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Nonlinear vibration analysis of MSGT boron-nitride micro ribbon based mass sensor using DQEM

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Abstract. In this research, the nonlinear free vibration analysis of boron-nitride micro ribbon (BNMR) on the Pasternak elastic foundation under electrical, mechanical and thermal loadings using modified strain gradient theory (MSGT) is studied. Employing the von Kármán nonlinear geometry theory, the nonlinear equations of motion for the graphene micro ribbon (GMR) using Euler-Bernoulli beam model with considering attached mass and size effects based on Hamilton's principle is obtained. These equations are converted into the nonlinear ordinary differential equations by elimination of the time variable using Kantorovich time-averaging method. To determine nonlinear frequency of GMR under various boundary conditions, and considering mass effect, differential quadrature element method (DQEM) is used. Based on modified strain MSGT, the results of the current model are compared with the obtained results by classical and modified couple stress theories (CT and MCST). Furthermore, the effect of various parameters such as material length scale parameter, attached mass, temperature change, piezoelectric coefficient, two parameters of elastic foundations on the natural frequencies of BNMR is investigated. The results show that for all boundary conditions, by increasing the mass intensity in a fixed position, the linear and nonlinear natural frequency of the GMR reduces. In addition, with increasing of material length scale parameter, the frequency ratio decreases. This results can be used to design and control nano/micro devices and nano electronics to avoid resonance phenomenon.

Keywords: nonlinear vibration analysis; boron-nitride micro ribbon; mass sensor; MSGT; DQEM

1. Introduction

The graphene micro ribbon (GMR) is one of the most common structures that at micro and nano-dimensions in micro sensors, biosensors, mass sensors, micro accelerometers and micro electro-mechanical systems are used. The existence of mass in a structure would have important effect on its vibration behavior and it is obvious that the rate of this effect is depend on location and quantity of mass, therefore, it is one of the most important issues that is considered in the fields of engineering recently. The size dependent effect has an important role at micro scale. Fleck and Hutchinson (1993, 1997, 2001) extended and reformulated the classical couple stress theory and renamed it as the strain gradient theory (SGT), in which for homogeneous isotropic and incompressible materials, three additional higher-order material length scale parameters are

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introduced. Lam, Yang *et al*. (2013) proposed a modified strain gradient elasticity theory (MSGT) in which a new additional equilibrium equations to govern the behavior of higher-order stresses, the equilibrium of moments of couples is introduced, in addition to the classical equilibrium equations of forces and moments of forces. Meanwhile, there are only three independent higher-order materials length scale parameters for isotropic linear elastic materials in the present theory. So far, many researches about buckling and vibration of structures at micro and nano scales are carried out. Kahrobaiyan, Asghari *et al*. (2013) examined longitudinal behavior of a micro bar using SGT. They used Hamilton's principle to obtain equilibrium equations and showed that there is a good coincidence between finite element method (FEM) and analytical method. They showed that the microbars modeled by the SGT are stiffer than those modeled by the classical theory (CT), the strain gradient bars have greater natural frequencies and smaller static deformations compared to the classical bars. Narender, Ravinder *et al.* (2012) presented strain gradient torsional vibration analysis of micro/nano rods. They derived the governing equation and both the classical and the non-classical boundary conditions by employing the Hamilton's principle. A numerical method to solve the differential equations is the differential quadrature method (DQM) which initially introduced by Bellman, Kashef *et al*. (1972), in which this method can employ only for continuous problems. Recently, differential quadrature element method (DQEM) has appeared as a numerical technique to analyze the structures with some local discontinuities in loading, attached mass, material properties, and geometry. Thus, this method is used to solve many problems especially in the vibration analysis. Ghorbanpour Arani, Atabakhshian *et al*. (2012) studied nonlinear vibration of boron nitride nanotube. Based on the von Kármán nonlinear geometry and nonlocal elasticity theories, they used Timoshenko beam theory, and finally they obtained high-order equations using Hamilton's principle and solved them using DQM. They examined the effect of length scale parameter, elastic coefficients of foundation, electrical potential amplitude and thermal changes on the natural frequency of boron nitride nano-beam. Ghayesh, Amabili *et al.* (2013) investigated nonlinear forced vibrations of a micro beam based on SGT, and they used Hamilton's principle to obtain equations and employed Galerkin method to solve them. In addition, they studied the effect of various values of damping coefficient on the curve of frequency response of system and found that for high values of viscosity, resonance domain decreases. Li, Feng *et al.* (2014) presented bending and free vibration of functionally graded piezoelectric micro beam based on MSGT. They considered material properties in the form of power function and variable in direction of thickness. Their results showed that by increasing variable factor of functionally graded material, the electrical potential decreases and vice versa for natural frequency. Shi, Ni *et al*. (2011) illustrated the buckling analysis of a double layer graphene nano ribbons. They used the nonlocal elasticity theory and plotted the buckling modes shapes for two states of in-phase and anti-phase. Their results indicated that the nonlocal effect is an inverse relationship with the buckling load, and the nonlocal effect decreases with increasing aspect ratio of double layer graphene nano ribbons (DLGNRs). Ke, Wang *et al*. (2012) analyzed the nonlinear vibration of a piezoelectric nano beam based on the nonlocal elasticity theory. They used Hamilton's principle and Timoshenko beam model to obtain the equations of motion subjected to voltage and uniform heat, also solved governing equations using DQM. Their results showed that a change in the external electric voltage from a positive value to a negative value leads to the decrease of the nonlinear frequency ratio. Mohammadimehr, Saidi *et al*. (2011) studied the buckling analysis of double walled carbon nanotubes (DWCNTs). They employed the nonlocal elasticity theory and Timoshenko beam model. Their results showed that the local beam model over estimates the critical buckling load if the small scale parameter for long nanotubes is overlooked. Reddy (2007) investigated the bending, buckling and free vibration analysis of beam using the nonlocal elasticity theory. He obtained analytical solution for the deflection, critical buckling load, and natural frequency and he examined the influence of small scale parameter on mentioned parameters. He showed that the considering nonlocal effect leads to increase deflections and decrease buckling loads and natural frequencies. Wang, Zhang *et al*. (2007) studied free vibration analysis of nano beam based on the nonlocal elasticity theory for Timoshenko beam model. They extracted governing equations and boundary conditions using Hamilton's principle and solved these equations for the natural frequencies of beam and various boundary conditions. The exact nonlocal Timoshenko beam solutions provided by them should be useful to engineers who are designing micro electromechanical and nano electromechanical devices. Chowdhury, Adhikari *et al*. (2009) examined ability the carbon nonotubes as a mass sensor and showed that new sensor equations are used for biosensors with acceptable accuracy. Using nonlocal elasticity theory, Murmu and Pradhan (2009) presented vibration response of a nano cantilever considering non-uniformity in cross section. They solved the governing equations by DQM. Khalili, Jafari *et al*. (2010) indicated a combined method to examine dynamic behavior of functionally graded beams under moving load. They extracted the governing equations using Euler Bernoulli beam theory and Lagrange method and they discrete existing derivatives using Ritz-DQ method. Comparing other single-phase methods such as Newmark and Wilson, their method had more accuracy even with large time distances. They found that the inertia effect of the moving load on the axial vibrations of the beam is more crucial than the corresponding one on the transverse vibrations of the beam. Xia, Wang *et al.* (2010) illustrated bending, post buckling and the nonlinear free vibration of a micro beam. They used non-classical continuum mechanics introducing length scale parameter. Marinaki, Marinakis *et al.* (2011) designed a mechanism to control vibration for a piezoelectric beam on which attached a particle swarm having sensor rule. They performed mechanical simulation system based on classical motion theory and they used FEM to solve it. They tested three different variants of the particle swarm optimization (PSO) and finally, showed that for different loadings, the PSO is very satisfactory. Kahrobaiyan, Asghari *et al.* (2011) presented a formulation of nonlinear Euler-Bernoulli beam based on SGT. They considered central plate extension as nonlinear terms in beam behavior. Their results showed that the difference between the obtained frequencies by the non-classical beam theories and those predicted by the classical beam theories is significant when the ratio of the beam thickness to the length scale parameter is low. Rahmati and Mohammadimehr (2014) studied the free vibrations of non-uniform Boron nitride nano rod on elastic foundation under electrical, thermal and mechanical loads. They used DQM to solve the governing equations; also they examined connected mass effects on natural frequencies. They showed that boundary conditions are effective on sensitivity of boron nitride nano rod to connected mass. Their results indicated that the non-dimensional frequency ratio of non-uniform and non-homogeneous boron nitride nano rods (BNNRs) decreases with an increase in the small scale parameter. Lei, He *et al*. (2013a) analyzed a new size dependent model for functionally graded beam using strain gradient elasticity theory and sinusoidal shear strain theory. They used Navier's type solution for simply supported boundary conditions and investigated the effect of various parameters such as length scale parameter and shear strain on free vibration of system. Their results showed that the dimensionless frequencies of the functionally graded (FG) micro beams larger than those of the metal micro beams and smaller than the ceramic micro beams. Lei, Murmu *et al*. (2013b) investigated dynamic behavior of nonlocal beams with viscoelastic damping and to obtain governing equations and viscoelastic modeling, they used Kelvin-voigt model for Euler-Bernoulli beam. They also presented a closed solution to analyze free vibration of beam using transition function method (TFM) and they studied nonlinear effects and viscoelastic coefficient on natural frequencies of system. Their results demonstrated that for the Kelvin-Voigt model, the imaginary part of the complex natural frequency increases almost linearly with the viscoelastic parameter. There is only a small dependence of the viscoelastic parameter on the real of the complex natural frequencies. Murmu and Adhikari (2013) examined vibration analysis of a single-layer graphene sheet considering the nonlocal elasticity theory. They introduced a mathematical framework for this problem and considered small scale effects. They observed that the performance of the sensor depends on the spatial distribution of the attached mass on the graphene sheet with and without nonlocal effects. Mohammadi and Mahzoon (2013) studied thermal effects on post buckling micro beam based on modified strain gradient theory. They presented a nonlinear model considered small scale effects and Poisson's ratio. They could examine analytically postbuckling behavior for various boundary conditions. Akgöz and Civalek (2011) illustrated stability of micro beams based on modified couple stress theory under various boundary conditions and effect of length scale parameters. Arefi and Allam (2015) investigated nonlinear analysis of an arbitrary functionally graded circular plate integrated with two functionally graded piezoelectric layers resting on the Winkler-Pasternak foundation. They showed the effect of different parameters such as parameters of foundation, non-homogenous index and boundary conditions on the mechanical and electrical results of the system. Akgöz and Civalek (2013) presented the vibration response of inhomogeneous and non-uniform micro beams based on Euler-Bernoulli beam theory and modified couple stress theory (MCST). They used Rayleigh-Ritz method to solve equations and examined effect of material properties on natural frequencies of functionally graded micro beam. Their results indicated that the dimensionless natural frequencies predicted by CT are always smaller than those obtained by MCST and also the values of material properties and tapered ratios are significant influences on vibration response of axially functionally graded (AFG) tapered micro beams. Wang, Lin *et al*. (2013) studied nonlinear free vibration analysis of micro beam based on Euler-Bernoulli beam theory, MCST, and von Kármán nonlinear theory. They transformed partial differential equations of problem to ordinary differential equations with eliminating of time variable using Kantorovich method. They showed that the natural frequency based on MCST is more than CT and the size effect on the nonlinear free vibration characteristics is only significant when the ratio of the beam height to material length scale is relatively small, it diminishes as this ratio increases. Tounsi *et al.* (2015) examined size effect on vibration and bending of micro beam made of functionally graded materials. They used MCST for considering size effect and Mori–Tanaka method to simulation of functionally graded material. They found that considering size effect and resulting structure’s hardness, leads to decrease vertical displacement and increase the natural frequency. Torabi, Al-Basyouni *et al.* (2013) investigated free vibrations of non-uniform Timoshenko beam considering several concentrated mass on it. They used DQEM to solve the governing equations. They examined effect of number, intensity and location of mass on system’s natural frequency. They found that by increasing of mass intensity and mass number, the natural frequency of beam decreases. In another work, Torabi, Afshari *et al*. (2014a) studied transverse vibrations of non-uniform Timoshenko beam with several crack under various boundary conditions using DQEM. Simsek (2014) presented nonlinear Euler-Bernoulli micro beam on nonlinear elastic foundation using MCST. He used Hamilton’s principle to obtain equations of motion and boundary conditions. He transformed nonlinear partial differential equations to ordinary differential equations using Galerkin method and he obtained system’s natural frequencies using variational method. He inferred that with an increase in the length scale parameters lead to decrease in the nonlinear frequency ratio although the linear and the nonlinear vibration frequencies increase with considering length scale parameters. Amiri Moghadam, Kouzani *et al.* (2015) investigated an effective modeling strategy for nonlinear large deformation (small strains and moderate rotations) dynamic analysis of polymer actuators. Mohammadimehr, Monajemi *et al.* (2015) studied free vibrations of viscoelastic micro rod with variable section on visco-Pasternak foundation using strain gradient theory. They used DQM to solve the equations of motion and they obtained natural frequencies for various boundary conditions. Their results showed that for higher modes, difference between non-dimensional natural frequencies related to MCST and SGT is more sensible than lower modes. Ying, Ni *et al.* (2015) investigated a micro-vibration response of the sandwich plate with magneto-rheological visco-elastomer (MRVE) core and supported mass under stochastic support motion excitations. Chen (2001) investigated the vibration of non-uniform shear deformable axisymmetric orthotropic circular plates. He used the DQEM to solve these equations. In other work, Chen (2002) considered free vibration analysis of non-prismatic beams resting on elastic foundations using the DQEM. He developed numerical algorithm to analyze the related pressure vessel and piping structures. Torabi, Afshari *et al*. (2014b) studied free vibration analysis of a rotating non-uniform blade with multiple open cracks using DQEM. They indicated that the damage is the highest effect on a natural frequency when the crack is located at a point that has maximum value of curvature in the corresponding normal mode. Chen (2005) presented DQEM analysis of in-plane vibration of curved beam structures. In the other work, Chen (2008) analyzed out-of-plane vibration of non-prismatic curved beam structures with considering the effect of shear deformation using DQEM. Malekzadeh, Karami *et al*. (2004) investigated semi-analytical free vibration analysis of thick plates with two opposite edges simply supported using DQEM. They achieved that reducing the thickness ratio, the fundamental natural frequencies are reduced. This is due to the fact that the stiffness of the plate reduces.

In this research, the nonlinear vibrations analysis of boron-nitride micro ribbon (BNMR) based on the modified strain gradient elasticity theory (MSGT) is studied. At micro and nano scales, the influences of temperature change, attached mass, piezoelectric coefficient, elastic foundation moduli and various boundary conditions on natural frequency of this system are investigated. Nonlinear equations of motion are solved using Kantorovich time averaging method and differential quadrature element method (DQEM). The results of this research compared with other methods and as can see there is a very good accordance with the other work. Finally, a comparison between various size dependent effect such as MSGT, modified couple stress theory (MCST), and classical theory (CT) is presented.

2. Basic equations

GMR is a strip made of graphene crystal, with limited width, that used in micro and nano systems because of its good electrical, thermal and mechanical properties. Fig. 1 shows a GMR with length of L and cross section $b\times h$ and height h in Cartesian coordination. x, y, and z coordinate axes are in length, width, and thickness (height) directions of GMR. In a GMR, the total displacements of u_1, u_2, and u_3 are assumed to be the functions of only the x, and z coordinates.

where x_m and m in Fig. 1 indicates location and mass of the attached mass, respectively.

2.1 Modified strain gradient theory

When dimensions of material are at micro and nano scales, classical elasticity theory cannot predicts behavior of materials. Therefore, it is necessary to use non-classical elasticity theories, which can consider of small-scale effects up to micro and nano meters and inherent discontinuity of nano structures. Among the nonclassical theories with respect of size effect, it can be point to the Eringen's nonlocal elasticity theory (Eringen and Edelen 1972), couple stress theory (CST) (Ejike 1969), modified couple stress theory (MCST) (Yang, Chong *et al.* 2002, Mohammadimehr *et al*. 2015, 2016a, b) and strain gradient theory (SGT) (Lam, Yang *et al*. 2003). One of the most appropriate nonclassical forms which have been presented is modified strain gradient theory (MSGT), containing three length scale parameters (l) in basic equations. The length scale parameters in the MSGT, shows that the behavior of material at micro scale is depend on dimensions of material length scale parameter. In MSGT, a new stress tensor is introduced, which is different from Cauchy stress tensor, and it can be used as general stress tensor in momentum equation. Considering MSGT suggested by (Lam, Yang *et al*. 2003), the strain energy U in linear elastic isotropic materials on area $\forall$ with small deformations is explained by Eq. (1).

$$U=\frac{1}{2}\int_{\forall}\left[\sigma_{ij}\varepsilon_{ij}+P_i\gamma_i+\tau_{ijk}\eta_{ijk}+m_{ij}\chi_{ij}-D_iE_i\right]d\forall \tag{1}$$

where ε_{ij}, η_{ijk}, χ_{ij}, γ_i, and E_i are the strain tensor, deviatoric stretch gradient tensor, symmetric rotation gradient tensor, dilatation gradient vector and electrical field, respectively (Lam, Yang *et al*. 2003).

$$\varepsilon_{ij}=\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}+\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \tag{2}$$

$$\eta_{ijk}=\frac{1}{3}\left[\varepsilon_{jk,i}+\varepsilon_{ki,j}+\varepsilon_{ij,k}\right]-\frac{1}{15}\delta_{ij}\left[\varepsilon_{mm,k}+2\varepsilon_{mk,m}\right]-\frac{1}{15}\delta_{jk}\left[\varepsilon_{mm,i}+2\varepsilon_{mi,m}\right]-\frac{1}{15}\delta_{ki}\left[\varepsilon_{mm,j}+2\varepsilon_{mj,m}\right] \tag{3}$$

$$\gamma_i=\varepsilon_{mm,i} \tag{4}$$

$$\theta_i=\frac{1}{2}\left[\nabla\times u\right]_i \tag{5}$$

$$\chi_{ij}=\frac{1}{2}\left[\theta_{i,j}+\theta_{j,i}\right] \tag{6}$$

$$E_m=-\phi_{,i} \tag{7}$$

where ϕ is the electrical potential.

σ_{ij}, p_i, τ_{ijk}, and m_{ij} the classical and higher order stress tensors, can be written as the following from (Lam, Yang *et al*. 2003).

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl}-e_{kij}E_k-\lambda_{ij}\Delta T \tag{8}$$

$$P_i = 2Gl_0^2\gamma_i \tag{9}$$

$$\tau_{ijk} = 2Gl_1^2\eta_{ijk} \tag{10}$$

$$m_{ij} = 2Gl_2^2\chi_{ij} \tag{11}$$

where e_{kij} and ΔT are the piezoelectric coefficient and temperature change, respectively. Also, in Eq. (8), λ_{ij} is equal to $C_{ijkl}\,\alpha_{kl}$ that C_{ijkl} and α_{kl} are fourth-order elastic modulus tensor of the classical isotropic elasticity and the thermal expansion coefficients, respectively. l_0 ,l_1 and l_2 are independent material length parameters related to dilatation gradients, deviatoric stretch gradients and symmetry rotation gradients, respectively. Also, G and θ denote shear modulus and rotation vector, respectively.

Electric displacement relation based on the piezoelasticity theory can be expressed as the following form

$$D_i = e_{imn}\varepsilon_{mn} + \in_{im} E_m \tag{12}$$

where D_i and $\in_{im}$ denote electric displacement and the dielectric permittivity constant, respectively.

2.2 Displacement fields and the constitutive equations

Based on the Euler-Bernoulli beam theory, the displacement field of GMR in x, y and z directions can be written as

$$\begin{aligned} u_1(x,z,t) &= U(x,t) - z\,\frac{\partial W(x,t)}{\partial x} \\ u_2(x,z,t) &= 0 \\ u_3(x,z,t) &= W(x,t) \end{aligned} \tag{13}$$

The components of the von Kármán strain tensor is used to obtain nonlinear strain-displacement relations. Therefore by substituting Eq. (13) into Eq. (2), the nonlinear strain-displacement relations are expressed as follows

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1(x,z,t)}{\partial x} + \frac{1}{2}\left(\frac{\partial}{\partial x}W(x,t)\right)^2 = \frac{\partial U(\mathrm{x},\mathrm{t})}{\partial x} - z\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}W(x,t)\right)^2 \\ \varepsilon_{12} &= \varepsilon_{13} = \varepsilon_{22} = \varepsilon_{23} = \varepsilon_{33} = 0 \end{aligned} \tag{14}$$

The components of dilatation gradient vector, deviatoric stretch gradient tensor and symmetric rotation gradient tensor are obtained by substituting Eqs. (13) and (14) into Eqs. (3)-(6).

$$\begin{aligned} \gamma_1 &= \frac{\partial^2}{\partial x^2}U(x,t) - z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right) + \left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) \\ \gamma_2 &= 0 \\ \gamma_3 &= -\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) \end{aligned} \tag{15}$$

$$\eta_{111}=\frac{2}{5}\frac{\partial}{\partial x}\varepsilon_{11}=\frac{2}{5}\frac{\partial^2}{\partial x^2}U(x,t)-\frac{2}{5}z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)+\frac{2}{5}\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\eta_{113}=\eta_{131}=\eta_{311}=\frac{4}{15}\frac{\partial}{\partial z}\varepsilon_{11}=-\frac{4}{5}\frac{\partial^2}{\partial x^2}W(x,t)$$

$$\eta_{212}=\eta_{122}=\eta_{221}=-\frac{1}{5}\frac{\partial}{\partial x}\varepsilon_{11}=-\frac{1}{5}\frac{\partial^2}{\partial x^2}U(x,t)+\frac{1}{5}z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)$$
$$-\frac{1}{5}\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\eta_{313}=\eta_{133}=\eta_{331}=-\frac{1}{5}\frac{\partial}{\partial x}\varepsilon_{11}=-\frac{1}{5}\frac{\partial^2}{\partial x^2}U(x,t)+\frac{1}{5}z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)$$
$$-\frac{1}{5}\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\eta_{333}=\frac{1}{5}\frac{\partial^2}{\partial x^2}W(x,t)$$

$$\eta_{223}=\eta_{232}=\eta_{322}=\frac{1}{15}\frac{\partial^2}{\partial x^2}W(x,t)$$

$$\eta_{112}=\eta_{121}=\eta_{211}=\eta_{213}=\eta_{132}=\eta_{321}=\eta_{312}=\eta_{123}=\eta_{231}=\eta_{222}=0 \tag{16}$$

$$\chi_{12}=\chi_{21}=-\frac{1}{2}\frac{\partial^2}{\partial x^2}W(x,t)$$
$$\chi_{11}=\chi_{13}=\chi_{22}=\chi_{23}=\chi_{33}=0 \tag{17}$$

Substituting Eqs. (14)-(17) into Eqs. (8)-(11), the components of Cauchy stress tensors and higher ordered stress tensors are determined as

$$\sigma_{11}=E\left(\varepsilon_{11}-\alpha\Delta T\right)+e_{11}\frac{\partial\phi}{\partial x} \tag{18}$$

where α and ϕ in Eq. (18) are the thermal expansion coefficient and the electrical potential, respectively.

$$P_1=2Gl_0^2\left(\left(\frac{\partial^2}{\partial x^2}U(x,t)\right)-z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)+\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)\right)$$
$$P_2=0$$
$$P_3=-2Gl_0^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) \tag{19}$$

$$\tau_{111}=\frac{4}{5}Gl_1^2\left(\left(\frac{\partial^2}{\partial x^2}U(\mathrm{x,t})\right)-z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)+\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)\right)$$

$$\tau_{113}=\tau_{131}=\tau_{311}=-\frac{8}{15}Gl_1^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\tau_{212}=\tau_{122}=\tau_{221}=\frac{2}{5}Gl_1^2\left(-\left(\frac{\partial^2}{\partial x^2}U(\mathrm{x,t})\right)+z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)-\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)\right)$$

$$\tau_{313}=\tau_{133}=\tau_{331}=\frac{2}{5}Gl_1^2\left(-\left(\frac{\partial^2}{\partial x^2}U(\mathrm{x,t})\right)+z\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)-\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)\right)$$

$$\tau_{223}=\tau_{232}=\tau_{322}=\frac{2}{15}Gl_1^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\tau_{333}=\frac{2}{5}Gl_1^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)$$

$$\tau_{112}=\tau_{121}=\tau_{211}=\tau_{213}=\tau_{132}=\tau_{321}=\tau_{312}=\tau_{123}=\tau_{231}=\tau_{222}=0 \tag{20}$$

$$m_{12} = m_{21} = -Gl_2^2\left(\frac{\partial^2}{\partial x^2}W\ (x,t)\right)$$
$$m_{11} = m_{13} = m_{22} = m_{23} = m_{33} = 0 \tag{21}$$

3. Equations of motion for GMR

The governing equations of the GMR in an elastic-Pasternak foundation for free vibration can be obtained using Hamilton's principle.

$$\delta\Pi = 0 \Rightarrow \int_0^t (\delta U - \delta K - \delta\Omega)dt = 0 \tag{22}$$

where U, K and Ω are the strain energy, kinetic energy and the work caused by external forces, respectively.

3.1 Strain energy of boron nitride GMR

The strain energy of GMR based on the modified strain gradient theory is defined by substituting the Eqs. (14)-(21) into Eq. (1) as follows

$$U = \frac{1}{2}\int_0^l \left(\begin{array}{l}
+EA\left(\frac{\partial}{\partial x}U(x,t)\right)^2 + \frac{4}{5}GAl_1^2\left(\frac{\partial^2}{\partial x^2}U(x,t)\right)^2 + 2GAl_0^2\left(\frac{\partial^2}{\partial x^2}U(x,t)\right)^2 \\
+2Ae_{11}\left(\frac{\partial}{\partial x}\phi(x,t)\right)\left(\frac{\partial}{\partial x}U(x,t)\right) + 4GAl_0^2\left(\frac{\partial^2}{\partial x^2}U(x,t)\right)\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) \\
+\frac{8}{5}GAl_1^2\left(\frac{\partial^2}{\partial x^2}U(x,t)\right)\left(\frac{\partial}{\partial x}W(x,t)\right)\left(\frac{\partial^2}{\partial x^2}W(x,t)\right) - EA\alpha\Delta T\left(\frac{\partial}{\partial x}U(x,t)\right) \\
+EA\left(\frac{\partial}{\partial x}U(x,t)\right)\left(\frac{\partial}{\partial x}W(x,t)\right)^2 + \frac{1}{4}EA\left(\frac{\partial}{\partial x}W(x,t)\right)^4 \\
+\frac{4}{5}GAl_1^2\left(\frac{\partial}{\partial x}W(x,t)\right)^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 + 2GIl_0^2\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)^2 \\
+2GAl_0^2\left(\frac{\partial}{\partial x}W(x,t)\right)^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 + \frac{8}{15}GAl_1^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 \\
+2GAl_0^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 + EI\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 + GAl_2^2\left(\frac{\partial^2}{\partial x^2}W(x,t)\right)^2 \\
+\frac{4}{5}GIl_1^2\left(\frac{\partial^3}{\partial x^3}W(x,t)\right)^2 - \frac{1}{2}EA\alpha\Delta T\left(\frac{\partial}{\partial x}W(x,t)\right)^2 \\
+Ae_{11}\left(\frac{\partial}{\partial x}\phi(x,t)\right)\left(\frac{\partial}{\partial x}W(x,t)\right)^2 - A\in_{11}\left(\frac{\partial}{\partial x}\phi(x,t)\right)^2 - A\alpha e_{11}\Delta T\left(\frac{\partial}{\partial x}\phi(x,t)\right)
\end{array}\right)dx \tag{23}$$

E is Young modulus and I is the moment of inertia of GMR's cross section.

3.2 Kinetic energy of GMR

Considering the displacement of the beam (U,W) in two directions x and z, the kinetic energy is an expression based on the following equation

$$K = \frac{1}{2}\int_0^L \int_A \rho\left((\frac{\partial u_1}{\partial t})^2 + (\frac{\partial u_2}{\partial t})^2 + (\frac{\partial u_3}{\partial t})^2 \right) dA\,dx$$
$$= \frac{1}{2}\int_0^L \left\{ \rho A \left(\frac{\partial U\,(\mathrm{x,t})}{\partial t} \right)^2 + \rho A \left(\frac{\partial W\,(\mathrm{x,t})}{\partial t} \right)^2 + \rho I \left(\frac{\partial^2 W\,(\mathrm{x,t})}{\partial x\,\partial t} \right)^2 \right\} dx \tag{24}$$

3.3 The work done by elastic foundation

The external work due to Winkler-Pasternak foundation can be calculated as the following from

$$\Omega = \frac{1}{2}\int_0^l (-k_w W\,(\mathrm{x,t}) + k_G \nabla^2 W\,(\mathrm{x,t})).\ W\,(\mathrm{x,t})dx \tag{25}$$

where k_w and k_G are the Winkler and shear moduli of the elastic medium, respectively. To simplify the equations of motion GMR, the dimensionless parameters are defined as follows

$$\zeta = \frac{x}{L} \qquad u(\zeta,\tau) = \frac{U\,(\mathrm{x,t})}{L} \qquad \mathrm{w}(\zeta,\tau) = \frac{\mathrm{W(x,t)}}{L} \qquad \tau = \frac{t}{L^2}\sqrt{\frac{E_0 I_0}{\rho A}}$$
$$\kappa = \frac{AL^2}{I_0} \qquad K_W = \frac{k_w L^4}{E_0 I_0} \qquad K_G = \frac{k_G L^2}{E_0 I_0} \qquad \delta_i = \frac{GA}{EI_0} l_i^2$$
$$\gamma = \frac{\in_{11} E_0}{(e_{11})^2} \qquad \phi(\zeta,\tau) = \frac{e_{11}}{E_0 L}\phi(x,t) \qquad \bar{M} = \frac{m}{\rho AL} \qquad \Delta\bar{T} = \alpha\Delta T \tag{26}$$

where $\bar{M}$ is the dimensionless attached mass. Substituting Eqs. (23)-(25) into Eq. (22), and using part-by-part integration technique, then making non-dimensional relations, the governing equations of motion GMR and boundary conditions are obtained as follows

δu :

$$-\left(\frac{\partial^2}{\partial\tau^2}u(\zeta,\tau)\right) + \kappa\left(\frac{\partial^2}{\partial\zeta^2}u(\zeta,\tau)\right) - \frac{4}{5}\delta_1\left(\frac{\partial^4}{\partial\zeta^4}u(\zeta,\tau)\right) - 2\delta_0\left(\frac{\partial^4}{\partial\zeta^4}u(\zeta,\tau)\right) - 6\delta_0\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)$$
$$+\kappa\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right) - \frac{12}{5}\delta_1\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right) - \frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)$$
$$-2\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right) + \frac{\kappa}{\gamma}\left(\frac{\partial^2}{\partial\zeta^2}u(\zeta,\tau)\right) + \frac{\kappa}{\gamma}\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right) = 0 \tag{27}$$

$$\delta w:$$

$$
\begin{aligned}
&-\left(\frac{\partial^2}{\partial\tau^2}w(\zeta,\tau)\right)+\frac{1}{\kappa}\left(\frac{\partial^4}{\partial\zeta^2\partial\tau^2}w(\zeta,\tau)\right)-\delta_2\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)+\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{\partial^6}{\partial\zeta^6}w(\zeta,\tau)\right)-\frac{8}{15}\delta_1\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)\\
&-\frac{4}{5}\delta_1\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^3-\frac{4}{5}\delta_1\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)-\frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)\\
&-\frac{4}{5}\delta_1\left(\frac{\partial^4}{\partial\zeta^4}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)-\frac{16}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)+2\frac{\delta_0}{\kappa}\left(\frac{\partial^6}{\partial\zeta^6}w(\zeta,\tau)\right)\\
&-2\delta_0\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^3-2\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^4}{\partial\zeta^4}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\\
&-2\delta_0\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)-8\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-\left(\frac{\partial^4}{\partial\zeta^4}w(\zeta,\tau)\right)\\
&+\kappa\left(\frac{\partial}{\partial\zeta}u(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)+\frac{3}{2}\kappa\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)+\kappa\left(\frac{\partial^2}{\partial\zeta^2}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\\
&+\frac{\kappa}{\gamma}\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}u(\zeta,\tau)\right)+\frac{3}{2}\frac{\kappa}{\gamma}\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)+\frac{\kappa}{\gamma}\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}u(\zeta,\tau)\right)\\
&-\frac{1}{2}\kappa\Delta\bar{T}\left(\frac{\partial^2}{\partial\zeta^2}W(\zeta,\tau)\right)-K_w w(\zeta,\tau)+K_G\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)=0
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&\delta u_{\zeta=0,l}=0\\
&\kappa\left(\frac{\partial}{\partial\zeta}u(\zeta,\tau)\right)-\frac{4}{5}\delta_1\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)-\frac{1}{2}\kappa\Delta\bar{T}+\kappa\left(\frac{\partial}{\partial\zeta}\varphi(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^2\\
&-2\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-\frac{4}{5}\delta_1\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^2-\frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)+\frac{1}{2}\kappa\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\\
&-N\left.\frac{L^2}{EI}\right|_{\zeta=0,l}=0
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\delta w_{\zeta=0,l}=0\\
&2\frac{\delta_0}{\kappa}\left(\frac{\partial^5}{\partial\zeta^5}w(\zeta,\tau)\right)-\frac{8}{15}\delta_1\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-\delta_2\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)\\
&+\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{\partial^5}{\partial\zeta^5}w(\zeta,\tau)\right)+\frac{1}{\kappa}\left(\frac{\partial^3}{\partial\zeta\partial\tau^2}w(\zeta,\tau)\right)-\frac{1}{2}\kappa\Delta\bar{T}\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\\
&-\frac{4}{5}\delta_1\left(\frac{\partial^3}{\partial\zeta^3}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)+\kappa\left(\frac{\partial}{\partial\zeta}u(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)+\frac{1}{2}\kappa\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^3-\frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^2\\
&-\frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)-2\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)\left(\frac{\partial^2}{\partial\zeta^2}w(\zeta,\tau)\right)^2-2\delta_0\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)^2\left(\frac{\partial^3}{\partial\zeta^3}w(\zeta,\tau)\right)\\
&+\kappa\left(\frac{\partial}{\partial\zeta}\varphi(\zeta,\tau)\right)\left(\frac{\partial}{\partial\zeta}w(\zeta,\tau)\right)-V\left.\frac{L^2}{EI}\right|_{\zeta=0,l}=0
\end{aligned}
\tag{30}
$$

$$\delta\left(\frac{\partial w}{\partial \zeta}\right)_{\zeta=0,l}=0$$

$$\begin{aligned}&-2\frac{\delta_0}{\kappa}\left(\frac{\partial^4}{\partial\zeta^4}w\left(\zeta,\tau\right)\right)+\frac{8}{15}\delta_1\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)+2\delta_0\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)+\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)+\delta_2\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)\\&-\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{\partial^4}{\partial\zeta^4}w\left(\zeta,\tau\right)\right)+2\delta_0\left(\frac{\partial^2}{\partial\zeta^2}u\left(\zeta,\tau\right)\right)\left(\frac{\partial}{\partial\zeta}w\left(\zeta,\tau\right)\right)+\frac{4}{5}\delta_1\left(\frac{\partial^2}{\partial\zeta^2}u\left(\zeta,\tau\right)\right)\left(\frac{\partial}{\partial\zeta}w\left(\zeta,\tau\right)\right)\\&+\frac{4}{5}\delta_1\left(\frac{\partial}{\partial\zeta}w\left(\zeta,\tau\right)\right)^2\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)+2\delta_0\left(\frac{\partial}{\partial\zeta}w\left(\zeta,\tau\right)\right)^2\left(\frac{\partial^2}{\partial\zeta^2}w\left(\zeta,\tau\right)\right)-M\left.\frac{L}{EI}\right|_{\zeta=0,l}=0\end{aligned} \tag{31}$$

4. The semi-analytical and numerical methods

For solving engineering problems, analytical methods with high accuracy are more interested, but in many cases, these methods are faced by limitations such as a complex geometry, existing discontinuity in the range of problem analyses or complexity of the boundary conditions. This issue caused to advent of semi-analytical and numerical methods, so that today, there are different numerical methods to solve engineering problems available.

4.1 Kantorovich time averaging method

In this study, the equations of motion for GMR are the coupled partial differential equations with nonlinear effects, that there are not known analytical solution to these equations. So, it should be used the semi-analytical to solve the equations. For this aim, the displacements of the GMR in the case of harmonic vibrations are expressed as Eq. (32). In this method, a time functions assumed and eliminated from the governing equations using the Kantorovich method (Wang *et al.* 2008). For an unbuckled GMR, in order to eliminate time variable, it is assumed that the essential vibration can be closely approximated by the following expressions (Wang *et al*. 2008)

$$\begin{aligned}u(\zeta,\tau)&=U(\zeta)\cos^2(\omega\tau)\\ \mathrm{w}(\zeta,\tau)&=W(\zeta)\cos(\omega\tau)\end{aligned} \tag{32}$$

If u and w are considered as the square and linear of cosωτ based on Kantorovich time averaging method, then the time variable is omitted from the Eqs. (27)-(31). On the other hands, with considering the above approximations (Eq. (32)), there is only a location term (ζ) in the governing equations of motion for GMR (Eqs. (33)-(37)). Moreover, with this approximation (Eq. (32)), the partial differential equations (Eqs. (27)-(31)) are converted to the ordinary differential equations (Eqs. (33)-(37)). Finally, the nonlinear governing equations of GMR (Eqs. (33)-(37) that have only a location term (ζ)) are analyzed using differential quadrature element method (DQEM).

In Eq. (32), ω is the dimensionless natural frequency and $U(\zeta)$ and $W(\zeta)$ are the undetermined mode shapes of a nonlinear vibration of the GMR (Wang, Lin *et al.* 2013). Substituting Eq. (32)

into Eqs. (27)-(31) and the Kantorovich time averaging method is applied to this equations, finally the governing equations of the GMR and boundary conditions are obtained as follows

δu :

$$U(\zeta)\omega^2+\frac{3}{4}\kappa\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)-\frac{3}{5}\delta_1\left(\frac{d^4}{d\zeta^4}U(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d^4}{d\zeta^4}U(\zeta)\right)-\frac{9}{2}\delta_0\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)$$
$$+\frac{3}{4}\kappa\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)-\frac{9}{5}\delta_1\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)$$
$$-\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)+\frac{3}{4}\frac{\kappa}{\gamma}\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)+\frac{3}{4}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)=0 \tag{33}$$

δw :

$$W(\zeta)\omega^2-\frac{1}{\kappa}\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\omega^2-\delta_2\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)+\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{d^6}{d\zeta^6}W(\zeta)\right)-\frac{8}{15}\delta_1\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)$$
$$-\frac{3}{5}\delta_1\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^3-\frac{3}{5}\delta_1\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)-\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)$$
$$-\frac{3}{5}\delta_1\left(\frac{d^4}{d\zeta^4}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)-\frac{12}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)+2\frac{\delta_0}{\kappa}\left(\frac{d^6}{d\zeta^6}W(\zeta)\right)$$
$$-2\delta_0\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^3-\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)$$
$$-\frac{3}{2}\delta_0\left(\frac{d^4}{d\zeta^4}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)$$
$$-6\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)+\frac{3}{4}\kappa\left(\frac{d}{d\zeta}U(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)$$
$$+\frac{9}{8}\kappa\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)+\frac{3}{4}\kappa\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)$$
$$+\frac{9}{16}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)+\frac{27}{32}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)$$
$$+\frac{9}{16}\frac{\kappa}{\gamma}\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\left(\frac{d}{d\zeta}U(\zeta)\right)-\frac{1}{2}\kappa\Delta\bar{T}\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)-K_wW(\zeta)+K_G\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)=0 \tag{34}$$

$\delta U_{\zeta=0,l}=0$

$$\frac{3}{4}\kappa\left(\frac{d}{d\zeta}U(\zeta)\right)-\frac{3}{5}\delta_1\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)-\frac{1}{2}\kappa\Delta\bar{T}-\frac{3}{2}\delta_0\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^2$$
$$-\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\frac{3}{5}\delta_1\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^2-\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)$$
$$+\frac{3}{8}\kappa\left(\frac{d}{d\zeta}W(\zeta)\right)^2+\frac{9}{16}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}U(\zeta)\right)+\frac{9}{32}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}W(\zeta)\right)^2-N\frac{L^2}{EI} \tag{35}$$

$$
\begin{aligned}
&\delta w_{\zeta=0,l}=0\\
&2\frac{\delta_0}{\kappa}\left(\frac{d^5}{d\zeta^5}W(\zeta)\right)-\frac{8}{15}\delta_1\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-2\delta_0\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\delta_2\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)\\
&+\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{d^5}{d\zeta^5}W(\zeta)\right)-\frac{1}{\kappa}\left(\frac{d}{d\zeta}W(\zeta)\right)\omega^2-\frac{1}{2}\kappa\Delta\bar{T}\left(\frac{d}{d\zeta}W(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)\\
&-\frac{3}{5}\delta_1\left(\frac{d^3}{d\zeta^3}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)+\frac{3}{4}\kappa\left(\frac{d}{d\zeta}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)+\frac{3}{8}\kappa\left(\frac{d}{d\zeta}W(\zeta)\right)^3\\
&-\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^2-\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)-\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)^2\\
&-\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^3}{d\zeta^3}W(\zeta)\right)+\frac{9}{16}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)+\frac{9}{32}\frac{\kappa}{\gamma}\left(\frac{d}{d\zeta}W(\zeta)\right)^3-V\,\frac{L^2}{EI}
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&\delta\left(\frac{\partial w}{\partial\zeta}\right)_{\zeta=0,L}=0\\
&-\frac{4}{5}\frac{\delta_1}{\kappa}\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)-2\frac{\delta_0}{\kappa}\left(\frac{d^4}{d\zeta^4}W(\zeta)\right)+\delta_2\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)+\frac{8}{15}\delta_1\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)+2\delta_0\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\\
&+\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)+\frac{3}{5}\delta_1\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)+\frac{3}{5}\delta_1\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)\\
&+\frac{3}{2}\delta_0\left(\frac{d^2}{d\zeta^2}U(\zeta)\right)\left(\frac{d}{d\zeta}W(\zeta)\right)+\frac{3}{2}\delta_0\left(\frac{d}{d\zeta}W(\zeta)\right)^2\left(\frac{d^2}{d\zeta^2}W(\zeta)\right)-M\,\frac{L}{EI}
\end{aligned}
\tag{37}
$$

4.2 Differential quadrature method

In this research, the governing equation of motion for boron-nitride micro ribbon is obtained by using differential quadrature method (DQM). Based on DQM, the derivative of the function with any order at any arbitrary point like $(x,y)=(x_i,y_i)$ can be written in terms of function values in all of the range.

$$
\begin{aligned}
\left.\frac{d^r f}{dx^r}\right|_{(x,y)=(x_i,y_j)}&=\sum_{n=1}^{N}A_{in}^{(r)}f_{nj}\\
\left.\frac{d^s f}{dy^s}\right|_{(x,y)=(x_i,y_j)}&=\sum_{m=1}^{M}A_{jm}^{(s)}f_{im}\\
\left.\frac{\partial^{r+s} f}{\partial x^r\partial y^s}\right|_{(x,y)=(x_i,y_j)}&=\sum_{n=1}^{N}\sum_{m=1}^{M}A_{in}^{(r)}A_{jm}^{(s)}f_{nm}
\end{aligned}
\tag{38}
$$

where

$$
f_{ij}=f\left(x_i,y_j\right)
\tag{39}
$$

And also, $A^{(r)}$ is the matrix of weighting coefficients, which are defined as follows

$$A_{ij}^{(1)} = \begin{cases} \dfrac{\prod_{\substack{m=1 \\ m\neq i,j}}^{N} (x_i - x_m)}{\prod_{\substack{m=1 \\ m\neq j}}^{N} (x_j - x_m)}, \ (i, j = 1,2,3,...,N; i \neq j) \\ \sum_{\substack{m=1 \\ m\neq i}}^{N} \dfrac{1}{(x_i - x_m)}, \ (i = j = 1,2,3,...,N) \end{cases}$$

$$A^{(r)} = A^{(r-1)} A^{(1)} \quad 2 \leq r \leq N-1 \tag{40}$$

A well-accepted set of the grid points is the Chebyshev–Gauss–Lobatto (CGL) points that these points for interval [0, 1] are given by

$$x_i = \frac{1}{2}\left\{1 - \cos[\frac{(i-1)\pi}{(N-1)}]\right\} \quad y_j = \frac{1}{2}\left\{1 - \cos[\frac{(j-1)\pi}{(M-1)}]\right\} \tag{41}$$

A numerical method to solve the differential equations is the differential quadrature method (DQM) in which this method can employ only for continuous problems. Recently, differential quadrature element method (DQEM) has appeared as a numerical technique to analyze the structures with some local discontinuities including attached mass, loading, material properties, and geometry. Thus, this method is used to solve many problems especially in the vibration analysis. Thus in this research, the DQEM is used for linear and nonlinear vibration analysis of micro ribbon. In this analysis, the attached mass as a local discontinuity on the micro ribbon is considered. We assume, at the coordinates ($\zeta=\zeta_m$) a discontinuity in the parameters of micro ribbon is created. Thus, the compatibility conditions for the discontinuity of the attached mass can be explained as follows

$$\begin{aligned} & U\left(\zeta_m^-\right) = U\left(\zeta_m^+\right) \\ & N\left(\zeta_m^+\right) - N\left(\zeta_m^-\right) = m\frac{\partial^2 u}{\partial t^2} \\ & W\left(\zeta_m^-\right) = W\left(\zeta_m^+\right) \\ & \frac{dW\left(\zeta_m^-\right)}{d\zeta} = \frac{dW\left(\zeta_m^+\right)}{d\zeta} \\ & M\left(\zeta_m^-\right) = M\left(\zeta_m^+\right) \\ & V\left(\zeta_m^+\right) - V\left(\zeta_m^-\right) = m\frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{42}$$

In this equation, m represents concentrated mass inertia and N, M and V are axial force, the bending moment and shear force, respectively. Based on DQEM, solving points should be divided to three categories of boundary points, common areas between sub-beams at discontinuities and domain points, and then by applying compatibility conditions and boundary conditions to equations of motion GMR, the natural frequency of the system is achieved which has been described for linear and nonlinear states of GMR with details in Appendices A and B, respectively.

5. Numerical results

In this section, the numerical results are presented to investigate the nonlinear free vibration of boron-nitride micro ribbon based mass sensor using MSGT. The effect of the material length scale parameters, attached mass, temperature change, piezoelectric coefficient and two parameters of elastic foundations on natural frequencies of the system are investigated. Also, if the all material length scale parameters are equal to zero ($l_0 = l_1 = l_2 = 0$), then this theory is named CT, while with considering $l_0 = l_1 = 0$ and $l_2 = l$, the equation of motions are reduced to those corresponding to the ribbon modelled based on the MCST. When all of the material length scale parameters (l_0, l_1, l_2) in the equations of motion are exist, this theory is named MSGT ($l_0 = l_1 = l_2 = l$). Therefore the results from the MSGT are compared with the obtained results of MCST and CT.

Mechanical and geometrical properties of the GMR as follows

$$E = 1.44GPa \qquad l = 17.6\,\mu m \qquad L = 20h \qquad b = 2h$$
$$\alpha = 1.2\times 10^{-6}\, 1/^{\circ} c \tag{43}$$

where h is considered as a fraction of l.

In Table 1, there is the slightly difference between the obtained present results (DQM) and Reddy (2011) (analytical solution) for linear frequencies, these differences are because the type of solution method. Thus in Table 1, the numerical results of this research based on DQM are compared with the obtained results by Reddy (2011) based on analytical solutions. He analyzed the linear vibration analysis of a micro beam with a simply supported using a MCST and analytical solution. Two methods have a good agreement together; while in Table 2 for nonlinear vibration, the method of solution in the present results and Wang *et al*. (2013) is the same in which DQM is used both them. They analyzed nonlinear vibrations of a micro beam with simply supported boundary conditions using the MCST and CT based on DQM. Thus two DQ methods have an excellent agreement.

Table 1 First three linear natural frequency of micro beam with simply supported and modified couple stress theory

h/ℓ	$\overline{\omega_1}$		$\overline{\omega_2}$		$\overline{\omega_3}$	
	Present work	Reddy, 2011	Present work	Reddy, 2011	Present work	Reddy, 2011
1	22.8004	22.80	90.9222	90.92	203.5397	203.54
5	10.6825	10.68	42.5989	42.60	95.3611	95.36
CT	9.8595	9.86	39.3170	39.32	88.0143	88.02

Table 2 First nonlinear natural frequency of micro beam with simply supported boundary condition, CT, and MCST

W(0.5,0)*10	h/ℓ=1		h/ℓ=3		CT	
	Present work	Wang *et al.* 2013	Present work	Wang *et al.* 2013	Present work	Wang *et al.* 2013
0.1	1.0084	1.0084	1.0299	1.0299	1.0440	1.0440
0.2	1.0330	1.0330	1.1147	1.1147	1.1661	1.1661
0.3	1.0729	1.0727	1.2433	1.2433	1.3452	1.3452
0.4	1.1263	1.1259	1.4036	1.4036	1.5617	1.5617
0.5	1.1915	1.1915	1.5860	1.5860	1.8022	1.8022

Table 3 First three linear natural frequency of GMR with simply supported boundary condition and different mass intensity ($\bar{M}$) in $\zeta_m = 0.5$ for MSGT ($l_0 = l_1 = l_2 = l = 17.6\mu m$) and MCST ($l_0 = l_1 = 0,\ l_2 = l = 17.6\mu m$)

h/ℓ	mode	$\bar{M}=0$		$\bar{M}=0.01$		$\bar{M}=0.05$		$\bar{M}=0.1$	
		MCST	MSGT	MCST	MSGT	MCST	MSGT	MCST	MSGT
1	$\overline{\omega_1}$	22.8004	39.9124	22.5761	39.5193	21.7400	38.0536	20.8131	36.4290
	$\overline{\omega_2}$	90.9222	159.5248	90.9222	159.5248	90.9222	159.5248	90.9222	159.5248
	$\overline{\omega_3}$	203.5397	358.4696	201.6012	355.0091	194.8925	343.0434	188.3211	331.3380
2	$\overline{\omega_1}$	14.2433	21.7062	14.1032	21.4924	13.5809	20.6955	13.0018	19.8121
	$\overline{\omega_2}$	56.7986	86.7261	56.7986	86.7261	56.7986	86.7261	56.7986	86.7261
	$\overline{\omega_3}$	127.1503	194.7696	125.9393	192.8933	121.7484	186.4046	117.6432	180.0557
3	$\overline{\omega_1}$	12.0071	16.2299	11.8890	16.0701	11.4487	15.4743	10.9605	14.8140
	$\overline{\omega_2}$	47.8812	64.8201	47.8812	64.8201	47.8812	64.8201	47.8812	64.8201
	$\overline{\omega_3}$	107.1876	145.4780	106.1667	144.0798	102.6338	139.2438	99.1732	134.5110

Continued-

4	$\overline{\omega_1}$	11.1187	13.8093	11.0093	13.6734	10.6016	13.1666	10.1496	12.6048
	$\overline{\omega_2}$	44.3384	55.1337	44.3384	55.1337	44.3384	55.1337	44.3384	55.1337
	$\overline{\omega_3}$	99.2566	123.6684	98.3113	122.4823	95.0398	118.3790	91.8352	114.3625
5	$\overline{\omega_1}$	10.6825	12.5316	10.5774	12.4083	10.1857	11.9485	9.7514	11.4387
	$\overline{\omega_2}$	42.5990	50.0192	42.5990	50.0192	42.5990	50.0192	42.5990	50.0192
	$\overline{\omega_3}$	95.3627	112.1469	94.4544	111.0729	91.3113	107.3574	88.2324	103.7199

Table 4 frequency ratio (ω_{NL}/ω_L) of GMR with simply supported boundary condition and different mass intensity ($\bar{M}$) in $\zeta_m = 0.5$ for MSGT ($l_0 = l_1 = l_2 = l = 17.6\mu m$) and MCST ($l_0 = l_1 = 0,\ l_2 = l = 17.6\mu m$)

h/ℓ	W(0.5,0)*10	$\bar{M} = 0$		$\bar{M} = 0.05$		$\bar{M} = 0.1$		$\bar{M} = 0.2$	
		MCST	MSGT	MCST	MSGT	MCST	MSGT	MCST	MSGT
2	0.3	1.1780	1.0802	1.1774	1.0800	1.1769	1.0797	1.1760	1.0794
	0.5	1.4411	1.2096	1.4393	1.2089	1.4377	1.2083	1.4350	1.2073
	0.7	1.7632	1.3810	1.7599	1.3796	1.7567	1.3783	1.7494	1.3762
3	0.3	1.2433	1.1395	1.2424	1.1391	1.2417	1.1387	1.2404	1.1381
	0.5	1.5860	1.3524	1.5835	1.3511	1.5812	1.3499	1.5772	1.3479
	0.7	1.9920	1.6198	1.9878	1.6172	1.9801	1.6148	1.9714	1.6097
4	0.3	1.2793	1.1885	1.2782	1.1879	1.2773	1.1873	1.2758	1.1864
	0.5	1.6635	1.4647	1.6606	1.4628	1.6579	1.4611	1.6521	1.4582
	0.7	2.1123	1.8010	2.1077	1.7975	2.0980	1.7941	2.0876	1.7861
5	0.3	1.2998	1.2251	1.2987	1.2244	1.2977	1.2237	1.2961	1.2225
	0.5	1.7072	1.5463	1.7041	1.5440	1.7012	1.5419	1.6947	1.5383
	0.7	2.1795	1.9300	2.1709	1.9259	2.1637	1.9192	2.1524	1.9113

In Tables 3 and 4, the effects of various parameters such as material length scale parameter for various theories such as MSGT ($l_0 = l_1 = l_2 = l = 17.6\mu m$) and MCST ($l_0 = l_1 = 0,\ l_2 = l = 17.6\mu m$), mass intensity and dimensionless amplitude on linear and nonlinear natural frequencies of GMR considering mass effect for the MCST and MSGT are investigated. By increasing mass intensity, the dimensionless natural frequency decreases.

The effect of the numbers of grid points on the accuracy and convergence of boron-nitride micro ribbon based mass sensor for first three dimensionless natural frequencies is shown in Fig. 2. Based on the convergence plots, the numbers of grid points according to the first three dimensionless natural frequencies (N) are equal to 15.

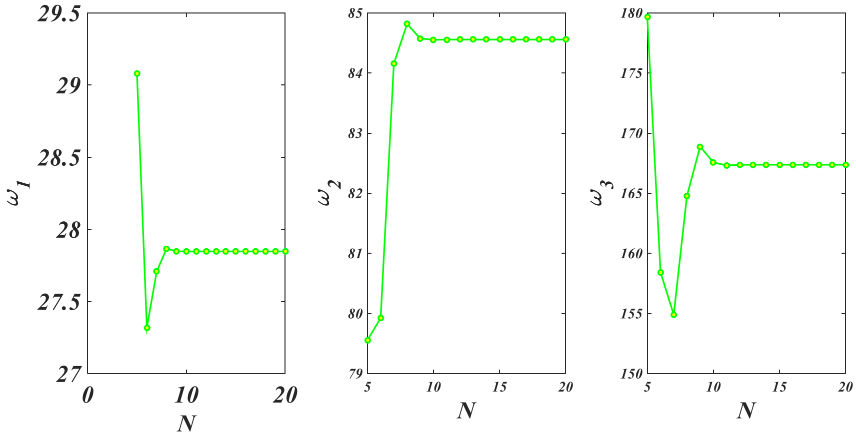
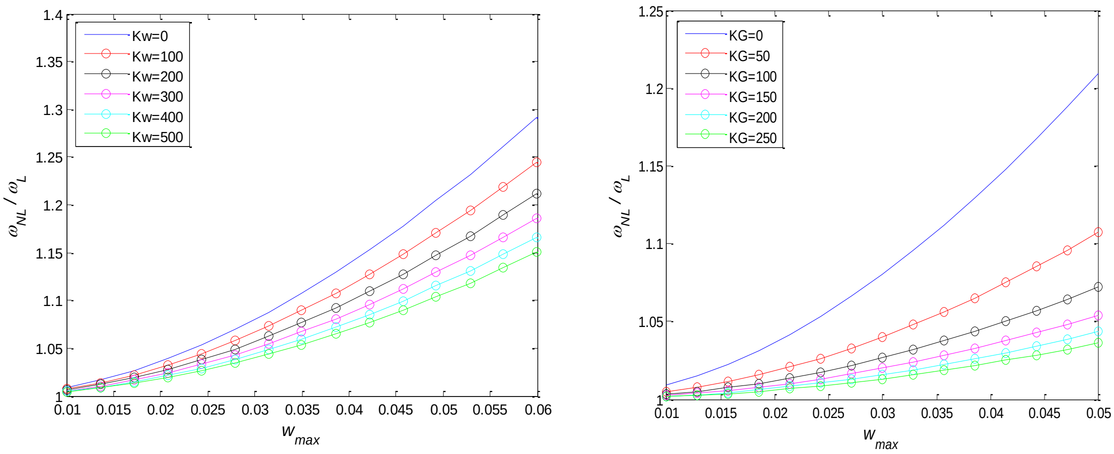


Fig. 2 The effect of the numbers of grid points on the accuracy and convergence of boron-nitride micro ribbon based mass sensor for first three dimensionless natural frequencies



(a) Winkler foundation (b) Pasternak foundation

Fig. 3 The effect of Winkler and Pasternak coefficient and dimensionless amplitude on frequency ratio of GMR based on MSGT ($l_0 = l_1 = l_2 = l = 17.6\mu m$) and $h/l = 2$

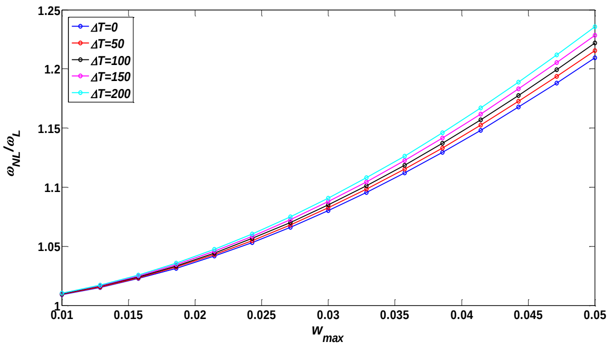


Fig. 4 The effect of temperature change on the frequency ratio of GMR versus dimensionless amplitude based on MSGT ($l_0 = l_1 = l_2 = l = 17.6\mu m$) and $h/l = 2$

Fig. 3(a) depicts the effect of Winkler elastic medium coefficient and dimensionless amplitude (W_{max}) on nonlinear frequency ratio of GMR. It can be seen that by increasing Winkler elastic medium coefficient, the frequency ratio (the nonlinear natural frequency to the linear natural frequency ratio) decreases. It should be noted that the increase of Winkler coefficient causes an increase in rigidity of the GMR and an increase in linear and nonlinear frequencies. However, the rate of increase in linear frequencies is much more that increase in nonlinear frequency and this matter causes to reduce the frequency ratio. Fig. 3(b) shows that with an increase in Pasternak coefficient, the frequency ratio of GMR decreases. At this state, with an increase in Pasternak coefficient causes to decrease in linear and nonlinear frequencies.

The effect of temperature change on the frequency ratio of GMR is examined in Fig. 4. In this figure is obvious that with an increase in the temperature change, both linear and nonlinear frequencies of the GMR reduce. This phenomenon leads to increase flexibility of GMR with increasing the temperature change. However, the decreasing linear natural frequency is more than the nonlinear natural frequency. Moreover, the frequencies ratio increases with an increase in the temperature.

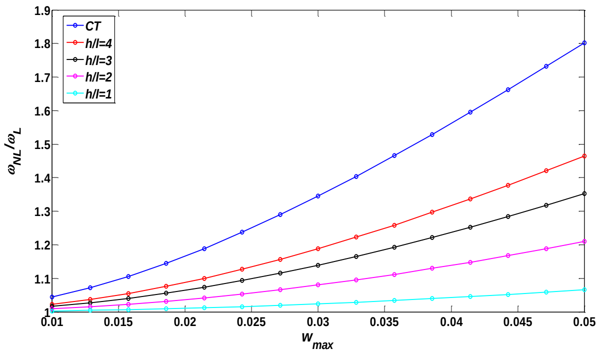


Fig. 5 The effect of dimensionless material length scale parameter on the frequency ratio of GMR with dimensionless amplitude for $h/l = 2$

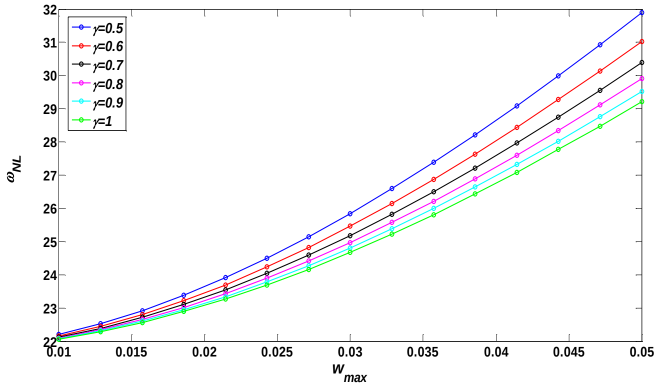


Fig. 6 The nonlinear natural frequency versus dimensionless amplitude at different piezoelectric coefficients based on MSGT ($l_0 = l_1 = l_2 = l = 17.6\,\mu m$)and $h/l = 2$

Fig. 5 displays the influence of the material length scale parameters on the nonlinear frequency ratio. As can be seen, by increasing dimensionless amplitude (W_{max}), the frequency ratio increases. Also, by increasing the material length scale parameter, the frequency ratio decreases. However, we know that by increasing the material length scale parameter, rigidity of the GMR increases and therefore, linear and nonlinear natural frequencies of system increases, but the increase in the linear natural frequency is larger than the nonlinear frequency.

Fig. 6 shows the influence of the piezoelectric coefficients on the nonlinear frequency of GMR. As can be seen, the nonlinear frequency of GMR increases with increasing of piezoelectric coefficient. In other words, if the selected material has higher piezoelectric coefficient, the nonlinear frequency of GMR increases.

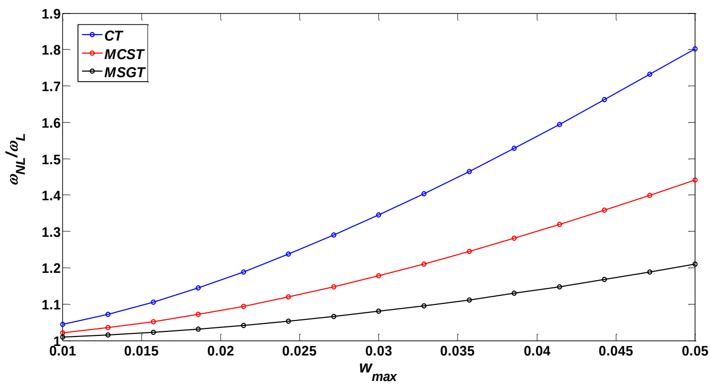


Fig. 7 The frequency ratio of GMR versus dimensionless amplitude for various theories of size dependent effect

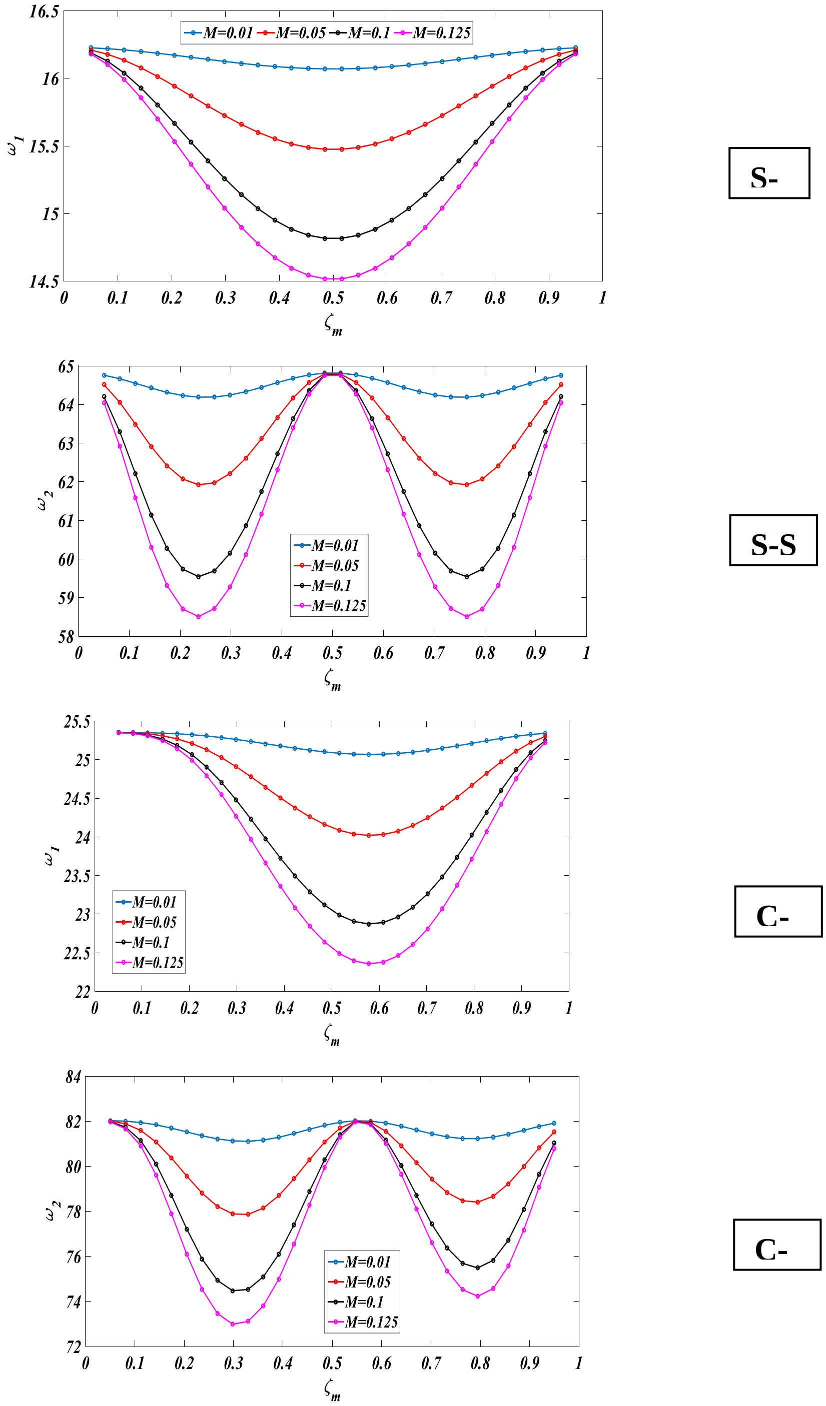


Fig. 8 The linear natural frequency of GMR under different boundary conditions and location mass intensity based on MSGT ($l_0 = l_1 = l_2 = l = 17.6\,\mu m$) and $h/l = 3$

The frequency ratio of GMR in various theories is inspected in Fig. 7. As can be seen, considering size effects and increasing rigidity of GMR, linear and nonlinear frequencies of system increase, but the increase of linear natural frequency is much more than nonlinear natural frequency and therefore, frequency ratio decreases. In CT, we observe the highest frequency ratio and in the MSGT, we see the lowest frequency ratio.

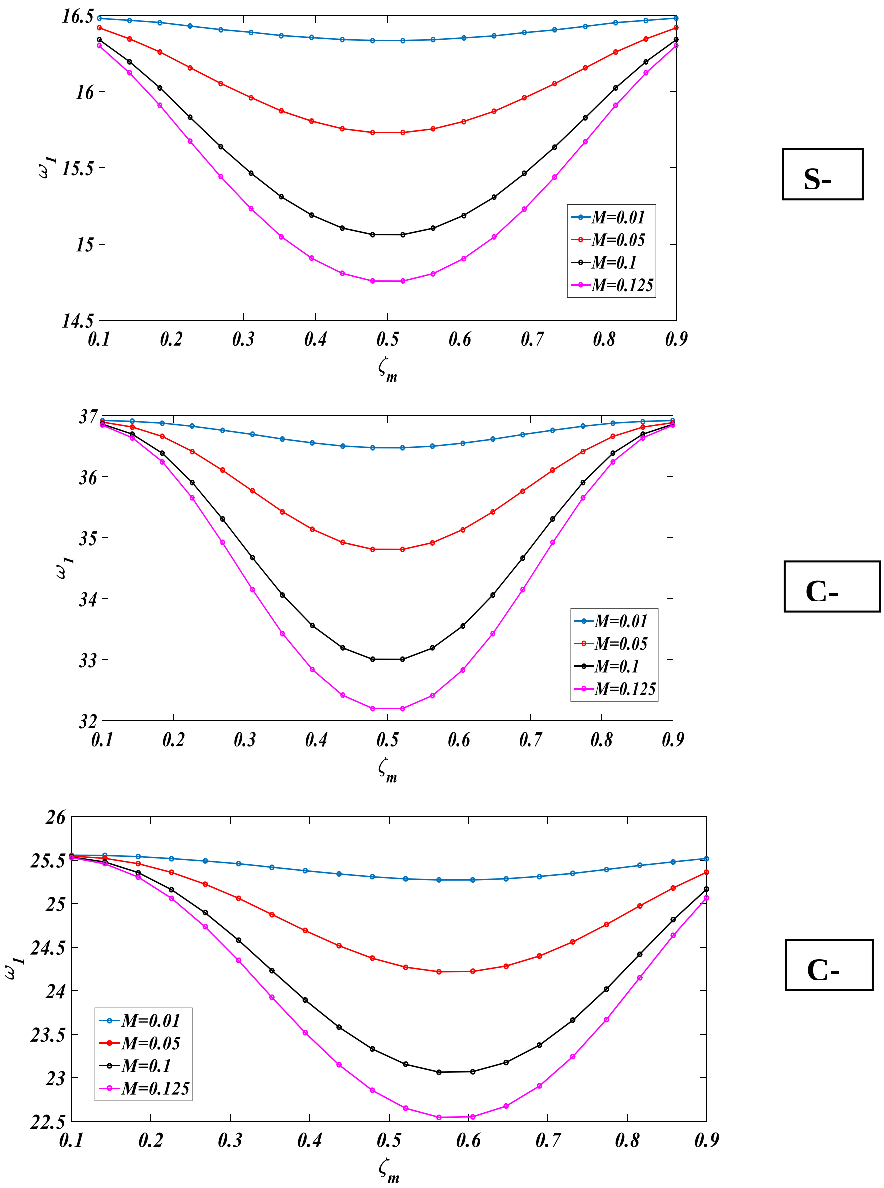


Fig. 9 The first nonlinear frequency of GMR with various boundary conditions versus different location mass intensity based on MSGT ($l_0 = l_1 = l_2 = l = 17.6\,\mu m$), $h/l = 3$, and $W_{\max} = 0.01$

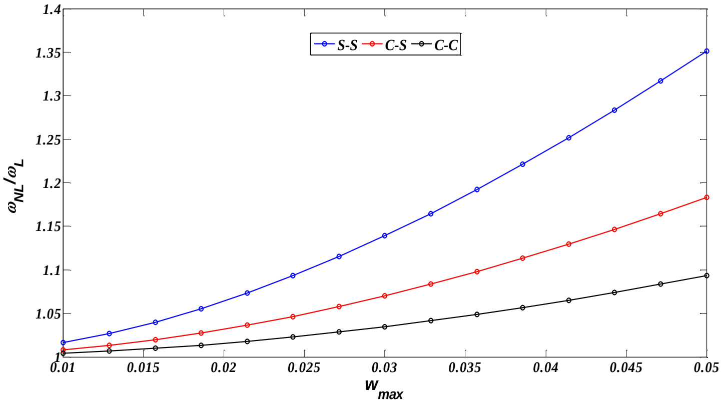


Fig. 10 The frequency ratio of GMR versus different dimensionless amplitude for various boundary conditions based on MSGT, $h/l=3$, $\zeta_m=0.5$ and $\bar{M}=0.05$

In Fig. 8, ω_1 and ω_2 are the first and second dimensionless natural frequency of micro ribbon. Figs. 8 indicated the effect of location mass (ζ_m) and mass intensity ($\bar{M}$) on the linear natural frequency of the GMR under different boundary conditions. These figures are shown at various boundary conditions such as simply-simply (S-S), and clamped-simply (C-S) boundary conditions, by increasing the mass intensity in a fixed position, the linear natural frequency of GMR is reduced. There are some points in any natural frequency that with any mass intensity in those points will not have any effect on the natural frequency; these points actually are corresponding inflection points of mode shape curve in which bending moment equals zero. There are also some points that mass intensity in those points will have the most effect on the natural frequency. Actually, these points are the points in which corresponding mode shape curve are maximum curvature, and bending moment is optimized value at these points.

The influence of location mass (ζ_m) and mass intensity on nonlinear frequency of GMR under various boundary conditions showed in Figs. 9. For all boundary conditions such as S-S, C-S and clamped-clamped (C-C) by increasing the mass intensity in a fixed position, the nonlinear frequency of the GMR is reduced.

Fig. 10 illustrates the effect of various boundary conditions on the frequency ratio of GMR. In the dimensionless linear natural frequency state, the natural frequency for C-C boundary conditions leads to an increase in the stiffness of GMR at the edges that the effect of this boundary condition is more than the other boundary conditions. Such a behavior is because the higher constraints at the edges increase the flexural rigidity of the GMR, leading to a higher dimensionless linear natural frequency; while to obtain the dimensionless nonlinear natural frequency, this frequency is dependent to W_{max} . Thus, C-C boundary conditions cause structure that is more constrained so the value of W_{max} with respect to the other boundary conditions is lower, and leads to decrease the dimensionless nonlinear natural frequency. According to the Fig. 9, the frequency ratio (the dimensionless nonlinear natural frequency to the dimensionless linear natural frequency ratio) for C-C boundary conditions is less than the other boundary conditions.

6. Conclusions

The nonlinear free vibrations of boron-nitride micro ribbon (BNMR) on the elastic Pasternak foundation under electrical, mechanical and thermal loadings are studied based on modified strain gradient theory (MSGT). A differential quadrature element method (DQEM) is developed to obtain the frequency ratio of GMR under different boundary conditions with considering the attached mass. The effects of various parameters such as material length scale parameters, attached mass, temperature change, piezoelectric coefficient, two parameters of elastic foundations on the frequency ratio of BNMR are investigated. This results can be used to design and control nano/micro devices and nanoelectronics to avoid resonance phenomenon. The following conclusions can be listed from present research:

- The linear and nonlinear natural frequency of GMR decreases with an increase in the mass intensity in a fixed position for various boundary conditions.
- With increasing Winkler and Pasternak modulli decreases the frequency ratio. The increase of Winkler and Pasternak coefficients lead to an increase in rigidity of the GMR and an increase in linear and nonlinear frequencies. However, the rate of increase in linear frequencies is much more that of nonlinear frequency and this matter cause to reduce the frequency ratio.
- The nonlinear frequency of GMR increases with increasing of piezoelectric coefficient.
- With an increase in the temperature change, both linear and nonlinear frequencies of the GMR decrease. This phenomenon leads to increase the flexibility of GMR with increasing the temperature change. However the decreasing linear natural frequency is more than the nonlinear natural frequency. Moreover, the frequencies ratio increases with an increase in the temperature change.
- The highest and lowest frequency ratio is related to CT and MSGT, respectively.
- By increasing dimensionless amplitude, the frequency ratio increases. Also, by increasing the material length scale parameter, the frequency ratio decreases. However, we know that by increasing the material length scale parameter, the rigidity of the GMR increases and therefore, linear and nonlinear natural frequencies of system increase, but the increase in the linear natural frequency is larger than the nonlinear frequency.
- In the dimensionless linear natural frequency state, the natural frequency for C-C boundary conditions leads to an increase in the stiffness of GMR at the edges that the effect of this boundary condition is more than the others. Such a behavior is because the higher constraints at the edges increase the flexural rigidity of the GMR, leading to a higher dimensionless linear natural frequency, while to obtain the dimensionless nonlinear natural frequency, this frequency is dependent to W_{max}. Thus, C-C boundary conditions cause structure that is more constrained so the value of W_{max} with respect to the other boundary conditions is lower, and leads to decrease the dimensionless nonlinear natural frequency. According to the results, the frequency ratio for C-C boundary conditions is less than the other boundary conditions.

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Appendix A

In this Appendix, for linear state of GMR, if there is the discontinuity because an attached mass, the compatibility conditions in Eq. (42) can apply into Eqs. (35)-(37) which is described as follows:

We have the axial, and shear forces and the bending moment as the following form

$$\begin{aligned}
N &= \frac{EI}{L^2}\left[\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U(\zeta)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U(\zeta)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\right]\\
V &= \frac{EI}{L^2}\left[\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W(\zeta)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W(\zeta)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W(\zeta)}{\partial\zeta}\right] \qquad \text{(A-1)}\\
M &= \frac{EI}{L}\left[-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W(\zeta)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W(\zeta)}{\partial\zeta^2}\right]
\end{aligned}$$

Substituting Eq. (A-1) into Eq. (42) yields the following relations

$$\begin{aligned}
&U(\zeta_m^-)=U(\zeta_m^+)\\
&\frac{EI}{L^2}\left[\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U(\zeta_m^+)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U(\zeta_m^+)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\right]\\
&=\frac{EI}{L^2}\left[\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U(\zeta_m^-)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U(\zeta_m^-)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\right]+m\,\frac{EI}{\rho AL^4}L\,\frac{\partial^2 U(\zeta_m^-)}{\partial\tau^2}\\
&W(\zeta_m^-)=W(\zeta_m^+)\\
&\frac{dW(\zeta_m^-)}{d\zeta}=\frac{dW(\zeta_m^+)}{d\zeta}\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{d^4 W(\zeta_m^-)}{d\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{d^2 W(\zeta_m^-)}{d\zeta^2}= \qquad \text{(A-2)}\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{d^4 W(\zeta_m^+)}{d\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{d^2 W(\zeta_m^+)}{d\zeta^2}\\
&\frac{EI}{L^2}\left[\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W(\zeta_m^+)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W(\zeta_m^+)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W(\zeta_m^+)}{\partial\zeta}\right]\\
&=\frac{EI}{L^2}\left[\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W(\zeta_m^-)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W(\zeta_m^-)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W(\zeta_m^-)}{\partial\zeta}\right]\\
&+m\,\frac{EI}{\rho AL^4}L\,\frac{\partial^2 W(\zeta_m^-)}{\partial\tau^2}
\end{aligned}$$

Eqs. (A-2) after simplifying can be written as

$$
\begin{aligned}
&U\left(\zeta_m^-\right)=U\left(\zeta_m^+\right)\\
&\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}\\
&=\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}+\frac{m}{\rho AL}\frac{\partial^2 U\left(\zeta_m^-\right)}{\partial\tau^2}\\
&W\left(\zeta_m^-\right)=W\left(\zeta_m^+\right)\\
&\frac{dW\left(\zeta_m^-\right)}{d\zeta}=\frac{dW\left(\zeta_m^+\right)}{d\zeta}\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W\left(\zeta_m^-\right)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}=\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W\left(\zeta_m^+\right)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\\
&\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W\left(\zeta_m^+\right)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\\
&=\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W\left(\zeta_m^-\right)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\\
&+\frac{m}{\rho AL}\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\tau^2}
\end{aligned}
\tag{A-3}
$$

By using $\bar{M}=\dfrac{m}{\rho AL}$ and Eq. (32), and applying them in Eq. (A-3), the following equations can be rewritten as follows

$$
\begin{aligned}
&U\left(\zeta_m^+\right)-U\left(\zeta_m^-\right)=0\\
&\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left[\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}-\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}\right]-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}\right]=-\bar{M}\,\omega^2 U\left(\zeta_m^-\right)\\
&W\left(\zeta_m^+\right)-W\left(\zeta_m^-\right)=0\\
&\frac{dW\left(\zeta_m^+\right)}{d\zeta}-\frac{dW\left(\zeta_m^-\right)}{d\zeta}=0\\
&\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\left[\frac{d^4 W\left(\zeta_m^+\right)}{d\zeta^4}-\frac{d^4 W\left(\zeta_m^-\right)}{d\zeta^4}\right]-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\left[\frac{d^2 W\left(\zeta_m^+\right)}{d\zeta^2}-\frac{d^2 W\left(\zeta_m^-\right)}{d\zeta^2}\right]=0\\
&\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\left[\frac{d^5 W\left(\zeta_m^+\right)}{d\zeta^5}-\frac{d^5 W\left(\zeta_m^-\right)}{d\zeta^5}\right]-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\left[\frac{d^3 W\left(\zeta_m^+\right)}{d\zeta^3}-\frac{d^3 W\left(\zeta_m^-\right)}{d\zeta^3}\right]=-\bar{M}\,\omega^2 W\left(\zeta_m^-\right)
\end{aligned}
\tag{A-4}
$$

Appendix B

In Appendix B, for nonlinear state of GMR, if there is the discontinuity because an attached mass, the compatibility conditions in Eq. (42) can apply into Eqs. (35)-(37) which is described as follows:

For nonlinear state of GMR, the axial, and shear forces and the bending moment are considered as the following form

$$N=\frac{EI}{L^2}\left\{\begin{array}{l}\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U(\zeta)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U(\zeta)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\\+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W(\zeta)}{\partial\zeta}\right)^2-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\left(\frac{\partial^2 W(\zeta)}{\partial\zeta^2}\right)^2+\frac{\partial W(\zeta)}{\partial\zeta}\frac{\partial^3 W(\zeta)}{\partial\zeta^3}\right]\end{array}\right\}$$

$$V=\frac{EI}{L^2}\left\{\begin{array}{l}\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W(\zeta)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W(\zeta)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W(\zeta)}{\partial\zeta}\\-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\frac{\partial^3 U(\zeta)}{\partial\zeta^3}\frac{\partial W(\zeta)}{\partial\zeta}+\left(\frac{\partial W(\zeta)}{\partial\zeta}\right)^2\frac{\partial^3 W(\zeta)}{\partial\zeta^3}+\frac{\partial W(\zeta)}{\partial\zeta}\left(\frac{\partial^2 W(\zeta)}{\partial\zeta^2}\right)^2\right]\\+\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U(\zeta)}{\partial\zeta}\frac{\partial W(\zeta)}{\partial\zeta}+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W(\zeta)}{\partial\zeta}\right)^3\end{array}\right\} \quad \text{(B-1)}$$

$$M=\frac{EI}{L}\left\{\begin{array}{l}-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W(\zeta)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W(\zeta)}{\partial\zeta^2}\\+3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\frac{\partial^2 U(\zeta)}{\partial\zeta^2}\frac{\partial W(\zeta)}{\partial\zeta}+\left(\frac{\partial W(\zeta)}{\partial\zeta}\right)^2\frac{\partial^2 W(\zeta)}{\partial\zeta^2}\right]\end{array}\right\}$$

By substituting Eq. (B-1) into Eq. (42), one can be obtained the following equations:

$$
\begin{aligned}
&U\left(\zeta_m^-\right)=U\left(\zeta_m^+\right)\\
&W\left(\zeta_m^-\right)=W\left(\zeta_m^+\right)\\
&\frac{dW\left(\zeta_m^-\right)}{d\zeta}=\frac{dW\left(\zeta_m^+\right)}{d\zeta}\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W\left(\zeta_m^-\right)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\\
&+3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\frac{\partial^2 U\left(\zeta_m^-\right)}{\partial\zeta^2}\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}+\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^2\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right]=\\
&-\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\frac{\partial^4 W\left(\zeta_m^+\right)}{\partial\zeta^4}+\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\\
&+3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\frac{\partial^2 U\left(\zeta_m^+\right)}{\partial\zeta^2}\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}+\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^2\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\right]\\
&\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\\
&+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^2-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\left(\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\right)^2+\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}\right]\\
&=\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}-\frac{1}{2}\kappa\alpha_1\Delta T\\
&+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^2-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\left(\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right)^2+\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}\right]\\
&+\frac{mL^2}{EI}\frac{EI}{\rho AL^4}L\frac{\partial^2 U}{\partial\tau^2}
\end{aligned}
\tag{B-2}
$$

$$\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W\left(\zeta_m^+\right)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}$$
$$-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}+\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^2\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}+\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\left(\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\right)^2\right]$$
$$+\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^3$$
$$=\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\frac{\partial^5 W\left(\zeta_m^-\right)}{\partial\zeta^5}-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}-\left(\frac{1}{2}\kappa\alpha_1\Delta T+\frac{1}{\kappa}\omega^2\right)\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta} \tag{B-3}$$
$$-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}+\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^2\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}+\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\left(\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right)^2\right]$$
$$+\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^3$$
$$+\frac{mL^2}{EI}\frac{EI}{\rho AL^4}L\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\tau^2}$$

Applying $\bar{M}=\frac{m}{\rho AL}$ and Eq. (32) into Eqs. (B-2) and (B-3), the following equations can be rewritten as follows

$$U\left(\zeta_m^+\right)-U\left(\zeta_m^-\right)=0$$
$$W\left(\zeta_m^+\right)-W\left(\zeta_m^-\right)=0$$
$$\frac{dW\left(\zeta_m^+\right)}{d\zeta}-\frac{dW\left(\zeta_m^-\right)}{d\zeta}=0$$
$$\frac{2}{\kappa}\left(\frac{2}{5}\delta_1+\delta_0\right)\left[\frac{\partial^4 W\left(\zeta_m^+\right)}{\partial\zeta^4}-\frac{\partial^4 W\left(\zeta_m^-\right)}{\partial\zeta^4}\right]-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\left[\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}-\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right] \tag{B-4}$$
$$-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\begin{array}{l}\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial^2 U\left(\zeta_m^+\right)}{\partial\zeta^2}-\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial^2 U\left(\zeta_m^-\right)}{\partial\zeta^2}\\+\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^2\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}-\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^2\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\end{array}\right]=0$$

$$\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left[\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}-\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}\right]-3\left(\frac{1}{5}\delta_1+\frac{1}{2}\delta_0\right)\left[\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}\right]$$
$$-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\left(\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\right)^2-\left(\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right)^2+\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}\right]$$
$$=-\bar{M}\,\omega^2 U\left(\zeta_m^-\right)$$

$$\frac{2}{\kappa}\left(\delta_0+\frac{2}{5}\delta_1\right)\left[\frac{\partial^5 W\left(\zeta_m^+\right)}{\partial\zeta^5}-\frac{\partial^5 W\left(\zeta_m^-\right)}{\partial\zeta^5}\right]-\left(1+2\delta_0+\frac{8}{15}\delta_1+\delta_2\right)\left[\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}\right]$$
$$-3\left(\frac{1}{2}\delta_0+\frac{1}{5}\delta_1\right)\left[\begin{array}{l}\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial^3 U\left(\zeta_m^+\right)}{\partial\zeta^3}-\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial^3 U\left(\zeta_m^-\right)}{\partial\zeta^3}\\ +\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^2\frac{\partial^3 W\left(\zeta_m^+\right)}{\partial\zeta^3}-\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^2\frac{\partial^3 W\left(\zeta_m^-\right)}{\partial\zeta^3}\\ +\left(\frac{\partial^2 W\left(\zeta_m^+\right)}{\partial\zeta^2}\right)^2\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}-\left(\frac{\partial^2 W\left(\zeta_m^-\right)}{\partial\zeta^2}\right)^2\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\end{array}\right]$$
$$+\frac{3}{4}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left[\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\frac{\partial U\left(\zeta_m^+\right)}{\partial\zeta}-\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\frac{\partial U\left(\zeta_m^-\right)}{\partial\zeta}\right]$$
$$+\frac{3}{8}\kappa\left(1+\frac{3}{4}\frac{1}{\gamma}\right)\left[\left(\frac{\partial W\left(\zeta_m^+\right)}{\partial\zeta}\right)^3-\left(\frac{\partial W\left(\zeta_m^-\right)}{\partial\zeta}\right)^3\right]=-\bar{M}\,\omega^2 W\left(\zeta_m^-\right)$$