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Mathematical solution for nonlinear vibration equations using variational approach

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Abstract. In this paper, we have applied a new class of approximate analytical methods called Variational Approach (VA) for high nonlinear vibration equations. Three examples have been introduced and discussed. The effects of important parameters on the response of the problems have been considered. Runge-Kutta's algorithm has been used to prepare numerical solutions. The results of variational approach are compared with energy balance method and numerical and exact solutions. It has been established that the method is an easy mathematical tool for solving conservative nonlinear problems. The method doesn't need small perturbation and with only one iteration achieve us to a high accurate solution.

Keywords: Variational Approach (VA); nonlinear vibrations; energy balance method; numerical method

1. Introduction

Differential equations arise in dynamical models and engineering problems and physical phenomena. Finding an analytical solution for structural mechanics problems and nonlinear mechanical systems is very important and also very interesting for civil and mechanical engine ringers. For many nonlinear problems, it is not always possible and sometimes not even advantageous to express exact solutions of nonlinear differential equations explicitly in terms of elementary functions, but it is possible to find elementary functions that are constant on solution curves.

Recently, many researchers have been working on the analytical and numerical methods in nonlinear vibrations such as : Homotopy perturbation method (Shaban *et al.* 2010, Bayat 2013a), Hamiltonian approach (Bayat *et al*. 2011a, 2012a, 2013a,b, 2014a,b), energy balance method (He 2002, Bayat *et al.* 2011b, Pakar *et al.* 2011a,b, Mehdipour 2010), variational iteration method (Dehghan 2010, Pakar *et al*. 2012), amplitude frequency formulation (Bayat 2011c, 2012b, Pakar *et al*. 2013a, He 2008), max-min approach (Shen *et al*. 2009, Zeng *et al.* 2009), variational approach (He 2007, Bayat *et al.* 2012c, 2013c, 2014c, Pakar *et al.* 2012b), and the other analytical and numerical (Xu 2009, Alicia *et al.* 2010, Bor-Lih *et al*. 2009, Wu 2011, Odibat *et al.* 2008).

Among these methods, variational approach is considered to solve the nonlinear vibration equations in this paper.

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The paper has been collocated as follows

First, we describe the basic concept of variational approach. Then the applications of variational approach have been studied to demonstrate the applicability and preciseness of the method for three examples. Some comparisons between analytical and numerical solutions are presented. Eventually we show that VA can converge to a precise cyclic solution for nonlinear systems. The basic idea of energy balance method and Runge-Kutta's algorithm are presented in Appendix A and Appendix B.

2. Basic concept of Variational Approach (VA)

He suggested a variational approach which is different from the known variational methods in open literature (He 2007). Hereby we give a brief introduction of the method

$$\ddot{\theta}+f\left(\theta\right)=0 \tag{1}$$

Its variational principle can be easily established utilizing the semi-inverse method (He 2007)

$$J\left(\theta\right)=\int_0^{T/4}\left(-\frac{1}{2}\dot{\theta}^2+F\left(\theta\right)\right)dt \tag{2}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial u=f$ .Assume that its solution can be expressed as

$$\theta\left(t\right)=A\cos\left(\omega t\right) \tag{3}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in

$$\begin{aligned} J\left(A,\omega\right)&=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F\left(A\cos\omega t\right)\right)dt \\ &=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2 t+F\left(A\cos t\right)\right)dt \\ &=-\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\ dt+\frac{1}{\omega}\int_0^{\pi/2}F\left(A\cos t\right)dt \end{aligned} \tag{4}$$

Applying the Ritz method, we require

$$\frac{\partial J}{\partial A}=0 \tag{5}$$

$$\frac{\partial J}{\partial \omega}=0 \tag{6}$$

But with a careful inspection, for most cases He find that

$$\frac{\partial J}{\partial \omega}=-\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\,dt-\frac{1}{\omega^2}\int_0^{\pi/2}F\left(A\cos t\right)dt<0 \tag{7}$$

Thus, He modify conditions Eqs. (5) and (6) into a simpler form

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. Application

In order to assess the advantages and the accuracy of the variational approach, we will consider the following examples:

3.1 Example 1

Consider the motion of a mass m moving without friction along a circle of radius R that is rotating with a constant angular velocity Ω about its vertical diameter as shown in Fig. 1. The forces acting on the mass are gravitational force mg, the centrifugal of the circle O and the reaction force. The following governing equation has been obtained (Nayfe 1973)

$$m\,R^2\ddot{\theta} - m\,R^2\Omega^2 \sin(\theta)\cos(\theta) + mgR\sin(\theta) = 0, \qquad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{9}$$

In order to apply the variational approach method to solve the above problem, the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$ and $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ is used. we can re-write Eq. (9) in the following form

$$\alpha\ddot{\theta} - \beta\left(\theta - \frac{1}{6}\theta^3\right)\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right) + \lambda\left(\theta - \frac{1}{6}\theta^3\right) = 0, \qquad \theta(0) = A, \quad \dot{\theta}(0) = 0 \tag{10}$$

Where

$$\alpha = mR^2,\ \beta = mR^2\Omega^2,\ \ \lambda = mgR \tag{11}$$

Its variational formulation can be readily obtained from Eq. (10) as follows

$$J(\theta) = \int_0^t \left(\frac{1}{2}\alpha\ddot{\theta}^2 - \frac{1}{2}\beta\theta^2 + \frac{1}{6}\beta\theta^4 - \frac{1}{48}\beta\theta^6 + \frac{1}{1152}\beta\theta^8 + \frac{1}{2}\lambda\theta^2 - \frac{1}{24}\lambda\theta^4\right)dt\,. \tag{12}$$

Choosing the trial function $\theta(t) = A\cos(\omega t)$ into Eq. (12) we obtain

$$J(A) = \int_0^{T/4} \begin{pmatrix} \frac{1}{2}\alpha A^2\omega^2 \sin^2(\omega t) - \frac{1}{2}\beta A^2\cos^2(\omega t) + \frac{1}{6}\beta A^4\cos^4(\omega t) - \frac{1}{48}\beta A^6\cos^6(\omega t) \\ + \frac{1}{1152}\beta A^8\cos^8(\omega t) + \frac{1}{2}\lambda A^2\cos^2(\omega t) - \frac{1}{24}\lambda A^4\cos^4(\omega t) \end{pmatrix} dt \tag{13}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\begin{pmatrix}\alpha A\omega^2\sin^2(\omega t)-\beta A\cos^2(\omega t)+\frac{2}{3}\beta A^3\cos^4(\omega t)-\frac{1}{8}\beta A^5\cos^6(\omega t)\\ +\frac{1}{144}\beta A^7\cos^8(\omega t)+\lambda A\cos^2(\omega t)-\frac{1}{6}\lambda A^3\cos^4(\omega t)\end{pmatrix}dt=0 \quad (14)$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\begin{pmatrix}\alpha A\omega^2\sin^2 t-\beta A\cos^2 t+\frac{2}{3}\beta A^3\cos^4 t-\frac{1}{8}\beta A^5\cos^6 t\\ +\frac{1}{144}\beta A^7\cos^8 t+\lambda A\cos^2 t-\frac{1}{6}\lambda A^3\cos^4 t\end{pmatrix}dt=0 \quad (15)$$

Solving Eq. (15), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\left(-\beta A\cos^2 t+\frac{2}{3}\beta A^3\cos^4 t-\frac{1}{8}\beta A^5\cos^6 t+\frac{1}{144}\beta A^7\cos^8 t+\lambda A\cos^2 t-\frac{1}{6}\lambda A^3\cos^4 t\right)}{\int_0^{\frac{\pi}{2}}\left(\alpha A\sin^2 t\right)} \quad (16)$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

$$\omega=\sqrt{\frac{\beta}{\alpha}-\frac{1}{2}\frac{\beta}{\alpha}A^2+\frac{5}{64}\frac{\beta}{\alpha}A^4-\frac{35}{9216}\frac{\beta}{\alpha}A^6-\frac{\lambda}{\alpha}+\frac{1}{8}\frac{\lambda}{\alpha}A^2} \quad (17)$$

By substituting Eq. (11) in to Eq. (17) we have

$$\omega_{VA}=\sqrt{\Omega^2-\frac{g}{R}+\frac{1}{8}\frac{g}{R}A^2-\frac{1}{2}\Omega^2A^2+\frac{5}{64}\Omega^2A^4-\frac{35}{9216}\Omega^2A^6} \quad (18)$$

Hence, the approximate solution can be readily obtained

$$\theta(t)=A\cos\left(\sqrt{\Omega^2-\frac{g}{R}+\frac{1}{8}\frac{g}{R}A^2-\frac{1}{2}\Omega^2A^2+\frac{5}{64}\Omega^2A^4-\frac{35}{9216}\Omega^2A^6}\ t\right) \quad (19)$$

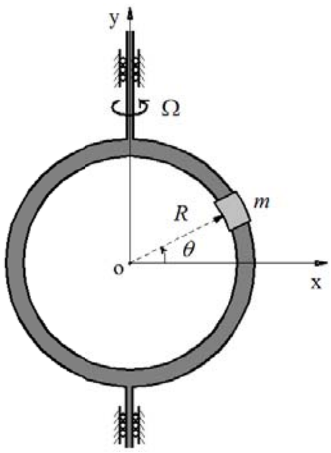

Fig. 1 Particle moving without friction on a rotating circular

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method (Appendix A) is

$$\omega_{EBM} = \frac{\sqrt{4R\left(-R\Omega^2\cos^2(\frac{\sqrt{2}}{2}A)+2g\cos(\frac{\sqrt{2}}{2}A)+R\Omega^2\cos^2(A)-2g\cos(A)\right)}}{RA} \tag{20}$$

The numerical solution by with 4th order Runge-Kutta method (Appendix B) for nonlinear equation is

$$\begin{aligned} \dot{\theta} &= y & \theta(0) &= A \\ \dot{y} &= \frac{mR^2\Omega^2\sin(\theta)\cos(\theta)-mgR\sin(\theta)}{mR^2} & y(0) &= 0 \end{aligned} \tag{21}$$

3.2 Example 2

An example of a single degree of freedom conservative system has been considered that is described by an equation as follows. A rigid rod is rigidly attached to the axle as shown in Fig. 2. The wheels roll without slip as the pendulum swings back and forth. The wheel is restrained by a spring which is fixed to a wall on the other side .Only the ball on the end of the pendulum has appreciable mass and it may be considered as a particle. The governing equation of the motion is (Nayfe 1973)

$$m\left(l^2+r^2-2rl\cos(\theta)\right)\ddot{\theta}+mrl\sin(\theta)\dot{\theta}^2+mgl\sin(\theta)+kr^2\theta=0 \tag{22}$$

With initial conditions

$$\theta(0)=A, \qquad \dot{\theta}(0)=0. \tag{23}$$

By using the Taylor's series expansion for $\cos(\theta(t))$, $\sin(\theta(t))$ and by some manipulation in Eq. (22) we can re-write Eq. (22) in the following form

$$\left(\alpha_1+\alpha_2-2\alpha_3(1-\tfrac{1}{2}\theta^2+\tfrac{1}{24}\theta^4)\right)\ddot{\theta}+\alpha_3\left(\theta-\tfrac{1}{6}\theta^3\right)\dot{\theta}^2+\alpha_4\left(\theta-\tfrac{1}{6}\theta^3\right)+\alpha_5\theta=0, \tag{24}$$

Where

$$\alpha_1=ml^2,\ \alpha_2=mr^2,\ \ \alpha_3=mrl,\ \ \alpha_4=mgl,\ \ \alpha_5=kr^2 \tag{25}$$

Its variational formulation can be readily obtained from Eq. (24) as follows

$$J(\theta)=\int_0^t\left(\frac{1}{2}\alpha_1\dot{\theta}^2+\frac{1}{2}\alpha_2\dot{\theta}^2-\alpha_3\dot{\theta}^2-\frac{1}{24}\alpha_4\theta^4\dot{\theta}^2+\frac{1}{2}\alpha_3\dot{\theta}^2\theta^2-\frac{1}{24}\alpha_4\theta^4+\frac{1}{2}\alpha_5\theta^2\right)dt\,. \tag{26}$$

Choosing the trial function $\theta(t)=A\cos(\omega t)$ into Eq. (26) we obtain

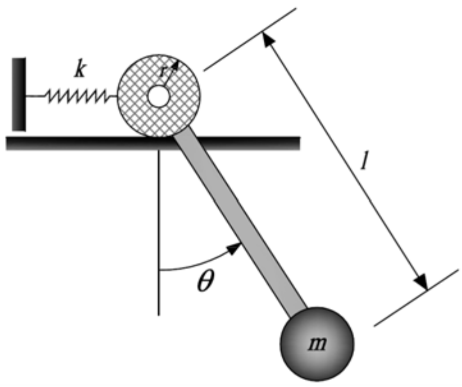

Fig. 2 Pendulum attached to rolling wheels that are restrained by a spring

$$J(A)=\int_0^{T/4}\begin{pmatrix}\frac{1}{2}\alpha_1 A^2\omega^2\sin^2(\omega t)+\frac{1}{2}\alpha_2 A^2\omega^2\sin^2(\omega t)-\alpha_3 A^2\omega^2\sin^2(\omega t)\\ -\frac{1}{24}\alpha_3 A^6\omega^2\cos^4(\omega t)\sin^2(\omega t)+\frac{1}{2}\alpha_3 A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)\\ +\frac{1}{2}\alpha_4 A^2\cos^2(\omega t)-\frac{1}{24}\alpha_4 A^4\cos^4(\omega t)+\frac{1}{2}\alpha_4 A^2\cos^2(\omega t)\end{pmatrix}dt \tag{27}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\begin{pmatrix}\alpha_1 A\omega^2\sin^2(\omega t)+\alpha_2 A\omega^2\sin^2(\omega t)-2\alpha_3 A\,\omega^2\sin^2(\omega t)\\ -\frac{1}{4}\alpha_3 A^5\omega^2\cos^4(\omega t)\sin^2(\omega t)+2\alpha_3 A^3\omega^2\sin^2(\omega t)\cos^2(\omega t)\\ +\alpha_4 A\cos^2(\omega t)-\frac{1}{6}\alpha_4 A^3\cos^4(\omega t)+\alpha_5 A\cos^2(\omega t)\end{pmatrix}dt=0 \tag{28}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\begin{pmatrix}\alpha_1 A\omega^2\sin^2 t+\alpha_2 A\omega^2\sin^2 t-2\alpha_3 A\,\omega^2\sin^2 t-\frac{1}{4}\alpha_3 A^5\omega^2\cos^4 t\sin^2 t\\ +2\alpha_3 A^3\omega^2\sin^2 t\cos^2 t+\alpha_4 A\cos^2 t-\frac{1}{6}\alpha_4 A^3\cos^4 t+\alpha_5 A\cos^2 t\end{pmatrix}dt=0 \tag{29}$$

Solving Eq. (29), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\alpha_4 A\cos^2 t-\frac{1}{6}\alpha_4 A^3\cos^4 t}{\int_0^{\frac{\pi}{2}}\left(\alpha_1 A\sin^2 t+\alpha_2 A^2\sin^2 t-2\alpha_3 A\sin^2 t+2\alpha_3 A^3\sin^2 t\cos t-\dfrac{1}{4}\alpha_3 A^5\sin^2 t\cos^4 t\right)} \tag{30}$$

Then we have

$$\omega_{VA}=2\sqrt{\frac{\alpha_4 A^3-A\alpha_4-8\alpha_5}{32\alpha_2-32\alpha_1+A^4\alpha_3-16A^2\alpha_3+64\alpha_3}} \tag{31}$$

By substituting Eq. (25) in to Eq. (31) we have

$$\omega_{VA} = 2\sqrt{\frac{mglA^3 - mglA - 8kr^2}{32mr^2 - 32ml^2 + A^4mrl - 16A^2mrl + 64mrl}} \tag{32}$$

According to $\theta(t) = A\cos(\omega t)$ and Eq. (32), we can obtain the following approximate solution

$$\theta(t) = A\cos\left(2\sqrt{\frac{mglA^3 - mglA - 8kr^2}{32mr^2 - 32ml^2 + A^4mrl - 16A^2mrl + 64mrl}}\,t\right) \tag{33}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method (Appendix A) is

$$\omega_{EBM} = \frac{\sqrt{-4mgl\cos(A) + 4mgl\cos(\frac{\sqrt{2}}{2}A) + kr^2A^2}}{\sqrt{A^2\left(-ml^2 - mr^2 + 2mrl\cos(\frac{\sqrt{2}}{2}A)\right)}} \tag{34}$$

The numerical solution by with 4th order Runge-Kutta method (Appendix B) for nonlinear equation is

$$\begin{aligned} \dot{\theta} &= y & \theta(0) &= A \\ \dot{y} &= -\frac{mrl\sin(\theta)\dot{\theta}^2 + mgl\sin(\theta) + kr^2\theta}{ml^2 + mr^2 - 2mrl\cos(\theta)} & y(0) &= 0 \end{aligned} \tag{35}$$

3.3 Example 3

The motion of a particle on a rotating parabola .The governing equation of motion and initial conditions can be expressed as (Nayfe1973)

$$(1 + 4q^2u^2)\ddot{u} + 4q^2u\dot{u}^2 + \Delta u = 0 \qquad u(0) = A, \quad \dot{u}(0) = 0 \tag{36}$$

Where $q > 0$ and $\Delta > 0$ are known positive constants.

Variational formulation of Eq. (36) can be readily obtained as follows

$$J(u) = \int_0^t \left(\frac{1}{2}\dot{u}^2 + 2q^2u^2\dot{u}^2 + \frac{1}{2}\Delta u^2\right)dt \tag{37}$$

Substituting the trial function $u(t) = A\cos(\omega t)$ into Eq. (37), we obtain

$$J(A) = \int_0^{T/4}\left(\frac{1}{2}A^2\omega^2 sin^2(\omega t) + 2q^2A^4\omega^2 sin^2(\omega t)cos^2(\omega t) + \frac{1}{2}\Delta A^2 cos^2(\omega t)\right)dt \tag{38}$$

The stationary condition with respect to *A* leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(A\,\omega^2 sin^2\left(\omega t\right)+8qA^3\omega^2 sin^2\left(\omega t\right)cos^2\left(\omega t\right)+\Delta A cos^2\left(\omega t\right)\right)dt\ =0 \tag{39}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(A\,\omega^2 sin^2 t+8qA^3\omega^2 sin^2 t\ cos^2 t+\Delta A cos^2 t\right)dt\ =0 \tag{40}$$

Solving Eq. (40), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\left(\Delta A cos^2 t\right)dt}{\int_0^{\frac{\pi}{2}}\left(A\,sin^2 t+8q\,A^3 sin^2 t\ cos^2 t\right)dt} \tag{41}$$

Then we have

$$\omega_{VA}=\sqrt{\frac{\Delta}{1+2A^2q^2}} \tag{42}$$

According to Eqs. (3) and (42), we can obtain the following approximate solution

$$u(t)=A\cos\left(\sqrt{\frac{\Delta}{1+2A^2q^2}}\,t\right) \tag{43}$$

The exact period is

$$\omega_{Exact}=2\pi\Big/\,4A\int_0^{\pi/2}\frac{\sqrt{1+4q^2A^2\cos^2 t}\ \sin t}{\sqrt{\Delta A^2\sin^2 t}}\,dt \tag{44}$$

For comparison of the approximate solution, frequency obtained from solution of nonlinear equation with the energy balance method (Appendix A) is

$$\omega_{EBM}=\sqrt{\frac{\Delta}{1+2A^2q^2}} \tag{45}$$

4. Results and discussions

In this section, to illustrate and verify the accuracy of this approximate analytical approach, some comparisons of the analytic responses with the numerical solutions and exact solutions are presented for these three examples.

In example 1, Table 1 is the comparison of the variational approach and energy balance method of different parameters of the problem. The results are very close together.

Figs. 3 and 4 are shown the comparison of Runge-kutta's algorithm and energy balance method with variational approach for time history response and phase plan curve of the problems for two cases: (Fig. 3) : $A=\pi/6,\ m=3,\ R=0.4,\ \Omega=1,\ g=10$,

(Fig. 4) : $A = \pi / 4, \; m = 3, \; R = 1.2, \; \Omega = 2, \; g = 10$.

The motion of the system is periodic and also functions of initial conditions. The effects of angular velocity (Ω) and radius(R) on nonlinear frequency is shown in Fig. 5.

In example 2, Table 2 represents the comparison of variational approach and energy balance method for various parameters of the system. An excellent agreement can be seen from this comparison.

Time history response and phase plan curves are shown in Figs. 6 and 7 for these two cases

(Fig. 6): $m = 5, \; l = 1.5, \; r = 0.5, \; g = 10, \; k = 500, \; A = \pi / 3$

(Fig. 7): $m = 5, \; l = 1.5, \; r = 0.5, \; g = 10, \; k = 500, \; A = \pi / 3$.

Figs. 8 and 9 are shown the Influence of important parameters on the nonlinear frequency of the systems. These important parameters are: axle length (l) and radius of wheel (r), mass (m) and spring stiffness (k) of system.

Table 1 Comparison of nonlinear frequency of two approximate VA and EBM solution corresponding to various parameters of system (example 1)

	Constant parameter					Approximate solution		
No	A	m	g	R	Ω	VA	EBM	Error %
1	$\pi/12$	2	10	0.5	1.5	4.2018	4.2018	0.0013
2	$\pi/8$	3	10	2	1.8	1.3811	1.3813	0.0165
3	$\pi/6$	5	10	1.2	2.5	1.6181	1.6192	0.0661
4	$\pi/4$	4	10	1.5	0.8	2.3857	2.3883	0.1104
5	$\pi/3$	1	10	0.4	0.5	4.6301	4.6456	0.3349
6	$\pi/2$	3	10	0.8	1	2.9086	2.9728	2.2099

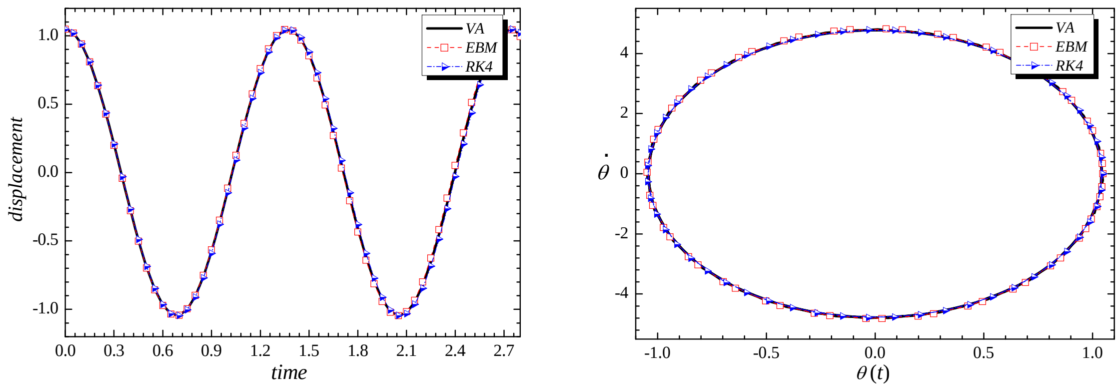

Fig. 3 Comparison of time history response and phase curve of variational approach and energy balance method with the numerical solution for $A = \pi / 6, \; m = 3, \; R = 0.4, \; \Omega = 1, \; g = 10$

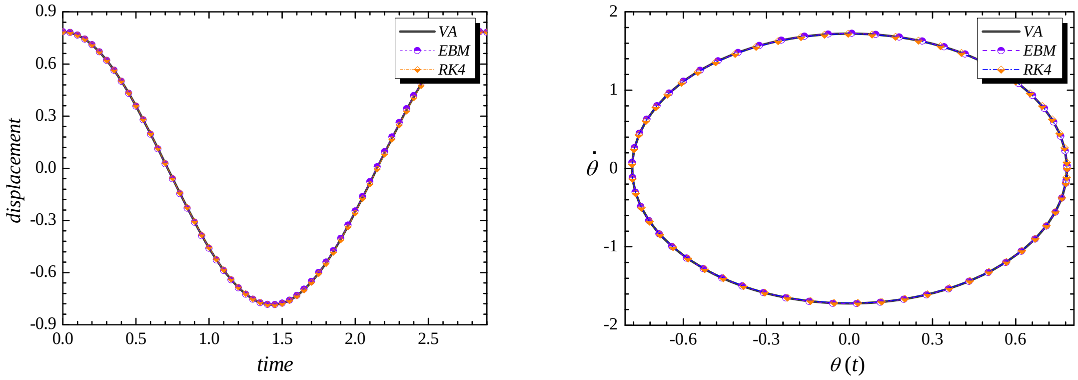

Fig. 4 Comparison of time history response and phase curve of variational approach and energy balance method with the numerical solution for $A=\pi/4,\ m=3,\ R=1.2,\ \Omega=2,\ g=10$

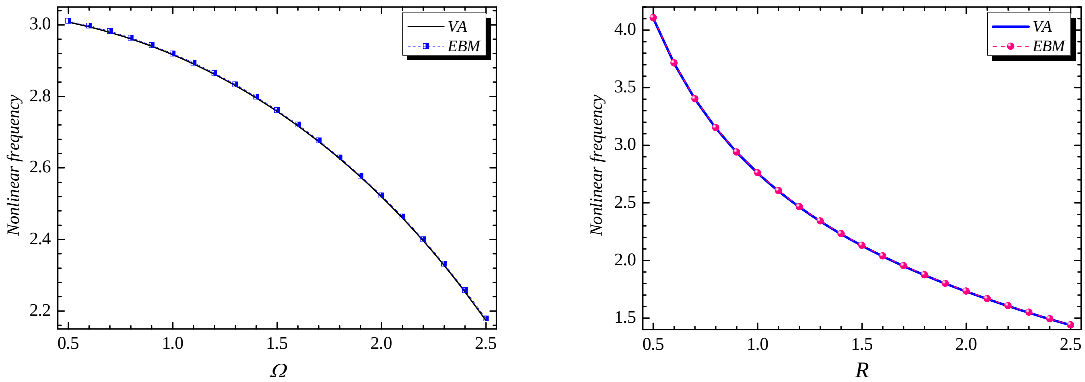

Fig. 5 Effect of angular velocity(Ω) and radius(R) on nonlinear frequency

Table 2 Comparison of nonlinear frequency of two approximate VA and EBM solution corresponding to various parameters of system (example 2)

No	Constant Parameter						Approximate solution		Error%
	A	l	r	m	k	g	VA	EBM	
1	$\pi/12$	3	0.5	8	1000	10	3.1112	3.1112	0
2	$\pi/8$	2	1	5	800	10	12.4827	12.4801	0.0209
3	$\pi/6$	1.5	0.5	3	1200	10	10.1970	10.1945	0.0243
4	$\pi/4$	1	0.2	10	1500	10	4.6666	4.6667	0.0022
5	$\pi/3$	2	1.5	8	500	10	9.4122	9.3144	1.0498
6	$\pi/2$	1	0.3	6	1800	10	6.4988	6.4438	0.8528

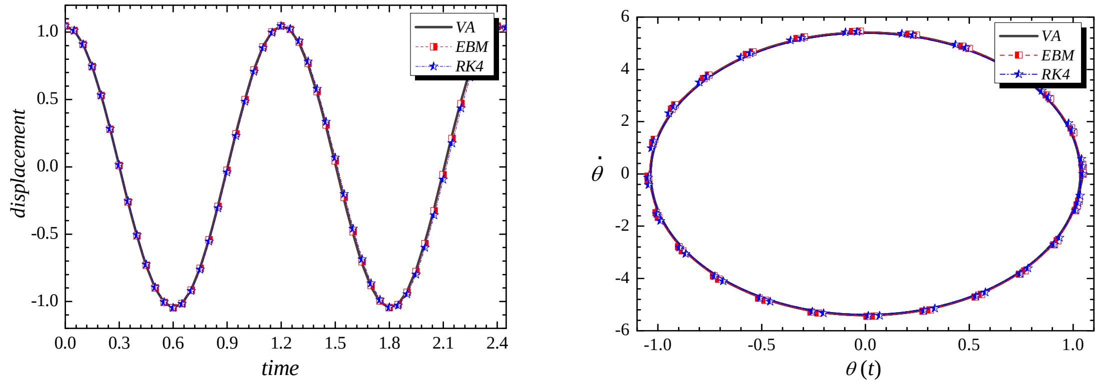

Fig. 6 Comparison of time history response and phase curve of variational approach and energy balance method with the numerical solution for $m=5$, $l=1.5$, $r=0.5$, $g=10$, $k=500$, $A=\pi/3$

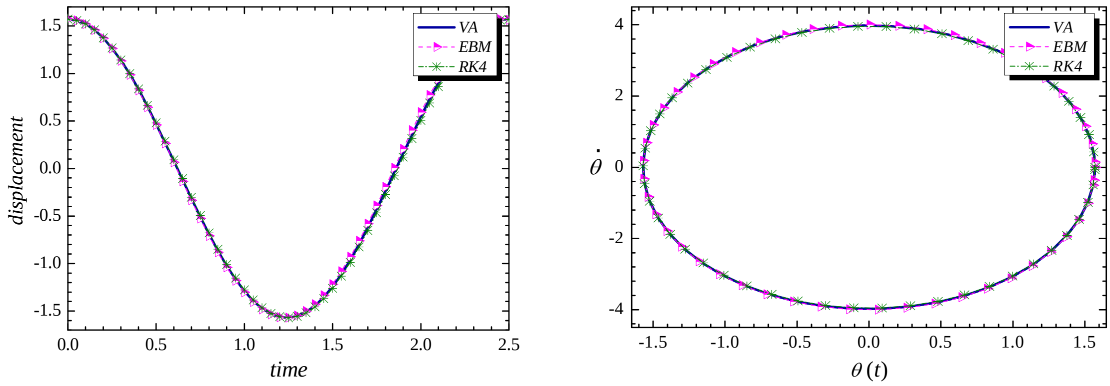

Fig. 7 Comparison of time history response and phase curve of variational approach and energy balance method with the numerical solution for $m=10$, $l=2$, $r=0.3$, $g=10$, $k=1000$, $A=\pi/2$

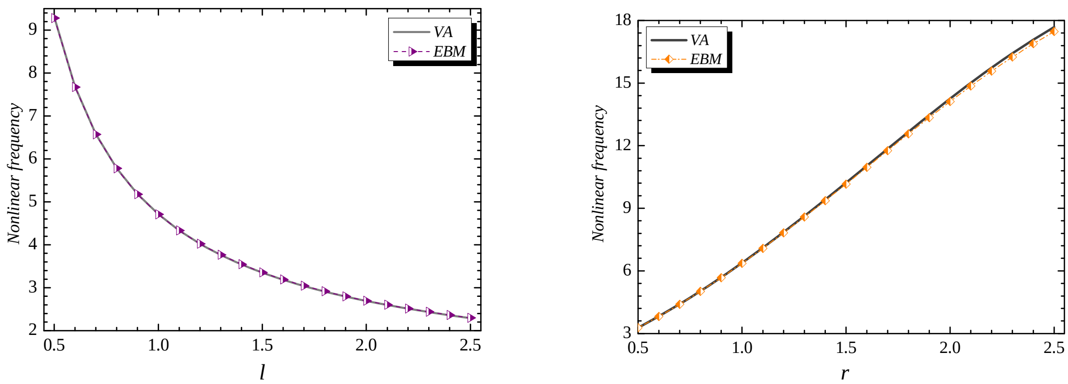

Fig. 8 Influence of axle length (l) and radius of wheel (r) on nonlinear frequency

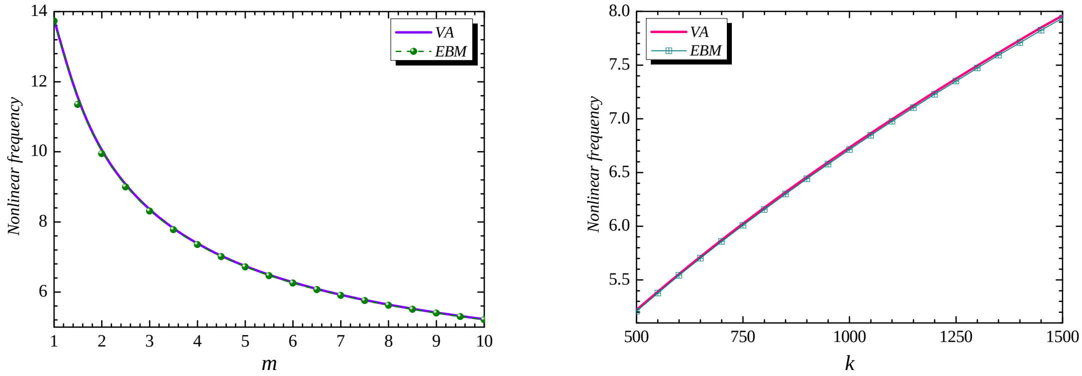

Fig. 9 Influence of mass (m) and spring stiffness (k) of system on nonlinear frequency

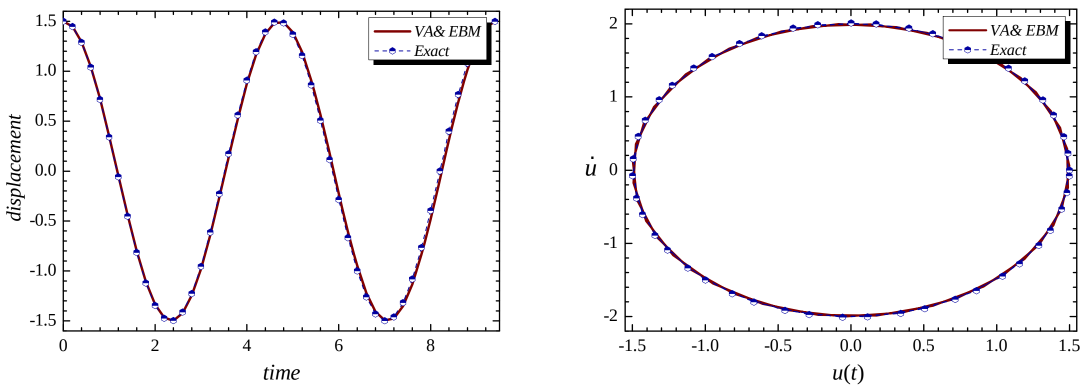

Fig. 10 Comparison of time history response and phase curve of variational approach and energy balance method with the Exact solution for $A = 1.5$, $\Delta = 2.5$, $q = 0.3$,

Table 3 Comparison of nonlinear frequency of two approximate VA and EBM solution with exact solution corresponding to various parameters of system (example 3)

A	q	Δ	ω_{VA} & ω_{EBM}	ω_{Exact}	Error%
0.5	1	0.5	0.5774	0.5815	0.7135
0.5	0.5	2	1.3333	1.3344	0.0774
1	0.8	1.5	0.8111	0.8288	2.1399
1	0.7	0.5	0.5025	0.5108	1.6300
1.5	0.5	2	0.9701	0.9888	1.8836
1.5	0.3	2.5	1.3339	1.3410	0.5298
2	0.2	4	1.7408	1.7473	0.3725
2	0.4	1	0.6623	0.6767	2.1399

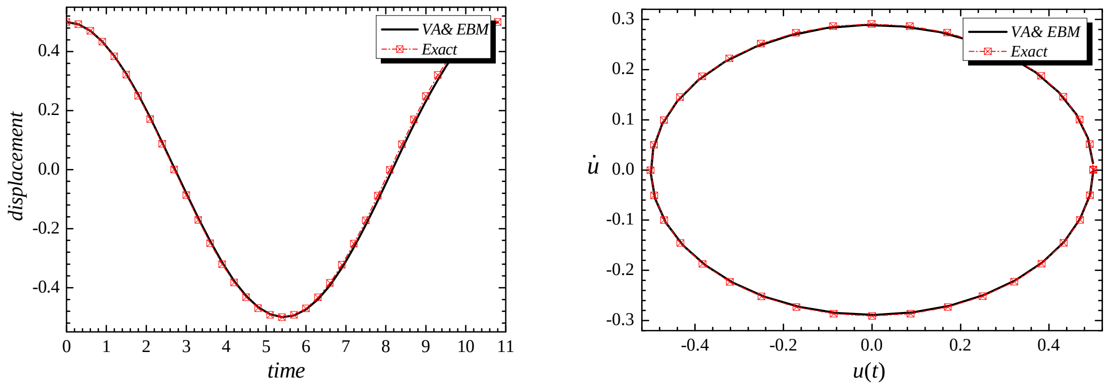

Fig. 11 Comparison of time history response and phase curve of variational approach and energy balance method with the Exact solution for $A = 0.5,\ \ \Delta = 0.5,\ \ q = 1$

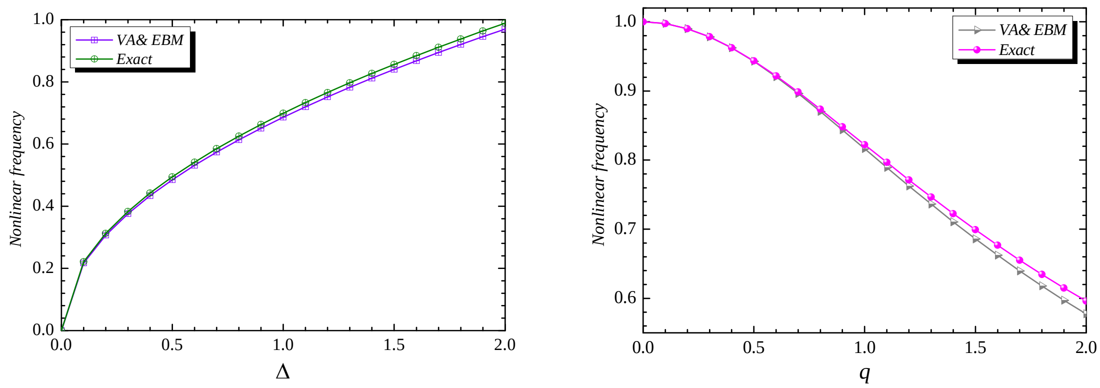

Fig. 12 Influence of various constant parameter (Δ) and (q) of system on nonlinear frequency

In example 3, frequency comparison for this example between variational approach and energy balance method and exact solution has been in Table 3.

Figs. 10 and 11 are the comparisons of variational approach and exact solutions for (Fig. 10): $A = 1.5,\ \ \Delta = 2.5,\ \ q = 0.3,$ and (Fig. 11) $A = 0.5,\ \ \Delta = 0.5,\ \ q = 1$.

The final figure is shown the influence of (Δ) and (q) on nonlinear frequency of the system.

It is obvious that VA shows high accuracy with the numerical solution and is quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the VA could be potentiality used for the analysis of strongly nonlinear oscillation problems.

5. Conclusions

In this study, a new application of variational approach has been completely presented and discussed. The results of variational approach and energy balance method and numerical solutions have been compared. Three different examples were presented and effects of important parameters on the response of the systems were studied. The present approach doesn't need any small perturbations and with only one iteration lead to a high accurate solution. It has been proven that the variational approach is very efficient, comfortable and sufficiently exact in engineering problems. Variational approach can be simply extended to any nonlinear conservative equations for the analysis of nonlinear systems.

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Appendix A: Basic Idea of Energy Balance Method (EBM)

Consider a general nonlinear oscillator in the form (He 2008)

$$\ddot{\theta} + f\left(\theta(t)\right) = 0 \tag{A.1}$$

In which θ and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(\theta) = \int_0^t \left(-\tfrac{1}{2}\dot{\theta}^2 + F(\theta)\right) dt \tag{A.2}$$

Where $T = 2\pi/\omega$ is period of the nonlinear oscillator, $F(\theta) = \int f(\theta) d\theta$.

Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}\dot{\theta}^2 + F(\theta) = F(A) \tag{A.3}$$

Or

$$\Re(t) = \frac{1}{2}\dot{\theta}^2 + F(\theta) - F(A) = 0 \tag{A.4}$$

Oscillatory systems contain two important physical parameters, i.e., The frequency ω and the amplitude of oscillation. A . So let us consider such initial conditions

$$\theta(0) = A, \qquad \dot{\theta}(0) = 0 \tag{A.5}$$

We use the following trial function to determine the angular frequency ω

$$\theta(t) = A\cos(\omega t) \tag{A.6}$$

Substituting (A.6) into θ term of (A.4), yield

$$\Re(t) = \frac{1}{2}\omega^2 A^2 \sin^2(\omega t) + F\left(A\cos(\omega t)\right) - F(A) = 0 \tag{A.7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make $\Re$ zero for all values of t by appropriate choice of ω. Since Eq. (A.6) is only an approximation to the exact solution, $\Re$ cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega = \sqrt{\frac{2\left(F(A)\right) - F\left(A\cos(\omega t)\right)}{A^2 \sin^2(\omega t)}} \tag{A.8}$$

Appendix B: Basic Idea of Runge-Kutta (RK)

The most often used method of the Runge-Kutta family is the Fourth-Order one, which extends the idea of the mid-point method, by jumping 1/4th of the way first, then going half-way, a la the mid-point method, then going 3/4th of the way and finally jumping all the way.

Consider an initial value problem be specified as follows

$$\dot{\theta} = f\ (t, \theta), \quad \theta(t_0) = \theta_0 \tag{B.1}$$

θ is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{6} h\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{B.2}$$

for n = 0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f\ (t_n, \theta_n), \\ k_2 &= f\ \left(t_n + \frac{1}{2}h,\ \ \theta_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\ \left(t_n + \frac{1}{2}h,\ \ \theta_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\ (t_n + h,\ \ \theta_n + hk_3). \end{aligned} \tag{B.3}$$

Where u_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. and the next value (θ_{n+1}) is determined by the present value (θ_n) plus the weighted average of four increments, where each increment is the product of the size of the interval, h, and an estimated slope specified by function f on the right-hand side of the differential equation.

- k_1 is the increment based on the slope at the beginning of the interval, using $\dot{\theta}$,
- k_2 is the increment based on the slope at the midpoint of the interval, using $\dot{\theta} + \frac{1}{2}hk_1$;
- k_3 is again the increment based on the slope at the midpoint, but now using $\dot{\theta} + \frac{1}{2}hk_2$;
- k_4 is the increment based on the slope at the end of the interval, using $\dot{\theta} + hk_3$.