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Vibration suppression of a double-beam system by a two-degree-of-freedom mass-spring system

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Abstract. This paper investigates the free vibration analysis of double-beam system coupled by a two-degree-of-freedom mass-spring system. In order to generalize the model, the main beams are assumed to be elastically restrained against translation and rotation at one end and free at the other. Furthermore, the mass-spring system is elastically connected to the beams at adjustable positions by means of four translational and rotational springs. The governing differential equations of the beams and the mass-spring system are derived and analytically solved by using the Fourier transform method. Moreover, as a second way, a finite element solution is derived. The frequency parameters and mode shapes of some diverse cases are obtained using both methods. Comparison of obtained results by two methods shows the accuracy of both solutions. The influence of system parameters on the free vibration response of the studied mechanical system is examined.

Keywords: double-beam; two-degree-of-freedom mass-spring system; vibration suppression; elastic supports; Fourier transform; free vibration; exact solution; finite element method

1. Introduction

Beams are versatile structural members with application in engineering. Therefore, many articles deal with analysis of such structures (Giunta *et al.* 2014, Hozhabrossadati *et al.* 2015, Ebrahimi and Shafiei 2016). Beams can be used to form a rather new structure called double-beam system. These systems are a type of vibration absorber. In these systems, an auxiliary beam is attached to the main beam to suppress excessive vibrations of the primary beam. Moreover, double-beam systems are used for modelling of flouting- slab tracks, which are widely used to control vibration from underground trains (Hussein and Hunt 2006). Therefore, it has been under consideration of numerous researchers (Kukla 1994, Vu *et al.* 2000, Inceoglu and Gurgoze 2001, Oniszczuk 2003, Abu-Hilal 2006, Li and Hua 2007, Lee 2009, Simsek and Cansiz 2012). Gurgoze *et al.* (2001) studied the lateral vibration problem of a double-beam system consisting of two cantilever Euler-Bernoulli beams coupled by a double mass-spring system attached to them in-span. Gurgoze and Erol (2004) dealt with the eigencharacteristics of a laterally vibrating system made up of two clamped-free Euler-Bernoulli beams carrying tip masses to which several double mass-spring systems were attached across the span. De Rosa and Lippiello (2007) used the differential quadrature method for free vibration analysis of a double-beam system embedded in soil of Winkler-type. The system of governing differential equations was discretized by means of the differential quadrature method. Zhang *et al.* (2008) investigated the properties of free transverse vibration and buckling of a double-beam system under compressive axial loading on the basis of Euler-Bernoulli beam theory. The two beams of the system were simply supported and continuously joined by a Winkler elastic layer. Later, Zhang *et al.* (2008) dealt with the forced vibration of an elastically connected simply-supported double-beam system under compressive axial loading. The dynamic response of the system caused by arbitrarily distributed continuous loads was obtained. It was concluded that the steady-state vibration amplitudes of the two beams are dependent on the axial compression. Jun and Hongxing (2008) analyzed a three-beam system by dynamic stiffness method. Stojanovic *et al.* (2011) considered the free transverse vibration and buckling of a double-beam continuously joined by a Winkler elastic layer under compressive axial loading with influence of rotary inertia and shear. The governing problem was analytically solved and natural frequencies as well as critical buckling load were found. Ariaei et al. (2011) investigated the dynamic response of an elastically connected multiple-beam system due to a moving load based on the Timoshenko beam theory. The solution method involved a change of variables and modal analysis to decouple and solve the governing differential equations. Stojanovic and Kozic (2012) studied the forced vibration and buckling of a double-beam system continuously joined by a Winkler elastic layer under compressive axial loading based on both Rayleigh and Timoshenko beam theories. The influence of rotary inertia, shear deformation and compressive load very thoroughly investigated. Lin and Yang (2013) considered the free vibration of a double-beam system consisting of two beams which were connected by a double mass-spring system. The free vibration response of the system for different boundary conditions of the beams was obtained, considering the compatibility of deformation

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of the springs of the mass-spring system. Mao and Wattanasakulpong (2015) employed the Adomian modified decomposition method to investigate the free vibration and stability of a cantilever double-beam system, which was continuously joined by a Winkler-type elastic layer. The free end of each beam was restrained by a translational spring and subjected to a combination of compressive axial and follower loads. Rezaiee-Pajand and Hozhabrossadati (2016) dealt with the free vibration of double-beam made up of axially functionally graded materials.

It is interesting to mention that though dynamic analysis of beams coupled with single-degree-of-freedom mass-spring system has been extensively treated by researchers (Laura *et al.* 1977, Magrab 2007, Cha and Zhou 2008, Yang *et al.* 2012, Rajabi *et al.* 2013, Hozhabrossadati *et al.* 2014, Hozhabrossadati *et al.* 2015), dynamic analysis of beams carrying two-degree-of-freedom mass-spring systems has been studied by a few researchers (Jen and Magrab 1993, Wu 2002, Wu 2005, El-Sayed and Farghaly 2016). Chang and Chang (1998) determined the natural frequencies and mode shapes of an Euler-Bernoulli beam with a two-degree-of-freedom mass-spring system. They used the well-known Laplace transform with respect to the spatial variable for the analysis of the mechanical system. Banerjee (2003) used the dynamic stiffness method for free vibration analysis of beams and frameworks carrying a two-degree-of-freedom mass-spring system. The effect of the mass-spring system was represented by replacing it with equivalent stiffness coefficients which were added to the appropriate stiffness coefficients of the bare beam. Chen (2006) studied the problem of free vibration of a uniform beam carrying multiple two-degree-of-freedom mass-spring system using the numerical assembly method. Combining the coefficient matrices for all two-degree-of-freedom mass-spring systems attached to the beam and the coefficient matrices for the boundary conditions of the beam, he obtained the overall coefficient matrix of the restrained beam. Cha and Chan (2009) employed one or more two-degree-of-freedom mass-spring systems to mitigate vibration of structures. They enforced nodes at specified locations along an arbitrary supported elastic structure subject to an external harmonic excitation.

The above-mentioned references show that there is no available study on the free vibration analysis of a double-beam system connected by a two-degree-of-freedom mass-spring system. Furthermore, in problems treated by researchers regarding free vibration of beams coupled with two-degree-of-freedom mass-spring systems, the two-degree-of-freedom mass-spring system is connected to the main beam by means of only translational springs. In this paper, we present a free vibration formulation for double-beam system connected by a two-degree-of-freedom mass-spring system. The two-degree-of-freedom mass-spring system is connected to each beam by means of a translational and a rotational spring. It is shown that the inclusion of the rotational springs tremendously increase the complexity of the problem.

2. Problem description

In this study, the mechanical system shown in Fig. 1 is investigated. The system consists of two elastically restrained cantilever beams which are connected by a two-degree-of-freedom mass-spring system. The length of each beam is L. The first ends of beams at x=0 are elastically restrained against rotation and translation. The other ends are assumed free. The two-degree-of-freedom mass-spring system is connected to the left beam at $x=\xi_1 L$ and to the right beam at $x=\xi_2 L$. The connection of the mass-spring system to each beam is made by means of a translational spring and a rotational spring.

As shown in Fig. 1, the double-beam system is divided into four segments. Therefore, four differential equations may be used to express the vibratory motion of the mechanical system. Besides, since the attached mass-spring system is assumed to be two-degree-of-freedom, two differential equations should be specified in order to describe the behavior of the mass-spring system.

In the next section, the eigenvalue problem governing the free vibration of the mechanical system under study is formulated. First, four differential equations of motion of the beams are introduced. Second, two differential equations which govern the dynamic behavior of the two-degree-of-freedom mass-spring system are derived. Third, the pertinent boundary and compatibility conditions of the problem are prescribed.

3. Formulation of eigenvalue problem

This section is concerned with the formulation of the governing eigenvalue problem. The differential equations of motion of the beams and the mass-spring system along with the pertinent boundary and compatibility conditions of the problem form the eigenvalue problem.

3.1 Equations of motion of beams

As mentioned previously, each beam is divided into two segments and therefore four differential equations should be written for stating the transverse vibrations of beams. These four equations of motion can be written as:

$$\begin{aligned} EI\,\frac{\partial^4 u_i}{\partial x^4}+\rho A\,\frac{\partial^2 u_i}{\partial t^2}=0 \\ EI\,\frac{\partial^4 v_i}{\partial x^4}+\rho A\,\frac{\partial^2 v_i}{\partial t^2}=0 \end{aligned} \tag{1}$$

with $i=1,2$. In Eq. (1) $u_i(x,t)$ and $v_i(x,t)$ are the deformation functions of the left and right beam, respectively. Furthermore, EI denotes the flexural rigidity, ρ shows the mass density and A indicates the cross-sectional area of the beams.

3.2 Equations of motion of mass-spring system

To derive the differential equations of the two-degree-of-freedom mass-spring system, the Newton's second law of

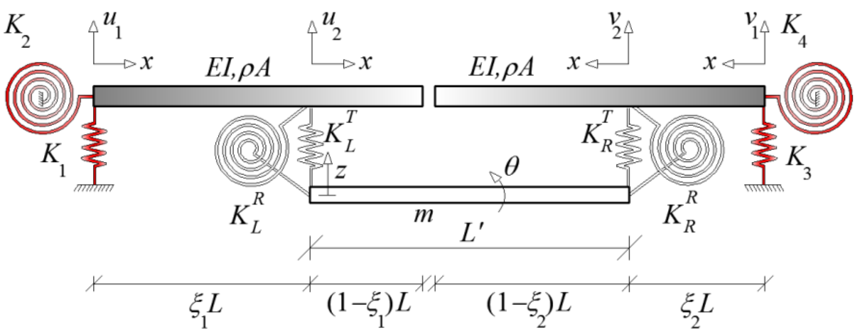


Fig. 1 The double-beam under study

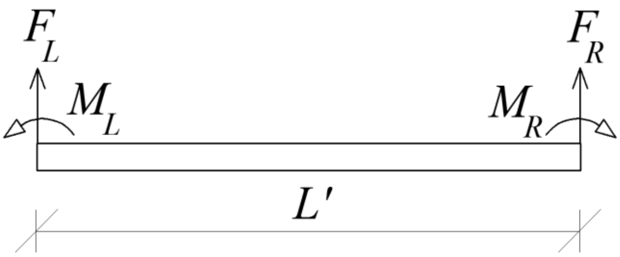


Fig. 2 The forces and moments acting on the mass-spring system

motion is utilized. Consider the mass-spring system. The system has a translational degree of freedom which is denoted by $z(t)$ and an angular degree of freedom indicated by $\theta(t)$. Furthermore, two translational springs and two rotational springs apply four forces and moments to the rigid mass as shown in Fig. 2. The following equations are considered

$$\begin{aligned} \sum F_y &= m\, a_G \\ \sum M_G &= I_G \ddot{\theta} \end{aligned} \tag{2}$$

in which m and I_G are the suspended mass and the mass moment of inertia of the mass with the axis of rotation at the center of the mass, respectively. Besides, a_G and $\ddot{\theta}$ indicate the translational and angular acceleration of the mass at the mass center. Herein, it is assumed that the suspended mass is a rigid bar of length L'. Therefore, I_G for the rigid bar can be written as

$$I_G = \frac{m L'^2}{12} \tag{3}$$

where $m = \gamma \rho A L$ and γ is a constant which denotes the ratio of the suspended mass to the mass of each beam.
After substituting the needed values into Eq. (2), it takes the next form

$$\begin{aligned} F_R + F_L &= m\left[\ddot{z} + \frac{L'}{2}\ddot{\theta}\right] \\ M_L - M_R + \frac{L'}{2}F_R - \frac{L'}{2}F_L &= I_G \ddot{\theta} \end{aligned} \tag{4}$$

where F_L and F_R are the forces of the translational springs and M_L and M_R are the moments of the rotational springs. Besides, an overdot shows a derivative with respect to time *t*. The values of these forces and moments will be found in the subsequent sections. The differential equations expressed in Eq. (4) are the governing equations of the behavior of the two-degree-of-freedom mass-spring system which are to be solved in the next sections.

3.3 Boundary conditions

In this subsection, the pertinent boundary and compatibility conditions of the problem are found. Two boundary conditions at each end support, two boundary conditions at each free end and four compatibility conditions at each connection point should be specified.

The boundary conditions of the end with elastic supports are of the form

$$\begin{aligned} K_1 u_1(0,t) + EIu_1'''(0,t) &= 0 \\ K_2 u_1'(0,t) - EIu_1''(0,t) &= 0 \\ K_3 v_1(0,t) + EIv_1'''(0,t) &= 0 \\ K_4 v_1'(0,t) - EIv_1''(0,t) &= 0 \end{aligned} \tag{5}$$

in which K_1 to K_4 are the stiffnesses of the end springs.

Boundary conditions of the free ends are

$$\begin{aligned} EIu_2''\left[(1-\xi_1)L, t\right] &= 0 \\ EIu_2'''\left[(1-\xi_1)L, t\right] &= 0 \\ EIv_2''\left[(1-\xi_2)L, t\right] &= 0 \\ EIv_2'''\left[(1-\xi_2)L, t\right] &= 0 \end{aligned} \tag{6}$$

At the location of the connected springs, the deflection and slope of the beam are continuous. Therefore

$$\begin{aligned} &u_1[\xi_1 L,t]=u_2[0,t] \\ &u_1'[\xi_1 L,t]=u_2'[0,t] \\ &v_1[\xi_2 L,t]=v_2[0,t] \\ &v_1'[\xi_2 L,t]=v_2'[0,t] \end{aligned} \tag{7}$$

The shearing forces of the segments of the beam at the connection must be in equilibrium with the force of the translational spring. Therefore

$$\begin{aligned} &EIu_1'''[\xi_1 L,t]-EIu_2'''[0,t]-F_L=0 \\ &EIv_2'''[\xi_2 L,t]-EIv_2'''[0,t]-F_R=0 \end{aligned} \tag{8}$$

in which

$$\begin{aligned} &F_L=K_L^T\left[u_2(0,t)-z(t)\right] \\ &F_R=K_R^T\left[v_2(0,t)-z(t)-L'\theta(t)\right] \end{aligned} \tag{9}$$

where K_L^T and K_R^T indicate the stiffness of the translational springs of the mass-spring system. Likewise, the equilibrium of the bending moments of the beam and the moments of the rotational spring gives

$$\begin{aligned} &EIu_1''[\xi_1 L,t]+M_L-EIu_2''[0,t]=0 \\ &EIv_2''[\xi_2 L,t]+M_R-EIv_2''[0,t]=0 \end{aligned} \tag{10}$$

where

$$\begin{aligned} &M_L=K_L^R\left[u_2'(0,t)-\theta(t)\right] \\ &M_R=K_R^R\left[v_2'(0,t)+\theta(t)\right] \end{aligned} \tag{11}$$

in which K_L^R and K_R^R demonstrate the stiffness of the rotational springs of the mass-spring system. Sixteen relations given by Eqs. (5)-(8) and (10) are the boundary and compatibility conditions of the problem. In the next section, the solution of the studied eigenvalue problem is presented.

4. Solution of problem by fourier transform

In the previous section, the governing eigenvalue problem was formulated. To solve the problem, the well-known Fourier transform with respect to the time is utilized. The Fourier transform and inverse Fourier transform have the following definitions

$$\begin{aligned} &F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt \\ &f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}\,dt \end{aligned} \tag{12}$$

The first equality in Eq. (12) is the Fourier transform of function $f(t)$ while the second one in Eq. (12(b)) is the inverse Fourier transform that converts a Fourier transform back to $f(t)$. Taking the Fourier transform of both sides of Eq. (1) gives

$$\begin{aligned} &F\left[EI\frac{\partial^4 u_i}{\partial x^4}+\rho A\frac{\partial^2 u_i}{\partial t^2}\right]=F[0] \\ &F\left[EI\frac{\partial^4 v_i}{\partial x^4}+\rho A\frac{\partial^2 v_i}{\partial t^2}\right]=F[0] \end{aligned} \tag{13}$$

Using the next properties of the Fourier transform

$$\begin{aligned} &F[u(x,t)]=U(x) \\ &F[\alpha u(x,t)+\beta v(x,t)]=\alpha U(x)+\beta V(x) \\ &F\left[\frac{\partial^n u}{\partial x^n}\right]=\frac{d^n U}{dx^n} \\ &F\left[\frac{\partial^n u}{\partial t^n}\right]=(i\omega)^n U \end{aligned} \tag{14}$$

results in

$$\begin{aligned} &\frac{d^4 U_i}{dx^4}-\lambda^4 U_i=0 \\ &\frac{d^4 V_i}{dx^4}-\lambda^4 V_i=0 \end{aligned} \tag{15}$$

wherein $U(x)$ and $V(x)$ are the Fourier transformed of $u(x,t)$ and $v(x,t)$. These equations are the governing differential of beams in the frequency domain. Besides, λ is the frequency parameter and has the next shape

$$\lambda^4=\frac{\rho A\,\omega^2}{EI} \tag{16}$$

Similarly, applying the Fourier transform on the governing equations of the rigid bar leads to

$$\begin{aligned} &F_R+F_L=-m\omega^2\left[Z+\frac{L'}{2}\Theta\right] \\ &M_L-M_R+\frac{L'}{2}F_R-\frac{L'}{2}F_L=-I_G\omega^2\Theta \end{aligned} \tag{17}$$

in which Z and Θ are the Fourier transformed of z and θ. Finally, the transformed boundary and compatibility conditions take the succeeding form

$$\begin{aligned} &K_1U_1(0)+EIU_1'''(0)=0 \\ &K_2U_1'(0)-EIU_1''(0)=0 \\ &K_3V_1(0)+EIV_1'''(0)=0 \\ &K_4V_1'(0)-EIV_1''(0)=0 \\ &EIU_2''[(1-\xi_1)L]=0 \\ &EIU_2'''[(1-\xi_1)L]=0 \\ &EIV_2''[(1-\xi_2)L]=0 \\ &EIV_2'''[(1-\xi_2)L]=0 \\ &U_1(\xi_1 L)=U_2(0) \\ &U_1'(\xi_1 L)=U_2'(0) \\ &V_1(\xi_2 L)=V_2(0) \\ &V_1'(\xi_2 L)=V_2'(0) \\ &EIU_1'''(\xi_1 L)-EIU_2'''(0)-F_L=0 \\ &EIV_2'''(\xi_2 L)-EIV_2'''(0)-F_R=0 \\ &EIU_1''(\xi_1 L)+M_L-EIU_2''(0)=0 \\ &EIV_2''(\xi_2 L)+M_R-EIV_2''(0)=0 \end{aligned} \tag{18}$$

F_L , F_R , M_L and M_R can be written as

$$F_L = K_L^T\left[U_2(0) - Z\right] \\ F_R = K_R^T\left[V_2(0) - Z - L'\Theta\right] \tag{19}$$

$$M_L = K_L^R\left[U_2'(0) - \Theta\right] \\ M_R = K_R^R\left[V_2'(0,t) + \Theta\right] \tag{20}$$

The solutions of Eq. (15) can be expressed as

$$\begin{aligned} U_1(x) &= C_1\sin\lambda x + C_2\cos\lambda x + C_3\sinh\lambda x + C_4\cosh\lambda x \\ U_2(x) &= C_5\sin\lambda x + C_6\cos\lambda x + C_7\sinh\lambda x + C_8\cosh\lambda x \\ V_1(x) &= C_9\sin\lambda x + C_{10}\cos\lambda x + C_{11}\sinh\lambda x + C_{12}\cosh\lambda x \\ V_2(x) &= C_{13}\sin\lambda x + C_{14}\cos\lambda x + C_{15}\sinh\lambda x + C_{16}\cosh\lambda x \end{aligned} \tag{21}$$

where C_1 and C_{16} are constants which can be determined utilizing the pertinent boundary and compatibility conditions of the problem.

Considering Eqs. (19) and (20), one needs the values of Z and Θ to use F_L , F_R , M_L and M_R . Thus, these two parameters are to be found. Substituting Eqs. (19) and (20) into Eqs. (17) results in

$$K_L^T\left[U_2(0) - Z\right] + K_R^T\left[V_2(0) - Z - L'\Theta\right] = -m\omega^2\left[Z + \frac{L'}{2}\Theta\right] \tag{22a}$$

$$K_L^R\left[U_2'(0) - \Theta\right] - K_R^R\left[V_2'(0,t) + \Theta\right] + \frac{L'}{2}K_R^T\left[V_2(0) - Z - L'\Theta\right] \\ -\frac{L'}{2}K_L^T\left[U_2(0) - Z\right] = -I_G\omega^2\Theta \tag{22b}$$

These two equations can be rewritten as

$$\left(m\omega^2 - K_L^T - K_R^T\right)Z + \left(\frac{L'}{2}m\omega^2 - L'K_R^T\right)\Theta \\ = -K_L^T U_2(0) - K_R^T V_2(0) \tag{23a}$$

$$\left(\frac{L'}{2}K_L^T - \frac{L'}{2}K_R^T\right)Z + \left(I_G\omega^2 - K_L^R - K_R^R - \frac{L'^2}{2}K_R^T\right)\Theta \\ = K_R^R V_2'(0) - K_L^R U_2'(0) + \frac{L'}{2}K_L^T U_2(0) - \frac{L'}{2}K_R^T V_2(0) \tag{23b}$$

Eqs. (23(a)) and (23(b)) can be solved for Z and Θ , but herein, for sake of brevity , we do not present the long result. Then, the obtained values for Z and Θ are substitute in Eqs. (19) and (20) and the values of F_L , F_R , M_L and M_R are in hand. Finally, substitution of F_L , F_R , M_L and M_R into Eq. (18) gives the pertinent boundary and compatibility conditions of the problem. Applying these conditions on functions U_1 to V_2 given by Eq. (21) leads to a set of algebraic equations in C_1 to C_{16} as unknowns. For a nontrivial solution, the determinant of the coefficient matrix is set to zero, giving the frequency equation of the problem. The frequency equation is numerically solved and the frequency parameters or the eigenvalues of the problem, i.e., λ are obtained.

5. Finite element solution

As another way, the studied problem is solved using the well-known finite element method. To do so, each beam is discretized into NE two-node Euler-Bernoulli beam elements. Fig. 3 shows the discretized beams. Using the Hermite shapes functions, the stiffness and mass matrices of each element are of the next form

$$\left[\mathbf{K}^e\right] = \frac{EI}{l^{e^3}}\begin{pmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 6l^{e^2} & -6l^e & 2l^{e^2} \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2l^{e^2} & -6l^e & 4l^{e^2} \end{pmatrix} \\ \left[\mathbf{M}^e\right] = \frac{ml^e}{420}\begin{pmatrix} 156 & 22l^e & 54 & -13l^e \\ 22l^e & 4l^{e^2} & 13l^e & -3l^{e^2} \\ 54 & 13l^e & 156 & -22l^e \\ -13l^e & -3l^{e^2} & -22l^e & 4l^{e^2} \end{pmatrix} \tag{24}$$

where l^e is the length of each element. It is worth mentioning that the inclusion of the attached two-degree-of-freedom in the finite element analysis gives rise to some changes in the some elements of the global stiffness matrix. Furthermore, two rows and columns are added to the global stiffness and mass matrices. Therefore, the stiffness and mass matrices become $(ndof + 2)\times(ndof + 2)$ matrices, where *ndof* demonstrates the number of degrees of freedom of the beams.

To find the changes in the stiffness and mass matrices, we act as follows. The degrees of freedoms of the connection points of the beam are renamed as (see Fig. 3)

$$2NE_1 + 1 = d_1 \quad ; \quad 2NE_1 + 2 = d_2 \\ 2NE_2 + 1 = d_3 \quad ; \quad 2NE_2 + 2 = d_4 \tag{25}$$

Moreover, the two degrees of freedom of the rigid bar are

$$d_5 = 4NE + 5 \\ d_6 = 4NE + 6 \tag{26}$$

Then, the forces and moments of the suspended mass-spring system is found based on theses degrees of freedoms

$$\begin{aligned} F_1 &= K_5(d_1 - d_5) \\ F_2 &= K_6(d_3 - d_6) \\ M_1 &= K_7\left(d_2 - \frac{d_6 - d_5}{L'}\right) \\ M_2 &= K_8\left(d_4 - \frac{d_6 - d_5}{L'}\right) \end{aligned} \tag{27}$$

where F_1 , F_2 , M_1 and M_2 are the forces and moments of the right and left spring, respectively.

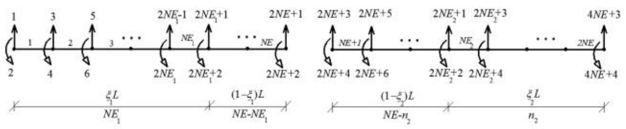


Fig. 3 The discretized system

Expanding the corresponding rows of d_1 and d_2 in the stiffness matrix gives the following relations

$$\begin{aligned} k_{d_1,1}d_1 + k_{d_1,2}d_2 + \cdots &= -F_1 \\ k_{d_2,1}d_1 + k_{d_2,2}d_2 + \cdots &= -M_1 \end{aligned} \tag{28}$$

or

$$\begin{aligned} k_{d_1,1}d_1 + k_{d_1,2}d_2 + \cdots &= -K_5(d_1 - d_5) \\ k_{d_2,1}d_1 + k_{d_2,2}d_2 + \cdots &= -K_7(d_2 - \frac{d_6 - d_5}{L'}) \end{aligned} \tag{29}$$

from which the modified elements of the stiffness matrix are found

$$\begin{aligned} \hat{k}_{d_1,d_1} &= k_{d_1,d_1} + K_5 \\ \hat{k}_{d_1,d_5} &= -K_5 \\ \hat{k}_{d_2,d_2} &= k_{d_2,d_2} + K_7 \\ \hat{k}_{d_2,d_5} &= \frac{K_7}{L'} \\ \hat{k}_{d_2,d_6} &= -\frac{K_7}{L'} \end{aligned} \tag{30}$$

Likewise, expanding the corresponding rows of d_3 and d_4 in the stiffness matrix yields

$$\begin{aligned} \hat{k}_{d_3,d_3} &= k_{d_3,d_3} + K_6 \\ \hat{k}_{d_3,d_6} &= -K_6 \\ \hat{k}_{d_4,d_4} &= k_{d_4,d_4} + K_8 \\ \hat{k}_{d_4,d_5} &= \frac{K_8}{L'} \\ \hat{k}_{d_4,d_6} &= -\frac{K_8}{L'} \end{aligned} \tag{31}$$

In order to find the new elements of the stiffness matrix, the Newton's second law of motion $\sum F_y = ma_G$ for the rigid bar is written

$$F_1 + F_2 = ma_G \tag{32}$$

or

$$K_5(d_1 - d_5) + K_6(d_3 - d_6) = -m\omega^2(\frac{d_5 + d_6}{2}) \tag{33}$$

This equation can be rearranged as

$$-K_5 d_1 - K_6 d_3 + K_5 d_5 + K_6 d_6 - \omega^2(\frac{m}{2}d_5 + \frac{m}{2}d_6) = 0 \tag{34}$$

Therefore, the elements of the new first row in the stiffness and mass matrices are found as

$$\begin{aligned} \hat{k}_{d_5,d_1} &= -K_5 \\ \hat{k}_{d_5,d_3} &= -K_6 \\ \hat{k}_{d_5,d_5} &= K_5 \\ \hat{k}_{d_5,d_6} &= K_6 \end{aligned} \tag{35}$$

$$\begin{aligned} \hat{m}_{d_5,d_5} &= \frac{m}{2} \\ \hat{m}_{d_5,d_6} &= \frac{m}{2} \end{aligned} \tag{36}$$

Similarly, writing the equation $\sum M_G = I_G\ddot{\theta}$ for the suspended mass-spring results in

$$\begin{aligned} \hat{k}_{d_6,d_1} &= \frac{K_5}{2} \\ \hat{k}_{d_6,d_2} &= -\frac{K_7}{L'} \\ \hat{k}_{d_6,d_3} &= -\frac{K_6}{2} \\ \hat{k}_{d_6,d_4} &= -\frac{K_8}{L'} \\ \hat{k}_{d_6,d_5} &= -\frac{K_5}{2} - \frac{K_7 + K_8}{L'^2} \\ \hat{k}_{d_6,d_6} &= \frac{K_6}{2} + \frac{K_7 + K_8}{L'^2} \\ \hat{m}_{d_6,d_6} &= \frac{mL'}{12} \\ \hat{m}_{d_5,d_6} &= -\frac{mL'}{12} \end{aligned} \tag{37}$$

Therefore, the modified stiffness and mass matrices are at hand. Finally, the natural frequencies of the mechanical system under study are obtained from the following relation

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0 \tag{38}$$

Fortunately, the obtained results by both exact and FE method are in an excellent agreement which indicates the accuracy of both solutions (coincide up to four decimal digits). However, herein the FE results for the sake of brevity are not presented.

6. Special case

In this section, it is assumed that the two-degree-of-freedom system is attached to the beams via two translational springs. In other words, the rotational springs are not considered in the analysis. In this case, the moments of rotational springs, i.e., M_L and M_R, are omitted in the last two relations in Eq. (18). Moreover, Eq. (2) may be written as following

$$\begin{aligned} F_R + F_L &= m\left[\ddot{z} + \frac{L'}{2}\ddot{\theta}\right] \\ \frac{L'}{2}F_R - \frac{L'}{2}F_L &= I_G\ddot{\theta} \end{aligned} \tag{39}$$

Substituting the values of translational springs forces given by Eq. (19) into Eq. (39) results in the following relations

$$\begin{aligned}&\left(m\omega^2-K_L^T-K_R^T\right)Z+\left(\frac{L'}{2}m\omega^2-L'K_R^T\right)\Theta\\&=-K_L^TU_2(0)-K_R^TV_2(0)\\&\left(\frac{L'}{2}K_L^T-\frac{L'}{2}K_R^T\right)Z+\left(I_G\omega^2-\frac{L'^2}{2}K_R^T\right)\Theta\\&=\frac{L'}{2}K_L^TU_2(0)-\frac{L'}{2}K_R^TV_2(0)\end{aligned} \tag{40}$$

or in matrix form

$$\begin{pmatrix} m\omega^2-K_L^T-K_R^T & \frac{L'}{2}m\omega^2-L'K_R^T \\ \frac{L'}{2}K_L^T-\frac{L'}{2}K_R^T & I_G\omega^2-\frac{L'^2}{2}K_R^T \end{pmatrix}\begin{pmatrix} Z \\ \Theta \end{pmatrix} = \begin{pmatrix} -K_L^TU_2(0)-K_R^TV_2(0) \\ \frac{L'}{2}K_L^TU_2(0)-\frac{L'}{2}K_R^TV_2(0) \end{pmatrix} \tag{41}$$

The solution of this inhomogeneous system of equations for Z and Θ is of the following form

$$\begin{cases} Z=\dfrac{K_L^T\left[4K_R^TL'^2-(4I_G+L'^2m)\omega^2\right]u_2(0)+K_R^T(-4I_G+L'^2m)\omega^2v_2(0)}{4K_L^TK_R^TL'^2-(K_L^T+K_R^T)(4I_G+L'^2m)\omega^2+4I_Gm\omega^4} \\ \Theta=\dfrac{2L'\left\{-K_R^Tm\omega^2v_2(0)+K_L^T\left(m\omega^2u_2(0)+2K_R^T\left[-u_2(0)+v_2(0)\right]\right)\right\}}{4K_L^TK_R^TL'^2-(K_L^T+K_R^T)(4I_G+L'^2m)\omega^2+4I_Gm\omega^4} \end{cases} \tag{42}$$

Substituting the obtained values for Z and Θ, one can find the amounts of F_R and F_L.

7. Numerical examples

In this section, frequency parameters and mode shapes of the mechanical system under study are presented. Due to the generality of the model, numerous cases can be investigated, but for sake of brevity, only a few cases are under consideration. First case is devoted to the double-beam system which consists of two cantilever beams coupled with the two-degree-of-freedom system with properties $K_L^T=K_R^T=10$, $K_L^R=K_R^R=1$, $\xi_1=\xi_2=0.5$ and $\gamma=1$. To simulate two cantilever beams, the values of the stiffness of the end springs are assumed infinity, i.e., $K_1,K_2,K_3,K_4\to\infty$. The first six mode shapes and dimensionless frequency parameters of the system are given in Fig. 4. The first dimensionless frequency parameter is $\lambda_1L=1.827982$ and the sixth one is $\lambda_6L=4.752937$. In order to study the influence of the suspended mass, the first case is examined, but when $\gamma=0.5$. Fig. 5 shows the first six mode shapes and dimensionless frequency parameters of the double-beam. From the result it is evident that decreasing the suspended mass results in an increase in the values of λL. For example, the value of first λL increases form 1.827982 to 1.956881 when value of γ decreases from 1 to 0.5.

As a third case, the effect of translational springs of the mass-spring system is investigated. In this case, a system with parameters $K_1,K_2,K_3,K_4\to\infty$, $K_L^T=K_R^T=1$, $K_L^R=K_R^R=1$, $\xi_1=\xi_2=0.5$ and $\gamma=0.5$ is considered. The first six mode shapes and dimensionless frequency parameters of the mechanical system is shown in Fig. 6. It can be easily seen that the values of λL have decreased owing to decreasing the stiffness of the translational springs.

Fourth case studies the influence of the rotational springs of the mass-spring system on the response of the double-beam. The previous system but when the values of K_L^R and K_R^R decreased to 0.1 is considered. Therefore, the properties of the system are: $K_1,K_2,K_3,K_4\to\infty$, $K_L^T=K_R^T=1$, $K_L^R=K_R^R=0.1$, $\xi_1=\xi_2=0.5$ and $\gamma=0.5$. Fig. 7 depicts the first six modes as well as the corresponding values of the dimensionless frequency parameters. As expected, decreasing the stiffness of the rotational springs results in decreasing the values of λL. It is interesting to note that the translational springs have greater effects on the dynamic behavior of the system, comparing the values of λL of the third and fourth case.

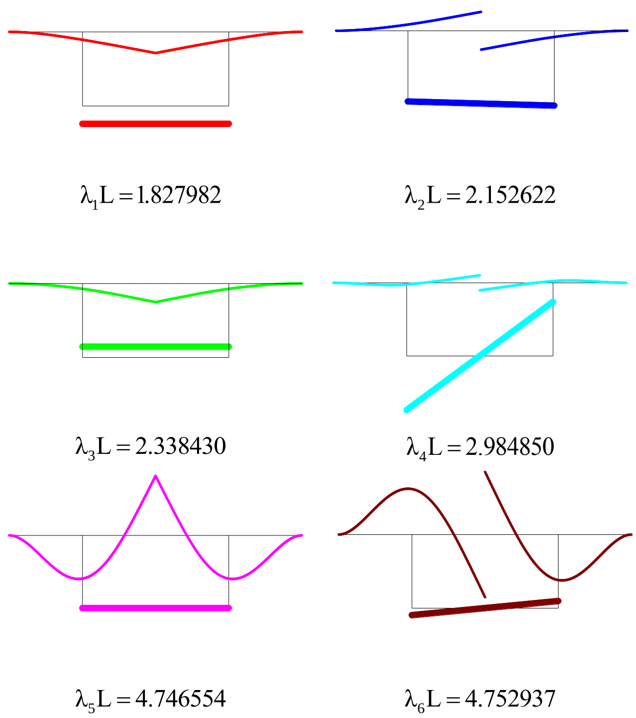


Fig. 4 The first six eigenfrequencies and eigenfunctions of case #1 for the following symmetric inputs

$$\begin{cases} K_1=10^6\dfrac{EI}{L^3};K_2=10^6\dfrac{EI}{L};K_3=10^6\dfrac{EI}{L^3};K_4=10^6\dfrac{EI}{L}; \\ K_L^T=10\dfrac{EI}{L^3};K_R^T=10\dfrac{EI}{L^3};K_L^R=1\dfrac{EI}{L};K_R^R=1\dfrac{EI}{L}; \\ \xi_1=0.5;\xi_2=0.5;L'=L;\gamma=1; \end{cases}$$

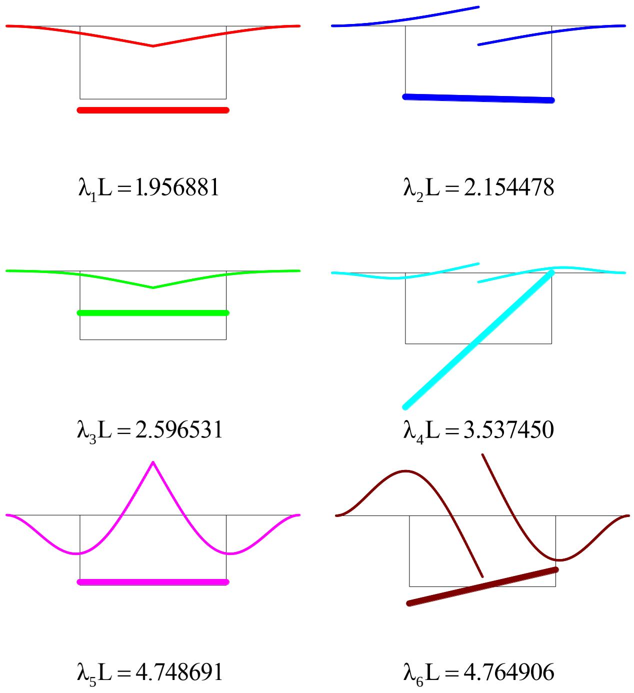


Fig. 5 The first six eigenfrequencies and eigenfunctions of case #2 for the following symmetric inputs

$$\begin{cases} K_1=10^6\dfrac{EI}{L^3};K_2=10^6\dfrac{EI}{L};K_3=10^6\dfrac{EI}{L^3};K_4=10^6\dfrac{EI}{L}; \\ K_L^T=10\dfrac{EI}{L^3};K_R^T=10\dfrac{EI}{L^3};K_L^R=1\dfrac{EI}{L};K_R^R=1\dfrac{EI}{L}; \\ \xi_1=0.5;\xi_2=0.5;L'=L;\gamma=\dfrac{1}{2}; \end{cases}$$

Last example of systems with symmetric spring conditions concerns the influence of the length of the rigid rod on the system. The chosen parameters are $K_1, K_2, K_3, K_4 \to \infty$, $K_L^T = K_R^T = 1$, $K_L^R = K_R^R = 0.1$, $\xi_1 = 0.3$, $\xi_2 = 0.5$ and $\gamma = 1.2$. The first six mode shapes and frequency parameters of the problem are presented in Fig. 8. Apparently, no general conclusion can be drawn from the results, since the position of the connections the mass-spring system to beams are not symmetric.

Next, a fully non-symmetric case is investigated. The properties of the system are assumed: $K_1 = 10$, $K_2 = 1$, $K_3 = 0.1$, $K_4 = 100$, $K_L^T = 1$, $K_R^T = 0.1$, $K_L^R = 0.01$, $K_R^R = 10$, $\xi_1 = 0.3$, $\xi_2 = 0.6$ and $\gamma = 2$. Fig. 9 presents the first six mode shapes and dimensionless frequency parameters of the system. It is evident that due to the non-symmetry of the model, no symmetric mode exists.

Finally, the special case studied in section 6 is examined. In this case, two beams are coupled with two-degree-of-freedom system via only translational springs, i.e., $K_L^R = K_R^R = 0$. The convergence of the finite element solution is studied for this example. Table 1 gives the values of the first six frequency parameters for $K_1, K_2, K_3, K_4 \to \infty$, $K_L^T = K_R^T = 10$, $\xi_1 = \xi_2 = 0.5$ and $\gamma = 1$. The excellent agreement between outcomes of both exact and finite element methods can be observed. It can be concluded from Table 1 that the exact results can be caught by lower number of elements for lower modes while for higher modes more elements is needed to reach exact solution.

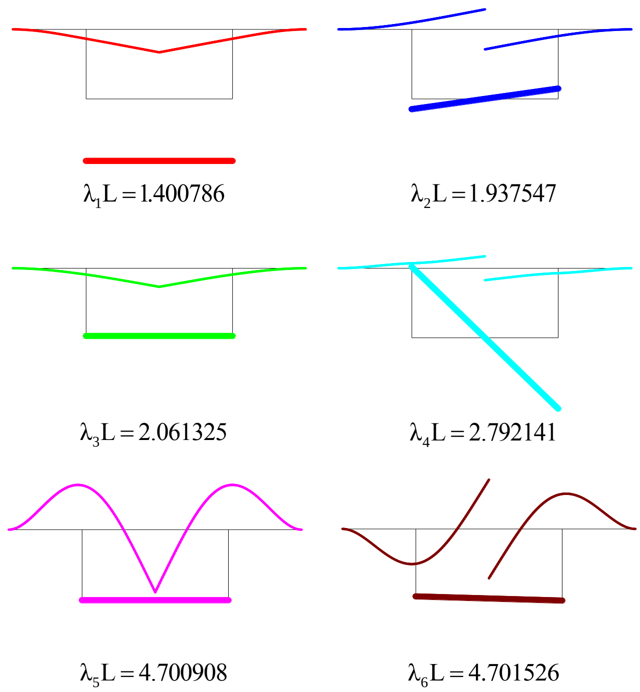
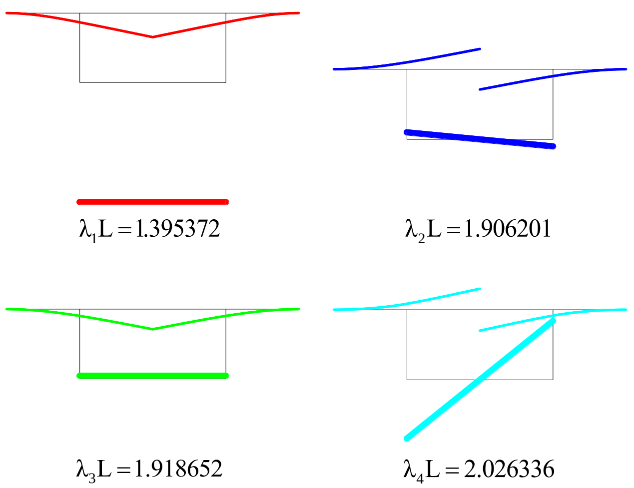


Fig. 6 The first six eigenfrequencies and eigenfunctions of case #3 for the following symmetric inputs

$$\begin{cases} K_1=10^6\dfrac{EI}{L^3};K_2=10^6\dfrac{EI}{L};K_3=10^6\dfrac{EI}{L^3};K_4=10^6\dfrac{EI}{L}; \\ K_L^T=1\dfrac{EI}{L^3};K_R^T=1\dfrac{EI}{L^3};K_L^R=1\dfrac{EI}{L};K_R^R=1\dfrac{EI}{L}; \\ \xi_1=0.5;\xi_2=0.5;L'=L;\gamma=\dfrac{1}{2}; \end{cases}$$



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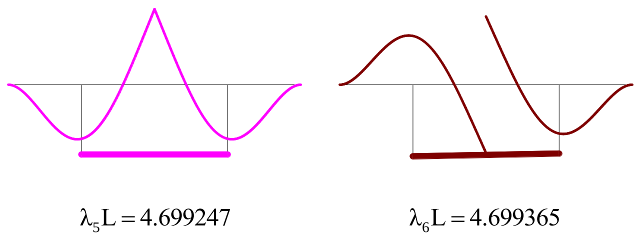

Fig. 7 The first six eigenfrequencies and eigenfunctions of case #4 for the following symmetric inputs

$$\begin{cases} K_1 = 10^6 \frac{EI}{L^3}; K_2 = 10^6 \frac{EI}{L}; K_3 = 10^6 \frac{EI}{L^3}; K_4 = 10^6 \frac{EI}{L}; \\ K_L^T = 1\frac{EI}{L^3}; K_R^T = 1\frac{EI}{L^3}; K_L^R = 0.1\frac{EI}{L}; K_R^R = 0.1\frac{EI}{L}; \\ \xi_1 = 0.5; \xi_2 = 0.5; L' = L; \gamma = \frac{1}{2}; \end{cases}$$

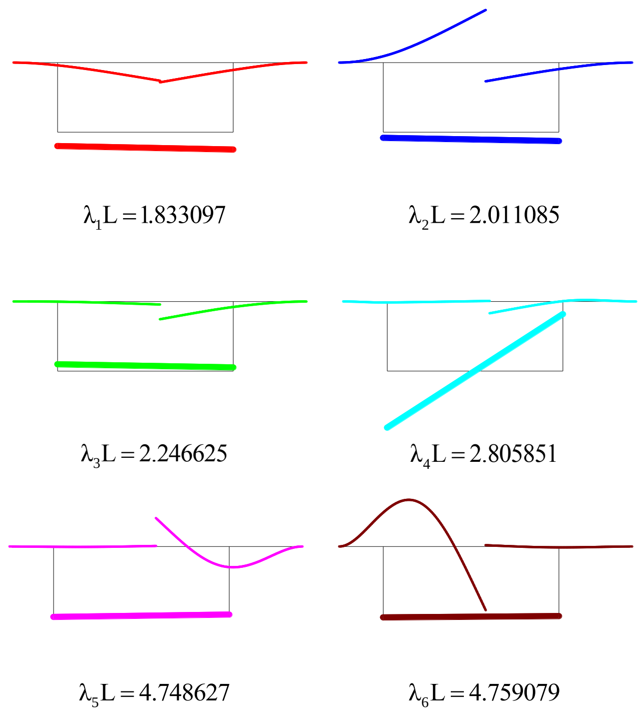

Fig. 8 The first six eigenfrequencies and eigenfunctions of case #5 for the following symmetric inputs

$$\begin{cases} K_1 = 10^6 \frac{EI}{L^3}; K_2 = 10^6 \frac{EI}{L}; K_3 = 10^6 \frac{EI}{L^3}; K_4 = 10^6 \frac{EI}{L}; \\ K_L^T = 10\frac{EI}{L^3}; K_R^T = 10\frac{EI}{L^3}; K_L^R = 1\frac{EI}{L}; K_R^R = 1\frac{EI}{L}; \\ \xi_1 = 0.3; \xi_2 = 0.5; L' = 1.2L; \gamma = 1.2; \end{cases}$$

8. Conclusions

This paper addresses the exact solution for the free vibration of a double-beam system coupled with a two-degree-of-freedom mass-spring system. The mass-spring system is connected to the main beams via two translational and two rotational springs. The governing eigenvalue problem is formulated and solved. Solution of the problem results in the frequency parameters and mode shapes of the mechanical system. Furthermore, developing a finite element solution, the accuracy of both exact and FE solutions is concluded. The effects of the mechanical parameters of the system are investigated. It can be concluded that increasing the stiffness of each spring results in increasing the values of the frequency parameters of the system. On the contrary, increasing the mass of the attached mass leads to decreasing the frequency parameters of the system. Some mode shapes are also presented.

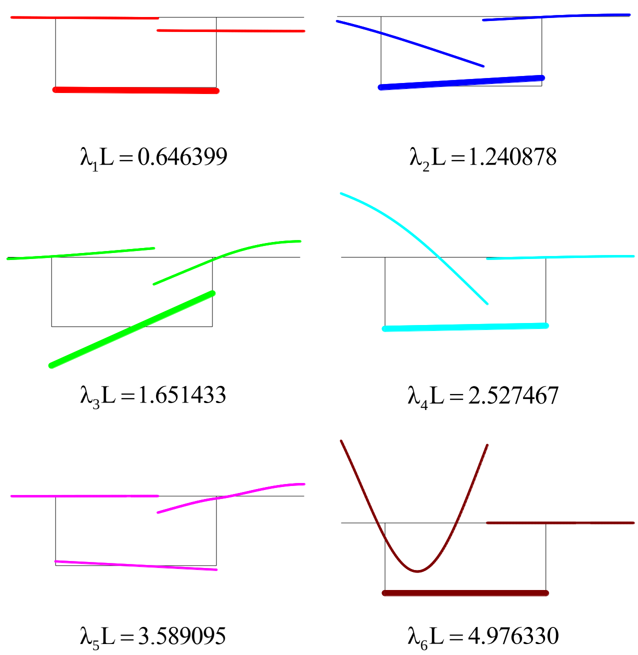

Fig. 9 The first six eigenfrequencies and eigenfunctions of case #6 for the following symmetric inputs

$$\begin{cases} K_1 = 10\frac{EI}{L^3}; K_2 = 1\frac{EI}{L}; K_3 = 0.1\frac{EI}{L^3}; K_4 = 100\frac{EI}{L}; \\ K_L^T = 1\frac{EI}{L^3}; K_R^T = 0.1\frac{EI}{L^3}; K_L^R = 0.01\frac{EI}{L}; K_R^R = 10\frac{EI}{L}; \\ \xi_1 = 0.3; \xi_2 = 0.6; L' = 1.1L; \gamma = 2; \end{cases}$$

Table 1 Comparison of finite element results and exact solution for $K_1, K_2, K_3, K_4 \to \infty$, $K_L^T = K_R^T = 10$, $\xi_1 = \xi_2 = 0.5$ and $\gamma = 1$

Mode #	Number of finite elements: 4	8	16	32	Exact
1	1.71722	1.71706	1.71706	1.71706	1.71706
2	1.83313	1.83279	1.83277	1.83277	1.83277
3	2.28418	2.28370	2.28367	2.28367	2.28367
4	2.81340	2.81309	2.81306	2.81306	2.81306
5	4.76624	4.74771	4.74502	4.74483	4.74493
6	4.77076	4.75219	4.74950	4.74931	4.74941

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