = \frac{1}{5}\left\{\frac{4G_m + 2k_r + l_r}{3(k_r + G_m)} + \frac{4G_m}{(p_r + G_m)} + \frac{2\left[G_m(3K_m + G_m) + G_m(3K_m + 7G_m)\right]}{G_m(3K_m + G_m) + m_r(3K_m + 7G_m)}\right\}, \qquad (36)$$

$$\delta_r = \frac{1}{3}\left[n_r + 2l_r + \frac{(2k_r - l_r)(3K_m + 2G_m - l_r)}{k_r + G_m}\right], \qquad (37)$$

$$\eta_r = \frac{1}{5}\left[\frac{2}{3}(n_r - l_r) + \frac{4G_m p_r}{(p_r + G_m)} + \frac{8G_m m_r(3K_m + 4G_m)}{3K_m(m_r + G_m) + G_m(7m_r + G_m)} + \frac{2(k_r - l_r)(2G_m + l_r)}{3(k_r + G_m)}\right], \qquad (38)$$

in which k_r، l_r، n_r، p_r , m_r are the Hills elastic modulus of the CNTs. Also, K_m and G_m are the bulk and shear moduli of the isotropic and elastic matrix material which can be written in terms of the Young's modulus E_m and the Poisson's ratio υ_m as

$$K_m = \frac{E_m}{3(1 - 2\upsilon_m)}, \qquad (39)$$

$$G_m = \frac{E_m}{2(1 + \upsilon_m)}. \qquad (40)$$

Moreover, β, α are considered in the following form

$$\alpha = \frac{(1 + \upsilon_{out})}{3(1 - \upsilon_{out})}, \qquad (41)$$

$$\beta = \frac{2(4 - 5\upsilon_{out})}{15(1 - \upsilon_{out})}, \qquad (42)$$

$$\upsilon_{out} = \frac{3K_{out} - 2G_{out}}{6K_{out} + 2G_{out}}. \qquad (43)$$

Eventually, the elastic modulus (E) and poison's ratio (υ) may be obtained as

$$E = \frac{9KG}{3K+G} \quad , \qquad (44)$$

$$\upsilon = \frac{3K-2G}{6K+2G}. \qquad (45)$$

Finally, substituting Eqs. (6), (23)-(25) into Eq. (5), the motion equations of the structure can be expressed as

$$\begin{aligned}
&\frac{\partial N_{xx}}{\partial x} + \frac{1}{r}\frac{\partial N_{x\theta}}{\partial\theta} - \frac{\partial^2 Y_{x}^{0}}{\partial x^2} - \frac{1}{r}\frac{\partial^2 Y_{\theta}^{0}}{\partial x\partial\theta} - \frac{2}{5}\frac{\partial^2 Y_{xxx}^{1}}{\partial x^2} \\
&+\frac{1}{5r^2}\frac{\partial^2 Y_{xxx}^{1}}{\partial\theta^2} + \frac{2}{5r}\frac{\partial^2 Y_{\theta\theta\theta}^{1}}{\partial x\partial\theta} - \frac{8}{5r}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial x\partial\theta} \\
&-\frac{4}{5r^2}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial\theta^2} + \frac{3}{5}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial x^2} + \frac{3}{5}\frac{\partial^2 Y_{zzx}^{1}}{\partial x^2} \\
&+\frac{1}{5r^2}\frac{\partial^2 Y_{zzx}^{1}}{\partial\theta^2} + \frac{2}{5r}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial x\partial\theta} + \frac{1}{2r^2}\frac{\partial^2 Y_{\theta\theta}^{2}}{\partial\theta} \\
&-\frac{1}{2r}\frac{\partial^2 Y_{xz}^{2}}{\partial x\partial\theta} \\
&-\frac{1}{2r^2}\frac{\partial^2 Y_{\theta z}^{2}}{\partial\theta^2} = I_1\frac{\partial^2 u_0}{\partial t^2} + I_2\frac{\partial^2 \phi_x}{\partial t^2},
\end{aligned} \qquad (46)$$

$$\begin{aligned}
&\frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial\theta} + \frac{\partial N_{x\theta}}{\partial x} + \frac{N_{\theta z}}{r} - \frac{1}{r}\frac{\partial^2 Y_{x}^{0}}{\partial x\partial\theta} - \frac{1}{r^2}\frac{\partial^2 Y_{\theta}^{0}}{\partial\theta^2} \\
&+\frac{2}{5r}\frac{\partial^2 Y_{xxx}^{1}}{\partial x\partial\theta} - \frac{2}{5r^2}\frac{\partial^2 Y_{\theta\theta\theta}^{1}}{\partial\theta^2} + \frac{1}{5}\frac{\partial^2 Y_{\theta\theta\theta}^{1}}{\partial x^2} - \frac{4}{5}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial x^2} \\
&+\frac{3}{5r^2}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial\theta^2} - \frac{8}{5r}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial x\partial\theta} + \frac{2}{5r}\frac{\partial^2 Y_{zzx}^{1}}{\partial x\partial\theta} + \frac{3}{5r^2}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial\theta^2} \\
&+\frac{1}{5}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial x^2} + \frac{1}{2r}\frac{\partial Y_{xx}^{2}}{\partial x} - \frac{1}{2r}\frac{\partial Y_{\theta\theta}^{2}}{\partial x} + \frac{1}{2r^2}\frac{\partial Y_{x\theta}^{2}}{\partial\theta} \\
&+\frac{1}{2}\frac{\partial^2 Y_{xz}^{2}}{\partial x^2} + \frac{1}{2r}\frac{\partial^2 Y_{\theta z}^{2}}{\partial x\partial\theta} + \frac{1}{5r^2}\frac{\partial Y_{zzz}^{1}}{\partial\theta} - \frac{1}{r}\frac{\partial Y_{x\theta z}^{1}}{\partial x} \\
&+\frac{1}{5r^2}\frac{\partial Y_{xxz}^{1}}{\partial\theta} - \frac{4}{5r^2}\frac{\partial Y_{\theta\theta z}^{1}}{\partial\theta} = I_1\frac{\partial^2 v_0}{\partial t^2} + I_2\frac{\partial^2 \phi_\theta}{\partial t^2},
\end{aligned} \qquad (47)$$

$$\begin{aligned}
&\frac{1}{2r}\frac{\partial^2 Y_{xx}^{2}}{\partial x\partial\theta} - \frac{1}{2r}\frac{\partial^2 Y_{\theta\theta}^{2}}{\partial x\partial\theta} - \frac{1}{2}\frac{\partial^2 Y_{x\theta}^{2}}{\partial x^2} \\
&+\frac{1}{2r^2}\frac{\partial^2 Y_{x\theta}^{2}}{\partial\theta^2} - \frac{1}{2r}\frac{\partial Y_{\theta z}^{2}}{\partial x} + \frac{1}{5}\frac{\partial^2 Y_{zzz}^{1}}{\partial x^2} \\
&+\frac{1}{5r^2}\frac{\partial^2 Y_{zzz}^{2}}{\partial\theta^2} - \frac{2}{r}\frac{\partial^2 Y_{x\theta z}^{1}}{\partial x\partial\theta} - \frac{3}{5r^2}\frac{\partial Y_{xx\theta}^{1}}{\partial\theta} \\
&+\frac{1}{5r^2}\frac{\partial^2 Y_{xxz}^{1}}{\partial\theta^2} - \frac{4}{5}\frac{\partial^2 Y_{xxz}^{1}}{\partial x^2} + \frac{4}{5r}\frac{\partial Y_{\theta\theta x}^{1}}{\partial x} \\
&-\frac{4}{5r^2}\frac{\partial^2 Y_{\theta\theta z}^{1}}{\partial\theta^2} + \frac{1}{5}\frac{\partial^2 Y_{\theta\theta z}^{1}}{\partial x^2} - \frac{1}{5r}\frac{\partial Y_{zzx}^{1}}{\partial x} \\
&-\frac{3}{5r^2}\frac{\partial Y_{zz\theta}^{1}}{\partial\theta} - \frac{N_{\theta\theta}}{r} + \frac{\partial N_{xz}}{\partial x} \\
&+\frac{1}{r}\frac{\partial N_{\theta z}}{\partial\theta} + \frac{1}{r}\frac{\partial Y_{x}^{0}}{\partial x} - \frac{1}{5r}\frac{\partial Y_{xxx}^{1}}{\partial x} \\
&+\frac{2}{5r^2}\frac{\partial Y_{\theta\theta\theta}^{1}}{\partial\theta} + \frac{1}{r^2}\frac{\partial Y_{\theta}^{0}}{\partial\theta} + \frac{2}{5r}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial x\partial\theta}\frac{\partial w_0}{\partial x} \\
&+\frac{1}{5r}\frac{\partial Y_{zz\theta}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{5r}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial x^2}\frac{\partial w_0}{\partial\theta} \\
&+\frac{2}{5r}\frac{\partial Y_{zz\theta}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta} - \frac{1}{5r}\frac{\partial Y_{zzz}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2} \\
&+\frac{1}{5r}\frac{\partial^2 Y_{zzz}^{1}}{\partial\theta^2}\frac{\partial w_0}{\partial x} + \frac{2}{5r}\frac{\partial Y_{zzx}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial x\partial\theta} \\
&-\frac{8}{5r}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial x\partial\theta}\frac{\partial w_0}{\partial x} - \frac{8}{5r}\frac{\partial Y_{xx\theta}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial x^2} \\
&+\frac{3}{5r^3}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial\theta^2}\frac{\partial w_0}{\partial\theta} + \frac{3}{5r^3}\frac{\partial Y_{xx\theta}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2} \\
&+\frac{4}{5r^2}\frac{\partial Y_{xx\theta}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{4}{5r^2}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial\theta^2}\frac{\partial w_0}{\partial x} \\
&+\frac{2}{5r^2}\frac{\partial^2 Y_{zzx}^{1}}{\partial x\partial\theta}\frac{\partial w_0}{\partial\theta} - \frac{8}{5r^2}\frac{\partial Y_{\theta\theta x}^{1}}{\partial\theta}\frac{\partial w_0}{\partial x\partial\theta} \\
&+\frac{2}{5r^2}\frac{\partial Y_{zzx}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2} + \frac{2}{5r}\frac{\partial^2 Y_{\theta\theta\theta}^{1}}{\partial x\partial\theta}\frac{\partial w_0}{\partial x} \\
&+\frac{1}{5r}\frac{\partial Y_{\theta\theta\theta}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial x^2} + \frac{1}{5r}\frac{\partial^2 Y_{\theta\theta\theta}^{1}}{\partial x^2}\frac{\partial w_0}{\partial\theta} \\
&+\frac{2}{5r}\frac{\partial Y_{\theta\theta\theta}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta} + \frac{4}{5r}\frac{\partial Y_{xx\theta}^{1}}{\partial\theta}\frac{\partial^2 w_0}{\partial x^2} \\
&-\frac{4}{5r}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial x^2}\frac{\partial w_0}{\partial\theta} - \frac{8}{5r}\frac{\partial Y_{xx\theta}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta} - \\
&\frac{8}{5r^2}\frac{\partial^2 Y_{xx\theta}^{1}}{\partial x\partial\theta}\frac{\partial w_0}{\partial\theta} - \frac{8}{5r^2}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2} \\
&+\frac{3}{5}\frac{\partial^2 Y_{\theta\theta x}^{1}}{\partial x^2}\frac{\partial w_0}{\partial x} + \frac{3}{5}\frac{\partial Y_{\theta\theta x}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + \frac{3}{5}\frac{\partial^2 Y_{zzx}^{1}}{\partial x^2}\frac{\partial w_0}{\partial x} \\
&+\frac{3}{5}\frac{\partial Y_{zzx}^{1}}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + \frac{3}{5r^3}\frac{\partial^2 Y_{zz\theta}^{1}}{\partial\theta^2}\frac{\partial w_0}{\partial\theta}
\end{aligned} \qquad (48)$$

$$
\begin{aligned}
&+\frac{3}{5r^3}\frac{\partial Y^1_{zz\theta}}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{\partial N_{xx}}{\partial x}\frac{\partial w_0}{\partial x}+N_{xx}\frac{\partial^2 w_0}{\partial x^2}\\
&+\frac{1}{r^2}\frac{\partial N_{\theta\theta}}{\partial\theta}\frac{\partial w_0}{\partial\theta}+\frac{N_{\theta\theta}}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\\
&+\frac{1}{r}\frac{\partial N_{x\theta}}{\partial\theta}\frac{\partial w_0}{\partial x}+\frac{2N_{x\theta}}{r}\frac{\partial^2 w_0}{\partial x\,\partial\theta}\\
&+\frac{1}{r}\frac{\partial N_{x\theta}}{\partial x}\frac{\partial w_0}{\partial\theta}-\frac{\partial^2 Y^0_x}{\partial x^2}\frac{\partial w_0}{\partial x}-\frac{\partial Y^0_x}{\partial x}\frac{\partial^2 w_0}{\partial x^2}-\\
&\frac{1}{r^2}\frac{\partial^2 Y^0_x}{\partial x\,\partial\theta}\frac{\partial w_0}{\partial\theta}-\frac{1}{r^2}\frac{\partial Y^0_x}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2}-\frac{1}{r}\frac{\partial^2 Y^0_\theta}{\partial x\,\partial\theta}\frac{\partial w_0}{\partial x}\\
&-\frac{1}{r}\frac{\partial Y^0_\theta}{\partial\theta}\frac{\partial^2 w_0}{\partial x^2}-\frac{1}{r^3}\frac{\partial^2 Y^0_\theta}{\partial\theta^2}\frac{\partial w_0}{\partial\theta}\\
&-\frac{1}{r^3}\frac{\partial Y^0_\theta}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}-\frac{2}{5}\frac{\partial^2 Y^1_{xxx}}{\partial x^2}\frac{\partial w_0}{\partial x}-\frac{2}{5}\frac{\partial Y^1_{xxx}}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\\
&-\frac{1}{5r^2}\frac{\partial Y^1_{xxx}}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{1}{5r^2}\frac{\partial^2 Y^1_{xxx}}{\partial\theta^2}\frac{\partial w_0}{\partial x}\\
&+\frac{2}{5r^2}\frac{\partial Y^1_{xxx}}{\partial\theta}\frac{\partial^2 w_0}{\partial x\,\partial\theta}-\frac{2}{5r^3}\frac{\partial^2 Y^1_{\theta\theta\theta}}{\partial\theta^2}\frac{\partial w_0}{\partial\theta}\\
&-\frac{2}{5r^3}\frac{\partial Y^1_{\theta\theta\theta}}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}\\
&+\eta\,\mathrm{h}H_x^2\frac{\partial^2 w}{\partial x^2}-f\,\cos(\omega\mathrm{t})=I_1\frac{\partial^2 w_0}{\partial t^2},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial M_{xx}}{\partial x}+\frac{1}{r}\frac{\partial M_{x\theta}}{\partial\theta}-N_{xz}-\frac{\partial^2 T^0_x}{\partial x^2}-\frac{1}{r}\frac{\partial^2 T^0_\theta}{\partial x\,\partial\theta}\\
&+\frac{\partial Y^0_z}{\partial x}-\frac{2}{5}\frac{\partial^2 T^1_{xxx}}{\partial x^2}+\frac{1}{5r^2}\frac{\partial^2 T^1_{xxx}}{\partial\theta^2}\\
&+\frac{2}{5r}\frac{\partial^2 T^1_{\theta\theta\theta}}{\partial x\,\partial\theta}-\frac{2}{5}\frac{\partial Y^1_{zzz}}{\partial x}+\frac{2}{r}\frac{\partial Y^1_{x\theta z}}{\partial\theta}-\frac{8}{5r}\frac{\partial^2 T^1_{xx\theta}}{\partial x\,\partial\theta}\\
&+\frac{8}{5}\frac{\partial Y^1_{xxz}}{\partial x}-\frac{4}{5r^2}\frac{\partial^2 T^1_{\theta\theta x}}{\partial\theta^2}+\frac{3}{5}\frac{\partial^2 T^1_{\theta\theta x}}{\partial x^2}\\
&-\frac{2}{5}\frac{\partial Y^1_{\theta\theta z}}{\partial x}+\frac{3}{5}\frac{\partial^2 T^1_{zzx}}{\partial x^2}+\frac{1}{5r^2}\frac{\partial^2 T^1_{zzx}}{\partial\theta^2}\\
&+\frac{2}{5}\frac{\partial^2 T^1_{zz\theta}}{\partial x\,\partial\theta}-\frac{1}{2r}\frac{\partial Y^2_{\theta\theta}}{\partial\theta}+\frac{1}{2r^2}\frac{\partial T^2_{\theta\theta}}{\partial\theta}\\
&+\frac{1}{2r}\frac{\partial Y^2_{zz}}{\partial\theta}-\frac{1}{2}\frac{\partial Y^2_{x\theta}}{\partial x}-\frac{1}{2r}\frac{\partial^2 T^2_{xz}}{\partial x\,\partial\theta}-\frac{1}{2r}Y^2_{\theta z}\\
&-\frac{1}{2r^2}\frac{\partial^2 T^2_{\theta z}}{\partial\theta^2}=I_2\frac{\partial^2 u_0}{\partial t^2}+I_3\frac{\partial^2\phi_x}{\partial t^2},
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta}+\frac{\partial M_{x\theta}}{\partial x}-N_{\theta z}+\frac{M_{\theta z}}{r}-\frac{1}{r}\frac{\partial^2 T^0_x}{\partial x\,\partial\theta}\\
&-\frac{1}{r^2}\frac{\partial^2 T^0_\theta}{\partial\theta^2}+\frac{1}{r}\frac{\partial Y^0_z}{\partial\theta}+\frac{2}{5r}\frac{\partial^2 T^1_{xxx}}{\partial x\,\partial\theta}-\\
&\frac{2}{5r^2}\frac{\partial^2 T^1_{\theta\theta\theta}}{\partial\theta^2}+\frac{1}{5}\frac{\partial^2 T^1_{\theta\theta\theta}}{\partial x^2}-\frac{2}{5r}\frac{\partial Y^1_{zzz}}{\partial\theta}\\
&+2\frac{\partial Y^1_{x\theta z}}{\partial x}+\frac{1}{5r^2}\frac{\partial^2 T^1_{zzz}}{\partial\theta}-\frac{4}{5}\frac{\partial^2 T^1_{xx\theta}}{\partial x^2}\\
&-\frac{1}{r}\frac{\partial T^1_{x\theta z}}{\partial x}+\frac{3}{5r^2}\frac{\partial^2 T^1_{xx\theta}}{\partial\theta^2}-\frac{2}{5r}\frac{\partial Y^1_{xxz}}{\partial\theta}\\
&+\frac{1}{5r^2}\frac{\partial T^1_{xxz}}{\partial\theta}-\frac{8}{5r}\frac{\partial^2 T^1_{\theta\theta x}}{\partial x\,\partial\theta}+\frac{8}{5r}\frac{\partial Y^1_{\theta\theta z}}{\partial\theta}\\
&-\frac{4}{5r^2}\frac{\partial T^1_{\theta\theta z}}{\partial\theta}+\frac{2}{5r}\frac{\partial^2 T^1_{zzx}}{\partial x\,\partial\theta}+\frac{3}{5r^2}\frac{\partial^2 T^1_{zz\theta}}{\partial\theta^2}\\
&+\frac{1}{5}\frac{\partial^2 T^1_{zz\theta}}{\partial x^2}+\frac{1}{2r}\frac{\partial T^2_{xx}}{\partial x}-\frac{Y^1_{zz\theta}}{5r}+\frac{1}{2}\frac{\partial Y^2_{xx}}{\partial x}\\
&-\frac{1}{2r}\frac{\partial T^2_{\theta\theta}}{\partial x}-\frac{1}{2}\frac{\partial Y^2_{zz}}{\partial x}+\frac{1}{2r^2}\frac{\partial T^2_{x\theta}}{\partial\theta}\\
&+\frac{1}{2r}\frac{\partial Y^2_{x\theta}}{\partial\theta}+\frac{1}{2}\frac{\partial^2 T^2_{xz}}{\partial x^2}-\frac{Y^2_{xz}}{2r}+\frac{1}{2r}\frac{\partial^2 T^2_{\theta z}}{\partial x\,\partial\theta}\\
&=I_2\frac{\partial^2 v_0}{\partial t^2}+I_3\frac{\partial^2\phi_\theta}{\partial t^2},
\end{aligned}
\tag{50}
$$

In superior equations, I_1, I_2 and I_3 are the moment inertia and are assumed as

$$
\left(I_1, I_2, I_3\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\left(1, z, z^2\right)dz\,. \tag{51}
$$

Furthermore, the stress resultants can be defined as

$$
\begin{aligned}
&N_{ij}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{ij}dz\,, \qquad M_{ij}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{ij}z dz\,,\\
&Y^{(2)}_{ij}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m^s_{ij}dz\,, \qquad T^{(2)}_{ij}=\int_{-\frac{h}{2}}^{\frac{h}{2}}m^s_{ij}z dz\,,\\
&Y^{(1)}_{ijk}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau^1_{ijk}dz\,, \qquad T^{(1)}_{ijk}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau^1_{ijk}z dz\,,\\
&Y^{(0)}_{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}P_i dz\,, \qquad T^{(0)}_{i}=\int_{-\frac{h}{2}}^{\frac{h}{2}}P_i z dz\,.
\end{aligned}
\tag{52}
$$

Using Eqs. (19(a))-(19(e)), (20(a))-(20(c)), (21(a))-(21(j)) and (22(a))-(22(f)), the expansion of above stress resultants can be derived which are

$$N_{xx}=\frac{Eh}{1-\nu^2}\left(\begin{array}{l}\dfrac{\partial u_0}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2\\ +\dfrac{\nu}{r}\left(w_0+\dfrac{\partial v_0}{\partial \theta}+\dfrac{1}{2r}\left(\dfrac{\partial v_0}{\partial \theta}\right)^2\right)\end{array}\right), \tag{53a}$$

$$N_{\theta\theta}=\frac{Eh}{1-\nu^2}\left(\begin{array}{l}\dfrac{w_0}{r}+\dfrac{1}{r}\dfrac{\partial v_0}{\partial \theta}+\dfrac{1}{2r^2}\left(\dfrac{\partial w_0}{\partial \theta}\right)^2\\ +\nu\left(\dfrac{\partial u_0}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2\right)\end{array}\right), \tag{53b}$$

$$N_{x\theta}=\mu h\left(\frac{\partial v_0}{\partial x}+\frac{1}{r}\frac{\partial u_0}{\partial \theta}+\frac{1}{r}\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial \theta}\right)\right), \tag{53c}$$

$$N_{xz}=\mu h\left(\Phi_x+\frac{\partial w_0}{\partial x}\right), \tag{53d}$$

$$N_{\theta z}=\mu h\left(\frac{1}{r}\frac{\partial w_0}{\partial \theta}+\Phi_\theta-\frac{v_0}{r}\right), \tag{53e}$$

$$M_{xx}=\frac{1}{12}\frac{Eh^3}{1-\nu^2}\left(\frac{\partial \Phi_x}{\partial x}+\frac{\nu}{r}\frac{\partial \Phi_\theta}{\partial \theta}\right), \tag{54a}$$

$$M_{\theta\theta}=\frac{1}{12}\frac{Eh^3}{1-\nu^2}\left(\nu\frac{\partial \Phi_x}{\partial x}+\frac{1}{r}\frac{\partial \Phi_\theta}{\partial \theta}\right), \tag{54b}$$

$$M_{x\theta}=\frac{1}{12}\mu h^3\left(\frac{\partial \Phi_\theta}{\partial x}+\frac{1}{r}\frac{\partial \Phi_x}{\partial \theta}\right), \tag{54c}$$

$$Y_x^{0}=2\mu l_0^2 h\left(\begin{array}{l}\dfrac{\partial^2 u_0}{\partial x^2}+\dfrac{1}{r}\dfrac{\partial w_0}{\partial x}+\dfrac{1}{r}\dfrac{\partial^2 v_0}{\partial x\partial \theta}\\ +\left(\dfrac{\partial w_0}{\partial x}\right)\left(\dfrac{\partial^2 w_0}{\partial x^2}\right)+\dfrac{1}{r^2}\left(\dfrac{\partial w_0}{\partial \theta}\right)\left(\dfrac{\partial^2 w_0}{\partial x\partial \theta}\right)\end{array}\right), \tag{55a}$$

$$Y_\theta^{0}=2\mu l_0^2 h\left(\begin{array}{l}\dfrac{1}{r}\dfrac{\partial^2 u_0}{\partial x\partial \theta}+\dfrac{1}{r^2}\dfrac{\partial w_0}{\partial \theta}+\dfrac{1}{r^2}\dfrac{\partial^2 v_0}{\partial \theta^2}\\ +\dfrac{1}{r}\left(\dfrac{\partial w_0}{\partial x}\right)\left(\dfrac{\partial^2 w_0}{\partial x\partial \theta}\right)+\dfrac{1}{r^3}\left(\dfrac{\partial w_0}{\partial \theta}\right)\left(\dfrac{\partial^2 w_0}{\partial \theta^2}\right)\end{array}\right), \tag{55b}$$

$$Y_z^{0}=2\mu l_0^2 h\left(\frac{\partial \Phi_x}{\partial x}+\frac{1}{r}\frac{\partial \Phi_\theta}{\partial \theta}\right), \tag{55c}$$

$$\begin{aligned}Y_{xxx}^{1}=\frac{2}{5}\mu l_1^2 h\Bigg(&2\frac{\partial^2 u_0}{\partial x^2}-\frac{2}{r^2}\frac{\partial w_0}{\partial \theta}\frac{\partial^2 w_0}{\partial x\partial \theta}\\ &+2\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}-\frac{1}{r}\frac{\partial^2 v_0}{\partial x\partial \theta}-\frac{1}{r}\frac{\partial w_0}{\partial x}\\ &-\frac{1}{r^2}\frac{\partial^2 u_0}{\partial \theta^2}-\frac{\partial^2 v_0}{\partial x\partial \theta}-\frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial \theta^2}\Bigg)\end{aligned} \tag{56a}$$

$$Y_{\theta\theta\theta}^{1}=\frac{2}{5}\mu l_1^2 h\left(\begin{array}{l}\dfrac{2}{r^2}\dfrac{\partial^2 v_0}{\partial \theta^2}+\dfrac{2}{r^2}\dfrac{\partial w_0}{\partial \theta}+\dfrac{2}{r^3}\dfrac{\partial w_0}{\partial \theta}\dfrac{\partial^2 w_0}{\partial \theta^2}\\ -\dfrac{2}{r}\dfrac{\partial w_0}{\partial x}\dfrac{\partial^2 w_0}{\partial x\partial \theta}-\dfrac{2}{r}\dfrac{\partial^2 u_0}{\partial x\partial \theta}-\dfrac{\partial^2 v_0}{\partial x^2}\end{array}\right), \tag{56b}$$

$$Y_{zzz}^{1}=\frac{2}{5}\mu l_1^2 h\left(\begin{array}{l}-2\dfrac{\partial \Phi_x}{\partial x}-\dfrac{2}{r}\dfrac{\partial \Phi_\theta}{\partial \theta}\\ -\dfrac{\partial^2 w_0}{\partial x^2}-\dfrac{1}{r^2}\dfrac{\partial^2 w_0}{\partial \theta^2}\end{array}\right), \tag{56c}$$

$$\begin{aligned}Y_{xx\theta}^{1}=\frac{2}{15}\mu l_1^2 h\Bigg(&4\frac{\partial^2 v_0}{\partial x^2}+\frac{8}{r}\frac{\partial^2 u_0}{\partial x\partial \theta}\\ &-\frac{3}{r^2}\frac{\partial w_0}{\partial \theta}-\frac{3}{r^2}\frac{\partial^2 v_0}{\partial \theta^2}+\frac{8}{r}\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial^2 w_0}{\partial x\partial \theta}\right)\\ &+\frac{4}{r}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial w_0}{\partial \theta}\right)-\frac{3}{r^3}\left(\frac{\partial w_0}{\partial \theta}\right)\left(\frac{\partial^2 w_0}{\partial \theta^2}\right)\Bigg),\end{aligned} \tag{56d}$$

$$Y_{xxz}^{1}=\frac{2}{15}\mu l_1^2 h\left(\begin{array}{l}4\dfrac{\partial^2 w_0}{\partial x^2}+8\dfrac{\partial \Phi_x}{\partial x}-\dfrac{2}{r}\dfrac{\partial \Phi_\theta}{\partial \theta}\\ -\dfrac{1}{r}\dfrac{\partial^2 w_0}{\partial \theta^2}+\dfrac{1}{r^2}\dfrac{\partial v_0}{\partial \theta}\end{array}\right), \tag{56e}$$

$$\begin{aligned}Y_{\theta\theta x}^{1}=\frac{2}{15}\mu l_1^2 h\Bigg(&\frac{8}{r}\frac{\partial^2 v_0}{\partial x\partial \theta}+\frac{4}{r^2}\frac{\partial^2 u_0}{\partial \theta^2}\\ &+\frac{4}{r}\frac{\partial w_0}{\partial x}-3\frac{\partial^2 u_0}{\partial x^2}+\frac{8}{r^2}\frac{\partial^2 w_0}{\partial x\partial \theta}\frac{\partial w_0}{\partial \theta}\\ &+\frac{4}{r^2}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial \theta^2}-3\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\Bigg)\end{aligned} \tag{56f}$$

$$Y^{1}_{\theta\theta z}=\frac{2}{15}\mu l_1^2 h\begin{pmatrix}\frac{4}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}+\frac{8}{r}\frac{\partial\Phi_\theta}{\partial\theta}\\ -\frac{2}{15}\frac{\partial\Phi_x}{\partial x}-\frac{1}{15}\frac{\partial^2 w_0}{\partial x^2}-\frac{4}{r^2}\frac{\partial v_0}{\partial\theta}\end{pmatrix}, \tag{56g}$$

$$\begin{aligned}Y^{1}_{zzx}&=\frac{2}{15}\mu l_1^2 h\Bigg(-3\frac{\partial^2 u_0}{\partial x^2}-\frac{1}{r}\frac{\partial w_0}{\partial x}\\&-\frac{1}{15r^2}\frac{\partial^2 u_0}{\partial\theta^2}-\frac{2}{r}\frac{\partial^2 v_0}{\partial x\partial\theta}-\frac{2}{15r^2}\frac{\partial^2 w_0}{\partial x\partial\theta}\frac{\partial w_0}{\partial\theta}\\&-3\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}-\frac{1}{r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial\theta^2}\Bigg)\end{aligned} \tag{56h}$$

$$\begin{aligned}Y^{1}_{zz\theta}&=\frac{2}{15}\mu l_1^2 h\Bigg(-\frac{2}{r}\frac{\partial^2 u_0}{\partial x\partial\theta}-\frac{3}{r^2}\frac{\partial w_0}{\partial\theta}\\&-\frac{3}{r^2}\frac{\partial^2 v_0}{\partial\theta^2}-\frac{\partial^2 v_0}{\partial x^2}-\frac{2}{r}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial\theta}\\&-\frac{3}{r^3}\frac{\partial w_0}{\partial\theta}\frac{\partial^2 w_0}{\partial\theta^2}-\frac{1}{r}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial\theta}\Bigg),\end{aligned} \tag{56i}$$

$$Y^{1}_{x\theta z}=\frac{1}{3}\mu l_1^2 h\begin{pmatrix}\frac{2}{r}\frac{\partial^2 w_0}{\partial x\partial\theta}+2\frac{\partial\Phi_\theta}{\partial x}\\ -\frac{1}{r}\frac{\partial v_0}{\partial x}+\frac{2}{r}\frac{\partial\Phi_x}{\partial\theta}\end{pmatrix}, \tag{56j}$$

$$Y^{2}_{xx}=\mu l_2^2 h\left(\frac{1}{r}\frac{\partial v_0}{\partial x}+\frac{\partial\Phi_\theta}{\partial x}-\frac{1}{r}\frac{\partial^2 w_0}{\partial x\partial\theta}\right), \tag{57a}$$

$$Y^{2}_{\theta\theta}=\mu l_2^2 h\begin{pmatrix}\frac{1}{r}\frac{\partial^2 w_0}{\partial x\partial\theta}-\frac{1}{r}\frac{\partial\Phi_x}{\partial\theta}\\ -\frac{1}{r}\frac{\partial v}{\partial x}+\frac{1}{r^2}\frac{\partial u_0}{\partial\theta}\end{pmatrix}, \tag{57b}$$

$$Y^{2}_{zz}=\mu l_2^2 h\left(\frac{1}{r}\frac{\partial\Phi_x}{\partial\theta}-\frac{\partial\Phi_\theta}{\partial x}\right), \tag{57c}$$

$$Y^{2}_{x\theta}=\frac{1}{2}\mu l_2^2 h\begin{pmatrix}\frac{\partial^2 w_0}{\partial x^2}-\frac{\partial\Phi_x}{\partial x}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\\ +\frac{1}{r}\frac{\partial\Phi_\theta}{\partial\theta}+\frac{1}{r^2}\frac{\partial v_0}{\partial\theta}\end{pmatrix}, \tag{57d}$$

$$Y^{2}_{xz}=\frac{1}{2}\mu l_2^2 h\left(-\frac{\partial^2 v_0}{\partial x^2}+\frac{1}{r}\frac{\partial^2 u_0}{\partial x\partial\theta}+\frac{1}{r}\Phi_\theta\right), \tag{57e}$$

$$Y^{2}_{\theta z}=\frac{1}{2}\mu l_2^2 h\begin{pmatrix}-\frac{1}{r}\frac{\partial w_0}{\partial x}+\frac{1}{r}\Phi_x\\ +\frac{1}{r^2}\frac{\partial^2 u_0}{\partial\theta^2}-\frac{1}{r}\frac{\partial^2 v_0}{\partial x\partial\theta}\end{pmatrix}, \tag{57f}$$

$$T^{0}_{x}=\frac{1}{6}\mu l_0^2 h^3\left(\frac{\partial^2\Phi_x}{\partial x^2}+\frac{1}{r}\frac{\partial^2\Phi_\theta}{\partial x\partial\theta}\right), \tag{58a}$$

$$T^{0}_{\theta}=\frac{1}{6}\mu l_0^2 h^3\left(\frac{1}{r}\frac{\partial^2\Phi_x}{\partial x\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi_\theta}{\partial\theta^2}\right), \tag{58b}$$

$$T^{0}_{z}=0, \tag{58c}$$

$$T^{1}_{xxx}=\frac{1}{30}\mu l_1^2 h^3\left(2\frac{\partial^2\Phi_x}{\partial x^2}-\frac{2}{r}\frac{\partial^2\Phi_\theta}{\partial x\partial\theta}-\frac{1}{r^2}\frac{\partial^2\Phi_x}{\partial\theta^2}\right), \tag{59a}$$

$$T^{1}_{\theta\theta\theta}=\frac{1}{30}\mu l_1^2 h^3\left(\frac{2}{r^2}\frac{\partial^2\Phi_\theta}{\partial\theta^2}-\frac{2}{r}\frac{\partial^2\Phi_x}{\partial x\partial\theta}-\frac{\partial^2\Phi_\theta}{\partial x^2}\right), \tag{59b}$$

$$T^{1}_{zzz}=0, \tag{59c}$$

$$T^{1}_{xx\theta}=\frac{1}{90}\mu l_1^2 h^3\left(4\frac{\partial^2\Phi_\theta}{\partial x^2}+\frac{8}{r}\frac{\partial^2\Phi_x}{\partial x\partial\theta}-\frac{3}{r^2}\frac{\partial^2\Phi_\theta}{\partial\theta^2}\right), \tag{59d}$$

$$T^{1}_{xxz}=0, \tag{59e}$$

$$T^{1}_{\theta\theta x}=\frac{1}{90}\mu l_1^2 h^3\left(\frac{8}{r}\frac{\partial^2\Phi_\theta}{\partial x\partial\theta}+\frac{4}{r}\frac{\partial^2\Phi_x}{\partial\theta^2}-3\frac{\partial^2\Phi_x}{\partial x^2}\right), \tag{59f}$$

$$T^{1}_{\theta\theta z}=0, \tag{59g}$$

$$T^{1}_{zzx} = \frac{1}{30}\mu l_1^2 h^3 \left(-3\frac{\partial^2 \Phi_x}{\partial x^2} - \frac{2}{r}\frac{\partial^2 \Phi_\theta}{\partial x\,\partial\theta} - \frac{1}{r^2}\frac{\partial^2 \Phi_x}{\partial\theta^2} \right), \tag{59h}$$

$$T^{1}_{zz\theta} = \frac{1}{30}\mu l_1^2 h^3 \left(-\frac{2}{r}\frac{\partial^2 \Phi_x}{\partial x\,\partial\theta} - \frac{3}{r^2}\frac{\partial^2 \Phi_\theta}{\partial\theta^2} - \frac{\partial^2 \Phi_\theta}{\partial x^2} \right), \tag{59i}$$

$$T^{1}_{x\theta z} = 0, \tag{59j}$$

$$T^{2}_{xx} = \frac{1}{12}\frac{\mu l_2^2 h^3}{r}\left(\frac{\partial \Phi_\theta}{\partial x} \right), \tag{60a}$$

$$T^{2}_{\theta\theta} = \frac{1}{12}\frac{\mu l_2^2 h^3}{r}\left(-\frac{\partial \Phi_\theta}{\partial x} + \frac{1}{r}\frac{\partial \Phi_x}{\partial\theta} \right), \tag{60b}$$

$$T^{2}_{zz} = 0, \tag{60c}$$

$$T^{2}_{x\theta} = \frac{1}{24}\frac{\mu l_2^2 h^3}{r^2}\left(\frac{\partial \Phi_\theta}{\partial\theta} \right), \tag{60d}$$

$$T^{2}_{xz} = \frac{1}{24}\frac{\mu l_2^2 h^3}{r}\left(-\frac{\partial^2 \Phi_x}{\partial x^2} + \frac{1}{r}\frac{\partial^2 \Phi_x}{\partial x\,\partial\theta} \right). \tag{60e}$$

It is notable that conniving l_0, l_1 and l_2 in the obtained motion equations leads to achieve the motion equations for the two reduced form of the SGT i.e. the modified couple stress theory ($l_0 = l_1 = 0$) and the classical cylindrical shell theory ($l_0 = l_1 = l_2 = 0$).

In addition, based on Hamilton's principle, the boundary conditions can be written as

$$\begin{aligned}
&\left(-\frac{N_{xx}}{2} + \frac{\partial Y^{0}_{x}}{\partial x} + \frac{1}{r}\frac{\partial Y^{0}_{\theta}}{\partial\theta} + \frac{2}{5}\frac{\partial Y^{1}_{xxx}}{\partial x} - \frac{2}{5r}\frac{\partial Y^{1}_{\theta\theta\theta}}{\partial\theta} \right.\\
&+ \frac{8}{5r}\frac{\partial Y^{1}_{xx\theta}}{\partial\theta} - \frac{3}{5}\frac{\partial Y^{1}_{\theta\theta x}}{\partial x} - \frac{3}{5}\frac{\partial Y^{1}_{zzx}}{\partial x}\\
&\left. - \frac{2}{5r}\frac{\partial Y^{1}_{zz\theta}}{\partial\theta} + \frac{1}{2r}\frac{\partial Y^{2}_{xz}}{\partial\theta} \right)\Bigg|_{x=0,L} = 0\\
&\qquad OR \qquad \delta u_0\big|_{x=0,L} = 0
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\Big(-N_{x\theta} + \frac{1}{r}\frac{\partial Y^{0}_{x}}{\partial\theta} - \frac{2}{5r}\frac{\partial Y^{1}_{xxx}}{\partial\theta} - \frac{1}{5}\frac{\partial Y^{1}_{\theta\theta\theta}}{\partial x} + \frac{4}{5}\frac{\partial Y^{1}_{xx\theta}}{\partial x}\\
&+ \frac{Y^{1}_{x\theta z}}{r} + \frac{8}{5r}\frac{\partial Y^{1}_{\theta\theta x}}{\partial\theta} - \frac{2}{5r}\frac{\partial Y^{1}_{zzx}}{\partial\theta}\\
&- \frac{1}{5}\frac{\partial Y^{1}_{zz\theta}}{\partial x} - \frac{Y^{2}_{xx}}{2r} + \frac{Y^{2}_{\theta\theta}}{2r} - \frac{1}{2}\frac{\partial Y^{2}_{xz}}{\partial x} - \frac{1}{2r}\frac{\partial Y^{2}_{\theta z}}{\partial\theta} \Big)\Bigg|_{x=0,L} = 0 \qquad OR \qquad \delta v_0\big|_{x=0,L} = 0
\end{aligned} \tag{62}$$

$$\begin{aligned}
&\left(-N_{xx}\frac{\partial w_0}{\partial x} - \frac{1}{r}N_{x\theta}\frac{\partial w_0}{\partial\theta} - N_{xz} + \frac{\partial Y^{0}_{x}}{\partial x}\frac{\partial w_0}{\partial x} - \right.\\
&\frac{Y^{0}_{x}}{r} + \frac{1}{r^2}\frac{\partial Y^{0}_{x}}{\partial\theta}\frac{\partial w_0}{\partial\theta} + \frac{Y^{0}_{x}}{r^2}\frac{\partial^2 w_0}{\partial\theta^2} + \frac{1}{r}\frac{\partial Y^{0}_{\theta}}{\partial\theta}\frac{\partial w_0}{\partial x}\\
&+ \frac{2}{5}\frac{\partial Y^{1}_{xxx}}{\partial x}\frac{\partial w_0}{\partial x} + \frac{Y^{1}_{xxx}}{5r} + \frac{Y^{1}_{xxx}}{5r^2}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{2}{5r}\frac{\partial Y^{1}_{\theta\theta\theta}}{\partial\theta}\frac{\partial w_0}{\partial x}\\
&- \frac{1}{5r}\frac{\partial Y^{1}_{\theta\theta\theta}}{\partial x}\frac{\partial w_0}{\partial\theta} - \frac{1}{5r}Y^{1}_{\theta\theta\theta}\frac{\partial^2 w_0}{\partial x\,\partial\theta} - \frac{1}{5}\frac{\partial Y^{1}_{zzz}}{\partial x}\\
&+ \frac{2}{r}\frac{\partial Y^{1}_{x\theta z}}{\partial\theta} + \frac{4}{5r}\frac{\partial Y^{1}_{xx\theta}}{\partial x}\frac{\partial w_0}{\partial\theta} + \frac{4}{5r}Y^{1}_{xxx}\frac{\partial^2 w_0}{\partial x\,\partial\theta} +\\
&\frac{4}{5}\frac{\partial Y^{1}_{xxz}}{\partial x} + \frac{8}{5r^2}\frac{\partial Y^{1}_{\theta\theta x}}{\partial\theta}\frac{\partial w_0}{\partial\theta} + \frac{8}{5r^2}Y^{1}_{\theta\theta x}\frac{\partial^2 w_0}{\partial\theta^2} - \frac{4}{5r}Y^{1}_{\theta\theta x}\\
&- \frac{3}{5}\frac{\partial Y^{1}_{\theta\theta x}}{\partial x}\frac{\partial w_0}{\partial x} - \frac{1}{5}\frac{\partial Y^{1}_{\theta\theta z}}{\partial x} - \frac{3}{5}\frac{\partial Y^{1}_{zzx}}{\partial x}\frac{\partial w_0}{\partial x} + \frac{1}{5r}Y^{1}_{zzx}\\
&- \frac{2}{5r}\frac{\partial Y^{1}_{zz\theta}}{\partial\theta}\frac{\partial w_0}{\partial x} - \frac{1}{5r}\frac{\partial Y^{1}_{zz\theta}}{\partial x}\frac{\partial w_0}{\partial\theta} - \frac{1}{5r}Y^{1}_{zz\theta}\frac{\partial^2 w_0}{\partial x\,\partial\theta}\\
&- \frac{1}{2r}\frac{\partial Y^{2}_{xx}}{\partial\theta} + \frac{1}{2r}\frac{\partial Y^{2}_{\theta\theta}}{\partial\theta} + \frac{1}{2}\frac{\partial Y^{2}_{x\theta}}{\partial x} + \frac{1}{2r}Y^{2}_{\theta z} +\\
&\frac{1}{5r}Y^{1}_{zzx}\frac{\partial^2 w_0}{\partial\theta^2} + \frac{8}{5r}\frac{\partial Y^{1}_{xx\theta}}{\partial\theta}\frac{\partial w_0}{\partial x} - \frac{4}{5r^2}Y^{1}_{\theta\theta x}\frac{\partial^2 w_0}{\partial\theta^2}\\
&\left.\vphantom{\frac{1}{1}}\right)\Big|_{x=0,L} = 0 \quad OR \quad \delta w_0\Big|_{x=0,L} = 0
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\Big(-M_{xx} + \frac{\partial T^{0}_{x}}{\partial x} + \frac{1}{r}\frac{\partial T^{0}_{\theta}}{\partial\theta} - Y^{0}_{z} + \frac{2}{5}\frac{\partial T^{1}_{xxx}}{\partial x} - \frac{2}{5r}\frac{\partial T^{1}_{\theta\theta\theta}}{\partial\theta}\\
&+ \frac{2}{5}Y^{1}_{zzz} + \frac{8}{5r}\frac{\partial T^{1}_{xx\theta}}{\partial\theta} - \frac{8}{5}Y^{1}_{xxz} - \frac{3}{5}\frac{\partial T^{1}_{\theta\theta x}}{\partial x}\\
&+ \frac{2}{5}Y^{1}_{\theta\theta z} - \frac{3}{5}\frac{\partial T^{1}_{zzx}}{\partial x} - \frac{2}{5r}\frac{\partial T^{1}_{zz\theta}}{\partial\theta} + \frac{Y^{2}_{x\theta}}{2} + \frac{1}{2r}\frac{\partial T^{2}_{xz}}{\partial\theta} \Big)\Bigg|_{x=0,L} = 0 \qquad OR \qquad \delta\phi_x\Big|_{x=0,L} = 0
\end{aligned} \tag{64}$$

$$\left(-M_{x\theta}+\frac{1}{r}\frac{\partial T_x^{0}}{\partial\theta}-\frac{2}{5r}\frac{\partial T_{xxx}^{1}}{\partial\theta}-\frac{1}{5}\frac{\partial T_{\theta\theta\theta}^{1}}{\partial x}-2Y_{x\theta z}^{1}+\frac{4}{5}\frac{\partial T_{xx\theta}^{1}}{\partial x}+\frac{8}{5r}\frac{\partial T_{\theta\theta x}^{1}}{\partial\theta}-\frac{2}{5r}\frac{\partial T_{zzx}^{1}}{\partial\theta}-\frac{1}{5}\frac{\partial T_{zz\theta}^{1}}{\partial x}-\frac{1}{2r}T_{xx}^{2}+\frac{1}{2r}T_{\theta\theta}^{2}+\frac{Y_{zz}^{2}}{2}-\frac{1}{2}\frac{\partial T_{xz}^{2}}{\partial x}-\frac{1}{2r}\frac{\partial T_{\theta z}^{2}}{\partial\theta}\right)\Bigg|_{x=0,L}=0 \quad OR \quad \delta\phi_\theta\Big|_{x=0,L}=0 \tag{65}$$

$$\left(\frac{Y_{\theta\theta\theta}^{1}}{5}-\frac{4Y_{xx\theta}^{1}}{5}+\frac{Y_{zz\theta}^{1}}{5}+\frac{Y_{xz}^{2}}{2}\right)\Bigg|_{x=0,L}=0 \quad OR \quad \delta\frac{\partial v_0}{\partial x}\Bigg|_{x=0,L}=0 \tag{66}$$

$$\left(-Y_x^{0}\frac{\partial w_0}{\partial x}-\frac{2Y_{xxx}^{1}}{5}\frac{\partial w_0}{\partial x}+\frac{Y_{\theta\theta\theta}^{1}}{5r}\frac{\partial w_0}{\partial\theta}+\frac{Y_{zzz}^{1}}{5}-\frac{4Y_{xx\theta}^{1}}{5r}\frac{\partial w_0}{\partial\theta}-\frac{4Y_{xxz}^{1}}{5}+\frac{3Y_{\theta\theta x}^{1}}{5}\frac{\partial w_0}{\partial x}+\frac{Y_{\theta\theta z}^{1}}{5}+\frac{3Y_{zzx}^{1}}{5}\frac{\partial w_0}{\partial x}+\frac{Y_{zz\theta}^{1}}{5r}\frac{\partial w_0}{\partial\theta}-\frac{Y_{x\theta}^{2}}{2}\right)\Bigg|_{x=0,L}=0 \quad OR \quad \delta\frac{\partial w_0}{\partial x}\Bigg|_{x=0,L}=0 \tag{67}$$

$$\left(\frac{T_{\theta\theta\theta}^{1}}{5}-\frac{4T_{xx\theta}^{1}}{5}+\frac{T_{zz\theta}^{1}}{5}+\frac{T_{xz}^{2}}{2}\right)\Bigg|_{x=0,L}=0 \quad OR \quad \delta\frac{\partial\phi_\theta}{\partial x}\Bigg|_{x=0,L}=0 \tag{68}$$

- **Simply supported-Simply supported (SS)**

$$@x=0,L \Rightarrow v=w=\phi_\theta=0,\quad \mathrm{E}qs.(61),(64),(66),(67),(68)=0 \tag{69}$$

- **Clamped- Clamped (CC)**

$$@x=0,L \Rightarrow u=v=w=\phi_x=\phi_\theta=0,\quad \mathrm{Eqs}.(66),(67),(68) \tag{70}$$

- **Clamped- Simply supported (CS)**

$$@x=0 \Rightarrow u=v=w=\phi_x=\phi_\theta=0,\quad \mathrm{Eqs}.(66),(67),(68),\quad @x=L \Rightarrow v=w=\phi_\theta=0,\quad \mathrm{E}qs.(61),(64),(66),(67),(68)=0. \tag{71}$$

3. Solution method

3.1 DQM

There is a lot of numerical method to solve the initialand/or boundary value problems which occur in engineering domain. Some of the common numerical methods are finite element method (FEM), Galerkin method, finite difference method, GDQM, etc. FEM and FD method for higher order modes require to a great number of grid points. Therefore, these solution methods for all these points need to more CPU time, while the GDQM has several benefits that are listed as below (Civalek 2004):

1. GDQM is a powerful method which can be used to solve numerical problems in the analysis of structural and dynamical systems.
2. The accuracy and convergence of the GDQM are higher than FEM.
3. GDQM is an accurate method for solution of nonlinear differential equations in approximation of the derivatives.
4. This method can easily and exactly satisfy a variety of boundary conditions and require much less formulation and programming effort.
5. Recently, GDQM has been extended to handle irregular shaped.

The DQM is utilized to solve the equation of motion. In this numerical method, the partial differential equations can be estimated by a first order algebraic equation using opportune weighting coefficients. Based on a mathematical point a view, the implementation of the DQM to a partial differential equation can be written as a function of x and θ as below (Kolahchi *et al.* 2016)

$$\frac{d^n f_x(x_i,\theta_j)}{dx^n}=\sum_{k=1}^{N_x}A_{ik}^{(n)}f(x_k,\theta_j) \quad n=1,\ldots,N_x-1, \tag{72}$$

$$\frac{d^m f_y(x_i,\theta_j)}{d\theta^m}=\sum_{l=1}^{N_\theta}B_{jl}^{(m)}f(x_i,\theta_l) \quad m=1,\ldots,N_\theta-1, \tag{73}$$

$$\frac{d^{n+m} f_{xy}(x_i,\theta_j)}{dx^n d\theta^m}=\sum_{k=1}^{N_x}\sum_{l=1}^{N_\theta}A_{ik}^{(n)}B_{jl}^{(m)}f(x_k,\theta_l), \tag{74}$$

where $A_{ik}^{(n)}$ and $B_{jl}^{(m)}$ are the weighting coefficients associated with nth-order partial derivative of $F(x,\theta)$ with regard to x and mth-order derivative with regard to θ at the discrete points x_i and θ_i, respectively. The election of the grid points positions is accomplished using Chebyshev polynomials as below (Kolahchi *et al.* 2016)

$$x_i=\frac{L}{2}\left[1-\cos\left(\frac{i-1}{N_x-1}\right)\pi\right] \quad i=1,\ldots,N_x \tag{75}$$

$$\theta_i = \frac{2\pi}{2}\left[1-\cos\left(\frac{i-1}{N_\theta - 1}\right)\pi\right] \qquad i = 1,...,N_\theta \tag{76}$$

So, the weighting coefficients can be calculated as (Kolahchi *et al*. 2016)

$$A_{ij}^{(1)} = \begin{cases} \dfrac{M(x_i)}{(x_i - x_j)M(x_j)} & for \quad i \neq j, \quad i, j = 1,2,..., N_x \\ -\sum\limits_{\substack{j=1 \\ i\neq j}}^{N_x} A_{ij}^{(1)} & for \quad i = j, \quad i, j = 1,2,..., N_x \end{cases} \tag{77}$$

$$B_{ij}^{(1)} = \begin{cases} \dfrac{P(\theta_i)}{(\theta_i - \theta_j)P(\theta_j)} & for \quad i \neq j, \quad i, j = 1,2,..., N_\theta, \\ -\sum\limits_{\substack{j=1 \\ i\neq j}}^{N_\theta} B_{ij}^{(1)} & for \quad i = j, \quad i, j = 1,2,..., N_\theta \end{cases} \tag{78}$$

in which

$$M(x_i) = \prod_{\substack{j=1 \\ j\neq i}}^{N_x} (x_i - x_j) \tag{79}$$

$$P(\theta_i) = \prod_{\substack{j=1 \\ j\neq i}}^{N_\theta} (\theta_i - \theta_j) \tag{80}$$

and for higher order derivatives we have

$$A_{ij}^{(n)} = n\left(A_{ii}^{(n-1)}A_{ij}^{(1)} - \frac{A_{ij}^{(n-1)}}{(x_i - x_j)}\right) \tag{81}$$

$$B_{ij}^{(m)} = m\left(B_{ii}^{(m-1)}B_{ij}^{(1)} - \frac{B_{ij}^{(m-1)}}{(\theta_i - \theta_j)}\right) \tag{82}$$

Eventually, the motion equations can be expressed in the matrix form as below

$$\left\{[K_L + K_{NL}][d] + [M][\ddot{d}]\right\} = [f\ \cos(\omega \text{t})], \tag{83}$$

where $[d] = [u_0\ v_0\ w_0\ \phi_x\ \phi_\theta]^T$ is the displacement vector; $[K_L]$ and $[K_{NL}]$ represent the linear and nonlinear stiffness matrices, respectively. Furthermore $[M]$ is the mass matrix.

3.2 Newmark time integration scheme

Here, the average acceleration method of Newmark-β (Simsek and Kocatu¨rk, 2009) in conjunction with an iteration method is used. Based on this method, the time domain Eq. (76) can be reduced to the following set of nonlinear algebraic equations

$$K^*(d_{i+1}) = Q_{i+1}, \tag{85}$$

where subscript i+1 shows the number of steps for the concerned time t=t_{i+1}; $K^*(d_{i+1})$ is the effective stiffness matrix and Q_{i+1} is the effective load vector, which can be written as

$$K^*(d_{i+1}) = K_L + K_{NL}(d_{i+1}) + \alpha_0 M, \tag{85}$$

$$Q_{i+1}^* = Q_{i+1} + M\left(\alpha_0 d_i + \alpha_1 \dot{d}_i + \alpha_2 \ddot{d}_i\right), \tag{86}$$

where (Simsek and Kocatűrk 2009)

$$\begin{aligned} &\alpha_0 = \frac{1}{\chi \Delta t^2}, \qquad \alpha_1 = \frac{1}{\chi \Delta t}, \qquad \alpha_2 = \frac{1}{2\chi} - 1, \\ &\alpha_3 = \Delta t\,(1-\gamma), \qquad \alpha_4 = \Delta t \gamma, \end{aligned} \tag{87}$$

and $\gamma = 0.5$ and $\chi = 0.25$ (Simsek and Kocatu¨rk, 2009). Based on this iteration method, Eq. (85) at any fixed time can be solved; and then, the new acceleration and velocity vectors can be obtained as follows

$$\ddot{d}_{i+1} = \alpha_0(d_{i+1} - d_i) - \alpha_1 \dot{d}_i - \alpha_2 \ddot{d}_i, \tag{88}$$

$$\dot{d}_{i+1} = \dot{d}_i + \alpha_3 \ddot{d}_i + \alpha_4 \ddot{d}_{i+1}, \tag{89}$$

This procedure should be repeated for each time step for obtaining the frequency–response curves of the structure.

4. Numerical results

In this section, the geometrical and mechanical properties of the structure are presented to obtain the numerical results and examine the effect of various parameters on the behavior of the system. The length to radius ratio (*L/R*) and thickness to radius ration of (*h/R*) of the micro cylindrical shell are considered to be 3 and 0.1, respectively. The structure is made of polystyrene with the Young's modulus of $E_m = 1190\ MPa$ and Poisson's ratio of $\nu_m = 0.3$ (Kolahchi *et al*. 2016) as the matrix material which is reinforced with CNTs with the Hills elastic modulus (Shi and Feng, 2004). In this section, initially, the validation of present work and convergence and accuracy of DQM will be implement and then the effects of diverse parameters on the dimensionless frequency ($\Omega = \omega R\sqrt{\rho / E}$) and nonlinear frequency-response curves of the structure will studied.

4.1 Validation

In order to ensure the accuracy and validity of the proposed numerical results, regardless of some terms including CNTs as reinforcer, magnetic field, l_0 and l_1, the present results are compared with the other available literature. For this purpose, based on modified couple stress theory, the vibrational behavior of a nano cylindrical shell with Young's modulus of $E = 1.06\,TPa$, mass density of $\rho = 2300\ Kg/m^3$, Poisson's ratio of $\nu = 0.3$, length to radius ratio of $L/R = 1$ and radius of $R = 2\ nm$, is considered. In Table 1, the variation of dimensionless frequency parameter for different thickness to radius ratio (*h/R*) and circumferential wave number is listed. The obtained results on the basis of DQM are compared with those reported by Tadi Beni *et al*. (2015). It can be seen that the results of present work have a good agreement with the obtained results by Tadi Beni *et al*. (2015).

4.2 DQM convergence

In Fig. 2, the convergence and accuracy of DQM are studied. According to the plotted diagram it can be stated, the appropriate number of grid points to achieve the accurate results, is fourteen for two cases of normalized frequency (excitation frequency to linear frequency ratio). Therefore, in order to investigate the normalized frequency of the structure, 15 grid points are selected.

4.3 Parametric study

Fig. 3 shows the effect of CNTs volume percent on the frequency-response curve of the system. As depicted, increasing the CNTs volume percent leads to the decreases of amplitude peak. Accordingly, the maximum amplitude happens at higher frequencies. The reason is that the increasing of the volume fraction of CNTs enhances the rigidity of the material.

In Fig. 4, the effect of various boundary conditions on the frequency-response curve of the system is shown. Three different boundary condition types are evaluated, including two ends simply-supported (SS), one end clamped and other one simply-supported (CS) and two ends clamped (CC). It is obvious that the amplitude frequency of the CC case is lower than two other boundary conditions. It can be inferred that the normalized frequency of the system in CC case occurs at lower amplitudes and thus, it can be said that choosing CC boundary condition type yields stiffer structure.

Table 1 Validation of this work

h/R	*n*	Classical theory (Tadi Beni 2015)	MCST (Tadi Beni 2015)	Classical theory Present work	MCST Present work
0.1	1	0.933	1.126	0.9335	1.1264
	2	0.776	1.0688	0.7764	1.0688
	3	0.713	1.207	0.7132	1.2071
0.2	1	1.048	1.537	1.0483	1.5373
	2	0.971	1.590	0.9714	1.5901
	3	1.052	1.928	1.0522	1.9284
0.3	1	1.181	1.878	1.1812	1.8787
	2	1.162	1.974	1.1626	1.9742
	3	1.330	2.415	1.3305	2.4153

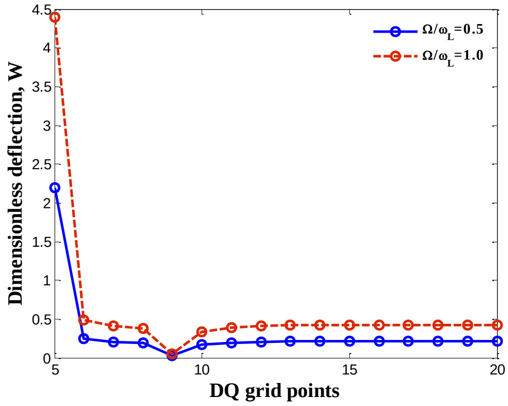

Fig. 2 The effect of DQ grid points number on the normalized frequency

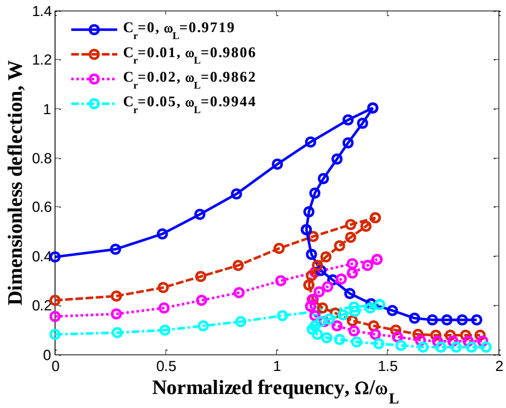

Fig. 3 Frequency–response curves of the structure for different CNTs volume percent

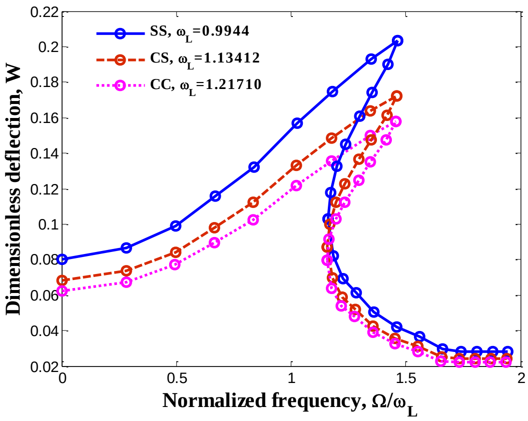

Fig. 4 Frequency–response curves of the structure for different boundary conditions

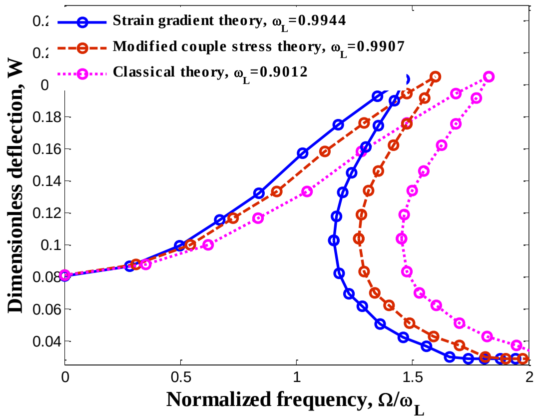

Fig. 5 Frequency–response curves of the structure for different theories

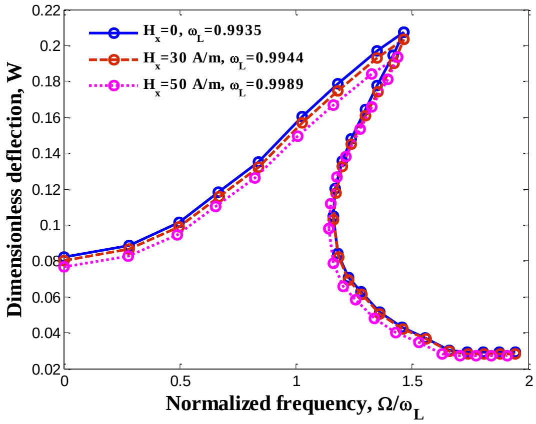

Fig. 7 Frequency–response curves of the structure for different magnetic field

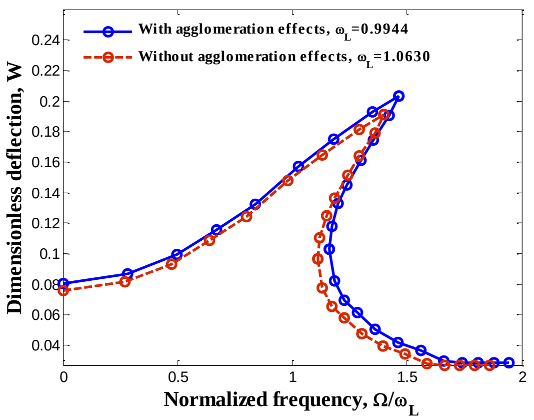

Fig. 6 Frequency–response curves of the structure for agglomeration of CNTs

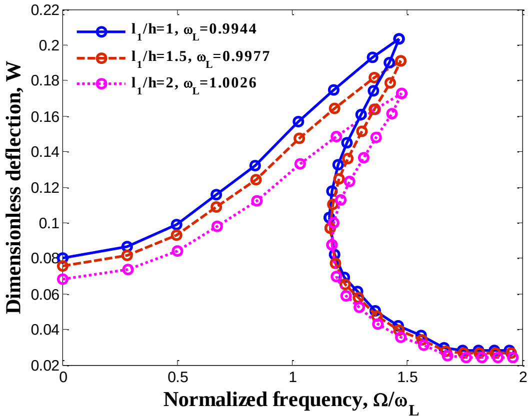

Fig. 8 Frequency–response curves of the structure for different material length scale parameter to thickness ratios

Fig. 5 is plotted to study the effect of different theories including of strain gradient, couple stress and classical theories. As observed the strain gradient theory has the lowest amplitude of the system than two other theories. This is due to the SGT has the three additional expression consisting of dilatation gradient tensor, the deviatoric stretch gradient tensor and the rotation gradient tensor while the MCST considered only the rotation gradient tensor.

Fig. 6 represents the effect of agglomeration of CNTs on the normalized frequency of the system. It can be seen that by considering agglomeration effects, the maximum amplitude increases and the linear frequency is decreased. It is since the CNTs agglomeration leads to decrease in the stability and rigidity of the structure.

The effect of magnetic field on the frequency-response curve of the structure is accomplished using Fig. 7. As can be seen, by applying magnetic field the amplitude of the system will be reduced. On the other word, by applying the magnetic field, the stiffness of the structure increases.

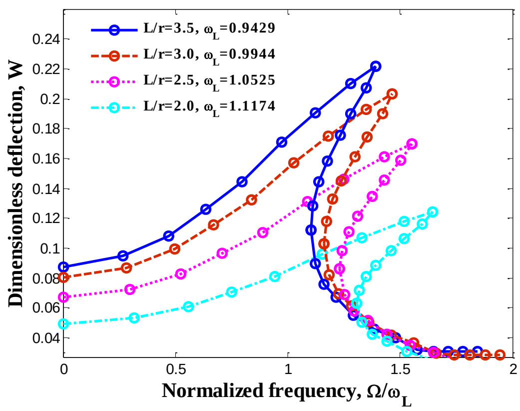

Fig. 9 Frequency–response curves of the structure for different length to radius ratios

The effects of various material length scale parameter to thickness ratios on the frequency-response curve are shown in Fig. 8. It is obvious that by increasing the material length scale parameter to thickness ratio, the structure becomes more tenuous, thus the stiffness of the structure decreases. So, the amplitude of the system will be decreased.

Fig. 9 demonstrates the effect of length to radius ratio of cylindrical shell on the frequency-response curve of the structure. As can be seen, with increasing the length to radius ratio, the amplitude of the structure is increased due to the reduction in the stiffness of structure.

5. Conclusions

Agglomeration effect on the forced vibration analysis of a micro cylindrical shell reinforced with CNTs is the main portion of the present study. Based on Mindlin theory, SGT and Hamilton's principle, the motion equations were derived. The DQM method was utilized to solve the problem and the effect of different parameters including magnetic field, CNTs volume percent and agglomeration effect, boundary conditions, size effect and length to thickness ratio on the nonlinear forced vibrational characteristic of the of the system was studied. Following results were obtained in this work:

1. By enhancing the CNTs volume percent, the amplitude of system happens at lower region while considering the CNTs agglomeration effect has an inverse effect.
2. The structure with CC boundary condition has the least displacement in comparison with others.
3. By assuming the size effect, the stiffness of the structure increases and thus, the amplitude of the system was decreased.
4. By considering the magnetic field, the amplitude of the system will be decreased.
5. With increasing the length to radius ratio, the amplitude of the structure was increased.

References

Ahouel, M., Houari, M.S.A., Adda Bedia, E.A. and Tounsi A. (2016) "Size-dependent mechanical behavior of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept", *Steel Compos. Struct.*, **20**(5), 963-981.

Akbas, S.D. (2016), "Forced vibration analysis of viscoelastic nanobeams embedded in an elastic medium", *Smart Struct. Syst.*, **18**(6), 1125-1143.

Alijani, F. and Amabili, M. (2014), "Non-linear static bending and forced vibrations of rectangular plates retaining non-linearities in rotations and thickness deformation", *Int. J. Nonlinear Mech.*, **67**, 394-404.

Assadi, A. (2013), "Size dependent forced vibration of nanoplates with consideration of surface effects", *Appl. Math. Model.*, **37**: 3575–3588.

Attia, A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2015), "Free vibration analysis of functionally graded plates with temperature-dependent properties using various four variable refined plate theories", *Steel Compos. Struct.*, **18**(1), 187-212.

Belabed, Z., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Bég, O.A. (2014), "An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates", *Compos.: Part B*, **60**, 274-283.

Beldjelili, Y., Tounsi, A. and Mahmoud, S.R. (2016), "Hygro-thermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four-variable trigonometric plate theory", *Smart Struct. Syst.*, **18**(4), 755-786.

Belkorissat, I., Houari, M.S.A., Tounsi, A. and Hassan, S. (2015), "On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model", *Steel Compos. Struct.*, **18**(4), 1063-1081.

Bellifa, H., Benrahou, K.H., Hadji, L., Houari, M.S.A. and Tounsi, A. (2016), "Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position", *J Braz. Soc. Mech. Sci. Eng.*, **38**(1), 265-275.

Bellifa, H., Benrahou, K.H., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2017), "A nonlocal zeroth-order shear deformation theory for nonlinear postbuckling of nanobeams", *Struct. Eng. Mech.*, **62**(6), 695 - 702.

Bennoun, M., Houari, M.S.A. and Tounsi, A. (2016), "A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates", *Mech. Advan. Mat. Struct.*, **23**(4), 423- 431.

Bessaim, A., Houari, M.S.A. and Tounsi, A. (2013), "A new higher-order shear and normal deformation theory for the static and free vibration analysis of sandwich plates with functionally graded isotropic face sheets", *J. Sandw. Struct. Mater.*, **15**(6), 671-703.

Besseghier, A., Houari, M.S.A., Tounsi, A. and Hassan, S. (2017), "Free vibration analysis of embedded nanosize FG plates using a new nonlocal trigonometric shear deformation theory", *Smart Struct. Syst.*, **19**(6), 601 - 614.

Bouafia, Kh., Kaci, A., Houari M.S.A. and Tounsi, A. (2017), "A nonlocal quasi-3D theory for bending and free flexural vibration behaviors of functionally graded nanobeams", *Smart Struct. Syst.*, **19**, 115-126.

Bousahla, A.A., Benyoucef, S., Tounsi, A. and Mahmoud, S.R. (2016a), "On thermal stability of plates with functionally graded coefficient of thermal expansion", *Struct. Eng. Mech.*, **60**(2), 313-335.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler–Pasternak elastic foundations", *Steel Compos. Struct.*, **14**(1), 85-104.

Bouderba, B., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2016b), "Thermal stability of functionally graded sandwich plates using a simple shear deformation theory", *Struct. Eng. Mech.*, **58**(3), 397-422.

Boukhari, A., Atmane, H.A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2016), "An efficient shear deformation theory for wave propagation of functionally graded material plates", *Struct. Eng. Mech.*, **57**(5), 837-859.

Bounouara, F., Benrahou, K.H., Belkorissat, I. and Tounsi A. (2016), "A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation", *Steel Compos. Struct.*, **20**(2), 227-249.

Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct.*, **18**(2), 409-423.

Chikh, A., Tounsi, A., Hebali, H. and Mahmoud, S.R. (2017), "Thermal buckling analysis of cross-ply laminated plates using a simplified HSDT", *Smart Struct. Syst.*, **19**(3), 289-297.

Civalek, Ö. (2004), "Application of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for buckling analysis of thin isotropic plates and elastic columns", *Eng. Struct.*, **26**, 171-186

Dai, H.L., Zhao, D.M., Zou, J.J. and Wang, L. (2016), "Surface effect on the nonlinear forced vibration of cantilevered nanobeams", *Physica E*, **80**, 25-30.

Dey, T. and Ramachandra, L.S. (2017), "Non-linear vibration analysis of laminated composite circular cylindrical shells", *Compos. Struct*., **163**, 89-100.

Draiche, K., Tounsi, A. and Mahmoud, S.R. (2016), "A refined theory with stretching effect for the flexure analysis of laminated composite plates", *Geomech. Eng*., **11**, 671-690.

El-Haina, F., Bakora, A., Bousahla, A.A. and Hassan, S. (2017), "A simple analytical approach for thermal buckling of thick functionally graded sandwich plates", *Struct. Eng. Mech*., **63**(5), 585-595.

Esawi, A.M.K. and Farag, M.M. (2007), "Carbon nanotube reinforced composites Potential and current challenges", *Mater. Des*., **28**, 2394-2401.

Farokhi, H. and Ghayesh, M.H. (2015a), "Thermo-mechanical dynamics of perfect and imperfect Timoshenko microbeams", *Int. J. Eng. Sci*., **91**, 12-33.

Farokhi, H. and Ghayesh, M.H. (2015b), "Nonlinear dynamical behaviour of geometrically imperfect microplates based on modified couple stress theory", *Int. J. Mech. Sci*., **90**, 133-144.

Fernandes, R., Mahmoud Mousavi, S. and El-Borgi, S. (2016), "Free and forced vibration nonlinear analysis of a microbeam using finite strain and velocity gradients theory", *Acta Mech*., **227**, 2657-2670.

Ghayesh, M.H., Amabili, M. and Farokhi, H. (2013a), "Nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory", *Int. J. Eng. Sci*., **63**, 52-60.

Ghayesh, M.H., Amabili, M. and Farokhi, H. (2013b), "Three-dimensional nonlinear size-dependent behaviour of Timoshenko microbeams", *Int. J. Eng. Sci*., **71**, 1-14.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2013c), "Nonlinear dynamics of a microscale beam based on the modified couple stress theory", *Compos. Part B: Eng*., **50**, 318-324.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2013d), "Nonlinear behaviour of electrically actuated MEMS resonators", *Int. J. Eng. Sci*., **71**, 137-155.

Ghayesh, M.H., Amabili, M. and Farokhi, H. (2013e), "Nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory*", Int. J. Eng. Sci*., **63**, 52-60.

Farokhi, H., Ghayesh, M.H. and Amabili, M. (2013f), "Nonlinear dynamics of a geometrically imperfect microbeam based on the modified couple stress theory", *Int. J. Eng. Sci*., **68**, 11-23.

Ghayesh, M.H., Farokhi, H. and Amabili, M. (2014), "In-plane and out-of-plane motion characteristics of microbeams with modal interactions", *Compos. Part B: Eng.*, **60**, 423-439.

Ghayesh, M.H. and Farokhi, H. (2015a), "Chaotic motion of a parametrically excited microbeam", *Int. J. Eng. Sci*., **96**, 34-45.

Ghayesh, M.H. and Farokhi, H. (2015b), "Nonlinear dynamics of microplates", *Int. J. Eng. Sci.*, **86**, 60-73.

Ghayesh, M.H., Farokhi, H. and Alici, G. (2016), "Size-dependent performance of microgyroscopes", *Int. J. Eng. Sci*., **100**, 99-111.

Ghayesh, M.H. (2018a), "Functionally graded microbeams: Simultaneous presence of imperfection and viscoelasticity", *Int. J. Mech. Sci*., **140**, 339-350.

International Journal of Mechanical Sciences,

Ghayesh, M.H. (2018b), "Nonlinear vibration analysis of axially functionally graded shear-deformable tapered beams", *Appl. Math. Model*., **59**, 583-596.

Ghayesh, M.H. (2018c), "Dynamics of functionally graded viscoelastic microbeams", *Int. J. Eng. Sci*., **124**, 115-131.

Gholipour, A., Farokhi, H. and Ghayesh, M.H. (2015), "In-plane and out-of-plane nonlinear size-dependent dynamics of microplates", *Nonlinear Dynam*., **79**, 1771-1785.

He, X.Q., Wang, J.B., Liu, B. and Liew, K.M. (2012), "Analysis of nonlinear forced vibration of multi-layered graphene sheets", *Comput. Mat. Sci.*, **61**, 194-199 .

Javanbakht, M., Daneshmehr, A.R., Shakeri, M. and Nateghi, A. (2012), "The dynamic analysis of the functionally graded piezoelectric (FGP) shell panel based on three-dimensional elasticity theory", *Appl. Math. Model*., **36**, 5320-5333.

Khetir, H., Bouiadjra, M.B., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2017), "A new nonlocal trigonometric shear deformation theory for thermal buckling analysis of embedded nanosize FG plates", *Struct. Eng. Mech*., **64**(4), 391-402.

Kolahchi, R., Hosseini, H. and Esmailpour, M. (2016), "Differential cubature and quadrature-Bolotin methods for dynamic stability of embedded piezoelectric nanoplates based on visco-nonlocal-piezoelasticity theories", *Compos. Struct*., **157**, 174-186.

Larbi Chaht, F., Kaci, A., Houari M.S.A. and Hassan, S. (2015), "Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect", *Steel Compos. Struct*., **18**(2), 425 –442.

Liew, K.M., Lei, Z.X. and Zhang, L.W. (2015), "Mechanical analysis of functionally graded carbon nanotube reinforced composites", *Compos. Struct.*, **120**, 90-97.

Mahi, A., Bedia, E.A.A. and Tounsi, A. (2015), "A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates", *Appl. Math. Model*., **39**, 2489–2508.

Mehri, M., Asadi, H. and Wang, Q. (2016), "Buckling and vibration analysis of a pressurized CNT reinforced functionally graded truncated conical shell under an axial compression using HDQ method", *Comput. Method. Appl. M.*, **303**, 75-100.

Menasria, A., Bouhadra, A., Tounsi, A. and Hassan, S. (2017), "A new and simple HSDT for thermal stability analysis of FG sandwich plates", *Steel Compos. Struct*., **25**(2), 157-175.

Meziane, M.A.A., Abdelaziz, H.H. and Tounsi, A.T. (2014), "An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions", *J. Sandw. Struct. Mater*., **16**(3), 293-318.

Motallebi, A., Irani, S. and Sazesh, S. (2016), "Analysis on jump and bifurcation phenomena in the forced vibration of nonlinear cantilever beam using HBM", *J. Braz. Soc. Mech. Sci. Eng*., **38**: 515-524.

Mouffoki, A., Adda Bedia, E.A., Houari M.S.A. and Hassan, S. (2017), "Vibration analysis of nonlocal advanced nanobeams in hygro-thermal environment using a new two-unknown trigonometric shear deformation beam theory", *Smart Struct. Syst*., **20**(3), 369-383.

Pasha Zanoosi, A.A., Haghpanahi, M. and Kalantarinejad, R. (2017), "Analyzing free and forced vibration of flexible polyurethane foam using multiple time scales method", *J. Braz. Soc. Mech. Sci. Eng*., **39**, 207-217.

Qian, D., Wagner, G.J., Liu, W.K., Yu, M.F. and Ruoff, R.S. (2002), "Mechanics of carbon nanotubes", *Appl. Mech. Rev*., **55**, 495-533.

Rafiee, M., Mohammadi, M., Sobhani Aragh, B. and Yaghoobi, H. (2013), "Nonlinear free and forced thermo-electro-aero-elastic vibration and dynamic response of piezoelectric functionally graded laminated composite shells", *Compos. Struct.*, **103**, 179-187.

Rougui, M., Moussaoui, F. and Benamar, R. (2007), "Geometrically non-linear free and forced vibrations of simply supported circular cylindrical shells", *Int. J. Nonlinear Mech*., **42**, 1102-1115.

Saito, R., Dresselhaus, G. and Dresselhaus, M.S. (1998), *Physical Properties of Carbon Nanotubes Imperial College Press*, London.

Shi, D.L. and Feng, X.Q. (2004), "The effect of nanotube waviness and agglomeration on the elastic property of carbon nanotube-reinforced composite", *J. Eng. Mat. Tech. ASME*, **126**,

250-270.
Shokravi, M. and Jalili, N. (2017), “Dynamic buckling response of temperature-dependent functionally graded-carbon nanotubes-reinforced sandwich microplates considering structural damping”, *Smart Struct. Syst*., **20**(5), 583-593.
Shokrollahi, H., Kargarnovin, M.H. and Fallah, F. (2015), “Deformation and stress analysis of sandwich cylindrical shells with a flexible core using harmonic differential quadrature method”, *J. Braz. Soc. Mech. Sci. Eng*., **37**, 325-337.
Shooshtari, A. and Rafiee, M. (2011), “Nonlinear forced vibration analysis of clamped functionally graded beams”, *Acta Mech*., **29**, 221-223.
Simsek, M. and Kocatűrk, T. (2009), "Nonlinear dynamic analysis of an eccentrically prestressed damped beam under a concentrated moving harmonic load", *J. Sound Vib*., **320**, 235-253.
Simsek, M. (2011), “Forced vibration of an embedded single-walled carbon nanotube traversed by a moving load using nonlocal Timoshenko beam theory”, *Steel Compos. Struct*., **11**(1), 59-76.
Şimşek, M. and Aydın, M. (2017), “Size-dependent forced vibration of an imperfect functionally graded (FG) microplate with porosities subjected to a moving load using the modified couple stress theory”, *Compos. Struct*., **160**, 408-421.
Sofiyev, A.H. (2016), “Nonlinear free vibration of shear deformable orthotropic functionally graded cylindrical shells”, *Comput. Struct*., **142**, 35-44.
Tadi Beni, Y., Mehralian, F. and Razavi, H. (2015), “Free vibration analysis of size-dependent shear deformable functionally graded cylindrical shell on the basis of the modified couple stress theory”, *Compos. Struct.*, **120**, 65-78.
Tagrara, S.H., Benachour, A., Bouiadjra, M.B. and Tounsi, A. (2015), “On bending, buckling and vibration responses of functionally graded carbon nanotube-reinforced composite beams”, *Steel Compos. Struct*., **19**(5), 1259-1277.
Xiaodong, Y. and Qun, C.L. (2006), “Non-linear forced vibration of axially moving viscoelastic beams”, *Act. Mech. Solid Sinic*., **19**, 365-373 .
Yakobson, B.I., Brabec, C.J. and Bernholc, J. (1996), “Nanomechanics of carbon tubes: instability beyond linear response”, *Phys. Rev Lett*., **76**, 2511-2514.
Yu, M.F., Files, B.S., Arepalli, S. and Ruoff, R.S. (2000), “Tensile loading of ropes of single wall carbon nanotubes and their mechanical properties”, *Phys. Rev. Lett.*, **84**: 5552–5555.
Zemri, A., Houari, M.S.A., Bousahla, A.A. and Tounsi, A. (2015), “A mechanical response of functionally graded nanoscale beam: an assessment of a refined nonlocal shear deformation theory beam theory”, *Struct. Eng. Mech*., **54**(4), 693-710.
Zidi, M., Tounsi, A., and Bég, O.A. (2014), “Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory”, *Aerosp. Sci. Tech*., **34**, 24-34.

CC