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Stochastic vibration suppression analysis of an optimal bounded controlled sandwich beam with MR visco-elastomer core

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Abstract. To control the stochastic vibration of a vibration-sensitive instrument supported on a beam, the beam is designed as a sandwich structure with magneto-rheological visco-elastomer (MRVE) core. The MRVE has dynamic properties such as stiffness and damping adjustable by applied magnetic fields. To achieve better vibration control effectiveness, the optimal bounded parametric control for the MRVE sandwich beam with supported mass under stochastic and deterministic support motion excitations is proposed, and the stochastic and shock vibration suppression capability of the optimally controlled beam with multi-mode coupling is studied. The dynamic behavior of MRVE core is described by the visco-elastic Kelvin-Voigt model with a controllable parameter dependent on applied magnetic fields, and the parameter is considered as an active bounded control. The partial differential equations for horizontal and vertical coupling motions of the sandwich beam are obtained and converted into the multi-mode coupling vibration equations with the bounded nonlinear parametric control according to the Galerkin method. The vibration equations and corresponding performance index construct the optimal bounded parametric control problem. Then the dynamical programming equation for the control problem is derived based on the dynamical programming principle. The optimal bounded parametric control law is obtained by solving the programming equation with the bounded control constraint. The controlled vibration responses of the MRVE sandwich beam under stochastic and shock excitations are obtained by substituting the optimal bounded control into the vibration equations and solving them. The further remarkable vibration suppression capability of the optimal bounded control compared with the passive control and the influence of the control parameters on the stochastic vibration suppression effectiveness are illustrated with numerical results. The proposed optimal bounded parametric control strategy is applicable to smart visco-elastic composite structures under deterministic and stochastic excitations for improving vibration control effectiveness.

Keywords: stochastic vibration; optimal bounded control; sandwich beam; magneto-rheological visco-elastomer; stochastic response reduction

1. Introduction

The vibration control for engineering structures subjected to strong and micro disturbance excitations is a significant research subject. Smart materials such as magneto-rheological liquid have been applied to the structural vibration suppression by semi-active control. Magneto-rheological dampers were installed on structures for vibration control (Dyke, Spencer *et al*. 1996, Spencer and Nagarajaiah 2003, Casciati, Rodellar *et al*. 2012, Zapateiro, Karimi *et al*. 2010, Cha, Zhang *et al*. 2013, Hernández, Marichal *et al*. 2015, Wang, Li *et al*. 2015, etc.).

The control forces act on structures as external inputs, and then the control law can be designed based on the control strategy such as linear quadratic control or linear quadratic Gaussian control. In addition, the vibration characteristics of sandwich beams with magneto-rheological liquid were studied (Rajamohan, Rakheja *et al*. 2010).

In recent years, magneto-rheological visco-elastomer (MRVE) has been fabricated and applied to the structural vibration control (Carlson and Jolly 2000, Ying, Ni *et al.* 2013, etc.). The MRVE based tunable vibration isolators, absorbers and dampers have been designed and tested (York, Wang *et al.* 2007, Jung, Eem *et al.* 2011). On the other hand, the MRVE is used to construct composite structures for mitigating vibration by optimizing dynamic characteristics. The vibration analysis of sandwich beams with uncontrollable viscoelastic damping has been given out early (Ditaranto 1965, Mead and Markus 1969, Yan and Dowell 1972, Frostig and Baruch 1994). The recent study on MRVE sandwich beams or plates has been presented, including the adjustable stiffness (Zhou and Wang 2006), frequency-response characteristics (Choi, Xiong *et al.* 2010), periodic vibration analysis using the finite element

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method (Yeh 2013), dynamic stability under periodic axial loads (Dwivedy, Mahendra *et al.* 2009, Nayak, Dwivedy *et al.* 2011) and micro-vibration response (Ni, Ying *et al.* 2011, Ying, Ni *et al*. 2015b, c). In these studies, the MRVE was considered as passive control under a deterministic magnetic field.

The periodic and stochastic vibrations of MRVE composite structures can be reduced by the area energy dissipation. However, the passive control effectiveness of the structural vibrations depends fully on the MRVE dynamic characteristics and then is confined, because the MRVE properties such as adjustable stiffness and damping are restricted within certain limits. The optimal adjustment of MRVE composite structures based on the dynamical programming principle can exert complete MRVE properties and achieve further vibration control effectiveness. However, the MRVE stiffness and damping adjustable by applied magnetic fields are represented as composite structure parameters (Ni, Ying *et al.* 2011, Ying, Ni *et al*. 2015b, c, etc.). The parametric control design differs from the non-parametric control design. In general, the non-parametric control is a structural external action such as external force, and the parametric control is a structural state-dependent action such as stiffness parameter. The linear quadratic Gaussian control is an optimal strategy for linear stochastic systems with external controls, but it is not an optimal strategy for the systems with parametric controls, that is, the linear quadratic Gaussian control cannot satisfy the dynamical programming principle for the parametric control and is unsuitable for use (Stengel 1994, Yong and Zhou 1999, Ying, Ni *et al*. 2015a). Therefore, the study on the optimal parametric control of MRVE composite structures, in particular, under stochastic excitation is necessary for further vibration suppression.

The parametric control adjustment of MRVE composite structures is bounded, because the controllable MRVE modulus has certain limits due to the magnetic-mechanical saturation. The bang-bang control is a simple and feasible bounded control with better effectiveness. The deterministic and stochastic bang-bang control strategy and its application have been presented for dynamic systems with external control inputs (Wu and Soong 1996, Lim, Chung *et al*. 2003, Dimentberg, Iourtchenko *et al.* 2000, Bratus, Dimentberg *et al.* 2000). Recently, the bang-bang parametric control of single degree-of-freedom dynamic systems has been studied (Dimentberg and Bratus 2000, Liu, Waters *et al.* 2005, Potter, Neild *et al*. 2010, Ying, Ni *et al.* 2015a). However, the optimal bounded parametric control of multi-degree-of-freedom dynamic systems such as MRVE composite structures is lack of research. As mentioned above, the control strategy used generally such as linear quadratic Gaussian control is unsuitable for MRVE composite structures with parametric controls. The optimal parametric control can exert complete MRVE properties and achieve better vibration control effectiveness. Thus the optimal bounded parametric control of MRVE composite structures such as sandwich beams with multi-mode coupling vibration under stochastic excitations and the stochastic vibration suppression capability of the nonlinear bounded parametric controlled structures need to be studied further.

In this paper, the optimal bounded parametric control of an MRVE sandwich beam with supported mass under stochastic and deterministic support motion excitations is designed, and the stochastic and shock vibration suppression capability of the beam with multi-mode coupling by using the optimal control is studied. Firstly, the partial differential equations for horizontal and vertical coupling motions of the sandwich beam in the time domain are obtained based on the dynamic equilibrium, constitutive and geometric relations. The dynamic behavior of the MRVE core is described by the visco-elastic Kelvin-Voigt model with a controllable parameter dependent on applied magnetic fields. The Galerkin method is applied to convert the partial differential equations into the multi-mode coupling vibration equations with bounded nonlinear parametric control. Secondly, the dynamical programming principle is applied to the optimal control problem described by the vibration equations and performance index for obtaining a dynamical programming equation. By the functional minimization in the programming equation with the bounded control constraint, the optimal bounded parametric control law is designed and determined finally by solving the value function equation. With the substitution of the optimal bounded control, the vibration equations of the optimally controlled sandwich beam under support excitations are obtained and solved to determine the controlled displacement responses which are used for evaluating the vibration suppression capability. Finally, numerical results on the displacement responses of the MRVE sandwich beam under stochastic and shock excitations by using the proposed optimal bounded parametric control are shown and compared with those of the passively controlled beam to illustrate the remarkable vibration suppression effectiveness of the proposed control. The results on the control parameter influence are given to illustrate the improvement of the stochastic vibration suppression capability through the reasonable design of the sandwich beam including MRVE properties.

2. Differential equations of motion of MRVE sandwich beam with supported mass

The MRVE with controllable dynamic characteristics is applied to construct a sandwich beam for vibration control. For example, to control the stochastic vibration of a vibration-sensitive instrument supported on a beam, the beam is designed as a sandwich structure with MRVE core and the instrument is modeled as a concentrated mass. The MRVE has dynamic properties such as stiffness and damping adjustable by applied magnetic fields. The horizontal MRVE sandwich beam with a supported concentrated mass is shown in Fig. 1, which is subjected to vertical support motion excitations. The length and width of the sandwich beam are L and b, respectively. The two non-magnetic facial layers are linearly elastic and have the identical elastic modulus of E_1, mass density of ρ_1 and thickness of h_1. The MRVE core layer is soft compared with the facial layers and has the shear modulus operator of G_2, mass density of ρ_2 and thickness of h_2. The supported mass is fixed on the beam and has the mass of $m_b \times b$, which size is small compared with the beam length and can be neglected. The supports have the identical vertical displacement of w_0, which is a deterministic or stochastic disturbance excitation.

The MRVE has the dynamic characteristics controllable by the applied magnetic field which is vertical and covers entire MRVE. Its shear strain depends linearly on the applied shear stress for finite deformation. Under certain magnetic field, the MRVE as visco-elastic material has the viscous and elastic combined properties, and then the shear stress can be divided equivalently into two corresponding parts (Ward and Sweeney 2013). Based on the Kelvin-Voigt model, the dynamic shear modulus G_2 can be expressed by differential operator in the time domain a

$$G_2 = G(1+\beta \frac{\partial}{\partial t}) \tag{1}$$

where G is the equivalent real shear modulus, β is the equivalent damping ratio, and t is the time variable. The damping ratio is regarded as a constant. The shear modulus G is controllable by the applied magnetic field and considered as the parametric control with certain limits. Before the magnetic saturation, the effect of the high-order nonlinear terms of the magnetic field intensity on the shear modulus is small and neglected. The shear modulus can be approximated to (Ying, Ni *et al.* 2013)

$$G = \alpha_0 + \alpha_1 B_\mathrm{m} + \alpha_2 B_\mathrm{m}^2 \tag{2}$$

where α_i (i=0, 1, 2) are constants, and B_m is the magnetic field intensity. The values of α_i and magnetic saturation intensity are determined by the MRVE properties. The magnetic field intensity B_m corresponding to the shear modulus G or parametric control can be calculated by using Eq. (2).

For the sandwich beam, it is assumed that: (1) the two elastic facial layers and MRVE core layer are respectively homogeneous and continuous; and the facial layer materials are isotropic while the core material is transversely isotropic under an applied magnetic field along z-axis; (2) the normal stress of the core layer is relatively small and neglected; (3) the normal stresses of the facial layers in the direction of z-axis are relatively small and neglected; (4) the vertical displacement of the sandwich beam is invariant along the thickness; (5) the cross section of each facial layer is perpendicular to its axis line in deformation; and the cross section of the core layer is a plane in deformation; (6) the longitudinal and rotational inertias of the beam are relatively small and neglected; (7) the interfaces between the facial layers and core layer are continuous all the time (Ni, Ying *et al.* 2011, Mead and Markus 1969, Yan and Dowell 1972).

Based on the above assumptions, the displacements and shear stresses on the interfaces between the facial layers and core layer are continuous. The vertical displacement of the beam relative to the supports is w=$w(x,t)$, where x is the horizontal coordinate. The horizontal displacements of the facial layers are expressed as

$$u_i(x, z_i, t) = u_{i0}(x,t) - z_i \frac{\partial w(x,t)}{\partial x}, \quad i=1, 3 \tag{3}$$

where u_{10} and u_{30} are respectively the horizontal displacements of the upper and lower facial mid layers, z_1 and z_3 are the vertical local coordinates of the two facial layers. The horizontal displacements on the two interfaces between the facial layers and core layer are

$$u_{1I} = u_{10} + \frac{h_1}{2}\frac{\partial w}{\partial x}, \quad u_{3I} = u_{30} - \frac{h_1}{2}\frac{\partial w}{\partial x} \tag{4}$$

Then the shear strain of the MRVE core layer is given by

$$\gamma_2 = \frac{u_{1I} - u_{3I}}{h_2} + \frac{\partial w}{\partial x} = \frac{u_{10} - u_{30}}{h_2} + \frac{h_a}{h_2}\frac{\partial w}{\partial x} \tag{5}$$

where h_a=h_1+h_2. By using the shear modulus Eq. (1), the shear stress of the MRVE core layer is obtained as

$$\tau_2(x,t) = G_2\gamma_2 = G(\frac{u_{10} - u_{30}}{h_2} + \frac{h_a}{h_2}\frac{\partial w}{\partial x}) + \beta G(\frac{\dot{u}_{10} - \dot{u}_{30}}{h_2} + \frac{h_a}{h_2}\frac{\partial \dot{w}}{\partial x}) \tag{6}$$

where $\dot{u}_{10} = \partial u_{10}/\partial t$, $\dot{u}_{30} = \partial u_{30}/\partial t$, and $\dot{w} = \partial w/\partial t$.

The horizontal normal strains ε_i of the two facial layers can be derived from the derivatives of displacements given in Eq. (3). The corresponding normal stresses of the facial layers are

$$\sigma_i(x, z_i, t) = E_1\varepsilon_i = E_1(\frac{\partial u_{i0}}{\partial x} - z_i \frac{\partial^2 w}{\partial x^2}), \quad i=1, 3 \tag{7}$$

By using the stress equilibrium equation of an element in the direction of x-axis, the shear stresses of the facial layers are obtained as

$$\begin{aligned}\tau_i(x, z_i, t) &= \int -\frac{\partial \sigma_i}{\partial x}\mathrm{d}z_i \\ &= E_1[(-1)^{(i-1)/2}\frac{h_1}{2} - z_i]\frac{\partial^2 u_{i0}}{\partial x^2} - E_1(\frac{h_1^2}{8} - \frac{z_i^2}{2})\frac{\partial^3 w}{\partial x^3},\end{aligned} \quad i=1, 3 \tag{8}$$

Based on the continuity conditions of shear stresses on the interfaces between the facial layers and core layer, the differential equation for the horizontal displacements of the sandwich beam is obtained as

$$E_1 h_1 h_2 \frac{\partial^2 u}{\partial x^2} - G(2u + h_a\frac{\partial w}{\partial x}) - \beta G(2\dot{u} + h_a \frac{\partial \dot{w}}{\partial x}) = 0 \tag{9}$$

where u=u_{10}=$-u_{30}$. With considering the vertical inertia, the dynamic equilibrium equation of a sandwich beam element with supported mass in the direction of z-axis yields

$$\int_{h_t} \frac{\partial \tau_i}{\partial x}\mathrm{d}z - [\rho h_t + m_b \delta(x - x_0)](\ddot{w} + \ddot{w}_0) = 0 \tag{10}$$

where ρ=$(2\rho_1 h_1+\rho_2 h_2)/h_t$ and h_t=$2h_1+h_2$, $\ddot{w} = \partial^2 w/\partial t^2$, $\ddot{w}_0 = \partial^2 w_0/\partial t^2$, $\delta(\cdot)$ is the Dirac delta function, and x_0 is the horizontal coordinate of the mass. By substituting shear stresses Eqs. (6) and (8) into Eq. (10), the differential equation for the vertical displacement of the beam is obtained as

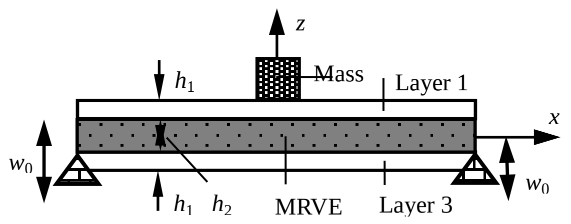

Fig. 1 Sandwich beam with MRVE core and supported mass

$$\frac{E_1 h_1^3}{6}\frac{\partial^4 w}{\partial x^4} - E_1 h_1^2 \frac{\partial^3 u}{\partial x^3} - G(h_a \frac{\partial^2 w}{\partial x^2} + 2\frac{\partial u}{\partial x}) - \beta G(h_a \frac{\partial^2 \dot{w}}{\partial x^2} + 2\frac{\partial \dot{u}}{\partial x}) + [\rho h_t + m_b \delta(x - x_0)](\ddot{w} + \ddot{w}_0) = 0 \tag{11}$$

The partial differential Eqs. (9) and (11) describe the horizontal and vertical coupling motions of the sandwich beam with supported mass under support motion excitations. The MRVE shear modulus G controllable by applied magnetic field appears in the parameters and then acts as the parametric control. For the simply supported beam, the boundary conditions of displacements obtained are (Mead and Markus 1969, Ni, Ying *et al.* 2011)

$$w(\pm\frac{L}{2}, t) = 0 \,,\quad \frac{\partial^2 w(\pm L/2, t)}{\partial x^2} = 0 \,,\quad \frac{\partial u(\pm L/2, t)}{\partial x} = 0 \tag{12}$$

Eqs. (9) and (11) for the MRVE sandwich beam with the boundary conditions Eq. (12) can be rewritten in the dimensionless form as follows

$$E_1 h_1 h_2 \frac{\partial^2 \overline{u}}{\partial y^2} - GL^2(2\overline{u} + \frac{h_a}{L}\frac{\partial \overline{w}}{\partial y}) - \beta G\sqrt{\frac{E_1 h_1^3}{\rho h_t}}(2\dot{\overline{u}} + \frac{h_a}{L}\frac{\partial \dot{\overline{w}}}{\partial y}) = 0 \tag{13}$$

$$\frac{E_1 h_1^3}{6}\frac{\partial^4 \overline{w}}{\partial y^4} - E_1 h_1^2 L \frac{\partial^3 \overline{u}}{\partial y^3} - GL^3(\frac{h_a}{L}\frac{\partial^2 \overline{w}}{\partial y^2} + 2\frac{\partial \overline{u}}{\partial y}) - \beta GL\sqrt{\frac{E_1 h_1^3}{\rho h_t}}(\frac{h_a}{L}\frac{\partial^2 \dot{\overline{w}}}{\partial y^2} + 2\frac{\partial \dot{\overline{u}}}{\partial y}) + E_1 h_1^3[1 + \frac{m_b}{\rho h_t L}\delta(y - y_0)](\ddot{\overline{w}} + \ddot{\overline{w}}_0) = 0 \tag{14}$$

$$\overline{w}(\pm\frac{1}{2}, \tau) = 0 \,,\quad \frac{\partial^2 \overline{w}(\pm 1/2, \tau)}{\partial y^2} = 0 \,,\quad \frac{\partial \overline{u}(\pm 1/2, \tau)}{\partial y} = 0 \tag{15}$$

where $\dot{\overline{w}} = \partial \overline{w} / \partial \tau$, $\dot{\overline{u}} = \partial \overline{u} / \partial \tau$, $\ddot{\overline{w}} = \partial^2 \overline{w} / \partial \tau^2$, $\ddot{\overline{w}}_0 = \partial^2 \overline{w}_0 / \partial \tau^2$, the amplitude of the support motion w_0 is W_a, the dimensionless coordinates, time and displacements are

$$y = \frac{x}{L} \,,\quad y_0 = \frac{x_0}{L} \,,\quad \tau = t\sqrt{\frac{E_1 h_1^3}{\rho h_t L^4}} \,,$$
$$\overline{u} = \frac{u}{W_a} \,,\quad \overline{w} = \frac{w}{W_a} \,,\quad \overline{w}_0 = \frac{w_0}{W_a} \tag{16}$$

3. Multi-mode coupling vibration equations with parametric control

The vibration displacements of the MRVE sandwich beam can be expanded into series in the modal space. Under the homogeneous boundary conditions given in Eq. (15), the expanded expressions are

$$\overline{u}(y, \tau) = \sum_{j=1}^{N} p_j(\tau) \sin(2j - 1)\pi y \tag{17}$$

$$\overline{w}(y, \tau) = \sum_{j=1}^{N} q_j(\tau) \cos(2j - 1)\pi y \tag{18}$$

where $p_j(\tau)$ and $q_j(\tau)$ are functions of dimensionless time τ, and N is an integer. According to the Galerkin method, substituting dimensionless displacements Eqs. (17) and (18) into Eqs. (13) and (14), multiplying the equations respectively by $\sin(2i-1)\pi y$ and $\cos(2i-1)\pi y$, and integrating them with respect to y yield ordinary differential equations for p_j and q_j. By neglecting less longitudinal beam damping and eliminating modal displacement p_j, the ordinary differential equations for modal displacement q_j corresponding to the dimensionless vertical displacement of the beam can be obtained and rewritten in the following matrix form

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{C}(G)\dot{\mathbf{Q}} + \mathbf{K}(G)\mathbf{Q} = \mathbf{F}(\tau) \tag{19}$$

where modal displacement vector $\mathbf{Q}=[q_1, q_2, \ldots, q_N]^{\mathrm{T}}$, $\dot{\mathbf{Q}} = \mathrm{d}\mathbf{Q}/\mathrm{d}\tau$, $\ddot{\mathbf{Q}} = \mathrm{d}^2\mathbf{Q}/\mathrm{d}\tau^2$, modal mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, stiffness matrix $\mathbf{K}$ and excitation vector $\mathbf{F}$ have elements as follows

$$M_{ij} = \delta_{ij} + \frac{2m_b}{\rho h_t L}\cos(2i-1)\pi y_0 \cos(2j-1)\pi y_0 \,.$$

$$C_{ij} = g_i C_{1ij} \,,\quad K_{ij} = K_{0ij} + g_i K_{1ij}$$

$$C_{1ij} = \frac{(2i-1)^4 \pi^4 \beta h_2 h_a}{L^2}\delta_{ij}\sqrt{\frac{E_1}{\rho h_t h_1}} \,,$$

$$K_{0ij} = (2i-1)^4 \pi^4 \delta_{ij} \,,\quad K_{1ij} = \frac{(2i-1)^4 \pi^4 h_a^2}{h_1^2}\delta_{ij} \tag{20}$$

$$g_i(G) = \frac{G}{2G + \alpha_i} \,,\quad \alpha_i = (2i-1)^2 \pi^2 \frac{E_1 h_1 h_2}{L^2} \,,$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$F_i = [\frac{4(-1)^i}{(2i-1)\pi} - \frac{2m_b}{\rho h_t L}\cos(2i-1)\pi y_0]\ddot{\overline{w}}_0 \,,\quad i, j = 1, 2, \ldots, N$$

Eq. (19) represents a multi-degree-of-freedom control system derived from the MRVE sandwich beam with supported mass, which has the damping $\mathbf{C}$ and stiffness $\mathbf{K}$ dependent on the parametric control G. By separating the control G from the damping and stiffness, Eq. (19) becomes

$$\mathbf{M}\ddot{\mathbf{Q}} + \mathbf{K}_0\mathbf{Q} + \mathbf{g}(G)(\mathbf{C}_1\dot{\mathbf{Q}} + \mathbf{K}_1\mathbf{Q}) = \mathbf{F}(\tau) \tag{21}$$

where control matrix $\mathbf{g}(G)=\mathrm{diag}[g_1, g_2, \ldots, g_N]$, the elements of matrices $\mathbf{K}_0$, $\mathbf{K}_1$ and $\mathbf{C}_1$ are K_{0ij}, K_{1ij} and C_{1ij}, respectively. To apply the dynamical programming principle and determining the optimal control, Eq. (21) is rewritten further as the state equation

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} - \mathbf{B}(G)\mathbf{Z} + \mathbf{W}(\tau) \quad (22)$$

where $\dot{\mathbf{Z}} = \mathrm{d}\mathbf{Z}/\mathrm{d}\tau$, state vector $\mathbf{Z}$, coefficient matrix $\mathbf{A}$, control matrix $\mathbf{B}$ and excitation vector $\mathbf{W}$ are

$$\mathbf{Z} = \begin{Bmatrix} \mathbf{Q} \\ \dot{\mathbf{Q}} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & 0 \end{bmatrix},$$
$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{g}\begin{bmatrix} 0 & 0 \\ \mathbf{K}_1 & \mathbf{C}_1 \end{bmatrix}, \quad \mathbf{W} = \mathbf{M}^{-1}\begin{Bmatrix} 0 \\ \mathbf{F} \end{Bmatrix} \quad (23)$$

in which $\mathbf{I}$ is the identity matrix.

4. Optimal bounded parametric control law and controlled response of MRVE sandwich beam

The vibration control of the MRVE sandwich beam or system given in Eq. (22) can be performed by actively adjusting the MRVE modulus G as a system parameter, and the system is a parametric control system. The parametric control G is bounded due to the applied magnetic field limits and MRVE magnetic-mechanical saturation, and then system given in Eq. (22) becomes a bounded parametric control system. The bounded control constraint is given by

$$G_l \le G \le G_h \quad (24)$$

where G_l and G_h are positive constants. Under the stochastic support motion excitation such as Gaussian white noise, the sandwich beam response is a stochastic process. The optimal bounded control of system given in Eq. (22) is to minimize a mean performance index related to the system response which is expressed as

$$J(G) = E[\int_0^{\tau_f} \mathbf{Z}^{\mathrm{T}}\mathbf{S}_c\mathbf{Z}\mathrm{d}\tau + \Psi(\mathbf{Z}(\tau_f))] \quad (25)$$

where $E[\cdot]$ is the expectation operator, $\mathbf{S}_c$ is a semi-positive definite symmetric weight matrix, $\Psi(\tau_f)$ is a terminal cost and τ_f is the terminal time. For the deterministic support motion excitation, the system response control has the performance index given in Eq. (25) without the expectation operation.

Eqs. (22), (24) and (25) construct a stochastic optimal bounded parametric feedback control problem. Based on the stochastic dynamical programming principle (Stengel 1994, Yong and Zhou 1999), the dynamical programming equation can be derived from system Eq. (22) with index Eq. (25). The equation obtained is

$$\frac{\partial V}{\partial \tau} + \min_G \{\frac{1}{2}\mathrm{tr}(\mathbf{D}_w \frac{\partial^2 V}{\partial \mathbf{Z}^2}) + [\mathbf{A}\mathbf{Z} - \mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + \mathbf{Z}^{\mathrm{T}}\mathbf{S}_c\mathbf{Z}\} = 0 \quad (26)$$

where $V(\mathbf{Z},\tau)$ is the value function, $\mathrm{tr}(\cdot)$ is the trace operator, and $\mathbf{D}_w$ is the intensity matrix of stochastic excitation $\mathbf{W}$. The minimization of the left side of Eq. (26) is equated with the maximization of $[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}}\partial V/\partial \mathbf{Z}$. The bounded parametric control G can be obtained by the maximization with the control constraint given in Eq. (24). Then the optimal bounded parametric control law is determined and expressed as

$$G^* = G\Big|_{\max\{[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}}\partial V/\partial \mathbf{Z}\},\ G \in [G_l, G_h]} \quad (27)$$

It is obtained from Eqs. (20) and (23) that the control G is a nonlinear parameter of system given in Eq. (22) and the $\mathbf{B}(G)$ is a nonlinear function of G. In general, the G^* corresponding to the maximum of $[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}}\partial V/\partial \mathbf{Z}$ may not be the bound G_h or G_l. In the case that G is much smaller than E_1, the $[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}}\partial V/\partial \mathbf{Z}$ is approximate to

$$[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} = (\mathbf{K}_1\mathbf{Q} + \mathbf{C}_1\dot{\mathbf{Q}})^{\mathrm{T}} \mathbf{g}\mathbf{M}^{-1} \frac{\partial V}{\partial \dot{\mathbf{Q}}}$$
$$\approx G\sum_{i=1}^{N} \frac{K_{1ii}q_i + C_{1ii}\dot{q}_i}{\alpha_i} (\mathbf{M}^{-1}\frac{\partial V}{\partial \dot{\mathbf{Q}}})_i \quad (28)$$

Substituting the optimal bounded control Eq. (27) into Eq. (26) yields the value function equation

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\mathrm{tr}(\mathbf{D}_w \frac{\partial^2 V}{\partial \mathbf{Z}^2}) + [\mathbf{A}\mathbf{Z} - \mathbf{B}(G^*)\mathbf{Z}]^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + \mathbf{Z}^{\mathrm{T}}\mathbf{S}_c\mathbf{Z} = 0 \quad (29)$$

By solving Eq. (29), V can be obtained and $[\mathbf{B}(G)\mathbf{Z}]^{\mathrm{T}}\partial V/\partial \mathbf{Z}$ can be calculated to determine the condition in expression Eq. (27). Eq. (29) has a quadratic stationary solution $V=\mathbf{Z}^{\mathrm{T}}\mathbf{P}_c\mathbf{Z}$, where $\mathbf{P}_c$ is the positive definite symmetric matrix determined by

$$\mathbf{S}_c + [\mathbf{A} - \mathbf{B}(G^*)]^{\mathrm{T}}\mathbf{P}_c + \mathbf{P}_c[\mathbf{A} - \mathbf{B}(G^*)] = 0 \quad (30)$$

The optimal bounded parametric control G^* is obtained consequently by solving Eqs. (30) and (27). For the deterministic support motion excitation, the dynamical programming equation (26) has not the second term. However, the optimal bounded parametric control G^* has the same expression as given in Eq. (27) and the quadratic stationary value function V can be obtained with the same equation as Eq. (30). Substituting the optimal control G^* into Eq. (22) yields the optimally controlled system equation

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} - \mathbf{B}(G^*)\mathbf{Z} + \mathbf{W}(\tau) \quad (31)$$

The controlled system response $\mathbf{Z}$ can be obtained by solving Eq. (31), and then the displacement of the controlled MRVE sandwich beam under support motion excitations can be calculated by the following expression

$$\overline{w}(y,\tau) = \sum_{j=1}^{N} q_j(\tau)\cos(2j-1)\pi y \quad (32)$$

The displacement response at the midpoint of the beam relative to the supports can be obtained by using Eq. (32) with y=0. The response statistics such as standard deviation of the stochastic beam vibration can be estimated by using the response.

5. Numerical results

To show the optimal bounded parametric control effectiveness, consider an MRVE sandwich beam with supported mass under stochastic and shock support motion excitations, which has basic parameter values: L=5 m, h_1=10 cm, h_2=20 cm, ρ_1=3000 kg/m^3, ρ_2=1200 kg/m^3, E_1=10 GPa, G_l=0.2 MPa, G_h=1.0 MPa, β=0.005, x_0=0, and m_b/L=80 kg/m^2. The optimal control weight is $\mathbf{S}_c$=diag[0, 0, 0, 5, 4, 3]. The control bounds G_l and G_h are determined based on the MRVE properties used. For comparison, the passively controlled system given in Eq. (22) is considered which has a constant control G. The passive parametric control is chosen as G=0.6 MPa which is the mean value of bounds G_l and G_h. The passive control cost is the approximate mean cost of the optimal control as given in the following results. Numerical results on the dimensionless displacement responses at the midpoint of the MRVE sandwich beam by using the proposed optimal control are obtained and shown in Figs. 2-17.

5.1 Optimal control for MRVE sandwich beam unde r shock excitation

Firstly, the MRVE sandwich beam is subjected to an initial shock excitation with unit amplitude and duration τ=0.04. Fig. 2 illustrates that the optimally controlled and passively controlled displacement responses $\overline{w}$ of the MRVE sandwich beam under the shock excitation vary with the dimensionless time τ.

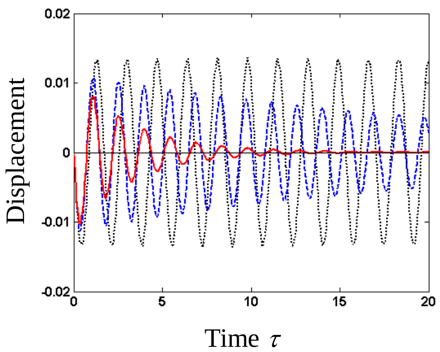

Fig. 2 Optimally controlled and passively controlled displacement responses of MRVE sandwich beam and displacement response of the beam without MRVE under shock excitation (dotted line: without MRVE; dashed line: passively controlled; solid line: optimally controlled)

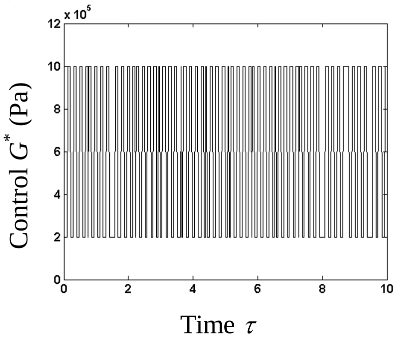

Fig. 3 Optimal bounded parametric control varying with time for shock excitation

Also the displacement response of the beam without MRVE is shown in Fig. 2. It is seen that the vibration response to the shock excitation is controlled effectively by using the proposed optimal bounded parametric control. The maximum dimensionless displacement is reduced from 0.0136 to 0.0080 by using the optimal control and from 0.0136 to 0.0107 by using the passive control. For the optimal control with time τ=14.9, the displacement response decreases to 0.00012, and to 0.0062 for the passive control with τ=14.8. The corresponding optimal bounded control G^* varying with time is shown in Fig. 3. The mean value of the optimal control is 0.63 MPa which approximates to the passive control of 0.6 MPa. Thus the proposed optimal bounded parametric control can achieve the remarkable control effectiveness and make better use of the MRVE than the passive control for the vibration suppression of the sandwich beam.

Fig. 4 shows the optimally controlled displacement responses of the MRVE sandwich beam under the shock excitation for different control bounds G_h. The optimal control bounds are G_h=0.8 MPa, 1.0 MPa, 1.4 MPa and the corresponding mean values are 0.52 MPa, 0.63 MPa, 0.84 MPa, respectively. It is seen that the vibration response is reduced with the increase of the control bound G_h, and however, the response decrement decreases nonlinearly with the bound increase. Fig. 5 illustrates that the optimal control time τ, in which the dimensionless displacement response is reduced from the maximum value of 0.0136 to the value smaller than 0.0001 and the dimensionless velocity response is reduced from 0.0591 to the value smaller than 0.0005, is shortened nonlinearly with the increase of the control bound G_h. The mean value of the corresponding optimal bounded control G^* increases linearly with the bound G_h as shown in Fig. 6. Therefore, the control effectiveness can be improved by enlarging the control bound greatly for small bound G_h and slightly for large bound G_h. Then the control bound can be chosen according to certain control effectiveness and MRVE fabrication.

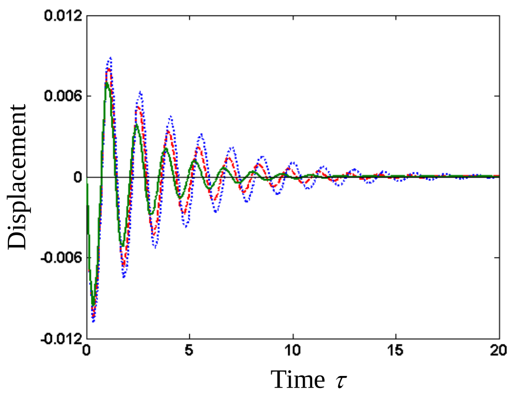

Fig. 4 Optimally controlled displacement responses of MRVE sandwich beam under shock excitation for different control bounds (dotted line: G_h=0.8 MPa; dashed line: G_h=1.0 MPa; solid line: G_h=1.4 MPa)

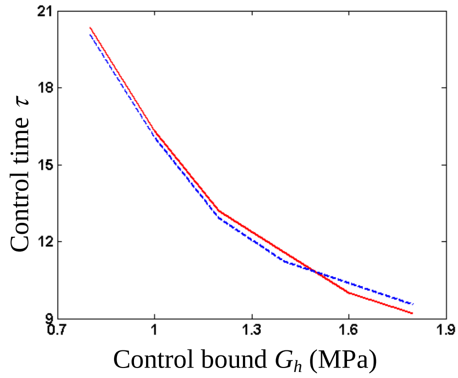

Fig. 5 Optimal control time τ versus control bound G_h of MRVE sandwich beam under shock excitation (dashed line: controlled velocity response $\dot{\overline{w}}$ <0.0005; solid line: controlled displacement response $\overline{w}$ <0.0001)

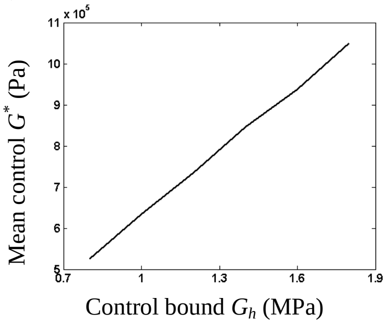

Fig. 6 Mean optimal control versus control bound G_h for shock excitation

5.2 Optimal control for MRVE sandwich beam under stochastic excitation

Secondly, the MRVE sandwich beam is subjected to a stochastic excitation of Guassian white noise with unit intensity as shown in Fig. 7. Fig. 8 illustrates that the optimally controlled and passively controlled displacement responses $\overline{w}$ of the MRVE sandwich beam under the stochastic excitation vary with the dimensionless time τ. Also the displacement response of the beam without MRVE is shown in Fig. 8. It is seen that the vibration response to the stochastic excitation is controlled effectively by using the proposed optimal bounded parametric control. The maximum dimensionless displacement is reduced from 0.4913 without MRVE to 0.0976 by using the optimal control and from 0.4913 to 0.1907 by using the passive control. The standard derivation of the displacement response is reduced from 0.2510 without MRVE to optimally controlled 0.0342 and passively controlled 0.0842, respectively. The relative reductions in the standard derivations of the optimally and passively controlled displacement responses (that is, the ratio of absolute difference of controlled and uncontrolled response standard derivations to uncontrolled response standard derivation) are 86.4% and 66.4%, respectively. A sample of the corresponding optimal bounded control G^* varying with time is shown in Fig. 9. The mean value of the optimal control is 0.62 MPa which approximates to the passive control of 0.6 MPa. Thus it is obtained again that the proposed optimal bounded parametric control can achieve the remarkable control effectiveness and make better use of the MRVE than the passive control for the stochastic vibration suppression of the sandwich beam.

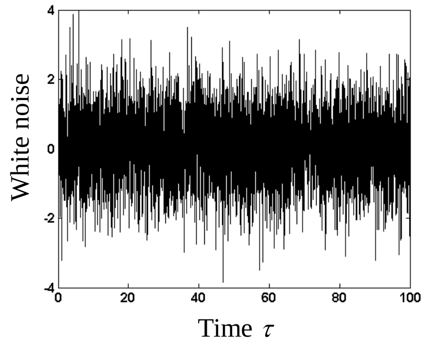

Fig. 7 Gaussian white noise excitation $\ddot{w}_0$

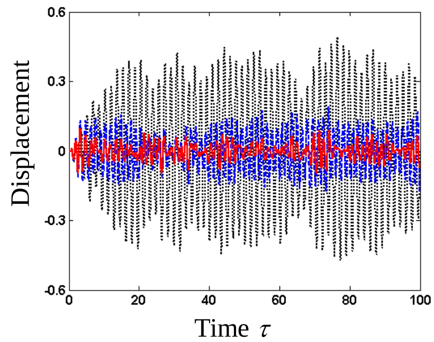

Fig. 8 Optimally controlled and passively controlled displacement responses of MRVE sandwich beam and displacement response of the beam without MRVE under stochastic excitation (dotted line: without MRVE; dashed line: passively controlled; solid line: optimally controlled)

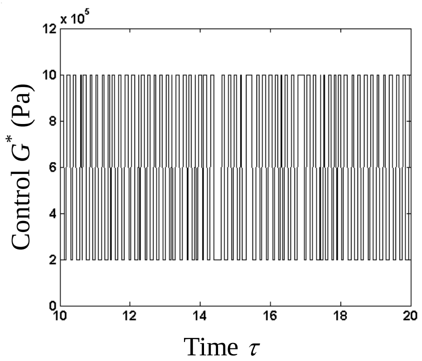

Fig. 9 Sample of optimal bounded parametric control varying with time for stochastic excitation

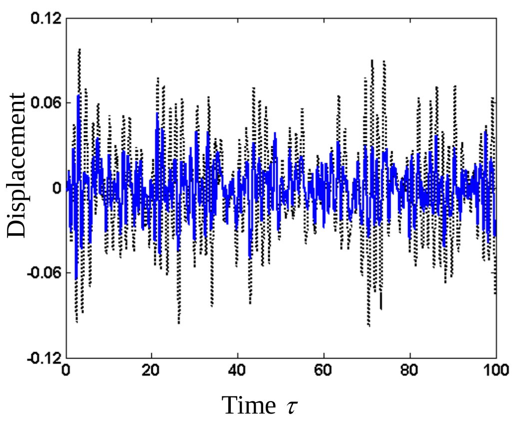

Fig. 10 Optimally controlled displacement responses of MRVE sandwich beam under stochastic excitation for different control bounds (dotted line: G_h=1.0 MPa; solid line: G_h=3.0 MPa)

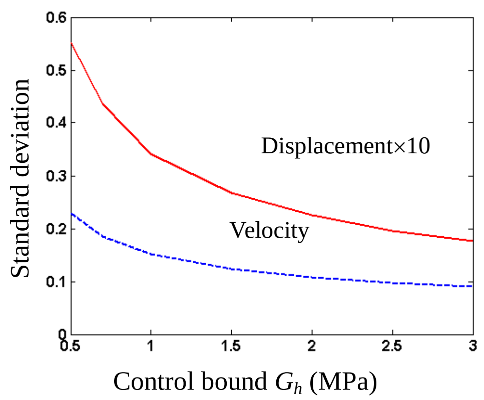

Fig. 11 Standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus control bound G_h (G_l=0.2 MPa) (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$)

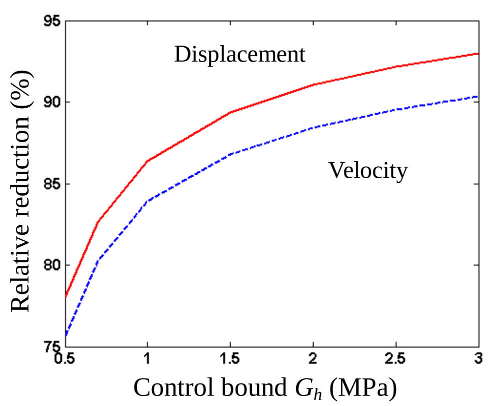

Fig. 12 Relative reductions in standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus control bound G_h (G_l=0.2 MPa) (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$)

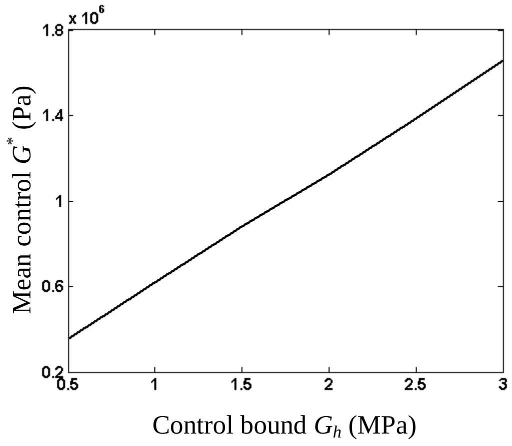

Fig. 13 Mean optimal control versus control bound G_h for stochastic excitation

The effects of the control bounds (G_h and G_l) on the displacement and velocity responses of the optimally controlled sandwich beam under the stochastic excitation are shown in Figs. 10-15. Fig. 10 shows the optimally controlled displacement responses of the MRVE sandwich beam for different control upper bounds G_h. With the bound G_h=1.0 MPa to 3.0 MPa, the maximum dimensionless displacement is reduced from 0.0976 to 0.0660. The corresponding standard derivation of the displacement response is reduced from 0.0342 to 0.0176, and the relative reductions of the standard derivations are 86.4% and 93.0%, respectively. The mean values of the optimal controls are 0.62 MPa and 1.65 MPa corresponding to the bounds G_h=1.0 MPa and 3.0 MPa, respectively. Fig. 11 illustrates that the standard deviations of the optimally controlled displacement and velocity responses are reduced nonlinearly with the increase of the upper bound G_h (G_l=0.2 MPa). The standard deviations of the vibration responses decrease with enlarging the upper bound greatly for small bound G_h and slightly for large bound G_h. For example, the further decrease of the vibration response is relatively small for the bound G_h>2.5 MPa. The optimal control G^* is a nonlinear parameter of the controlled system and then the optimally controlled response depends nonlinearly on the control. As the control upper bound G_h increases, the mean control with its mean square value is heightened correspondingly and then the optimally controlled response or response standard deviation is reduced nonlinearly. Fig. 12 illustrates that the relative reductions in the standard deviations of the displacement and velocity responses increase with the upper bound. The further increase of the relative response reduction (that is, the relative reduction of the response standard deviations) requires a larger increment of the control bound. The mean value of the corresponding optimal bounded control G^* increases linearly with the bound G_h as shown in Fig. 13.

Fig. 14 illustrates that the standard derivations of the optimally controlled displacement and velocity responses vary with the lower bound G_l (G_h=1.5 MPa). Fig. 15 illustrates that the relative reductions in the standard derivations of the displacement and velocity responses vary with the lower bound. It is seen that the standard derivations of the displacement and velocity responses are reduced slightly with the lower bound G_l and the relative response reductions (that is, the relative reduction of the response standard deviations) have a small increase. Although the lower bound reduction enlarges the control domain of G, the average MRVE stiffness and damping are reduced. The mean value of the corresponding optimal bounded control G^* decreases linearly with the lower bound G_l. It is obtained by comparing Figs. 11 with 14 that the control effectiveness is better by increasing the upper bound G_h than decreasing the lower bound G_l of the optimal bounded parametric control.

The effect of the damping ratio β on the displacement and velocity responses of the optimally controlled sandwich beam under the stochastic excitation is shown in Figs. 16 and 17. Fig. 16 illustrates that the standard deviations of the optimally controlled displacement and velocity responses are reduced with the increase of the damping ratio. Fig. 17 illustrates that the relative reductions in the standard deviations of the displacement and velocity responses increase with the damping ratio. The mean value of the corresponding optimal bounded control G^* varies slightly with the damping ratio, for example, from 0.62 MPa for β=0.01 to 0.63 MPa for β=0.05. However, the control effectiveness improved by enlarging the upper control bound is generally better than that by heightening the damping ratio based on the comparison of Figs. 11 with 16. The optimal feedback control for the beam vibration can produce larger artificial damping than the material damping. In consequence, the proposed optimal bounded parametric control can achieve the remarkable effectiveness of the vibration suppression of the MRVE sandwich beam with supported mass under the stochastic support motion excitation, and the stochastic vibration suppression capability of the MRVE sandwich beam is improved further.

6. Conclusions

The optimal bounded parametric control of MRVE sandwich beams with supported mass under stochastic and deterministic excitations, and the response mitigation capability of the stochastic and shock multi-mode coupling vibration by using the optimal control have been studied to illustrate the further vibration control effectiveness.

The partial differential equations for the horizontal and vertical coupling motions of the sandwich beam in the time domain have been obtained, and then converted into the multi-mode coupling vibration equations with the bounded nonlinear parametric control by using the Galerkin method.

The dynamical programming equation for the optimal vibration control of the beam under the stochastic excitation has been obtained based on the dynamical programming principle.

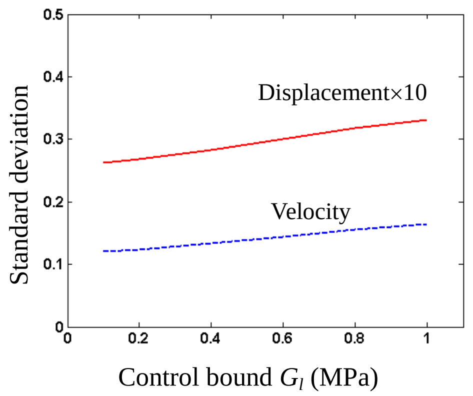

Fig. 14 Standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus control bound G_l (G_h=1.5 MPa) (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$).

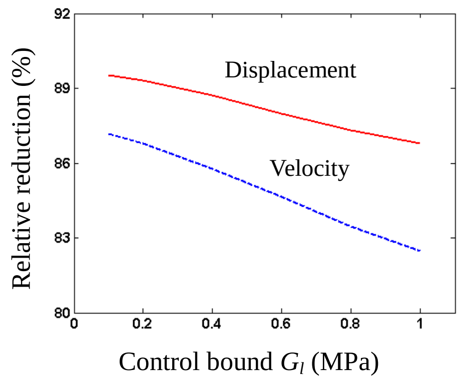

Fig. 15 Relative reductions in standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus control bound G_l (G_h=1.5 MPa) (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$)

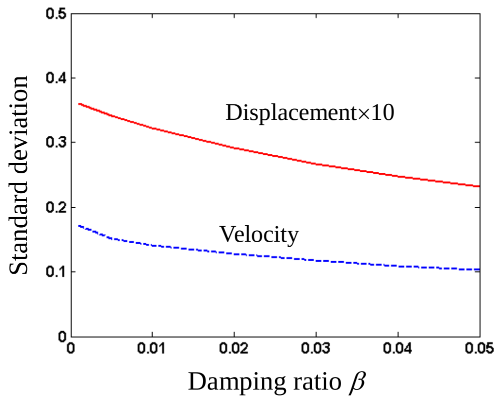

Fig. 16 Standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus damping ratio β (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$)

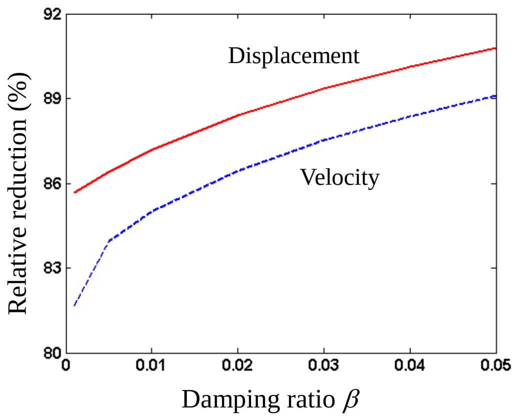

Fig. 17 Relative reductions in standard deviations of optimally controlled responses of MRVE sandwich beam under stochastic excitation versus damping ratio β (solid line: displacement $\overline{w}$; dashed line: velocity $\dot{\overline{w}}$)

The optimal bounded parametric control law has been determined by the functional minimization in the programming equation with the bounded control constraint. Then the optimally controlled vibration responses of the MRVE sandwich beam with supported mass under support motion excitations have been analyzed and calculated to evaluate the vibration suppression capability. The developed optimal bounded parametric control strategy is applicable to smart visco-elastic composite structures with bounded parametric controls under deterministic and stochastic excitations for improving vibration control effectiveness. Numerical results illustrate that (1) the vibration responses of the MRVE sandwich beam under stochastic and shock excitations can be suppressed greatly by using the proposed optimal bounded parametric control; (2) the optimal bounded parametric control effectiveness of the MRVE sandwich beam under stochastic and shock excitations is better than the corresponding passive control effectiveness, and the controllable MRVE characteristics can be used completely by the optimal control; (3) the optimal bounded parametric control effectiveness can be improved further by heightening the upper bound of the controllable stiffness or MRVE modulus, and be influenced slightly by the damping ratio. The above results are valuable for the vibration control design of MRVE composite structures and the MRVE fabrication based on application.

Acknowledgments

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