Smart Structures and Systems, *Vol. 17, No. 2 (2016) 231-256*
DOI: http://dx.doi.org/10.12989/sss.2016.17.2.231

Closed-form optimum tuning formulas for passive Tuned Mass Dampers under benchmark excitations

Jonathan Salvi and Egidio Rizzi*

Università di Bergamo, Dipartimento di Ingegneria e Scienze Applicate, viale G. Marconi 5, I-24044 Dalmine (BG), Italy

(Received December 6, 2014, Revised December 1, 2015, Accepted December 2, 2015)

Abstract. This study concerns the derivation of optimum tuning formulas for a passive Tuned Mass Damper (TMD) device, for the case of benchmark ideal excitations acting on a single-degree-of-freedom (SDOF) damped primary structure. The free TMD parameters are tuned first through a non-linear gradient-based optimisation algorithm, for the case of harmonic or white noise excitations, acting either as force on the SDOF primary structure or as base acceleration. The achieved optimum TMD parameters are successively interpolated according to appropriate analytical fitting proposals, by non-linear least squares, in order to produce simple and effective TMD tuning formulas. In particular, two fitting models are presented. The main proposal is composed of a simple polynomial relationship, refined within the fitting process, and constitutes the optimum choice. A second model refers to proper modifications of literature formulas for the case of an undamped primary structure. The results in terms of final (interpolated) optimum TMD parameters and of device effectiveness in reducing the structural dynamic response are finally displayed and discussed in detail, showing the wide and ready-to-use validity of the proposed optimisation procedure and achieved tuning formulas. Several post-tuning trials have been carried out as well on SDOF and MDOF shear-type frame buildings, by confirming the effective benefit provided by the proposed optimum TMD.

Keywords: Tuned Mass Damper (TMD); harmonic excitation; white noise excitation; tuning formulas; optimisation; nonlinear least squares; fitting models

1. Introduction

This work deals with the optimum tuning of the free parameters of a passive Tuned Mass Damper (Rizzi *et al.* 2009, Salvi and Rizzi 2011, Salvi and Rizzi 2012, Salvi *et al.* 2013, Salvi and Rizzi 2014, Salvi *et al.* 2014a, b, c) for benchmark ideal excitations.

Since their original introduction, likely represented by the patent of Frahm (1911), TMDs have constituted one of the most studied vibration control devices, applied to either mechanical or structural systems. Ormondroyd and Den Hartog (1928), Hahnkamm (1933), Brock (1946) and Den Hartog (1956) established firm theoretical bases for the best selection of the tuning parameters (specifically frequency ratio f and TMD damping ratio ζ_T, as defined below), for a damped TMD ($\zeta_T \neq 0$) added to an undamped ($\zeta_S = 0$) SDOF primary structure subjected to harmonic force. In these studies, both TMD parameters have been determined analytically, but

*Corresponding author, Professor, E-mail: egidio.rizzi@unibg.it

http://www.techno-press.org/?journal=sss&subpage=8 ISSN: 1738-1584 (Print), 1738-1991 (Online)

while the obtained frequency ratio f^{opt} represents a true optimum value, the TMD damping ratio ζ_T^{opt} is slightly approximated and actually taken as a mean value (Brock 1946, Den Hartog 1956).

Afterwards, many contributions have considered the optimum tuning for damped main structures ($\zeta_S \neq 0$), despite higher difficulties in treating the governing dynamical equations (Asami *et al.* 2002, Bakre and Jangid 2007). For this reason, in a large number of works, the adoption of numerical optimisation approaches has been proposed as a suitable way to achieve optimum TMD tuning. Among those, one of the first examples is probably represented by that of Ioi and Ikeda (1978), where optimum TMD tuning formulas have been pointed out, as a result of an optimisation process developed through a Newton's method. Within subsequent studies which shared a similar *modus operandi*, of great interest appear those of Randall *et al.* (1981), which obtained the optimum TMD parameters in the form of graphical representations.

Additionally, Thompson (1981) achieved the optimum TMD parameters via an analytical and graphical way, i.e., by means of the root locus method, in the frequency domain. Tsai and Lin (1993, 1994) provided design formulas for the loading cases of harmonic force and base acceleration. Fujino and Abé (1994) obtained tuning formulas for different loading conditions through a perturbation method. Pennestrì (1998) further investigated the Minimax optimisation method for the case of harmonic force on a damped primary structure. A similar work was proposed by Rana and Soong (1998), again based on a Minimax algorithm, where the resulting optimum TMD parameters have been condensed in design abaci. Furthermore, Asami *et al.* (2002) provided tuning formulas for several loading cases, by analytical approaches.

Moreover, the tuning concept proposed by Den Hartog, has been revisited in different works (Krenk 2005, Liu and Liu 2005) especially with the consideration of the TMD damping ratio. Ghosh and Basu (2007) considered the fixed-point theory of Den Hartog (1956) also for the case of lightly damped structures, and proposed a closed-form formula for the optimum frequency ratio in case of harmonic force. Bakre and Jangid (2007) developed polynomial tuning formulas for the case of white noise loading, acting as either force on the primary structure or as base acceleration, with the consideration of different response indexes as objective function, such as displacement or velocity of the primary structure. Wong and Cheung (2008) investigated the TMD tuning in a different configuration, i.e. with the TMD damper linked to the ground, in order to minimise the effect of a possible ground motion. Brown and Singh (2011) assumed the Minimax method in order to assess the TMD tuning in case of uncertainties in the excitation frequency, related to harmonic loading. Bisegna and Caruso (2012) provided an optimum TMD conceived through the root locus method, so that to maximise the exponential decay of the transient response, for an undamped primary structure. Tigli (2012) further explored the TMD optimisation in case of random (white noise) loading, providing compact closed-form formulas, and pointed out that a better overall TMD performance for the considered cases should be obtained when the velocity response is minimised. The work of Zilletti *et al.* (2012) addressed the TMD tuning problem with a different task, which consisted in the minimisation of the kinetic energy of the host structure and the maximisation of the TMD power dissipation.

Finally, recent studies proposed new algorithms as better means for the solution of the tuning problem with respect to classical numerical methods. In this sense, relevant works appear those of Leung and Zhang (2009), which presented a numerical tuning procedure based on a Particle Swarm Optimisation method, and of Bekdaş and Nigdeli (2011), where a Harmony Search algorithm was tested in case of harmonic loading on a multi-degree-of-freedom (MDOF) primary structure.

From a general point of view, from all these studies it appears that the tuning parameters may change according to the applied dynamic loading (Bandivadekar and Jangid 2013, Marano and Greco 2011, Setareh 2001, Sun *et al.* 2014, Warnitchai and Hoang 2006). This issue is also ascertained in different works where wind (Aly 2014, Morga and Marano 2014) or earthquake (Adam and Fürtmuller 2010, Desu *et al.* 2007, Matta 2011) excitations have been considered.

Moreover, in the presence of structural damping ($\zeta_S \neq 0$), tuning formulas and relevant estimates may take quite elaborate forms (see e.g., Asami *et al.* 2002, Bakre and Jangid 2007).

The present investigation has been developed in *two main phases*, and represents a comprehensive extension of a previous preliminary study (Salvi and Rizzi 2012), with presentation of further and final analyses and results, including detailed investigations on the different relevant features involved within the optimisation process and on several post-tuning trials on reliable structural systems, for the final assessment of the effectiveness of the proposed tuning method and formulas. *First*, a wide-range numerical tuning based on an original non-linear gradient-based optimisation algorithm has been pursued. Particularly, the tuning has been developed for a TMD added on a SDOF primary structure, subjected to Harmonic or White Noise loading, applied as both input Force or base Acceleration on the primary structure, for a total of four considered loading cases (HF, HA, WNF, WNA). In the present tuning approach, the optimisation variables are taken as the frequency ratio f and the TMD damping ratio ζ_T, as a function of two free given parameters, i.e., mass ratio μ and damping ratio of the primary structure ζ_S, both fixed *a priori* within a wide range of values, including those suitable for engineering applications. *Second*, a subsequent interpolation process was carried out for all the four cases, where the optimum TMD parameters have been fitted with proper unifying analytical models, calibrated through non-linear least squares, in order to obtain final compact closed-form TMD tuning formulas, in view of effective and practical use. Such final output has been compared to that from the relevant literature, in terms of both optimum TMD parameters and achieved dynamic response reduction of the primary structure.

The present paper is structured as follows. The context represented by the structural system, the dynamic excitations and the related response indexes is firstly presented (Section 2). The tuning process, preceded by a preliminary investigation on the well-posedness of the optimisation problem (Section 3), is then stated in detail in all its features, including for the optimisation algorithm and the optimisation parameters and variables. The tuning results, i.e., the optimum TMD parameters are displayed with 3D comprehensive plots. Afterwards, in Section 4 two fitting models are presented and analysed in detail. The first Proposed Model (PM) represents the main proposal of this work, based on polynomial expressions, while the second one, a Literature-based Model (LM), acts as a sort of sparring partner model based on literature formulas for the case of an undamped main structure. Finally, the validity of PM is tested within different stages. First, an investigation in terms of both values of the optimum TMD parameters and efficiency of the related tuned TMD in reducing the dynamic response is proposed, also with comparison to well-known literature tuning formulas (Section 5). Second, post-tuning trials developed on benchmark SDOF and MDOF frame structures taken from the literature (Leung *et al.* 2008, Villaverde and Koyama 1993) are reported, including outcomes in the form of time history plots and tables, with further discussion on the level of benefit achieved by the proposed optimum TMD (Section 6). Finally, closing considerations are outlined in Section 7.

2. Structural and dynamic context

The structural system assumed as a benchmark in this study is composed of a SDOF primary structure and a TMD added on it (Fig. 1), subjected to either point force on the primary structure $F(t)$ or base acceleration $\ddot{x}_g(t)$. The primary structure is characterised by a mass m_S, a constant linear elastic stiffness k_S and a linear viscous damping coefficient c_S. The natural frequency ω_S and damping ratio ζ_S of the primary structure are defined as usual, i.e., respectively

$$\omega_S = \sqrt{\frac{k_S}{m_S}}\,, \qquad \zeta_S = \frac{c_S}{2\sqrt{k_S m_S}}\,. \tag{1}$$

Likewise, the parameters of the TMD device are an added secondary mass m_T, a constant stiffness k_T of an added elastic spring and a damping TMD coefficient c_T of an added viscous damper. As above, the TMD natural frequency ω_T and damping ratio ζ_T are respectively

$$\omega_T = \sqrt{\frac{k_T}{m_T}}\,, \qquad \zeta_T = \frac{c_T}{2\sqrt{k_T m_T}}\,. \tag{2}$$

The free TMD parameters, which play the role of tuning variables, are defined in terms of mass ratio μ, tuning frequency ratio f and TMD damping ratio ζ_T itself, as

$$\mu = \frac{m_T}{m_S}\,, \qquad f = \frac{\omega_T}{\omega_S} = \sqrt{\frac{1}{\mu}\frac{k_T}{k_S}}\,. \tag{3}$$

The tuning concept is based on the minimisation of a given dynamic response index, which basically depends on both the structural system and the applied dynamic loading. In this sense, four dynamic loading cases have been considered (Fig. 1), namely Harmonic Force on the primary structure (HF), Harmonic base Acceleration (HA), White Noise Force on the primary structure (WNF), White Noise base Acceleration (WNA).

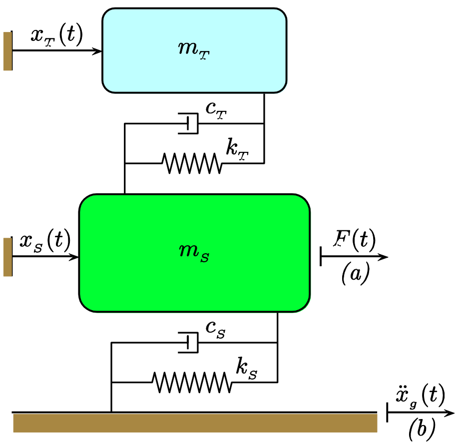


Fig. 1 Structural parameters and absolute (relative to the ground) dynamic degrees of freedom of a 2DOF mechanical system composed of a SDOF primary structure (S) equipped with an added TMD (T), subjected to: (a) point force $F(t)$, (b) base acceleration $\ddot{x}_g(t)$

The corresponding response indexes, in terms of displacement of the main structure x_S, which is taken as objective function in the optimisation process, are reported in Tables 1 and 2 below (Crandall and Mark 1963, Den Hartog 1956, Warburton 1982). Such dimensionless frequency response functions are in the form of dynamic amplification factors R for the case of harmonic loading (with excitation frequency ω, frequency ratio $g = \omega/\omega_S$ (in the present study, g = [0:0.0005:2], in MATLAB vector notation), force amplitude $\bar{F}$ or acceleration magnitude $\bar{X}_g$ (Warburton 1982) and in the form of mean square response indices N for the case of stationary Gaussian white noise loading (with constant power spectral density of the loading S_0 and variance of the displacement structural response σ_{xs}) (Warburton 1982).

Table 1 Objective functions for Harmonic loading in terms of displacement of the primary structure (Warburton 1982)

Harmonic Force (HF)	$R_F = \left\lvert \dfrac{x_S}{\bar{F}/k_S} \right\rvert = \sqrt{\dfrac{A_F^2 + B_F^2}{C^2 + D^2}}$
Harmonic Acceleration (HA)	$R_A = \left\lvert \dfrac{x_S}{\bar{X}_g/\omega_S^2} \right\rvert = \sqrt{\dfrac{A_A^2 + B_A^2}{C^2 + D^2}}$

$$A_F = f^2 - g^2, \quad B_F = 2g\zeta_T f, \quad A_A = f^2(1+\mu) - g^2, \quad B_F = 2g\zeta_T f(1+\mu)$$

$$C = \left(f^2 - g^2\right)\left(1 - g^2\right) - \mu f^2 g^2 - 4\zeta_S \zeta_T f g^2, \qquad D = 2g\zeta_T f\left[1 - g^2(1+\mu)\right] + 2\zeta_S g\left(f^2 - g^2\right)$$

Table 2 Objective functions for White Noise loading in terms of displacement of the primary structure (Warburton 1982)

White Noise Force (WNF)	$N_F = \dfrac{\sigma_{x_S}^2}{2\pi S_{0,F}\,\omega_S/k_S^2} = \dfrac{1}{4}\dfrac{I_F}{L}$
White Noise Acceleration (WNA)	$N_A = \dfrac{\sigma_{x_S}^2}{2\pi S_{0,A}/\omega_S^3} = \dfrac{1}{4}\dfrac{I_A}{L}$

$$I_F = f^4\left[\zeta_T(1+\mu)^2\right] + f^3\left[\zeta_S\mu + 4\zeta_S\zeta_T^2(1+\mu)\right] + f^2\left[-\zeta_T(2+\mu) + 4\zeta_S^2\zeta_T + 4\zeta_T^3(1+\mu)\right] + \\ + f\left(4\zeta_S\zeta_T^2\right) + \zeta_T$$

$$I_A = f^4\left[\zeta_T(1+\mu)^4\right] + f^3\left[\zeta_S\mu(1+\mu)^2 + 4\zeta_S\zeta_T^2(1+\mu)^3\right] + f^2\left[-\zeta_T(2-\mu)(1+\mu)^2 + \right. \\ \left. + 4\zeta_S^2\zeta_T(1+\mu)^2 + 4\zeta_T^3(1+\mu)^3\right] + f\left[\zeta_S\mu^2 + 4\zeta_S\zeta_T^2(1+\mu)^2\right] + \zeta_T$$

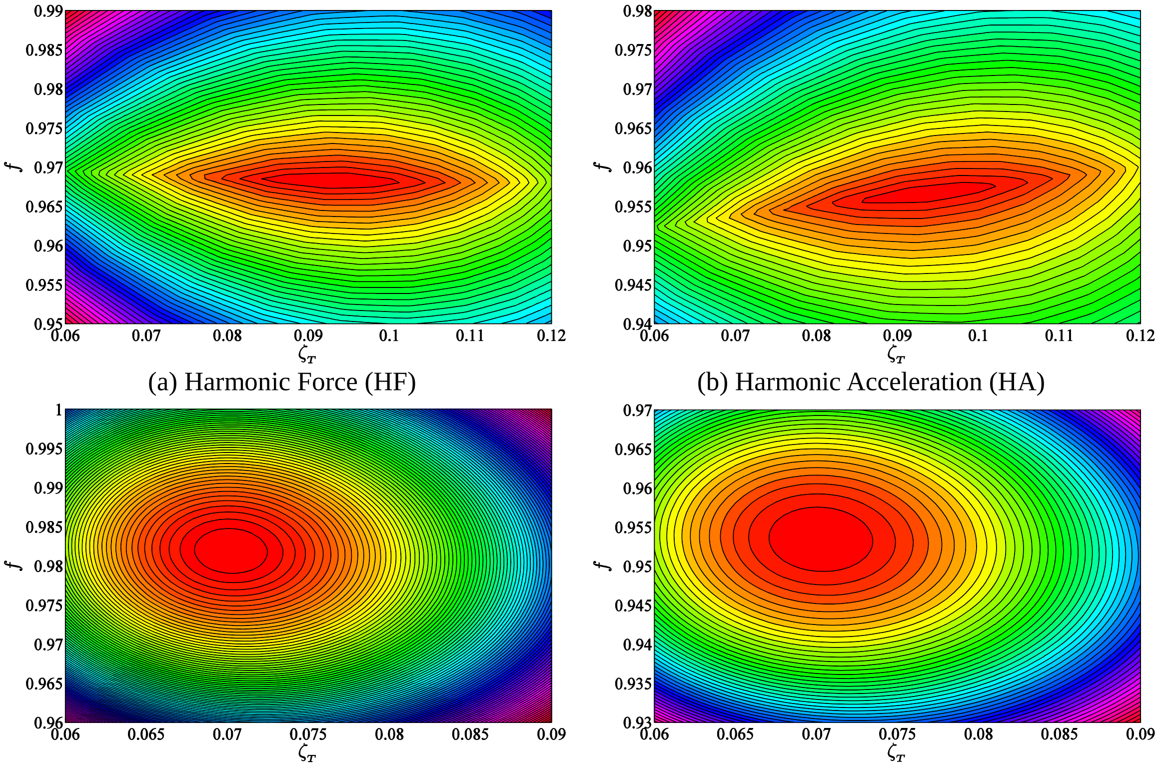
$$L = f^4\left[\zeta_S\zeta_T(1+\mu)^2\right] + f^3\left[\zeta_S^2\mu + 4\zeta_S^2\zeta_T^2(1+\mu)\right] + f^2\left[-2\zeta_S\zeta_T + 4\zeta_S^3\zeta_T + 4\zeta_S\zeta_T^3(1+\mu)\right] + \\ + f\left(\zeta_T^2\mu + 4\zeta_S^2\zeta_T^2\right) + \zeta_S\zeta_T$$

3. Tuning process

3.1 Preliminary analysis on the objective functions

A preliminary investigation on the characteristics of the selected objective functions, for the different loading cases, has been developed. The response index assumed as objective function has been evaluated nearby the expected optimum region, i.e., where it is expected to take the smallest values. An extract of the outcomes of this study is reported in Fig. 2, where the response indexes reported in Tables 1 and 2 have been evaluated for the case of $\mu = 0.02$, $\zeta_S = 0.05$, leading to the following considerations, which however would hold as well for generic values of the structural parameters.

The main feature, quite positive in view of TMD tuning, is the presence of a clear region with a global minimum of the considered function, that allows, in principle, for a robust optimisation process. Indeed, these regions of minimum denote a quite convex shape of the objective function, and therefore it is expected that the optimisation algorithm could easily get to the optimum values of the TMD parameters, corresponding to the smallest amplitude of the response index.



(a) Harmonic Force (HF)

(b) Harmonic Acceleration (HA)

(c) White Noise Force (WNF)

(d) White Noise Acceleration (WNA)

Fig. 2 Optimum region of the objective function for the considered four loading cases, for $\mu = 0.02$, $\zeta_S = 0.05$

In general, as expected the location of the global minimum is close to coordinate $f = 1$, i.e., to the resonance conditions, and with a TMD damping ratio of about $\zeta_T = 0.05$, but with some differences between the loading cases. In this sense, the objective function related to Harmonic excitations (HF, HA) exhibits a quite narrow shape as a function of f and lengthened along ζ_T, which gives more relevance to the precision of the detection of the optimum value of the frequency ratio. On the other hand, the region of minimum of the response indexes for the White Noise loading cases (WNF, WNA) display an almost equal width in all directions, i.e. it looks quite convex with respect to both TMD parameters.

Besides these specific considerations, from this investigation one could point out that the tuning process for the considered loading cases turns out to be well posed, and therefore it should be possible to provide suitable optimum TMD parameters.

3.2 Main features of the optimisation methodology

The optimisation process has been carried out for each dynamic loading case, through a dedicated non-linear gradient-based algorithm (Salvi and Rizzi 2011, 2012, Salvi *et al.* 2013, Salvi and Rizzi 2014, Salvi *et al.* 2014a, b, c), developed within a MATLAB environment (The Mathworks Inc. 2011) by the creation of *ad-hoc* numerical codes.

In this sense, different non-linear numerical methods have been preliminary tested, so that to assess their ability in finding the global minimum of the objective function and therefore to assume the most suitable algorithm for the tuning purposes. In particular, Interior Point, Trust Region and Sequential Quadratic Programming methods have been analysed in their performance, finding that all of them could easily detect the optimum region, for all the considered objective functions, proving once again the well-posedness of the present tuning problem. The final choice of a Sequential Quadratic Programming (SQP) optimisation method is mainly due to its wide and successful use in the TMD tuning literature (Bandivadekar and Jangid 2013, Randall *et al.* 1981, Rana and Soong 1998, Tsai and Lin 1993, 1994) and to its proven effectiveness in the present previous experiences (Rizzi *et al.* 2009, Salvi and Rizzi 2011).

The TMD tuning, stated as optimisation problem, can be written as follows

$$\min_{\mathbf{p}} \ \mathbf{J}(\mathbf{p}), \qquad \mathbf{l}_b \le \mathbf{p} \le \mathbf{u}_b , \tag{4}$$

where $\mathbf{p}$ is the vector of the tuning variables, $\mathrm{J}(\mathbf{p})$ is the objective function, $\mathbf{l}_b$ and $\mathbf{u}_b$ are the lower and upper bound vectors of the tuning variables. The optimisation context explained below ensured a suitable trade-off between convergence and accuracy. Also, from the numerical tests it has been noticed that the optimisation process converges promptly and smoothly.

The task of the numerical algorithm consists in the minimisation of the maximum value of the previously reported response functions (Tables 1 and 2), which obviously depend, given the fixed primary structure parameters, on the free TMD parameters. It is worth noting that the optimisation problem related to the minimisation of the frequency peak response for the cases of harmonic loading (HF and HA), as displayed e.g. in Figs. 3 and 4 for HF, can be easily reinterpreted as a Minimax principle, often presented in the literature (Brown and Singh 2011, Ioi and Ikeda 1978, Pennestrì 1998).

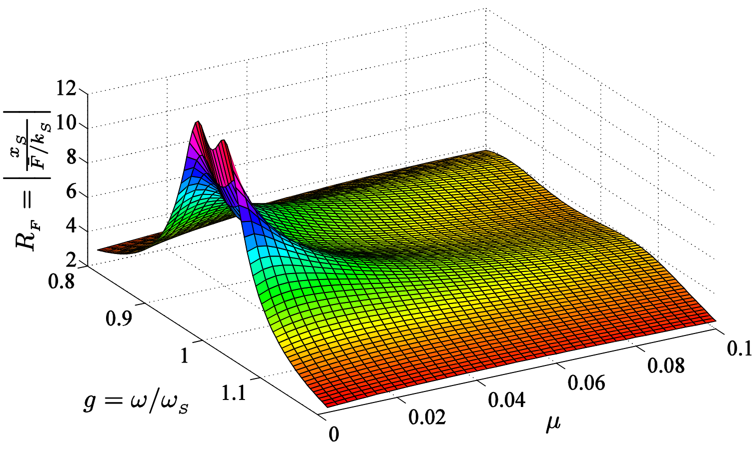


Fig. 3 Frequency response of the primary structure as a function of excitation frequency ratio g and of mass ratio μ (reported here for the HF case)

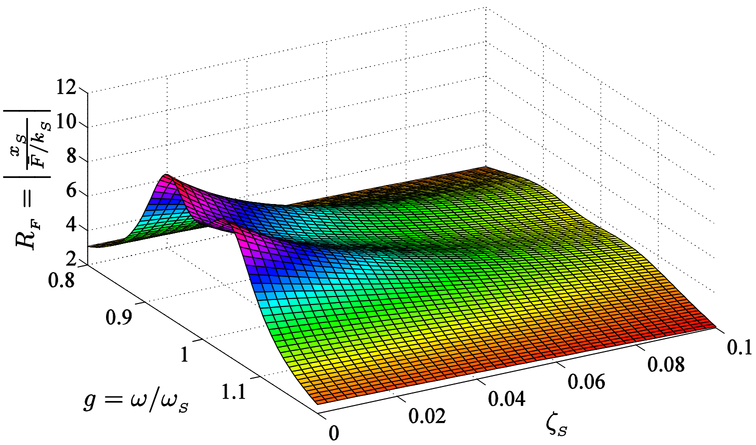


Fig. 4 Frequency response of the primary structure as a function of the excitation frequency ratio g and of primary structure damping ratio ζ_S (reported here for the HF case)

Although in principle the method would allow for the optimisation of all three TMD parameters μ, f, ζ_T, the following typical *modus operandi* has been adopted, i.e., for a given fixed mass ratio μ and a primary structure damping ratio ζ_S, whose values have been assumed within the following range (in MATLAB vector notation)

$$\boldsymbol{\mu} = [0.0025 : 0.0025 : 0.1], \qquad \zeta_S = [0 : 0.0025 : 0.05,\ 0.055 : 0.005 : 0.1], \tag{5}$$

the SQP algorithm seeks the optimum frequency ratio f^{opt} and TMD damping ratio ζ_T^{opt}, leading to best tuning, subjected to the assumed bounds reported here

$$\mathbf{l}_b = \left[0.85;\ 10^{-3}\right], \qquad \mathbf{u}_b = [1.05;\ 0.3], \tag{6}$$

Thus, f and ζ_T are here the two assumed free variables of the optimisation process, listed in (2 × 1) vector **p**. The optimisation process begins with initial values of the two variable parameters f and ζ_T, provided here through literature tuning formulas (Den Hartog 1956, Warburton 1982), referring to the case of undamped primary structures ($\zeta_S = 0$), gathered in Table 3 below.

Table 3 Optimum tuning formulas from the literature for undamped primary structures ($\zeta_S = 0$)

Loading	Author (ref.)	f^{opt}	ζ_T^{opt}
Harmonic Force	Den Hartog (1956)	$\frac{1}{1+\mu}$	$\sqrt{\frac{3}{8}\frac{\mu}{1+\mu}}$
Harmonic Acceleration	Warburton (1982)	$\frac{1}{1+\mu}\sqrt{\frac{2-\mu}{2}}$	$\sqrt{\frac{3\mu}{4(1+\mu)(2-\mu)}}$
White Noise Force	Warburton (1982)	$\frac{1}{1+\mu}\sqrt{\frac{2+\mu}{2}}$	$\sqrt{\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}}$
White Noise Acceleration	Warburton (1982)	$\frac{1}{1+\mu}\sqrt{\frac{2-\mu}{2}}$	$\sqrt{\frac{\mu(4-\mu)}{8(1+\mu)(2-\mu)}}$

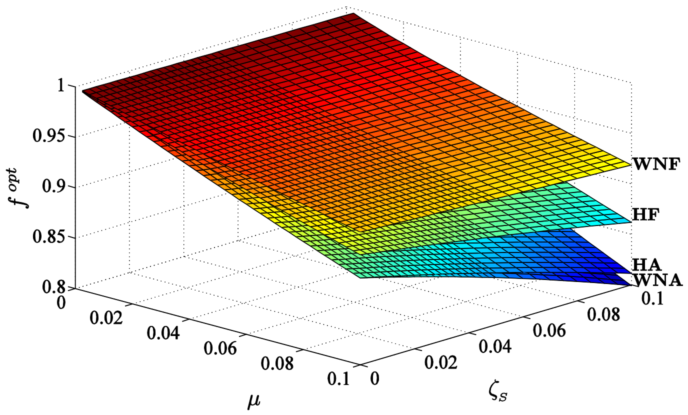


Fig. 5 Optimum frequency ratio f^{opt} from the numerical optimisation process for the four considered excitations

3.3 Tuning results and relevant considerations

The tuning results obtained from the numerical optimisation process for the four considered loading cases (HF, HA, WNF, WNA) are displayed by surface (3D) plots in Figs. 5 and 6 and by line section (2D) plots in Figs. 7 and 8, respectively in terms of optimum frequency ratio f^{opt} and TMD damping ratio ζ_T^{opt}. Also, further line section plots drawn out from the surface maps in Figs. 5 and 6 will be presented later for further quantitative analysis (Section 5).

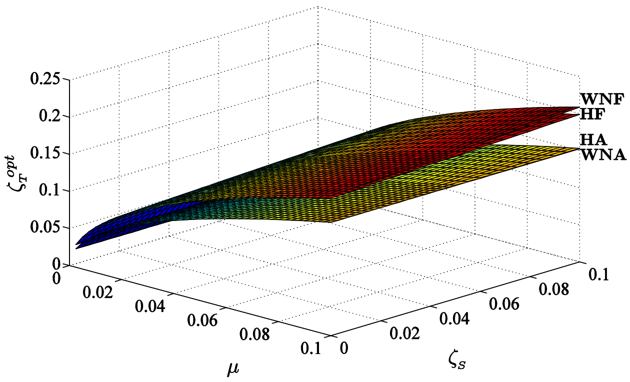


Fig. 6 Optimum TMD damping ratio ζ_T^{opt} from the numerical optimisation process for the four considered excitations

From the obtained tuning plots in Figs. 5 and 6, the following basic considerations arise (to be noted for the subsequent interpolation process). First, from Fig. 5 on f^{opt}, three main trends of f^{opt} may be mainly observed, out of the four loading cases. The higher values of f^{opt} belong to the WNF case, while lower values of f^{opt} are recovered in the HF case and lowest tight values are obtained for HA and WNA, which display an almost similar trend along both μ and ζ_S directions.

At the same time, from WNF to WNA cases, it appears that f^{opt} becomes more variable with respect to structural damping ratio ζ_S. Particularly, for the WNF case, the variability on ζ_S appears almost negligible; for the HF case just a bit more visible and almost bilinear; for HA and WNA cases more apparent and with increasing non-linearity. Also, the outcomes for HF (reference case in Den Hartog's analysis (Den Hartog 1956) of undamped structures, $\zeta_S = 0$) are almost halfway to those from WNF and HA/WNA.

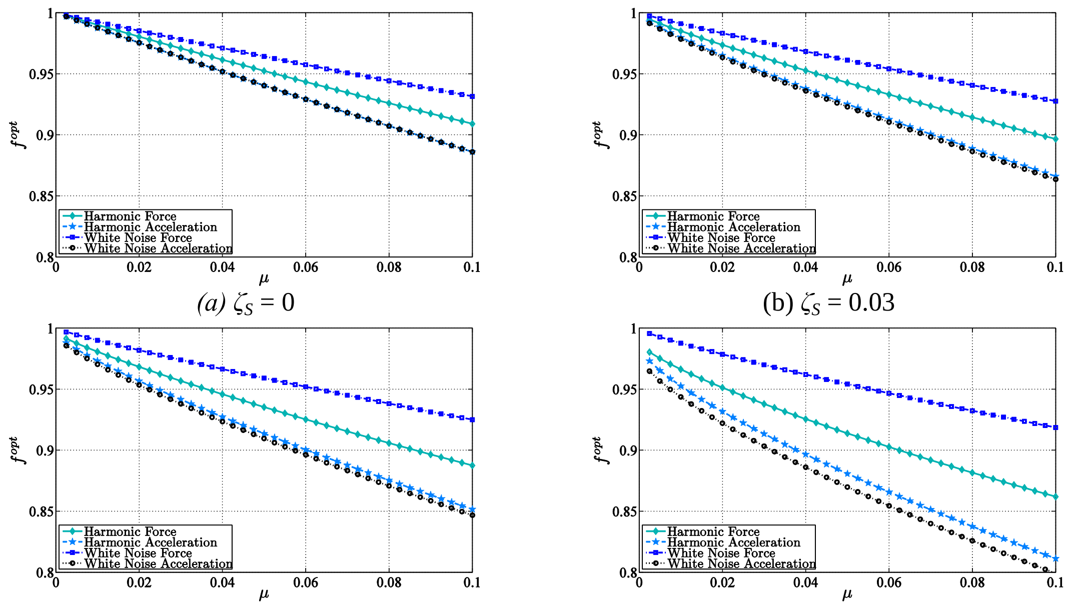
The context represented by the optimum TMD damping ratio ζ_T^{opt} in Fig. 6 is quite different, as clearly pointed out by the surface plot. Two main trends are actually displayed, which separate the cases of Harmonic and White Noise loadings (both for point Force and base Acceleration). The common and important feature of these two trends is the quite negligible variation of ζ_T^{opt} as a function of structural damping ratio ζ_S. This holds true especially for White Noise loading.

In short, from a visual comparison between the two series of plots in Figs. 5 and 6, it should be said that trends sort out as follows:

- by point of load application (Force vs. Acceleration) for f^{opt};
- by type of loading (Harmonic vs. White Noise) for ζ_T^{opt}.

Results on both f^{opt} and ζ_T^{opt} recall those obtainable for the case of undamped primary structures ($\zeta_S = 0$), as reported by the tuning formulas listed in Table 3.

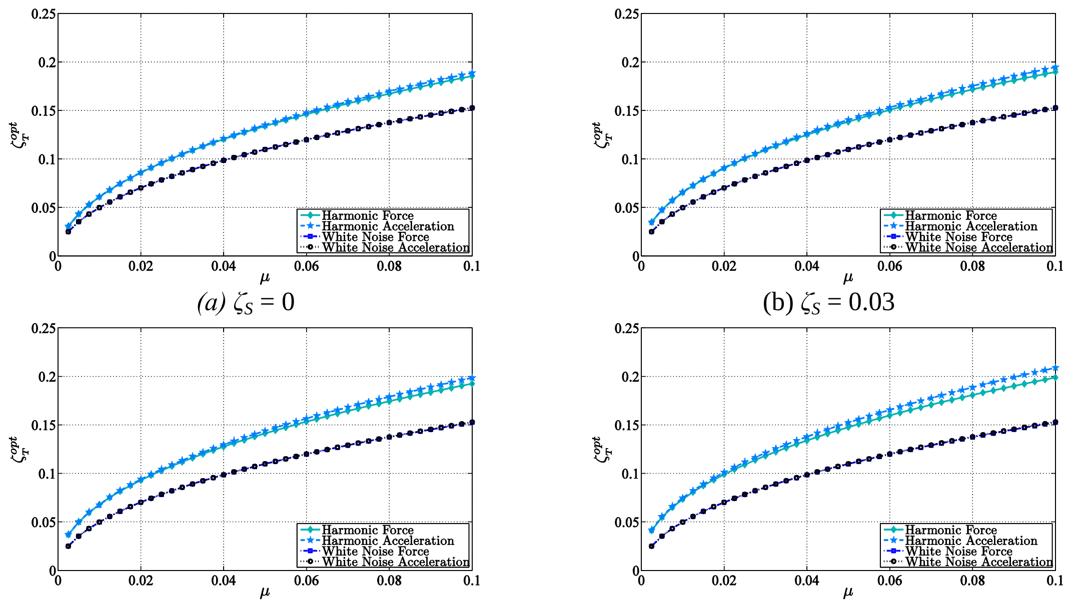
The considerations outlined above get clearer if observed in Figs. 7 and 8, which refer to the optimum frequency ratio and TMD damping ratio, respectively, with line sections extracted from Figs. 5 and 6, for particular values of the structural damping ratio, namely $\zeta_S = [0, 0.03, 0.05, 0.1]$. The optimum frequency ratio f^{opt} (Fig. 7) looks quite insensitive to variations of inherent damping for small values of this parameter, while more spread trends are recovered for $\zeta_S > 0.05$, for all the considered excitation cases. These remarks can be extended to the optimum TMD damping ratio ζ_T^{opt} (Fig. 8), with slight differences. Indeed, if for $\zeta_S = [0, 0.03]$ the plots display almost the same trends, few differences arise for a higher structural damping, but related only to harmonic excitations, since for white noise loading results appear very similar for both cases of point of application.



(a) $\zeta_S = 0$ (b) $\zeta_S = 0.03$

(c) $\zeta_S = 0.05$ (d) $\zeta_S = 0.10$

Fig. 7 Optimum frequency ratio f^{opt} from the numerical optimisation process, for different values of primary structure damping ratio ζ_S



(a) $\zeta_S = 0$ (b) $\zeta_S = 0.03$

(c) $\zeta_S = 0.05$ (d) $\zeta_S = 0.10$

Fig. 8 Optimum TMD damping ratio $\zeta_T{}^{opt}$ from the numerical optimisation process, for different values of primary structure damping ratio ζ_S

4. Tuning formulas

4.1 Fitting process

The numerically-obtained optimum TMD parameters f^{opt}, $\zeta_T^{\,opt}$ have been post-processed through a proper interpolation method, seeking the best match of the achieved results to possible unifying analytical fitting proposals that may be elaborated as described in the following. Direct 2D surface fittings on both variables μ and ζ_S have been attempted. The numerical interpolation method relies on non-linear least squares estimates based on the minimisation of the sum of the residuals between optimum and fitting values (The Mathworks Inc. 2011). The fitting model coefficients have been evaluated iteratively.

First, a starting estimate of the model coefficients is attempted. The so-obtained fitting is assessed, and its Jacobian evaluated. Then, the model coefficients are adjusted by an optimisation algorithm based on a Trust Region method (The Mathworks Inc. 2011), by improving the achieved interpolation until appropriate convergence criteria are met.

Once a best fitting is obtained with a proposed model, its reliability may be assessed by various error indices, such as the *Summed Square of Error* (SSE) or the *Root Mean Square of Error* (RMSE), and accuracy indices, such as the *R-square* correlation between optimum and fitting values (The Mathworks Inc. 2011). This last index has been reported in the results that follow. In general, the closer the R-square index to 1, the smaller and nearer to zero the error indices, and the better the fitting estimate.

4.2 Fitting models

The following contents stand as selected outcomes of a wider study, where different analytical fitting models have been considered and assessed, within the task of seeking a model apt to match the best compromise among appropriate representation of optimum results and simpleness of the tuning formulas. Specifically, *two fitting models* are presented here, for each optimised TMD parameter f^{opt} and $\zeta_T^{\,opt}$.

The *first model*, which outlines the present main proposal (validated also in following Section 5, will be denoted as *Proposed Model (PM)*. This model does not explicitly refer to the literature tuning formulas for the case of undamped primary structures (Table 3). Instead, it represents an original contribution and just refers to polynomial expressions in the variables μ, ζ_S. Conversely, the *second model*, denoted as *Literature-based Model (LM)*, follows a similar approach but basically relies on the tuning formulas in Table 3, which may be recovered exactly as particular cases for $\zeta_S = 0$.

4.3 Frequency ratio

The fitting model for frequency ratio $\tilde{f}^{opt}$ and related tuning formulas are introduced and discussed in detail in this section. Based on the direct inspection of the surface plots in Fig. 5, it considers first a fitting model expressed by a polynomial expansion in the variables μ, ζ_S, generalising a bilinear dependency, endowed with proper exponents *e, f, g, h* accounting for possible non-linearity and with additional coefficients *a*, *b*, *c*, *d* ruling the importance of each term

$$\tilde{f}^{opt} = a - b\,\mu^{e} - c\mu^{f}\,\zeta_S^{g} - d\,\zeta_S^{h}\,. \tag{7}$$

The signs appearing in Eq. (7) have been assigned by taking into account results obtained from the tuning procedure itself. Indeed, it may be noticed from Fig. 5 that the values assumed by f^{opt} are largest for small values of mass ratio μ, and decrease at increasing values of both free variables μ, ζ_S.

A first estimate of the model coefficients, for the four different loading cases (HF, HA, WNF, WNA), has been reported in Table 4, whose results lead to the following considerations. For all the loading cases, parameter *a* takes, as expected, values near 1 (meaning that a Tuned Mass Damper characterised by a small mass and attached to a lightly-damped main structure should be resonant with the structure itself). Coefficients *e* and *g* are also near 1, while coefficient *f* approaches 1/2. The remaining coefficients are quite different for each loading case but, as a general observation, it may be pointed out that very similar values have been obtained for the cases of Harmonic Acceleration and White Noise Acceleration, while Harmonic Force and White Noise Force cases follow different trends.

These outcomes lead, after further refinements, to the final tuning formulas for f^{opt} of the Proposed Model reported in Table 5, based on square-root dependencies, which display an important feature: with quite a simple model and slight changes of the coefficients among the different loading cases, it is possible to achieve a good general fitting in terms of f^{opt}. Also, the fittings for the two acceleration loading cases are unified and actually set the same.

Table 4 Optimum coefficients for the fitting model of frequency ratio f^{opt} in Eq. (7)

Loading	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	R-square
HF	1.003	0.7365	0.9475	0.8791	0.8959	0.4214	0.9794	1.936	1.0000
HA	1.003	0.9272	1.696	1.229	0.8977	0.4223	1.004	2.014	1.0000
WNF	1.002	0.5610	0.3969	0.01145	0.9018	0.5057	1.033	0.7986	1.0000
WNA	1.004	0.9219	1.787	1.659	0.8948	0.4307	0.9872	1.926	1.0000

Table 5 Final tuning formulas for the Proposed Model of frequency ratio $\tilde{f}^{opt}$

Loading	$\tilde{f}^{opt}$	R-square
Harmonic Force	$1-\sqrt{3\mu}\left(\frac{1}{2}\sqrt{\mu}+\zeta_S\right)$	0.9916
Harmonic Acceleration	$1-\sqrt{3\mu}\left(\frac{2}{3}\sqrt{\mu}+\frac{3}{2}\zeta_S\right)$	0.9947
White Noise Force	$1-\sqrt{3\mu}\left(\frac{2}{5}\sqrt{\mu}+\frac{1}{4}\zeta_S\right)$	0.9974
White Noise Acceleration	$1-\sqrt{3\mu}\left(\frac{2}{3}\sqrt{\mu}+\frac{3}{2}\zeta_S\right)$	0.9924

Table 6 Optimum coefficients for the fitting model of frequency ratio f^{opt} in Eq. (8)

Loading	a	b	c	d	e	f	g	h	R-square
HF	1.0000	0.003704	1.229	1.181	9.662	0.4632	0.9916	2.106	1.0000
HA	1.0000	0.2065	2.262	1.260	3.516	0.4989	1.003	1.987	1.0000
WNF	1.0000	0.006310	0.3795	0.1529	9.208	0.4439	0.9875	4.370	1.0000
WNA	1.0000	10.0000	2.456	1.780	5.687	0.5041	0.9943	1.947	1.0000

Table 7 Final tuning formulas for the Literature-based Model of frequency ratio $\tilde{f}^{opt}$

Loading	$\tilde{f}^{opt}$	R-square
Harmonic Force	$\frac{1}{1+\mu}\left(1-\sqrt{3\mu}\,\zeta_S\right)$	0.9969
Harmonic Acceleration	$\frac{1}{1+\mu}\sqrt{\frac{2-\mu}{2}}\left(1-\frac{3}{2}\sqrt{3\mu}\,\zeta_S\right)$	0.9988
White Noise Force	$\frac{1}{1+\mu}\sqrt{\frac{2+\mu}{2}}\left(1-\frac{1}{4}\sqrt{3\mu}\,\zeta_S\right)$	0.9998
White Noise Acceleration	$\frac{1}{1+\mu}\sqrt{\frac{2-\mu}{2}}\left(1-\frac{3}{2}\sqrt{3\mu}\,\zeta_S\right)$	0.9943

Based on such an experience, fitting formulas may be further refined by taking into account, as basis, the tuning formulas for undamped structures ($\zeta_S = 0$) listed in Table 3. This allows to recover them exactly, when structural damping ratio ζ_S is set to zero. Namely, by combining formulas in Table 3 and fittings originated from the analytical model in Eq. (7), one may re-state the fitting as

$$\tilde{f}^{opt} = f^{opt}_{ref(\zeta_S=0)} \cdot \left(a - b\,\mu^e - c\mu^f \zeta_S^g - d\,\zeta_S^h\right). \tag{8}$$

Best fitting on this further interpolation model leads then to results presented in Tables 6 and 7, which are homologous to those derived earlier (Tables 4 and 5). The final Literature-based Model tuning proposal in Table 7 unifies again cases HA and WNA.

4.4 TMD damping ratio

Easier interpretation is achieved for possible analytical fittings $\tilde{\zeta}_T{}^{opt}$ of the optimum TMD damping ratio $\zeta_T{}^{opt}$, which are approached again in two ways. Following what stated above for $\tilde{f}^{opt}$ in Eq. (7), a similar fitting model is attempted

$$\tilde{\zeta}_T{}^{opt} = a - b\,\mu^e - c\mu^f \zeta_S^g - d\,\zeta_S^h\,, \tag{9}$$

where the signs take into account the trends in Fig. 6. Table 8 reports the results of a first estimate of the model coefficients, which leads to the following observations. First, for all the loading cases, parameter a takes values near 0 (meaning that an optimum TMD should be lightly damped for small μ, ζ_S), while coefficient e approaches 1/2; parameter b results slightly lower than 3/5 and 1/2, respectively for Harmonic and White Noise loadings; the values assumed by coefficients c, d, f, g, h seem to point out that the contribution of ζ_S may be negligible overall.

Table 9 reports the final obtained Proposed Model tuning formulas for TMD damping ratio $\zeta_T^{\,opt}$, based on square-root dependencies on μ similar to those for $\tilde{f}^{opt}$, according to the following considerations. First, $\zeta_T^{\,opt}$ is weakly influenced by structural damping ratio ζ_S, basically just by mass ratio μ, especially for the case of White Noise loading. Instead, a slight dependence on ζ_S is displayed in the case of Harmonic loading.

Second, the reported tuning formulas are quite simple and display at the same time high accuracy in the considered ranges. Third, the four loading cases are unified by two common formulas, differing just by a single coefficient (besides for an additional term related to ζ_S) and coupled in two by the type of acting loading (Harmonic vs. White Noise).

Also for TMD damping ratio ζ_T^{opt}, a Literature-based Model may be further provided, as reported in Eq. (10) below, whose optimum coefficients and final obtained tuning formulas are respectively gathered in Tables 10-11, respectively

$$\tilde{\zeta}_T^{opt} = \zeta_{T,\,ref(\zeta_S=0)}^{opt} \cdot \left(a - b\mu^e - c\mu^f\,\zeta_S^g - d\zeta_S^h \right). \tag{10}$$

Table 8 Optimum coefficients for the fitting model of TMD damping ratio ζ_T^{opt} in Eq. (9)

Loading	a	b	c	d	e	f	g	h	R-square
HF	-0.01166	0.539	0.01705	0.1024	0.4371	0.202	1.045	0.9029	1.0000
HA	-0.001	0.5734	0.2994	0.001005	0.4795	0.1832	0.9973	10	1.0000
WNF	-0.005521	0.4534	0.03684	0.08062	0.4572	4.999	5.818	8.749	1.0000
WNA	-0.005614	0.4548	0.1584	0.001293	0.4579	1.245	1.485	9.587	1.0000

Table 9 Final tuning formulas for the Proposed Model of TMD damping ratio $\tilde{\zeta}_T^{opt}$

Loading	$\tilde{\zeta}_T^{opt}$	R-square
Harmonic Force	$\frac{3}{5}\sqrt{\mu}+\frac{1}{6}\zeta_S$	0.9947
Harmonic Acceleration	$\frac{3}{5}\sqrt{\mu}+\frac{1}{6}\zeta_S$	0.9987
White Noise Force	$\frac{1}{2}\sqrt{\mu}$	0.9920
White Noise Acceleration	$\frac{1}{2}\sqrt{\mu}$	0.9928

Table 10 Optimum coefficients for the fitting model of TMD damping ratio ζ_T^{opt} in Eq. (10)

Loading	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	R-square
HF	1.002	0.2549	0.1614	0.6359	10	8.196	8.12	0.8518	0.9988
HA	0.9972	0.2419	0.3477	1.079	9.344	5.298	7.73	0.9602	0.9987
WNF	1.000	0.006248	0.02605	0.1006	1.808	7.893	7.901	8.833	1.0000
WNA	1.000	0.001	1.140	1.339	1.757	1.457	1.238	4.560	1.0000

Table 11 Final tuning formulas for the Literature-based Model of TMD damping ratio $\tilde{\zeta}_T^{opt}$

Loading	$\tilde{\zeta}_T^{opt}$	R-square
Harmonic Force	$\sqrt{\frac{3}{8}\frac{\mu}{1+\mu}}(1+\zeta_S)$	0.9985
Harmonic Acceleration	$\sqrt{\frac{3\mu}{4(1+\mu)(2-\mu)}}(1+\zeta_S)$	0.9981
White Noise Force	$\sqrt{\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}}$	1.0000
White Noise Acceleration	$\sqrt{\frac{\mu(4-\mu)}{8(1+\mu)(2-\mu)}}$	1.0000

Such Literature-based Model tuning formulas in Table 11 confirm the quite independence of ζ_T^{opt} on ζ_S (in fact, just a slight contribution occurs in the case of Harmonic loading). Therefore, the formulas valid for the undamped case have proved to provide good predictions also for the case of damped primary structures.

4.5 Considerations on the fitting results

Globally, for both PM and LM fitting models the tuning proposals are characterised by a good matching of the results from the numerical optimisation, together with quite a low level of complexity.

Also, a unified way of tuning is foreseen, by switching from the various loading cases through changes of few coefficients. Main results are condensed in Tables 5 ($\tilde{f}^{opt}$) and 9 ($\tilde{\zeta}_T^{opt}$) for the PM proposal and in Tables 7 ($\tilde{f}^{opt}$) and 11 ($\tilde{\zeta}_T^{opt}$) for the LM proposal.

Particularly, the PM proposal clearly shows that the optimum TMD parameters are matched by quite simple relations. On the other hand, the LM proposal allows to match, with a simple additional term, the optimum results with good accuracy, referring also to the case of damped primary structures. The remarks pointed out above are valid specifically for TMD damping ratio ζ_T^{opt}, with optimum values that can be obtained by very simple formulas. Moreover, an interesting consideration arises for $\tilde{\zeta}_T^{opt}$ from the PM tuning formulas for the cases of White Noise loadings (Table 9), since they confirm those obtained, with a different approach, by Krenk and Høgsberg (2008).

Specifically, identical fittings are proposed, in couples, for $\tilde{f}^{opt}$ in HA and WNA cases, and for ζ_T^{opt} in HF and HA cases and in WNF and WNA cases. Thus, while for $\tilde{f}^{opt}$, formulas rather group for the point of application of the loading action (Force vs. Acceleration), for ζ_T^{opt} they rather group for the type of loading (Harmonic vs. White Noise), see trends in Figs. 5 and 6.

Obviously, validity and accuracy of the various tunings are attached to the assumed range of free variables ($0 < \mu \leq 0.1$, $0 \leq \zeta_S \leq 0.1$). However, for different ranges of μ and ζ_S, it would be possible to adjust the calibration coefficients, within the same proposed fitting models, by keeping reasonable levels of accuracy in the achieved predictions.

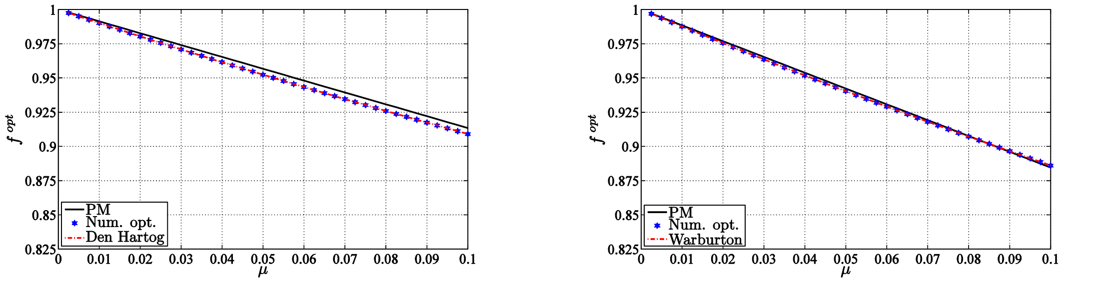
5. Comparisons to the tuning literature

In this section, the PM fitting assumption (Table 5 for $\tilde{f}^{opt}$, Table 9 for $\tilde{\zeta}_T^{opt}$), has been inspected and validated through a series of line plots, reported in the following. They concern both the optimum TMD parameters f^{opt}, ζ_T^{opt} and the optimised response functions of the primary structure (Tables 1 and 2), as a function of mass ratio μ. Two cases have been reported here, i.e. those of undamped ($\zeta_S = 0$) and damped ($\zeta_S = 0.05$) primary structures. The literature formulas adopted for comparison purposes are those in Table 3, for the case of undamped primary structures, or come from additional literature works (Asami *et al.* 2002, Bakre and Jangid 2007, Ioi and Ikeda 1978, Krenk and Høgsberg 2008, Leung and Zhang 2009, Sadek *et al.* 1997, Tsai and Lin 1993, 1994), for the case of damped main structures. Results are reported in following Figs. 9-14.

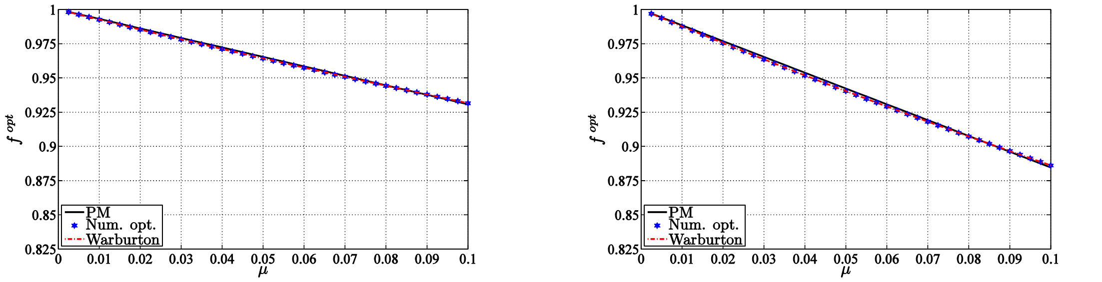
First, the case of undamped primary structures ($\zeta_S = 0$) is considered and represented in Figs. 9, 10 and 13, respectively for optimum frequency ratio f^{opt}, TMD damping ratio ζ_T^{opt} and corresponding structural response indices R, N (Tables 1 and 2).

Despite that the tuning formulas from the literature display clear non-linear trends on f^{opt}, Fig. 9 shows that the trends of frequency ratio f^{opt} may be considered almost linear, at least in the considered range of μ. This remark supports the validity of the proposed fitting formulas (Table 5), which reduce to linear functions in the case of undamped main structures. Except for the fitting case of HF loading, which shows little discrepancy with respect to classical Den Hartog's formula $f^{opt} = 1/(1+\mu)$, all the other fitting cases point out a good agreement between the proposed and the corresponding literature formulas (Fig. 9). For TMD damping ratio ζ_T^{opt} (Fig. 10), a similar situation may be noticed. A general correspondence with output from literature formulas is achieved, particularly for the case of HA, where an accurate matching is recovered. Also, Fig. 13 shows a very good agreement among all represented trends and supports a high effectiveness of the proposed TMD in reducing the primary structure response.

From Figs. 11, 12 and 14, representative of the case of a damped primary structure ($\zeta_S = 0.05$), further important considerations may be noted. First, a higher spread of the various trends is generally obtained, with respect to those of the undamped case, most of all for frequency ratio f^{opt}. In this sense, the proposed tuning seems to imply lower trends, except for the WNF case. On the other hand, the proposed TMD damping ratio ζ_T^{opt} appears to take almost the same values as those from most literature formulas, except for the case of Sadek *et al.* (1997) formula, which leads (intentionally) to quite higher values of ζ_T^{opt}.

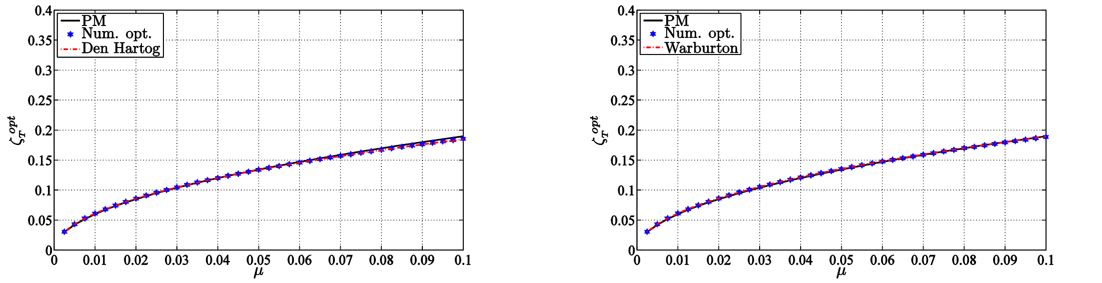


(a) Harmonic Force (b) Harmonic Acceleration

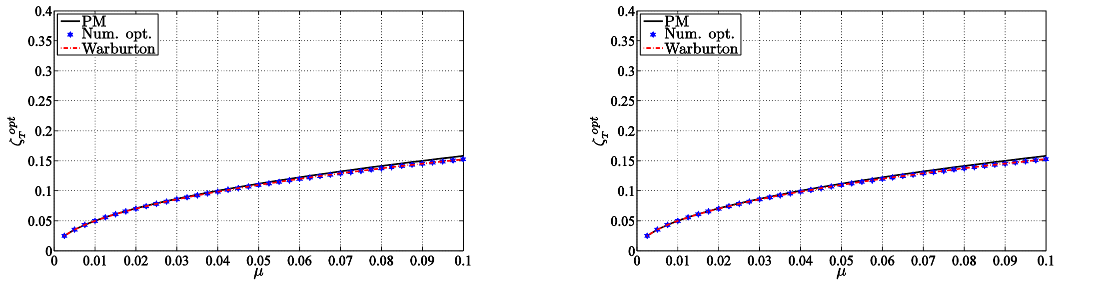


(c) White Noise Force (d) White Noise Acceleration

Fig. 9 Optimum frequency ratio f^{opt} in the case of undamped primary structure ($\zeta_S = 0$) for the considered four loading cases, compared to results from PM formulas (Table 5) and from tuning formulas in the literature

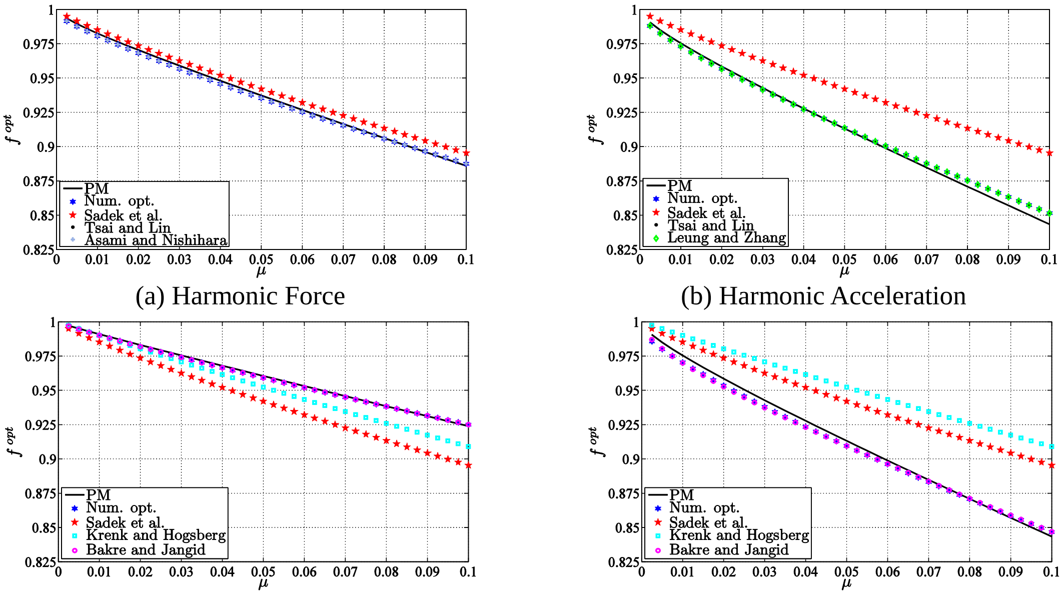


(a) Harmonic Force (b) Harmonic Acceleration



(c) White Noise Force (d) White Noise Acceleration

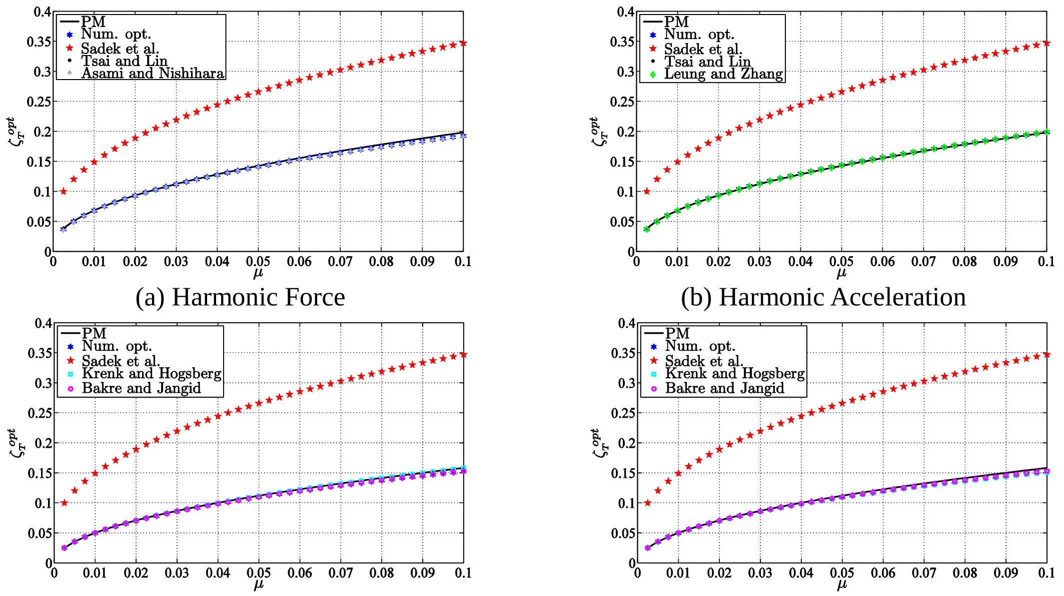
Fig. 10 Optimum TMD damping ratio ζ_T^{opt} in the case of undamped primary structure ($\zeta_S = 0$) for the considered four loading cases, compared to results from PM formulas (Table 9) and from tuning formulas in the literature



(a) Harmonic Force (b) Harmonic Acceleration

(c) White Noise Force (d) White Noise Acceleration

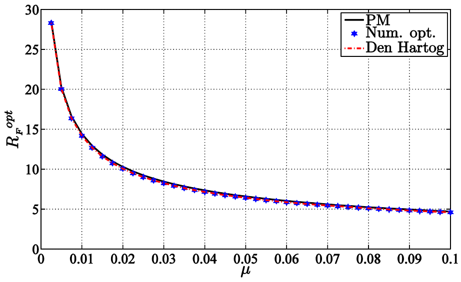
Fig. 11 Optimum frequency ratio f^{opt} in the case of damped primary structure ($\zeta_S = 0.05$) for the considered four loading cases, compared to results from PM formulas (Table 5) and from tuning formulas in the literature



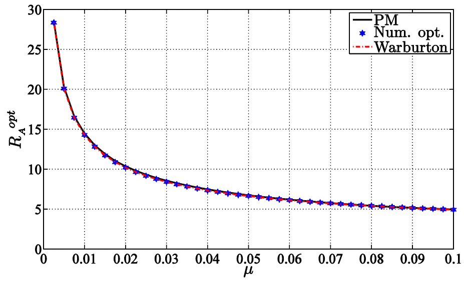
(a) Harmonic Force (b) Harmonic Acceleration

(c) White Noise Force (d) White Noise Acceleration

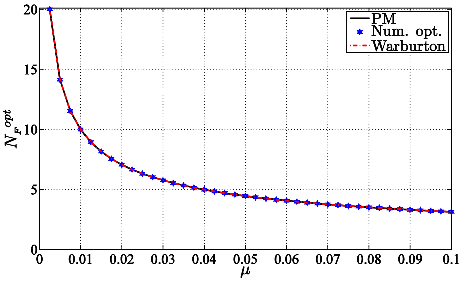
Fig. 12 Optimum TMD damping ratio ζ_T^{opt} in the case of damped primary structure ($\zeta_S = 0.05$) for the considered four loading cases, compared to results from PM formulas (Table 9) and from tuning formulas in the literature



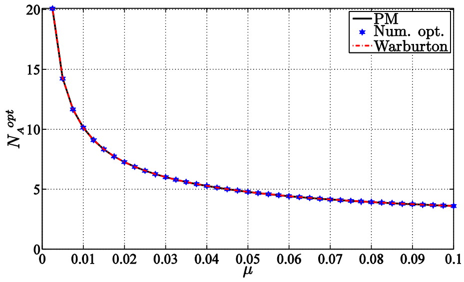
(a) Harmonic Force



(b) Harmonic Acceleration

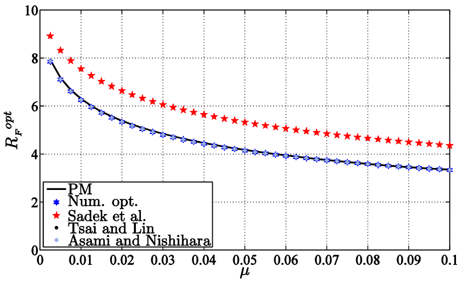


(c) White Noise Force

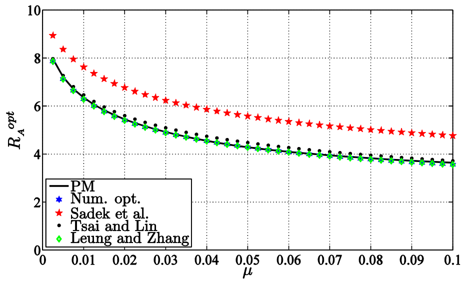


(d) White Noise Acceleration

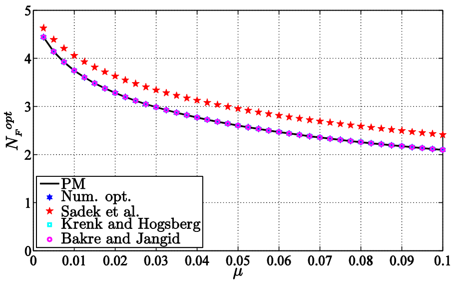
Fig. 13 Maximum response displacement of the primary structure in the case of undamped primary structure ($\zeta_S = 0$) for the considered four loading cases, compared to results from PM formulas (Tables 5 and 9) and from tuning formulas in the literature



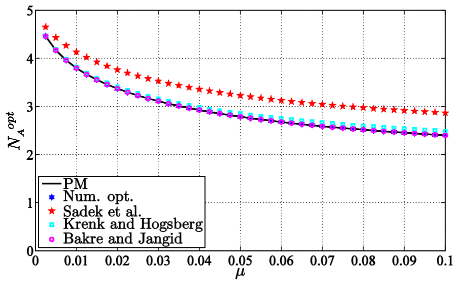
(a) Harmonic Force



(b) Harmonic Acceleration



(c) White Noise Force



(d) White Noise Acceleration

Fig. 14 Maximum response displacement of the primary structure in the case of damped primary structure ($\zeta_S = 0.05$) for the considered four loading cases, compared to results from PM formulas (Tables 5 and 9) and from tuning formulas in the literature

Finally, important observations arise from frequency response functions *R*, *N* in Fig. 14, depicting the achieved optimum response of the damped structure. Indeed, it may be noted that the proposed tuning formulas enable to achieve the most effective Tuned Mass Damper, for all the four considered loading cases.

6. Optimum TMD performance for SDOF and MDOF structures

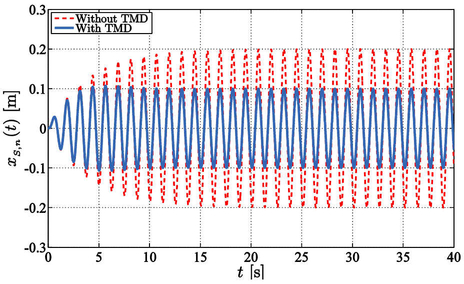
In this section, the performance of the optimum TMD, tuned following the PM approach (see Tables 5 and 9) for all the considered excitations, is further assessed with post-tuning numerical trials developed on different shear-type frames, by measuring the amount of achieved response reduction. Specifically, the considered structures are the following:

- The SDOF structure stated in Leung *et al.* (2008), with structural damping ratio $\zeta_{S,I} = 0.05$;
- The 10-storey shear-type frame proposed by Villaverde and Koyama (1993), with a dominant first bending mode and modal damping $\zeta_{S,I} = 0.02$.

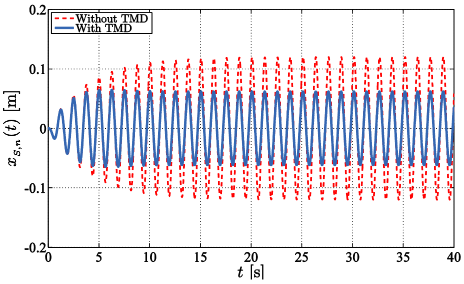
A given value of the mass ratio $\mu = 0.02$ has been assumed within the tests, which is compatible with possible practical engineering applications. The results in terms of response reduction have been gathered in Table 12 and in Figs. 15 and 16, based *inter alia* on given values of loading magnitude for all the four cases (HF, HA WNF, WNA), so that to obtain a reliable structural response amplitude. Relevant considerations on these tests have been outlined in the following.

In Table 12, the percentage reduction of the primary structure top storey displacement is listed for the considered cases, evaluated in terms of both H_∞ and H_2 norms. In general, a higher TMD performance is obtained for the 10-storey building, likely due to its lower modal damping ratio $\zeta_{S,I} = 0.02$ with respect to that of the SDOF frame ($\zeta_{S,I} = 0.05$). Indeed, about one half of response reduction is achieved for the 10-storey structure in case of white noise loading, while even a larger abatement, namely three quarters of the total displacement, occurs in case of harmonic loading. On the other hand, smaller but again remarkable response decrease is recovered for the SDOF structure, i.e., about 45% for harmonic loading and from 40% to 20% for white noise excitation. Hence, the TMD turns out more effective in case of harmonic loading; however, being the results concerning the white noise excitation a sample, quite variable outcomes would be obtained in case of different trials, since it is a random vibration loading.

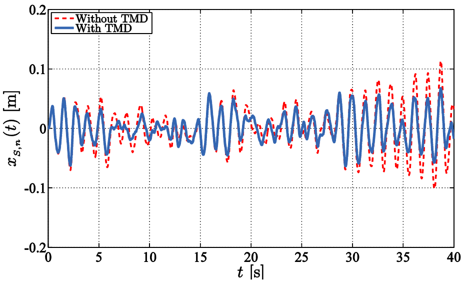
Figs. 15 and 16 further confirm these remarks. In general, the effect of the control device becomes noticeable corresponding to the beginning of the steady-state response, independently on the structure and on its damping. This fact is likely due to the pure inertial nature of the passive TMD, and perhaps could be improved only with the further addition of an active controller (Salvi *et al*. 2015a). An outstanding TMD benefit is recovered for the case of harmonic excitation, especially for the 10-storey building, whose low inherent damping emphasises the benefit of the control device. On the other hand, the plots related to the white noise loading show a few intervals where the response is clearly decreased, in particular when a large oscillation amplitude occurs, especially after a considerable excitation time; again, this fact is more observable for the 10-storey building. Hence, from these tests one may outline a substantial effectiveness of the TMD tuned with the proposed model, which could further take advantage from favourable circumstances (in the sense of the control device), such as from low structural damping, where the presence of the control device is more effective.



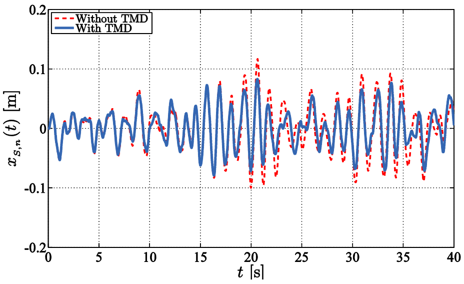
(a) Harmonic Force



(b) Harmonic Acceleration

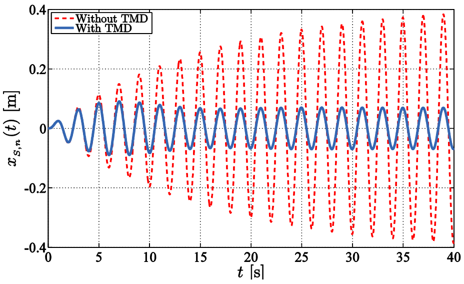


(c) White Noise Force

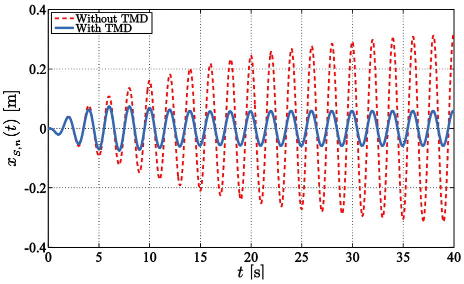


(d) White Noise Acceleration

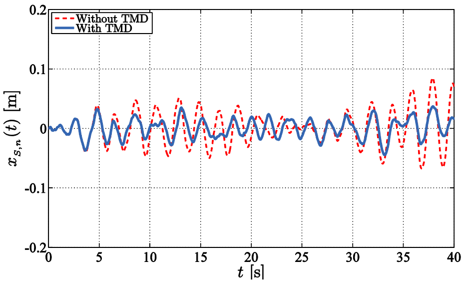
Fig. 15 Time response in terms of displacement of the SDOF primary structure (Leung *et al.* 2008), for the considered four loading cases, without and with the TMD optimised with PM formulas (Tables 5 and 9), with $\mu = 0.02$



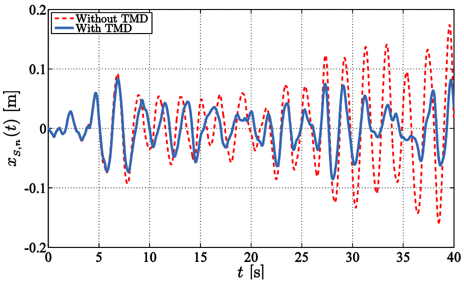
(a) Harmonic Force



(b) Harmonic Acceleration



(c) White Noise Force



(d) White Noise Acceleration

Fig. 16 Time response in terms of top-storey displacement of the 10-storey primary structure (Villaverde and Koyama 1993), for the considered four loading cases, without and with the TMD optimised with PM formulas (Tables 5 and 9), with $\mu = 0.02$

Table 12 Percentage response reduction for the considered primary structures and excitations, with TMD optimum coefficients for the fitting model of TMD damping ratio ζ_T^{opt} in Eq. (10)

Primary structure (ref.)	$\Delta\|x_{S,n}\|_\infty$ [%]				$\Delta\|x_{S,n}\|_2$ [%]			
	HF	HA	WNF	WNA	HF	HA	WNF	WNA
1-storey (Leung *et al.* 2008)	46.36	45.96	40.50	29.03	46.77	45.23	31.20	20.01
10-storey (Villaverde and Koyama 1993)	76.46	76.39	47.24	50.88	75.23	74.62	44.58	46.16

7. Conclusions

This work dealt with the attempt of unification of different tuning formulas, with relevant consideration of the structural damping, based on a common polynomial model, towards obtaining the best TMD parameters, in the framework of four typical ideal loading cases (Harmonic or White Noise, applied as point Force or base Acceleration).

Optimum TMD parameters f^{opt} and ζ_T^{opt} are derived first, through a numerical SQP optimisation approach, based on the direct minimisation of the associated displacement response functions. The results of the numerical optimisation process displayed three main trends for frequency ratio f^{opt} and two main trends for TMD damping ratio ζ_T^{opt}. From a general point of view, it appears that the trends of the optimum frequency ratio gather by point of application of the dynamic loading, i.e., point force vs. base acceleration (particularly, quite similar results have been obtained for the two cases of base acceleration). This should highlight the main effect of TMD insertion. Conversely, the trends of the optimum TMD damping ratio appear to gather according to the type of loading, i.e., harmonic vs. white noise loading. A further difference between the two optimum TMD parameters regards the variability of the tuning results on the primary structure damping ratio ζ_S, which appears to be higher for f^{opt} and lower, and almost negligible, for ζ_T^{opt}.

The so-obtained results have been then interpolated by various analytical fitting models and reference ones are obtained, in view of deriving simple tuning formulas that would resemble the optimum numerical tuning results. In particular, the main proposal of this study consisted in a Proposed Model (PM) based on a pure polynomial fitting model, whose coefficients are refined and shaped to meet almost exactly the trends outlined by the numerical optimisation process. Indeed, the analysis carried out proved that the deviation of the fitted curves with respect to the numerical results is quite negligible. The result of this process is a series of tuning formulas, specific for each considered loading, that exhibit a similar and very simple form, which in principle should favour their assumption for a general TMD tuning. A complementary Literature-based Model (LM) is also presented, based on the refinement and the extension of literature formulas for the case of undamped primary structure, which could represent a sort of alternative solution for an efficient TMD optimisation.

The outcomes obtained with PM have been then compared to those from typical tuning formulas in the literature, in terms of both optimum TMD parameters and reduction of the assigned response index, for each loading case, in both cases of undamped and damped main structures. Such comparisons have led to the following considerations. First, for all the considered loading cases, both optimum TMD parameters obtained with PM were found to be in line with main trends outlined by much involved literature tunings. Furthermore, it may be noted that the proposed tuning leads to the design of the most effective Tuned Mass Damper, since the trends of structural response are always the lowest, for all the loading cases.

Finally, several post-tuning trials have been developed on SDOF and MDOF shear-type frame buildings taken from the literature, so that to measure the benefit achieved with the proposed optimum TMD, specifically in terms of displacement response reduction in time. The outcomes of this investigation provided convincing indications, represented by a significant abatement of dynamic response, even in case of quite high structural damping. This latter parameter has been shown to influence the TMD performance. The achieved benefit from the control device is however positive, and even outstanding, for lower levels of inherent damping. These results, in principle, promote both the assumption of adopting the TMD as a vibration control device and its tuning with the proposed formulas (PM) put forward in the present paper.

As a conclusion, a study on a comprehensive tuning of Tuned Mass Damper has been attempted. Four characteristic dynamic loadings have been considered. Simple TMD tuning formulas have been derived and calibrated, which result highly predictive of the true numerical optimum results and gather all loading cases through a common model. Such formulas have proved to allow for the tuning of a best effective TMD, as assessed by comparisons to various literature proposals and by relevant post-tuning trials on various frame structures.

These outcomes should support the proposed tuning as an easy reference for a wide and general design of Tuned Mass Damper devices. Further investigations would be scheduled to enquire the validity of the present tuning proposals in the context of different real loading scenarios, as e.g., those of wind force and earthquake excitation.

Acknowledgements

The authors would like to acknowledge public research funding from *"Fondi di Ricerca d'Ateneo ex 60%"* and a former ministerial doctoral grant at the University of Bergamo, School of Engineering (Dalmine), MITIMM Doctoral School.

References

Adam, C. and Fürtmuller, T. (2010), "Seismic performance of tuned mass dampers", (Eds., H. Irschik, M. Krommer, K. Watanabe, T. Furukawa), *Mechanics and Model-Based Control of Smart Materials and Structures*, Springer-Verlag, 1st Ed., 11-18.

Aly, A.M. (2014), "Vibration control of high-rise buildings for wind: a robust passive and active tuned mass damper", *Smart Struct. Syst.*, **13**(3), 473-500.

Asami, T., Nishihara, O. and Baz, A.M. (2002), "Analytical solutions to H_∞ and H_2 optimization of dynamic vibration absorber attached to damped linear systems", *J. Vib. Acoust. - ASME*, **124**(2), 284-295.

Bandivadekar, T.P. and Jangid, R.S. (2013), "Optimization of multiple tuned mass dampers for vibration control of system under external excitation", *J. Vib. Control*, **19**(12), 1854-1871.

Bakre, S.V. and Jangid, R.S. (2006), "Optimum parameters of tuned mass damper for damped main system", *Struct. Control Health.*, **14**(3), 448-470.

Bekdaş, G. and Nigdeli, S.M. (2011), "Estimating optimum parameters of tuned mass dampers using harmony search", *Eng. Struct.*, **33**(9), 2716-2723.

Bisegna, P. and Caruso, G. (2012), "Closed-form formulas for the optimal pole-based design of tuned mass

dampers", *J. Sound Vib.*, **331**(10), 2291-2314.
Brock, J.E. (1946), "A note on the damped vibration absorber", *J. Appl. Mech. - ASME*, **13**(4), A-284.
Brown, B. and Singh, T. (2011), "Minimax design of vibration absorbers for linear damped systems", *J. Sound Vib.*, **330**(11), 2437-2448.
Crandall, S.H. and Mark,W.D. (1963), *Random Vibration in Mechanical Systems*, Academic Press, 1st Ed..
Den Hartog, J.P. (1956), *Mechanical Vibrations*, McGraw-Hill, 4th Ed..
Desu, N.B., Dutta, A. and Deb, S.K. (2007), "Optimal assessment and location of tuned mass dampers for seismic response control of a plan-asymmetrical building", *Struct. Eng. Mech.*, **26**(4), 459-477.
Frahm, H. (1911), *Device for damping vibrations of bodies*, U.S.Patent No.989958.
Fujino, Y. and Abé, M. (1993), "Design formulas for tuned mass dampers based on a perturbation technique", *Earthq. Eng. Struct. D.*, **22**(10), 833-854.
Ghosh, A. and Basu, B. (2007), "A closed-form optimal tuning criterion for TMD in damped structures", *Struct. Control Health.*, **14**(4), 681-692.
Hahnkamm, E. (1933), "Die Dämpfung von Fundamentschwingungen bei veränderlicher Erregerfrequenz", *Ing.-Archiv*, **4**, 192-201.
Ioi, T. and Ikeda, K. (1978), "On the dynamic vibration damped absorber of the vibration system", *Bull. Jap. Soc. of Mech. Eng.*, **21**(151), 64-71.
Krenk, S. (2005), "Frequency analysis of the Tuned Mass Damper", *J. Appl. Mech.- ASME*, **72**(6), 936-942.
Krenk, S. and Høgsberg, J. (2008), "Tuned mass absorbers on damped structures under random load", *Probabilist. Eng. Mech.*, **23**(4), 408-415.
Leung, A.Y.T. and Zhang, H. (2009), "Particle Swarm Optimization of tuned mass dampers", *Eng. Struct.*, **31**(3), 715-728.
Leung, A.Y.T., Zhang, H., Cheng, C.C. and Lee, Y.Y. (2008), "Particle Swarm Optimization of TMD by non-stationary base excitation during earthquake", *Earthq. Eng. Struct. D.*, **37**(9), 1223-1246.
Liu, K. and Liu, J. (2005), "The damped dynamic vibration absorbers: revisited and new results", *J. Sound Vib.*, **284**(3-5),1181-1189.
Marano, G.C. and Greco, R. (2011), "Optimization criteria for tuned mass dampers for structural vibration control under stochastic excitation", *J. Vib. Control*, **17**(5), 679-688.
Matta, E. (2011), "Performance of tuned mass dampers against near-field earthquakes", *Struct. Eng. Mech.*, **39**(5), 621-642.
Morga, M. and Marano, G.C. (2014), "Optimization criteria of TMD to reduce vibrations generated by the wind in a slender structure", *J. Vib. Control*, **20**(16), 2404-2416.
Ormondroyd, J. and Den Hartog, J.P. (1928), "The theory of the dynamic vibration absorber", *J. Appl. Mech. - ASME*, **50**(7), 9-22.
Pennestrì, E. (1998), "An application of Chebyshev's Min-Max criterion to the optimal design of a damped dynamic vibration absorber", *J. Sound Vib.*, **217**(4), 757-765.
Rana, R. and Soong, T.T. (1998), "Parametric study and simplified design of tuned mass dampers", *Eng. Struct.*, **20**(3), 193-204.
Randall, S.E., Halsted, D.M. and Taylor, D.L. (1981), "Optimum vibration absorbers for linear damped systems", *J. Mech. Des. - ASME*, **103**(4), 908-913.
Rizzi, E., Brescianini, D. and Scotti, M. (2009), "On the optimal tuning of Tuned Mass Dampers in structural systems", *Proceedings of the ECCOMAS Thematic Conference - 2nd Int. Conf. on Computational Methods in Structural Dynamics and Earthquake Engineering (COMPDYN 2009)*, Rhodes, Greece, 22-24 June 2009, (Eds., M. Papadrakakis, N.D. Lagaros, M. Fragiadakis), 24 pages.
Sadek, F., Mohraz, B., Taylor, A.W. and Chung, R.M. (1997), "A method of estimating the parameters of Tuned Mass Dampers for seismic applications", *Earthq. Eng. Struct. D.*, **26**(6), 617-635.
Salvi, J. and Rizzi, E. (2011), "Minimax optimization of Tuned Mass Dampers under seismic excitation", *Proceedings of the 8th Int. Conf. on Structural Dynamics (EURODYN 2011)*, Leuven, Belgium, 4-7 July 2011, (Eds., G. De Roeck, G. Degrande, G. Lombaert, G. Müller), Book of Abstract, ISBN: 978-90-760-1931-4, p. 68; CD-ROM Proceedings, p. 1892-1899, 8 pages, 2011.
Salvi, J. and Rizzi, E. (2012), "A numerical approach towards best tuning of Tuned Mass Dampers",

Proceedings of the 25th Int. Conf. on Noise and Vibration Engineering (ISMA 2012), Leuven, Belgium, 17–19 September 2012, (Eds., P. Sas, D. Moens, S. Jonckheere), Book of Abstracts, ISBN: 978-90-738-0289-6, p. 141; CD-ROM Proceedings. p. 2419-2434, 16 pages.

Salvi, J., Rizzi, E., Rustighi, E. and Ferguson, N.S. (2013), "Analysis and optimisation of Tuned Mass Dampers for impulsive excitation", *Proceedings of the 11th Int. Conf. on Recent Advances in Structural Dynamics (RASD 2013)*, Pisa, Italy, July 1-3, 2013, (Ed., E. Rustighi), Book of Abstracts, p. 64, CD-ROM Proceedings, ISBN: 9780854329649, Paper ID 1002, p. 1-15, 15 pages.

Salvi, J., Rizzi, E. and Gavazzeni, M. (2014), "Analysis on the optimum performance of Tuned Mass Damper devices in the context of earthquake engineering", *Proceedings of the 9th International Conference on Structural Dynamics (EURODYN 2014)*, Porto, Portugal, 30 June–2 July 2014, A. Cunha, (Eds., E. Caetano, P. Ribeiro, G. Müller), Book of Abstracts, ISBN: 978-972-752-166-1, p. 85; CDROM Proceedings, ISSN: 2311-9020, ISBN: 978-972-752-165-4, p. 1729-1736, 8 pages, 2014.

Salvi, J. and Rizzi, E. (2015), "Optimum tuning of Tuned Mass Dampers for frame structures under earthquake excitation", *Struct. Control Health.*, doi:10.1002/stc.1710, **22**(4), 707-715.

Salvi, J., Rizzi, E., Rustighi, E. and Ferguson, N.S. (2015), "On the optimisation of a hybrid Tuned Mass Damper for impulse loading", *Smart Mater. Struct.*, doi:10.1088/0964-1726/24/8/085010, **24**(8)(2015)085010, 15 pages.

Salvi, J., Rizzi, E., Rustighi, E. and Ferguson, N.S. (2015), "Optimum tuning of passive Tuned Mass Dampers for the mitigation of pulse-like responses", Submitted for publication.

Setareh, M. (2001), "Use of semi-active mass dampers for vibration control of force-excited structures", *Struct. Eng. Mech.*, **11**(4), 341-356.

Sun, C., Nagarajaiah, S. and Dick, A.J. (2014), "Family of smart tuned mass dampers with variable frequency under harmonic excitations and ground motions: closed-form evaluation", *Smart Struct. Syst.*, **13**(2), 319-341.

The MathWorks Inc. (2011), *MATLAB User's Guide and Optimization Toolbox*, USA.

Thompson, A.G. (1981), "Optimum tuning and damping of a dynamic vibration absorber applied to a force excited and damped primary system", *J. Sound Vib.*, **77**(3), 403-415.

Tigli, O.F. (2012), "Optimum vibration absorber (tuned mass damper) design for linear damped systems subjected to random loads", *J. Sound Vib.*, **331**(13), 3035-3049.

Tributsch, A. and Adam, C. (2012), "Evaluation and analytical approximation of Tuned Mass Damper performance in an earthquake environment", *Smart Struct. Syst.*, **10**(2),155-179.

Tsai, H.C. and Lin, G.C. (1993), "Optimum tuned-mass dampers for minimizing steady-state response of support-excited and damped systems", *Earthq. Eng. Struct. D.*, **22**(11), 957-973.

Tsai, H.C. and Lin, G.C. (1994), "Explicit formulae for optimum absorber parameters for force-excited and viscously damped systems", *J. Sound Vib.*, **176**(5), 585-596.

Villaverde, R. and Koyama, L.A. (1993), "Damped resonant appendages to increase inherent damping in buildings", *Earthq. Eng. Struct. D.*, **22**(6), 491-507.

Warburton, G.B. (1982), "Optimum absorber parameters for various combinations of response and excitation parameters", *Earthq. Eng. Struct. D.*, **10**(3), 381-401.

Warnitchai, P. and Hoang, N. (2006), "Optimal placement and tuning of multiple tuned mass dampers for suppressing multi-mode structural response", *Smart Struct. Syst.*, **2**(1), 1-24.

Wong, W.O. and Cheung, Y.L. (2008), "Optimal design of a damped dynamic vibration absorber for vibration control of structure excited by ground motion", *Eng. Struct.*, **30**(1), 282–286.

Zilletti, M., Elliott, S.J. and Rustighi, E. (2012), "Optimisation of dynamic vibration absorbers to minimise kinetic energy and maximise internal power dissipation", *J. Sound Vib.*, **331**(18), 4093-4100.

CC