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Nonlinear responses of an arbitrary FGP circular plate resting on the Winkler-Pasternak foundation

Mohammad Arefi[*1] and M.N.M. Allam[2a]

[1]*Department of Solid Mechanics, Faculty of Mechanical Engineering, University of Kashan, Kashan 87317-51167, Iran*
[2]*Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

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Abstract. This paper presents nonlinear analysis of an arbitrary functionally graded circular plate integrated with two functionally graded piezoelectric layers resting on the Winkler-Pasternak foundation. Geometric nonlinearity is considered in the strain-displacement relation based on the Von-Karman assumption. All the mechanical and electrical properties except Poisson's ratio can vary continuously along the thickness of the plate based on a power function. Electric potential is assumed as a quadratic function along the thickness direction. After derivation of general nonlinear equations, as an instance, numerical results of a functionally graded material integrated with functionally graded piezoelectric material obeying two different functionalities is investigated. The effect of different parameters such as parameters of foundation, non homogenous index and boundary conditions can be investigated on the mechanical and electrical results of the system. A comprehensive comparison between linear and nonlinear responses of the system presents necessity of this study. Furthermore, the obtained results can be validated by using previous linear and nonlinear analyses after removing the effect of foundation.

Keywords: Winkler-Pasternak foundation; nonlinear responses; functionally graded material; piezoelectric; circular plate

1. Introduction

Application of material for different environments and barring the opposite conditions was one of frequently encountered problems for engineers and material scientists. A group of material scientist in Japan has proposed solution of this problem. They proposed materials with variable properties. The property of these materials can be changed continuously and gradually along the thickness direction. These materials named FGM's. For application of these materials in electromechanical systems as sensor or actuator, structure made of these materials can be integrated with piezoelectric layers. Pierre and Jacques Curie have presented the piezoelectric effect scientifically in 1880. Piezoelectric structures are very applicable in the industrial systems as sensor or actuator in various geometries such as plates, cylinders and shells. Derivation of the relation between the applied loads and displacement in a piezoelectric structure such as circular

*Corresponding author, Assistant Professor, E-mail: arefi63@gmail.com, arefi@kashanu.ac.ir
[a] Professor, E-mail: mallam@mans.edu.eg

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plate may be considered as an important subject especially when the plate undergoes large deformation. A foundation has important effect on the responses of the electromechanical system and needs more investigations.

Woo and Meguid (2001) investigated the nonlinear analysis of a functionally graded plates and shallow shells. They proposed an analytical solution for the coupled large deflection of the FG plates and shallow shells. Von Karman theory is employed for considering the large transverse deflection. GhannadPour and Alinia (2006) investigated the large deflection analysis of a rectangular FG plate based on the Von Karman theory for simulation of the large deflection. The solution was obtained using minimization of the total potential energy with respect to unknown parameters. Hui-Shen (2007) considered the nonlinear response of a FG plate due to heat conduction. It was assumed that the plate to be shear deformable. Higher order shear deformation theory was employed for analysis of the problem. Classical plate theory (CPT) has been used to study free axisymmetric vibrations of orthotropic annular plates with variable thickness resting on a Pasternak-type elastic foundation (Gupta *et al.* 2008).

Huang *et al.* (2008) presented exact solutions for functionally graded thick plates resting on Winkler–Pasternak elastic foundations using the three-dimensional theory of elasticity. The effects of stiffness of the foundation, loading and non-homogenous index on mechanical responses of the plates were investigated. Ebrahimi and Rastgo (2008) investigated the free vibration of smart circular thin FG plate using the classical plate theory. The power function is employed for simulation of the material properties distribution along the thickness direction. Plate was composed of a FG layer and two FGP layers at top and bottom of that. The obtained results were verified by those obtained results from three dimensional finite element analyses. Alinia and GhannadPour (2009) investigated the large deflection analysis of a rectangular FG plate with logarithmic distribution of material properties. Sarfaraz Khabbaz *et al.* (2009) investigated the nonlinear analysis of FG plates under pressure based on the higher-order shear deformation theory. The first and higher order shear deformation theories were employed to investigate the large deflection of FG plate. The effect of the thickness and non-homogenous index were investigated on the distribution of the displacements and stresses. Khoshgoftar *et al.* (2009) investigated thermo elastic analysis of a FGP cylinder under pressure. It was assumed that all mechanical and electrical properties except Poisson ratio vary continuously along the thickness direction based on a power function. Arefi and Rahimi (2010) studied thermo-elastic analysis of a functionally graded cylindrical shell using first order shear deformation theory.

Behravan *et al.* (2010) presented static analysis of functionally graded annular plate resting on elastic foundation with various boundary conditions by using a semi-analytical approach. Three dimensional theory of elasticity has been used for derivation of governing equations. The effect of the nonhomegenous index, the elastic foundation coefficients (Winkler-Pasternak), the thickness to radius ratio and edge supports have been discussed on the bending behavior of the FGM annular plate. Alipur and Shariyat (2010) employed first-order shear-deformation theory to formulate bending and stress analyses of two-directional functionally graded (FG) circular plates resting on non-uniform two-parameter foundations. For increasing the accuracy of the results, they accounted shear stresses using three dimensional theory of elasticity. Benyoucef *et al.* (2010) presented static analysis of simply supported functionally graded plates subjected to a transverse uniform load resting on an elastic foundation. The material properties of the plate are assumed to be graded in the thickness direction according to a simple power-law distribution in terms of volume fractions of material constituents. The foundation is modeled as a two-parameter Pasternak-type foundation. Ait Atmane *et al.* (2010) studied Free vibration analysis of simply supported functionally graded plates (FGP) resting on a Winkler–Pasternak elastic foundation by a new higher shear deformation theory. The equation of motion for FG rectangular plates resting on elastic foundation was obtained through Hamilton's principle. Zenkour *et al*. (2011a) studied the bending response of an orthotropic rectangular plate resting on two-parameter elastic foundations. Analytical solutions for deflection and stresses were developed by means of the simple and mixed first-order shear deformation plate theories. The results were compared with those obtained in the literature using three-dimensional elasticity theory or higher-order shear deformation plate theory to check the accuracy of the simple and mixed first-order shear deformation theories.

Rahimi *et al.* (2011) investigated a functionally graded piezoelectric rotating cylinder as mechanical sensor under pressure and thermal loads analytically for evaluation of angular velocity of rotary devices. Zenkour (2011b) investigated on the bending response of simply supported orthotropic plates. The mixed first-order shear deformation plate theory (MFPT) was employed to study the bending responses. The foundation was modeled using Winkler elastic foundation. Zenkour *et al*. (2013a) presented bending response of an orthotropic rectangular plate resting on two-parameter elastic foundations under thermo-mechanical loadings using a unified shear deformation plate theory. Zenkour *et al.* (2013b) also studied bending responses of a functionally graded plate resting on elastic foundations and subjected to a transverse mechanical load. A relationship between the simple and mixed first-order transverse shear deformation theories was presented as the main result of that study. The obtained results using both simple and mixed first-order theories were compared with them. Arefi *et al.* (2011), Arefi and Rahimi (2011, 2012a, b, c, d, e 2014a, b), Arefi (2013) and Arefi and Nahas (2014) have presented some linear and nonlinear analysis of functionally graded piezoelectric structures.

This paper tries to present nonlinear electromechanical responses of an arbitrary functionally graded piezoelectric circular plate resting on the Winkler-Pasternak foundation. This problem has been considered with general distribution of material properties along the thickness direction. In the other word, no functionality has been considered throughout the derivation of final equations. The author can solve this problem for different functionalities without loss of generality. The effect of non-homogeneity and Winkler-Pasternak foundation is considered on the responses of the system. To understand the effect of nonlinear analysis on the responses of the system, a comparison between linear and nonlinear responses is performed.

2. Formulation

This paper tries to presents general formulation for studying the nonlinear behavior of an arbitrary functionally graded circular plate integrated with smart layers. The circular plate is constrained with Winkler-Pasternak foundation.

General functionality along the thickness direction is considered for this problem. This generality can be considered using a function $\mathrm{F(z)}$ as follows

$$P(z) = P_0 \mathrm{F(z)} \tag{1}$$

where, $P(z)$ is any mechanical and electrical property except Poisson ratio and P_0 is same property at known surface such as inner or outer surfaces. Nonlinear electromechanical analysis of a FGM circular plate embedded with two smart layers at top and bottom is analyzed using the CPT. Based on the CPT, the displacement of every layer is defined by two terms including the displacement of mid-plane and rotation about the mid-plane (Ugural 1981, Ebrahimi and Rastgo 2008, Arefi and Rahimi 2012). Therefore, we will have

$$\left.\begin{aligned} u(r,z) &= u_0(r) - z\frac{dw_0(r)}{dr} \\ w(r,z) &= w_0(r) \end{aligned}\right\} \tag{2}$$

where, u_0, w_0 are displacement components of the plate mid-plane ($z=0$) and $\vec{u}=(u,0,w)$ is displacement vector. Considering Eq. (2), the nonlinear strain components are obtained as (Lai *et al*. 1999)

$$\{\varepsilon\} = \tfrac{1}{2}\left\{\nabla\vec{\mathbf{u}} + \nabla\vec{\mathbf{u}}^T + (\nabla\vec{\mathbf{u}})^T(\nabla\vec{\mathbf{u}})\right\} \tag{3}$$

where, ε_{ij} are the strain components and ∇ is del operator. The nonlinear components of the strains are obtained using substituting Eq. (2) into Eq. (3) as follows

$$\varepsilon_{rr} = \frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2 \quad , \quad \varepsilon_{\theta\theta} = \frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr} \quad , \quad \varepsilon_{r\theta} = 0 \tag{4}$$

Fig. 1 shows the schematic figure of a functionally graded circular plate embedded with functionally graded piezoelectric layers at top and bottom resting on the Winkler-Pasternak foundation. Stress-strain relations for FGM and FGPM sections in general state are (Khoshgoftar *et al*. 2009)

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}E_k \tag{5}$$

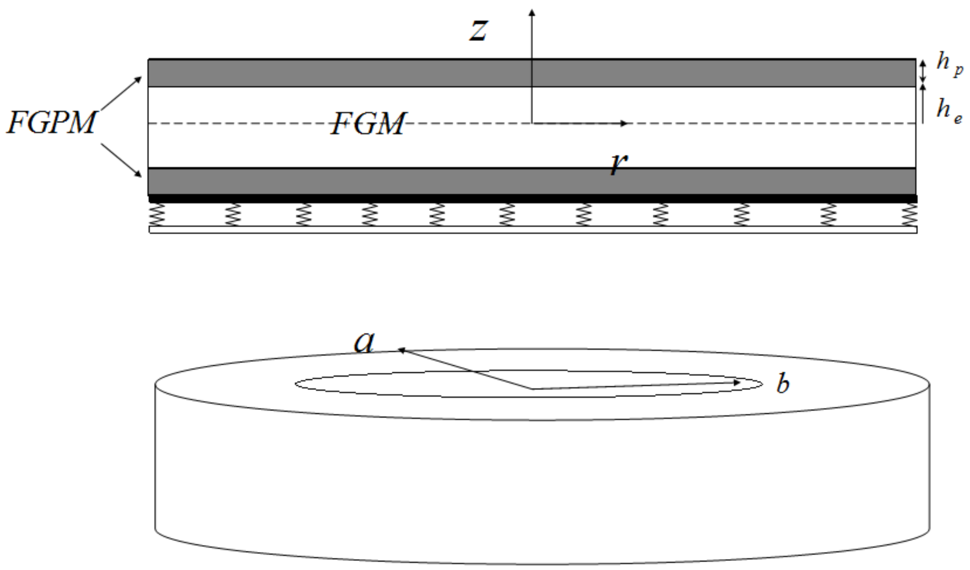

Fig. 1 FG circular plate with piezoelectric layers resting on Winkler-Pasternak foundation

where, σ_{ij} and ε_{kl} are the stress and strain components, E_k is electric field, C_{ijkl} and e_{ijk} are the stiffness and piezoelectric coefficients.

For FGM section $-h_e \le z \le h_e$ where electric potential hasn't effect on stress components, the constitutive equations may expressed as follows

$$\left.\begin{aligned} \sigma_{rr} &= C^e{}_{rrrr}\varepsilon_{rr} + C^e{}_{rr\theta\theta}\varepsilon_{\theta\theta} \\ \sigma_{\theta\theta} &= C^e{}_{\theta\theta rr}\varepsilon_{rr} + C^e{}_{\theta\theta\theta\theta}\varepsilon_{\theta\theta} \end{aligned}\right\} \tag{6}$$

In Eq. (6), normal stresses are depending on the normal strains and shear stress is a function of shear strain only. Due to these assumptions $C^e{}_{rrr\theta} = C^e{}_{\theta\theta r\theta} = C^e{}_{rr\theta r} = C^e{}_{\theta\theta\theta r} = 0$. Furthermore, due to small ratio of the plate thickness with respect to the length and width of the plate, the normal stress σ_{zz} and shear stresses $\sigma_{rz}, \sigma_{\theta z}$ is ignorable.

The constitute equations for piezoelectric sections of the plate (FGP) $h_e < |z| \le h_e + h_p$ are expressed as

$$\left.\begin{aligned} \sigma_{rr} &= C^p{}_{rrrr}\varepsilon_{rr} + C^p{}_{rr\theta\theta}\varepsilon_{\theta\theta} - e_{rrr}E_r - e_{rrz}E_z \\ \sigma_{\theta\theta} &= C^p{}_{\theta\theta rr}\varepsilon_{rr} + C^p{}_{\theta\theta\theta\theta}\varepsilon_{\theta\theta} - e_{\theta\theta r}E_r - e_{\theta\theta z}E_z \end{aligned}\right\} \tag{7}$$

The assumptions expressed after Eq. (6) must be considered for piezoelectric section $C^p{}_{rrr\theta} = C^p{}_{\theta\theta r\theta} = C^p{}_{rr\theta r} = C^p{}_{\theta\theta\theta r} = 0$. Electric field E_k is obtained using gradient of a potential function $\phi(r,z)$ with minus sign as follows (Khoshgoftar *et al*. 2009)

$$\phi = \phi(r,z) \rightarrow \begin{cases} E_r = -\dfrac{\partial\phi(r,z)}{\partial r} \\ E_\theta = 0 \\ E_z = -\dfrac{\partial\phi(r,z)}{\partial z} \end{cases} \tag{8}$$

Short circuit condition of piezoelectric layers can be satisfied if we use $f(z)$ as follows: (Ebrahimi and Rastgo 2008)

$$\begin{aligned} &\phi(r,z) = \phi_r(r) \times f(z) \\ &\phi(z = h_e) = \phi(z = h_e + h_p) = 0 \rightarrow f(z) = \left(1 - \left\{\frac{2z - 2h_e - h_p}{h_p}\right\}^2\right) \end{aligned} \tag{9}$$

The electric displacement D_i, that must satisfy Maxwell's equations in the electromechanical systems is defined as (Khoshgoftar *et al.* 2009)

$$D_i = e_{ijk}\,\varepsilon_{jk} + \eta_{ik}E_k \tag{10}$$

where, η_{ik} are the dielectric coefficients.

The electric displacement equations for the piezoelectric sections of the FGP plate $h_e \le z \le h_e + h_p$ are

$$\left.\begin{aligned} D_r &= e_{rrr}\varepsilon_{rr} + e_{r\theta\theta}\varepsilon_{\theta\theta} + \eta_{rr}E_r + \eta_{rz}E_z \\ D_z &= e_{zrr}\varepsilon_{rr} + e_{z\theta\theta}\varepsilon_{\theta\theta} + \eta_{zr}E_r + \eta_{zz}E_z \end{aligned}\right\} \tag{11}$$

After definition of necessary mechanical and electrical components, we can employ energy method to evaluate the nonlinear mechanical and electrical responses of the system.

The energy per unit volume of the plate $\bar{u}$ is given by

$$\bar{u} = \frac{1}{2}\left(\boldsymbol{\varepsilon}^T\boldsymbol{\sigma} - \mathbf{E}^T\mathbf{D}\right) \rightarrow \bar{u} = \frac{1}{2}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} - D_rE_r - D_zE_z\right) \tag{12}$$

In order to evaluate the energy per unit volume of the structure, the components of ε, σ, E, D must be specified. These components can be expressed as follows

$-h_e \le z \le h_e$:

$$\left.\begin{aligned} \sigma_{rr} &= C^e{}_{rrrr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^e{}_{rr\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} \\ \sigma_{\theta\theta} &= C^e{}_{\theta\theta rr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^e{}_{\theta\theta\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} \end{aligned}\right\}$$

$h_e \le |z| \le h_e + h_p$:

$$\left.\begin{aligned} \sigma_{rr} &= C^p{}_{rrrr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^p{}_{rr\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} + e_{rrr}\frac{\partial\phi(r,z)}{\partial r} + e_{rrz}\frac{\partial\phi(r,z)}{\partial z} \\ \sigma_{\theta\theta} &= C^p{}_{\theta\theta rr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^p{}_{\theta\theta\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} + e_{\theta\theta r}\frac{\partial\phi(r,z)}{\partial r} + e_{\theta\theta z}\frac{\partial\phi(r,z)}{\partial z} \end{aligned}\right\}$$

$$\left.\begin{aligned} D_r &= e_{rrr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + e_{r\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} - \eta_{rr}\frac{\partial\phi(r,z)}{\partial r} - \eta_{rz}\frac{\partial\phi(r,z)}{\partial z} \\ D_z &= e_{zrr}\{\frac{du_0}{dr} - z\frac{d^2w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + e_{z\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} - \eta_{zr}\frac{\partial\phi(r,z)}{\partial r} - \eta_{zz}\frac{\partial\phi(r,z)}{\partial z} \end{aligned}\right\}$$

(13)

By introduction of the potential energy, the total energy equation of the plate under uniform or non-uniform pressure is expressed by (Ugural 1981)

$$U = \int_A \int_{-(h_e+h_p)}^{(h_e+h_p)} \bar{u}(r,z)\,dz\,dA - \iint_A p(r)w\,dA - \frac{1}{2}\iint_A L(r)w\,dA \tag{14}$$

where, $L(r)$ is distributed pressure on the plate. This pressure includes normal pressure and the effect of Winkler- Pasternak foundation. In the general state, for mentioned foundation we will have

$$L(r) = f_f \tag{15}$$

where, f_f is force due to Winkler-Pasternak foundation which in general form have direct and shear effects as follows

$$f_f = -kw + G\nabla^2 w \tag{16}$$

where, in radial coordinate system, $\nabla^2 w = \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r}$. The energy equation for two different sections of the plate takes the form

$$\begin{aligned} U = &\iint_A \left[\int_{-h_e}^{h_e} \frac{1}{2}\left(\sigma^e{}_{rr}\varepsilon_{rr} + \sigma^e{}_{\theta\theta}\varepsilon_{\theta\theta}\right)dz + 2\int_{h_e}^{h_e+h_p} \frac{1}{2}\left(\sigma^p{}_{rr}\varepsilon_{rr} + \sigma^p{}_{\theta\theta}\varepsilon_{\theta\theta} - D_r E_r - D_z E_z\right)dz \right] dA \\ &- \int\int_A p(r) w dA - \frac{1}{2}\iint_A \left[-kw + G\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right] w\, dA \end{aligned} \tag{17}$$

In order to present matrix of coefficients, we have to constitute total energy as follow

$$U = \iint_A \left[\begin{array}{l} \displaystyle\int_{-h_e}^{h_e} \frac{1}{2} \left(\begin{array}{l} \{C^e{}_{rrrr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^e{}_{rr\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\}\}\{\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\}\} \\ + \{C^e{}_{\theta\theta rr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^e{}_{\theta\theta\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\}\}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} \end{array} \right) dz \\ + 2\displaystyle\int_{h_e}^{h_e+h_p} \frac{1}{2} \left(\begin{array}{l} \{C^p{}_{rrrr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^p{}_{rr\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} + e_{rrr}\frac{\partial\phi(r,z)}{\partial r} \\ + e_{rrz}\frac{\partial\phi(r,z)}{\partial z}\}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} \\ + \{C^p{}_{\theta\theta rr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + C^p{}_{\theta\theta\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} + e_{\theta\theta r}\frac{\partial\phi(r,z)}{\partial r} \\ + e_{\theta\theta z}\frac{\partial\phi(r,z)}{\partial z}\}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} \\ + \{e_{rrr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + e_{r\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} - \eta_{rr}\frac{\partial\phi(r,z)}{\partial r} \\ - \eta_{rz}\frac{\partial\phi(r,z)}{\partial z}\}\frac{\partial\phi(r,z)}{\partial r} + \\ \{e_{zrr}\{\frac{du_0}{dr} - z\frac{d^2 w_0}{dr^2} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2\} + e_{z\theta\theta}\{\frac{u_0}{r} - \frac{z}{r}\frac{dw_0}{dr}\} - \eta_{zr}\frac{\partial\phi(r,z)}{\partial r} \\ - \eta_{zz}\frac{\partial\phi(r,z)}{\partial z}\}\frac{\partial\phi(r,z)}{\partial z} \end{array} \right) dz \end{array} \right] dA \tag{18}$$

$$- \int\int_A p_0 w dA - \frac{1}{2}\iint_A \left[-kw_0 + G\left(\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r}\right)\right] w_0\, dA$$

3. Derivation of the governing equations of the system

Using Eq. (1) and extending that for all mechanical and electrical properties except Poisson ratio yields

$$\left\{E(z), C(z), e(z), \eta(z)\right\} = \left\{E_0, C_0, e_0, \eta_0\right\} \mathrm{F(z)}, \quad -(h_e + h_p) \le z \le h_e + h_p \tag{19}$$

Substitution of this distribution in energy equation presents

$$U = \iint_A \overline{U}_A dA \tag{20}$$

where, $\overline{U}_A$ is energy functional per unit area of the structure given by

$$\overline{U}_A = -\left\{ p_0 - \frac{1}{2}kw + \frac{1}{2}G\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right\} w + \Lambda \tag{21}$$

where, Λ is defined as follows

$$\Lambda = \sum_{i=1}^{10} P_i L_i \tag{22}$$

where, P_i, L_i are given in Appendix A.

Final differential equations of the system may be obtained by using the Euler equation as follows

$$\frac{\partial \overline{U}}{\partial \Psi} - \frac{\partial}{\partial r}\left(\frac{\partial \overline{U}}{\partial \Psi_{,r}}\right) = 0, \tag{23}$$

where, $\Psi = \left[u_0, w_0, \phi_r\right], \Psi_{,r} = \left[\frac{du_0}{dr}, \frac{dw_0}{dr}, \frac{d\phi_r}{dr}\right]$

where, $\overline{U}$ is integration of $\overline{U}_A$ on the area of the structure. $\overline{U}_A$ is presented as follows

$$\begin{aligned}
\overline{U}_A = {} & [\left(\frac{Mu_0}{r} + \frac{u_0^2}{r^2}\right)C_{rr\theta\theta 0} + M^2 C_{rrrr0} + \frac{Mu_0}{r}C_{\theta\theta\theta\theta 0}]L_1 \\
& -[\left[2MC_{rrrr0} + \frac{u_0}{r}(C_{rr\theta\theta 0} + C_{\theta\theta\theta\theta 0})\right]\frac{d^2 w_0}{dr^2} - \left[\left(\frac{2u_0}{r^2} + \frac{M}{r}\right)C_{rr\theta\theta 0} + \frac{M}{r}C_{\theta\theta\theta\theta 0}\right]\frac{dw_0}{dr}]L_2 \\
& +[C_{rrrr0}\left(\frac{d^2 w_0}{dr^2}\right)^2 + \frac{u_0}{r}(C_{rr\theta\theta 0} + C_{\theta\theta\theta\theta 0})\frac{d^2 w_0}{dr^2} + \frac{1}{r^2}C_{rr\theta\theta 0}\left(\frac{dw_0}{dr}\right)^2]L_3 \\
& +[2\left(Me_{rrr0} + \frac{u_0}{r}e_{rr\theta 0}\right)\frac{d\phi}{dr}]L_4 + [2\left(Me_{rrz0} + \frac{u_0}{r}e_{\theta\theta z0}\right)\phi]L_5 \\
& [-2\left(e_{rrr0}\frac{d^2 w_0}{dr^2} + \frac{e_{rr\theta 0}}{r}\frac{dw_0}{dr}\right)\frac{d\phi}{dr}]L_6 + [-2\left(e_{rrz0}\frac{d^2 w_0}{dr^2} + \frac{e_{\theta\theta z0}}{r}\frac{dw_0}{dr}\right)\phi]L_7 \\
& [-\left(\frac{d\phi}{dr}\right)^2 \eta_{rr0}]L_8 + [-\eta_{zz0}\phi^2]L_9 + [-2\eta_{rz0}\phi\frac{d\phi}{dr}]L_{10} - \left\{p(r) - \frac{1}{2}kw + \frac{1}{2}G\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right\}w
\end{aligned} \tag{24}$$

By substitution of functional from Eq. (20) into Euler equation (Eq. (23)), we will have three fundamental equations as follows

$$
\begin{aligned}
&\left(\frac{M}{r}+2\frac{u_0}{r^2}\right)C_{rr\theta\theta0}L_1-[\left[\frac{1}{r}(C_{rr\theta\theta0}+C_{\theta\theta\theta\theta0})\right]\frac{d^2w_0}{dr^2}-\left[\left(\frac{2}{r^2}\right)C_{rr\theta\theta0}\right]\frac{dw_0}{dr}]L_2\\
&+\frac{1}{r}(C_{rr\theta\theta0}+C_{\theta\theta\theta\theta0})\frac{d^2w_0}{dr^2}L_3+2\left(\frac{1}{r}e_{\theta\theta z0}\right)\phi]L_5\\
&-\frac{\partial}{\partial r}\begin{pmatrix}[\frac{u_0}{r}C_{rr\theta\theta0}+2MC_{rrrr0}+\frac{u_0}{r}C_{\theta\theta\theta\theta0}]L_1-[2C_{rrrr0}\frac{d^2w_0}{dr^2}-\frac{1}{r}[C_{rr\theta\theta0}+C_{\theta\theta\theta\theta0}]\frac{dw_0}{dr}]L_2\\ +2e_{rrr0}\frac{d\phi}{dr}L_4+2e_{rrz0}\phi L_5\end{pmatrix}=0
\end{aligned}
$$

$$
\begin{aligned}
&-\left\{p(r)-kw_0+\frac{1}{2}G\left(\frac{\partial^2w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}\right)\right\}\\
&-\frac{\partial}{\partial r}\begin{pmatrix}[\left(\frac{Mu_0}{r}\right)C_{rr\theta\theta0}+2MC_{rrrr0}+\frac{u_0}{r}C_{\theta\theta\theta\theta0}]L_1\left(\frac{dw_0}{dr}\right)-[2C_{rrrr0}\frac{d^2w_0}{dr^2}-\frac{1}{r}[C_{rr\theta\theta0}+C_{\theta\theta\theta\theta0}]\\ \frac{dw_0}{dr}]L_2\frac{dw_0}{dr}+\frac{2}{r^2}C_{rr\theta\theta0}\left(\frac{dw_0}{dr}\right)L_3+2e_{rrr0}\frac{d\phi}{dr}\frac{dw_0}{dr}L_4+2e_{rrz0}L_5\phi\frac{dw_0}{dr}\\ -2\frac{e_{rr\theta0}}{r}L_6\frac{d\phi}{dr}-2\frac{e_{\theta\theta z0}}{r}\phi L_7-G\frac{1}{2r}w\end{pmatrix}=0
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&2\left(Me_{rrz0}+\frac{u_0}{r}e_{\theta\theta z0}\right)L_5-2\left(e_{rrz0}\frac{d^2w_0}{dr^2}+\frac{e_{\theta\theta z0}}{r}\frac{dw_0}{dr}\right)L_7-2\eta_{zz0}\phi L_9-2\eta_{rz0}\frac{d\phi}{dr}L_{10}\\
&-\frac{\partial}{\partial r}\left(2\left(Me_{rrr0}+\frac{u_0}{r}e_{rr\theta0}\right)L_4-2\left(e_{rrr0}\frac{d^2w_0}{dr^2}+\frac{e_{rr\theta0}}{r}\frac{dw_0}{dr}\right)L_6-2\left(\frac{d\phi}{dr}\right)\eta_{rr0}L_8-2\eta_{rz0}\phi L_{10}\right)=0
\end{aligned}
$$

Above three nonlinear differential equations of the system can be solved analytically by supposing three unknown fields. Due to existence of nonlinear terms in above equations, presentation of equations in matrix form is not useful for readers.

4. Results and discussion

Derived governing differential equations can be solved for a special functionally graded material as numerical solution. For numerical solution of the problem, power function distribution may be considered for material. This distribution may be considered for both FGM and FGPM sections as follows

$$
F(z)=\begin{cases}(\frac{E_c}{E_m}-1)\left(\frac{1}{2}+\frac{z}{2h_e}\right)^n+1 & -h_e\le z\le h_e\\ \left(\frac{z}{h_e}\right)^n & h_e\le|z|\le h_e+h_p\end{cases}
\tag{26}
$$

where, E_m and E_c is properties at bottom and top of plate, respectively. The nonlinear differential equations of the system may obtained by taking variations with respect to employed unknown functions. Alternatively, we can propose series solution using trigonometric terms for displacement components and electric potential.

The plate is fixed to inner and outer edges and therefore, the displacements and the slope of that at these two edges are considered zero. Furthermore, the homogenous boundary conditions are considered for the electric potential (Ebrahimi and Rastgo 2008). Top and bottom of piezoelectric layers are short circuited. The solution procedure may continue with assumption of three fields for the displacements and electric potential. The polynomial function is employed for these assumptions as follows (GhannadPour and Alinia 2006, Alinia and GhannadPour 2009)

$$\left[u_0(r), w_0(r), \phi(r)\right] = \left(1-\left(\frac{r}{a}\right)^2\right)^2 \left(1-\left(\frac{r}{b}\right)^2\right)^2 \left[\sum_{m=0} U_m r^m, \sum_{m=0} W_m r^m, \sum_{m=0} \Phi_m r^m\right] \tag{27}$$

where, U_m, W_m and Φ_m describes the amplitudes of the displacement components and electric potential, *m* defines the number of the required terms for definition of the three fields.

By substituting the material distribution from Eq. (26) into Eqs. (19) and then substituting the three fields from Eq. (27) into Eq. (25), we will have three equations of the system.

The effect of parameters of Winkler-Pasternak parameters and non-homogeneity can be considered on the responses of the system. Shown in Fig. 2 is the distribution of maximum nonlinear dimensionless transverse displacement in terms of stiffness parameter of foundation (*k*) for different values of non-homogenous index.

The obtained results in Fig. 2 indicate that the dimensionless transverse displacement increases with increasing the non-homogenous index. Furthermore, it can be concluded that with increasing the stiffness parameter of foundation, the maximum displacement decreases.

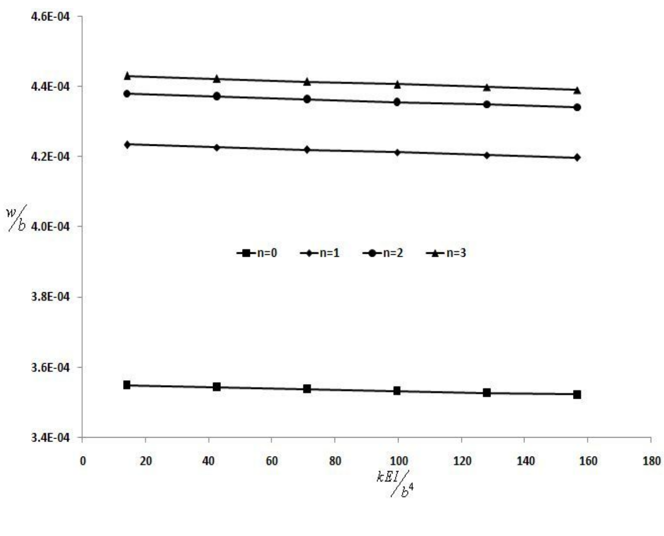

Fig. 2 The distribution of dimensionless transverse displacement in terms of dimensionless stiffness parameter for different non-homogenous indexes

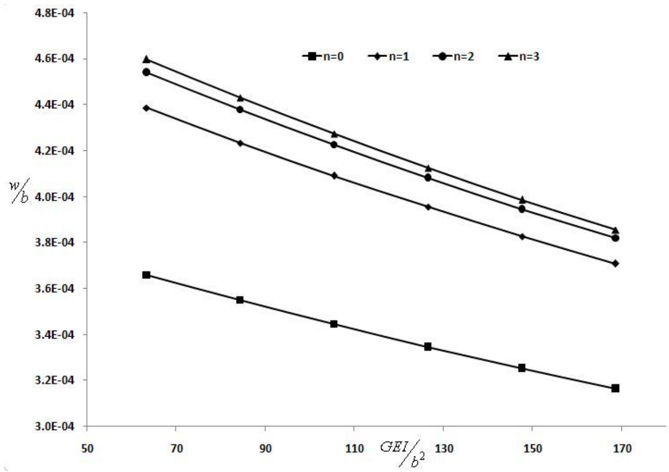

Fig. 3 The distribution of dimensionless transverse displacement in terms of dimensionless shear parameter for different non-homogenous indexes

The same investigation has performed to evaluate the effect of shear parameter G on the dimensionless transverse displacement of the plate. Shown in Fig. 3 is the distribution of maximum dimensionless transverse displacement in terms of shear parameter of foundation (G) for different values of non-homogenous index. These figures indicate that increasing the shear parameter of foundation decreases maximum displacement.

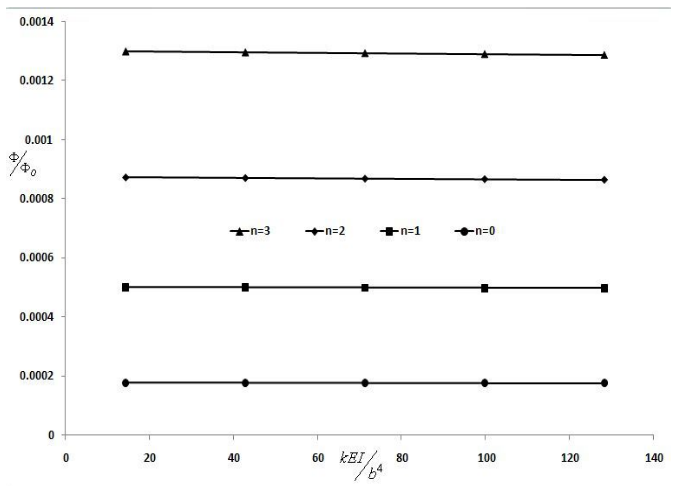

Fig. 4 The distribution of dimensionless electric potential in terms of dimensionless stiffness parameter for different non-homogenous indexes

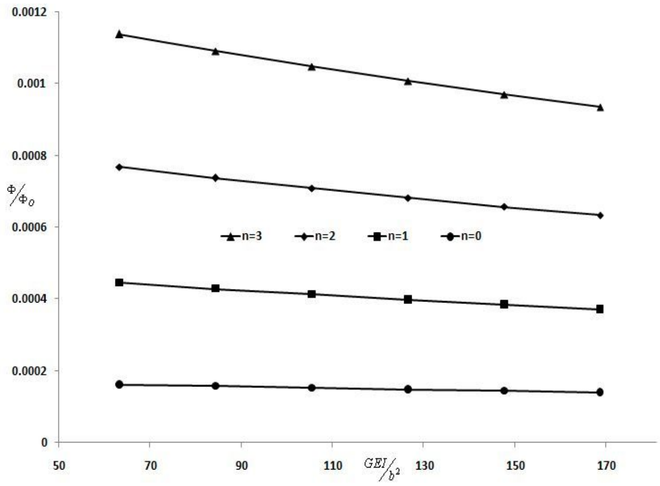

Fig. 5 The distribution of dimensionless electric potential in terms of dimensionless shear parameter for different non-homogenous indexes

The effect of Winkler-Pasternak parameters can be studied on the maximum dimensionless electric potential of the system. Shown in Fig. 4 is distribution of dimensionless electric potential in terms of stiffness parameter of foundation for different values of non-homogenous index. It is obvious that with increasing the non-homogenous index, dimensionless electric potential monotonically and considerably increases.

Shown in Fig. 5 is distribution of dimensionless electric potential in terms of shear parameter of foundation for different values of non-homogenous index. It can be concluded that with increasing the non-homogenous index, dimensionless electric potential increases. Furthermore, the obtained results indicate that increasing the non-homogenous index amplifies decreasing manner of dimensionless electric potential in terms of increasing shear parameter.

4.1 Linear analysis, comparison with nonlinear responses

In this section, the effect of used nonlinear analysis is investigated rather than a linear analysis. This investigation can performed using comparison between responses of the linear and nonlinear analyses in terms of two parameters of foundation. Shown in Figs. 6 and 7 are linear and nonlinear dimensionless transverse displacements in terms of stiffness and shear parameters of foundation, respectively.

4.2 Convergence of the results

The convergence of the results can be investigated for different values of stiffness parameters and in terms of non homogenous index. Fig. 8 show this convergence. You can find that with increasing the value of non homogenous index, the amplitude of dimensionless transverse displacement converges to an asymptotic value.

Considering Eq. (24), it is important to discuss on the required terms (m) of polynomial in Eq. (24). Table 1 presents the trend of convergence for three fields (two displacements and one electric potential).

It can be concluded that considering two terms is sufficient in order to obtain acceptable results.

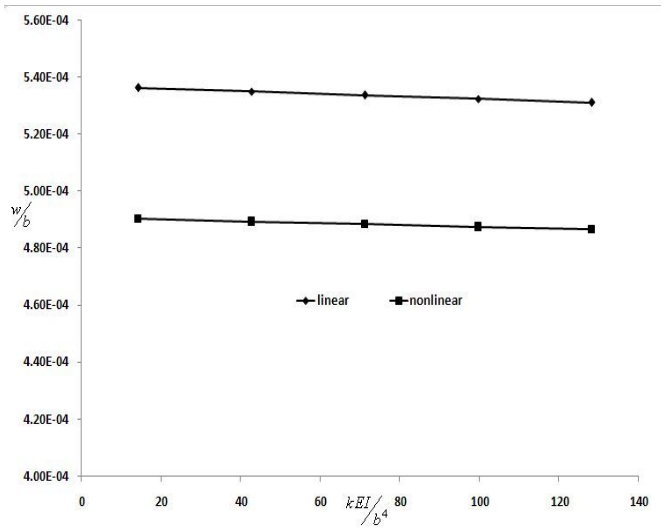

Fig. 6 Comparison between linear and nonlinear dimensionless transverse displacement in terms of dimensionless stiffness parameter

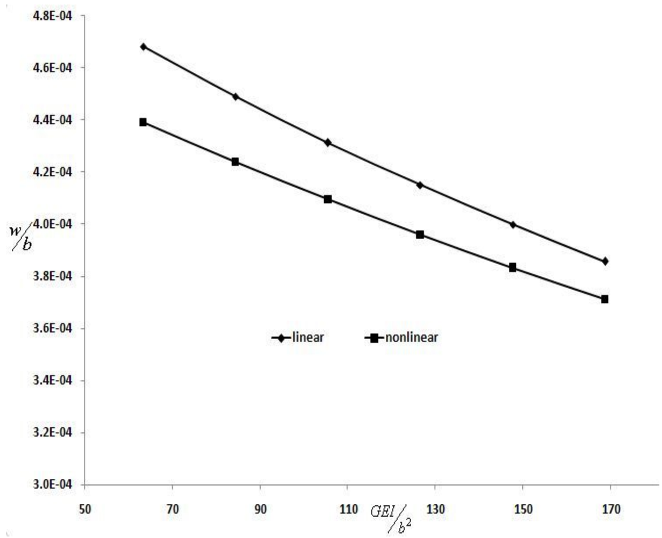

Fig. 7 Comparison between linear and nonlinear dimensionless transverse displacement in terms of dimensionless shear parameter

Table 1 The required terms for considered polynomials in Eq. (24)

		n=1		
	1 term	2 terms	3 terms	4 terms
$w\times 10^3(m)$	2.32	4.07	4.05	4.05
$\phi(V)$	189	285	280	280
		n=2		
	1 term	2 terms	3 terms	4 terms
$w\times 10^3(m)$	2.42	4.03	4.02	4.02
$\phi(V)$	328	496	495	495

4.3 The effect of boundary conditions on the results

This section evaluates the effect of boundary conditions on the results of the system. Simply support and fixed boundary conditions have been considered for this comparison. The all previous results have been considered for fixed boundary condition (as presented in Eq. (27)). The simply-supported boundary conditions can be modeled by following distributions

$$\left[u_0(r),w_0(r),\phi(r)\right]=\left(1-\left(\frac{r}{a}\right)^2\right)\left(1-\left(\frac{r}{b}\right)^2\right)\left[\sum_{m=0}U_m r^m,\sum_{m=0}W_m r^m,\sum_{m=0}\Phi_m r^m\right] \tag{28}$$

Shown in Figs. 9 and 10 are dimensionless electric potential and transverse displacement of annular plate in terms of shear parameter of the foundation, respectively.

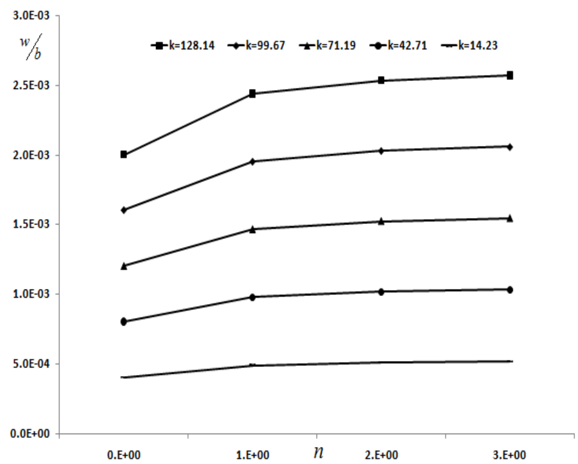

Fig. 8 The convergence of dimensionless transverse displacement for increasing the non homogenous index

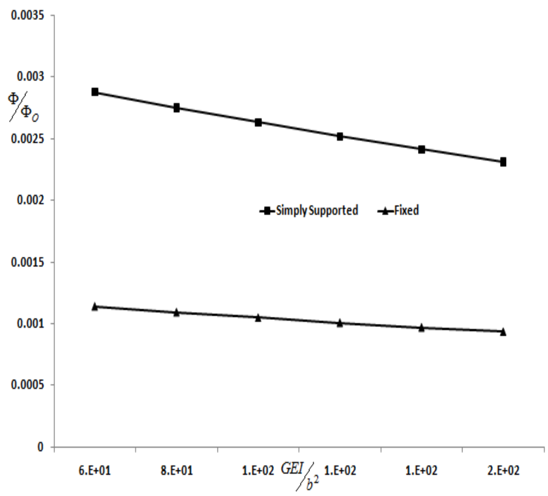

Fig. 9 The comparison between dimensionless electric potential of simply supported and fixed plate

4.4 Comparison between present and previous results

A comprehensive comparison between present and previous (Arefi and Rahimi 2012) results is performed in this section. For this comparison, the effect of foundation in this study has been disregarded. Fig. 11 shows comparison maximum dimensionless transverse displacement between present and previous (Arefi and Rahimi 2012a) results. The same comparison can be performed for dimensionless electric potential in Fig. 12.

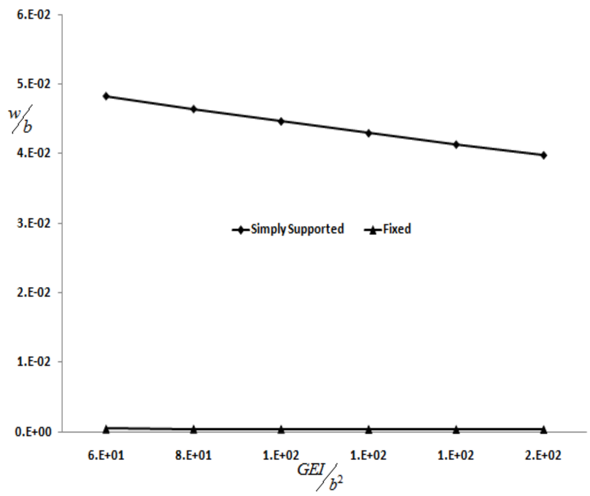

Fig. 10 The comparison between dimensionless transverse displacement of simply supported and fixed plate

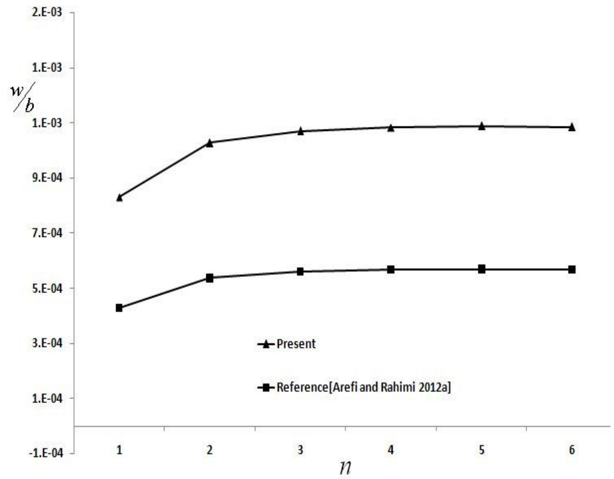

Fig. 11 The comparison between dimensionless transverse displacement of present and previous (Arefi and Rahimi 2012) a results

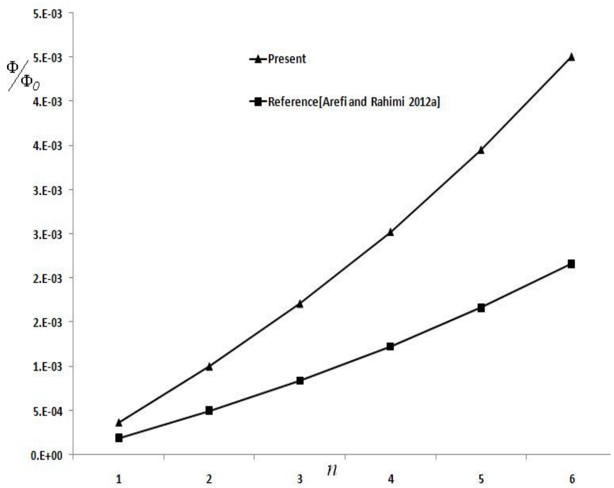

Fig. 12 The comparison between dimensionless electric potential of present and previous (Arefi and Rahimi 2012a) results

5. Conclusions

Nonlinear electromechanical analysis of an arbitrary functionally graded circular plate integrated with functionally graded piezoelectric layers resting on the Winkler-Pasternak foundation has been performed in this paper. Functional of the system has been derived by considering stress-strain and electric displacement equations. The obtained differential equations of the system can be applied for solution of problems with arbitrary functionality. The effect of different parameters such as non-homogenous index and foundation parameters was studied on the mechanical and electrical responses. In order to evaluate the effect of a nonlinear analysis on the responses of the system rather than a linear analysis, the comparisons between linear and nonlinear responses has performed for mechanical and electrical parameters. A linear analysis is performed and the obtained results are compared with those results that are extracted from the nonlinear analysis. This comparison indicates that employing a nonlinear analysis has important effect on improvement of the results rather than a linear analysis.

Furthermore a convergence test has been performed for evaluation of the effect of necessary used terms that is needed for exact solution.

Acknowledgements

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CC

Appendix

$$P_1 = \left(\frac{Mu_0}{r} + \frac{u_0^2}{r^2}\right)C_{rr\theta\theta} + M^2 C_{rrrr} + \frac{Mu_0}{r}C_{\theta\theta\theta\theta},$$

$$P_2 = -\left[2MC_{rrrr} + \frac{u_0}{r}\left(C_{rr\theta\theta} + C_{\theta\theta\theta\theta}\right)\right]\frac{d^2 w_0}{dr^2} - \left[\left(\frac{2u_0}{r^2} + \frac{M}{r}\right)C_{rr\theta\theta} + \frac{M}{r}C_{\theta\theta\theta\theta}\right]\frac{dw_0}{dr},$$

$$P_3 = C_{rrrr}\left(\frac{d^2 w_0}{dr^2}\right)^2 + \frac{u_0}{r}\left(C_{rr\theta\theta} + C_{\theta\theta\theta\theta}\right)\frac{d^2 w_0}{dr^2} + \frac{1}{r^2}C_{rr\theta\theta}\left(\frac{dw_0}{dr}\right)^2,$$

$$P_4 = 2\left(Me_{rrr} + \frac{u_0}{r}e_{rr\theta}\right)\frac{d\phi}{dr}, \qquad P_5 = 2\left(Me_{rrz} + \frac{u_0}{r}e_{\theta\theta z}\right)\phi,$$

$$P_6 = -2\left(e_{rrr}\frac{d^2 w_0}{dr^2} + \frac{e_{rr\theta}}{r}\frac{dw_0}{dr}\right)\frac{d\phi}{dr}, \quad P_7 = -2\left(e_{rrz}\frac{d^2 w_0}{dr^2} + \frac{e_{\theta\theta z}}{r}\frac{dw_0}{dr}\right)\phi,$$

$$P_8 = -\left(\frac{d\phi}{dr}\right)^2 \eta_{rr}, \quad P_9 = -\eta_{zz}\phi^2, \quad P_{10} = -2\eta_{rz}\phi\frac{d\phi}{dr},$$

$$\left[L_1, L_2, ..., L_{10}\right] = \frac{1}{2}\int_{-\bar{h}}^{\bar{h}} F(z)\left[1, \mathrm{z}, \mathrm{z}^2, \mathrm{f}, \mathrm{f}', \mathrm{zf}, \mathrm{zf}', \mathrm{f}^2, \mathrm{f}'^2, \mathrm{f\,f}'\right]dz,$$

where

$$\mathrm{f} = f(z), \; \mathrm{f}' = \frac{df}{dz}, \; M = \frac{du_0}{dr} + \frac{1}{2}\left(\frac{dw_0}{dr}\right)^2,$$

$$u_0 = u_0(r), \; w_0 = w_0(r), \; \phi = \phi(r),$$

Nomenclature

C_{ijkl}	Stiffness coefficient
C^{e}_{ijkl} , C^{p}_{ijkl}	Stiffness coefficient for FG and FGP layer
K, G	Parameters of foundation
D_i	Electric displacement
e_{ijk}	Piezoelectric coefficient
E_k	Electric field components
$2h_e$	Thickness of FG layer
h_p	Thickness of FGP layer
a,b	Inner and outer radii
p	Applied pressure
$E(z)$	Distribution of material properties
n	Non-homogeneity index
r,z	Components of coordinate system
u,w	Displacement components
u_0,w_0	Displacement components at mid-plane
ε_{ij}	Strain components
σ_{ij}	Stress components
$\bar{u}$	Energy per unit volume
$\bar{U}_A$	Energy per unit area
U	Total energy of system
η_{ik}	Dielectric coefficient
U_m, W_m, Φ_m	Amplitude of assumed function
ϕ	Electric potential
m	Number of required terms for displacement and electric potential field

FGM: functionally graded material
FG: functionally graded
FGP: functionally graded piezoelectric
FGPM: functionally graded piezoelectric material
CPT: classical plate theory