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Experimental determination of the buckling load of a flat plate by the use of dynamic parameters

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Abstract. After manufacturing a structure, the assembly of structural components is often not as perfect as expected due to the immaturity of current engineering techniques. Thus the actual buckling load for an element is sometimes not consistent with that predicted in the design. For design considerations, it is necessary to establish an analytical method for determining the buckling load experimentally. In this paper, a dynamic method is described for determining the linear buckling loads for elastic, perfectly flat plates. The proposed method does not require the application of in-plane loads and is feasible for arbitrary types of boundary conditions. It requires only the vibrational excitation of the plate. The buckling load is determined from the measured natural frequencies and vibration mode shapes.

Key words: buckling load; natural frequency; mode shape.

1. Introduction

Thin plate elements in engineering structures are often subjected to in-plane loads of normal and shearing forces. If these in-plane forces are sufficiently small, the equilibrium is stable and the resulting deformations are characterized by the absence of lateral displacement. As the magnitude of these in-plane forces increases to a certain load intensity, the stable equilibrium becomes unstable and a marked change in the character of the deformation pattern takes place (Szilard 1974). That is, simultaneous to the application of a disturbing force, (with the exception of in-plane deformations), lateral displacements are introduced. In this condition, the plate is buckled. The importance of the buckling load is the initiation of a deflection pattern, which, if the load is further increased, rapidly leads to very large lateral deflections and eventually to structure failure (Ghali and Neville 1978). In many practical situations it is necessary to determine the magnitude of the in-plane loads at which a plate buckles. Conventionally, the buckling load is determined by measuring the plate's deflections while increasing the imposed in-plane loads. However, an in-plane force may be difficult to impose experimentally in many cases. Thus, an alternative method for determining the buckling load without imposing in-plane loads must be developed. Research efforts (e.g., Lurie 1952, Baruch 1973, Sweet *et al.* 1971, 1976, 1977, Segall 1980, 1986, Sang 1988, Laura 1989) in this field of buckling measurement may be classified into two categories: the static approach and the dynamic approach. For the static approach, the implementation of a simulated load on the structure may not be easy. For the dynamic approach, the characteristic dynamic parameters related to the bucking

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load are the major considerations. In 1997, Go *et al.* proposed an experimental method to determine the buckling load of a straight structural member using dynamic parameters (Go *et al.* 1997). In this paper, Go's proposal is extended to the plate problem for determining the buckling load. This method requires vibrational excitation of the plate. The buckling load is then determined from the vibratory data, natural frequencies and mode shapes.

2. Analysis model

Consider a flat plate subjected to the action of in-plane forces, N_x, N_y, and N_{xy}, as shown in Fig. 1. Its strain energy, U, may be related to its deflected shape W (Timoshenko 1984), viz.

$$U=\frac{1}{2}\iint_{(A)}\left[N_x\left(\frac{\partial W}{\partial X}\right)^2+2N_{xy}\left(\frac{\partial W}{\partial X}\frac{\partial W}{\partial Y}\right)+N_y\left(\frac{\partial W}{\partial Y}\right)^2\right]dx\,dy \tag{1}$$

The buckling shape of the plate may be expressed by the use of Lagrange's interpolation function $W(x, y)$ (Zienkiewicz 1989),

$$W(x, y)=\sum_{i=1}^{n}\sum_{j=1}^{n}L_{ij}D_{ij} \tag{2}$$

Where $L_{ij}=L_i(x)\times L_j(y)$

$$=\frac{(x-x_1)..(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_1)..(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}\times\frac{(y-y_1)..(y-y_{j-1})(y-y_{j+1})...(y-y_n)}{(y_j-y_1)..(y_j-y_{j-1})(y_j-y_{j+1})...(y_j-y_n)} \tag{3}$$

and D_{ij} denotes the deflection at the interpolation point (x_i, y_j).

The equivalent nodal force F_{ij}, as shown in Fig. 2, which is associated with the displacement D_{ij}, may be obtained by using Castigliano's First Theorem (Ghali 1978)

$$F_{ij}=\frac{\partial U}{\partial D_{ij}} \qquad i, j=1, 2,...n \tag{4}$$

By introducing the definitions of in-plane forces N_X, N_Y and N_{XY} as shown in Fig. 1

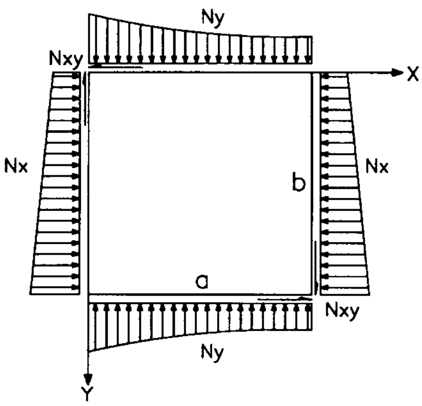

Fig. 1 Flat plate subjected to forces N_x, N_y and N_{xy}

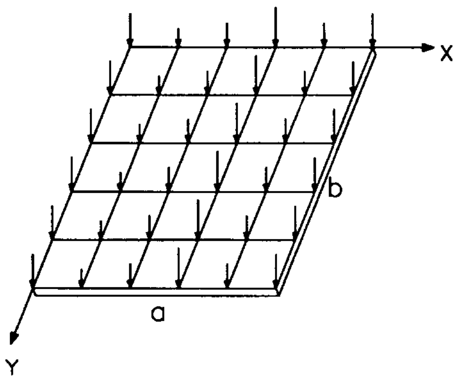

Fig. 2 Scheme for the coordination of equivalent nodal forces

$$N_X = Nf_1(x, y)$$
$$N_Y = Nf_2(x, y)$$
$$N_{XY} = Nf_3(x, y) \tag{5}$$

then

$$F_{ij} = N\sum_{k=1}^{n}\sum_{l=1}^{n}\Big[\iint_{(A)}[f_1(x, y)L_i'(x)L_j(y)L_k'(x)L_l(y)+f_2(x, y)L_i(x)L_j'(y)L_k(x)L_l'(y)$$
$$+f_3(x, y)L_i(x)L_j'(y)L_k'(x)L_l(y)+f_3(x, y)L_i'(x)L_j(y)L_k(x)L_l'(y)]dxdy\Big]D_{kl} \tag{6}$$

D_{kl} denotes the deflection at the point (K, l). Eq. (6) may be represented in matrix form as

$$\{F^*\} = \mathbf{N}[B]\{D^*\} \tag{7}$$

where the matrix $\{F^*\}$ and $\{D^*\}$ are column matrices. Also, the equivalent force $\{F^*\}$, may be related to the deflection $\{D^*\}$ as

$$\{D^*\} = [G]\{F^*\} \tag{8}$$

where $[G]$ is the flexibility matrix and its element G_{ij} the flexibility influence coefficient defined as the displacement at node i due to a unit load applied at node j. Substituting Eq. (7) into Eq. (8) yields

$$\{D^*\} = [G]\{F^*\} = \mathbf{N}[G][B]\{D^*\},\ \lambda\{D^*\} = [G][B]\{D^*\} \tag{9}$$

where $\lambda = \dfrac{1}{N}$

For a non -trivial solution one must have

$$|[G][B] - \lambda[I]| = 0 \tag{10}$$

The solution for the maximum eigenvalue $\lambda_{\max}$, is related to the bucking load $\mathbf{N}_{cr}$

using
$$N_{cr} = \frac{1}{\lambda_{\max}} \tag{11}$$

3. Establishment of the flexibility matrix

The differential equation of free vibration for a structural flat plate may be stated as (Timoshenko 1984):

$$D\nabla^4 W + \rho\frac{\partial^2 W}{\partial t^2} = 0 \tag{12}$$

where $D = \dfrac{Eh^3}{12(1-\upsilon^2)}$

Assuming that the solution of Eq. (12) is separable into time and space factors, one may write

$$W(x,y,t) = \sum_{i=1}^{\infty}\phi_i(x,y)T_i(t) \tag{13}$$

Substituting Eq. (13) into Eq. (12), leads to two differential equations

$$\mathbf{T}_i''+\omega_i^2\mathbf{T}_i=0 \tag{14}$$

$$\mathbf{D}\nabla^4\phi_i-\rho\omega_i^2\phi_i=0 \tag{15}$$

where ω_i and $\phi_i(x,y)$ are the natural frequency and corresponding mode shape of the ith mode respectively. The model shape, $\phi_i(x,y)$ must satisfy the orthogonal condition

$$\iint_{(A)}\phi_i\phi_j\rho dxdy=M_i\delta_{ij} \tag{16}$$

where δ_{ij} is the Kronecker delta, ρ is the area density, and M_i is the generalized mass determined using

$$M_i=\iint_{(A)}\phi_i^2\rho dxdy \tag{17}$$

The equivalent equation of the above-mentioned system, which is subjected to a static unit load $F=\delta(x-\xi)\delta(y-\eta)$, at the position $x=\xi$, $y=\eta$, may be expressed as

$$\mathbf{D}\nabla^4 W=\delta(x-\xi)\delta(y-\eta) \tag{18}$$

By Galerkin's method, $W(x, y)$ may be approximated by a linear combination of function (ϕ_1, ϕ_2, ϕ_3,….) as

$$W(x,y)=\sum_{k=1}^{\infty}a_k\phi_k(x,y) \tag{19}$$

where a_k are unknown constants that can be determined, and thus

$$\sum_{k=1}^{\infty}a_k\iint_{(A)}(D\nabla^4\phi_k(x,y))\cdot\phi_i(x,y)dxdy=\iint_{(A)}\delta(x-\xi)\delta(y-\eta)\phi_i(x,y)dxdy \tag{20}$$

By comparing Eq. (15) with Eq. (20), the unknown constants a_k may be simplified by the use of Eq. (17) to

$$a_k=\frac{\phi_k(\xi,\eta)}{M_k\omega_k^2} \tag{21}$$

The deflection curve $W(x, y)$ may thus be obtained by substituting Eq. (21) into Eq. (19) to yield

$$W(x,y)=\sum_{k=1}^{\infty}\frac{\phi_k(\xi,\eta)\phi_k(x,y)}{M_k\omega_k^2} \tag{22}$$

For $x=\xi$, $y=\eta$, the flexibility influence coefficient $\mathbf{G}_{ij}$ in Eq. (8) can then be determined:

$$\mathbf{G}_{ij}=\mathbf{G}(x,y;\ \xi,\eta)=W(x,y)=\sum_{k=1}^{\infty}\frac{\phi_k(\xi,\eta)\phi_k(x,y)}{M_k\omega_k^2} \tag{23}$$

The parameter $\mathbf{G}(x,y;\ \xi,\eta)$ denotes the deflection at point (x,y) due to a unit lateral load at point (ξ,η) (Bisplinghoff 1955). All of the information regarding the boundary conditions and material properties are contained implicitly in the parameter $\mathbf{G}$ (Segall 1986).

4. Solution consideration

Theoretically, the interpolation function of Eq. (3) for the buckling shape plays a dominating role in the accuracy. With proper selection of the interpolation points (i.e., stations in the experimentation) one may obtain a well-defined shape, which leads to satisfactory solutions. Examples with extreme end constraints (e.g., free end and clamped end constraints) are chosen to illustrate the feasibility of this method. The feasibility of the proposed approach was investigated using the analysis of a rectangular flat plate with various end constraints, as shown in Fig. 4(a-f), where S, C, and F represent simply supported, clamped and free ends, respectively. A simplified and practical case, which still represents the characteristics of the general problem, is a rectangular plate subjected to in-plane forces (Segall 1986). For example, in the case N_x=$Nf_1(x, y)$ but N_y and N_{xy}=0, the first five modes of vibration were considered. By referring to Eq. (3) and Eq. (12), the minimal number of stations (5×5), were implemented as shown in Fig. 3. At loading conditions $f_1(x, y)$=1, N_y=N_{xy}=0 and a/b=1.0, the results were very close to the exact solutions, as shown in Table 1

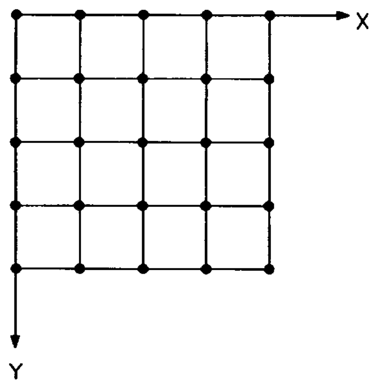

Fig. 3 Uniform distribution of 5×5 stations for the analysis of buckling load

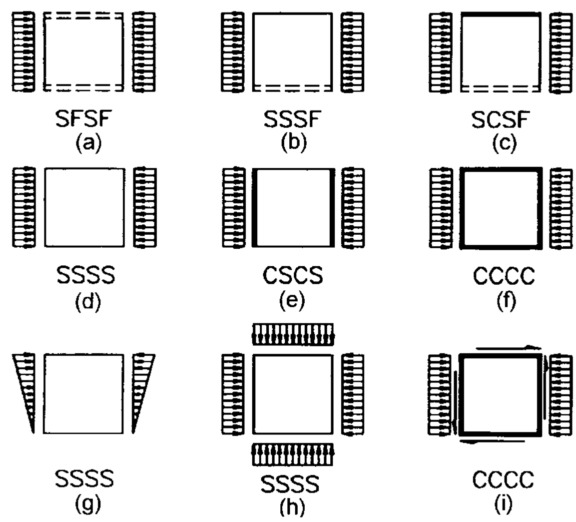

Fig. 4 Flat plates for analysis on solution consideration

(Timoshenko 1984), except for the case of C-C-C-C. Theoretically, the buckling load is significantly affected by the distance between the inflection points of a structural member, viz. the effective length. Thus an increasing number of test stations provides a better representation of the buckling shape and may improve the results. This improvement is also shown in Table 1 for 7×7 stations instead of 5×5 stations. For the case of a non-uniform load distribution, as shown in Fig. 4(g-h), the proposed approach also gives a very good result in comparison with the existing result, as shown in Table 2. The presence of shearing force, (as shown in Fig. 4(i)), leads to good results when the placement of stations is 7×7, as shown in Table 3. All of these examples lead to the conclusion that the 7×7 station arrangement is sufficient for dependable accuracy.

5. Feasibility of experimental identification

In practice, the accuracy of the experimental determination of the buckling load is always affected

Table 1 Buckling load for varied boundary conditions

Boundary condition	Exact solution	5×5 stations		7×7 stations	
	$N_{cr}\left(\frac{D}{b^2h}\right)$	$N_{cr}\left(\frac{D}{b^2h}\right)$	Error%	$N_{cr}\left(\frac{D}{b^2h}\right)$	Error%
S-F-S-F	9.695	9.585	−1.13	9.570	−1.28
S-S-S-F	14.212	14.240	−0.20	14.193	−0.15
S-C-S-F	16.778	16.829	+0.30	16.801	+0.13
S-S-S-S	39.478	35.509	+0.08	39.482	0.00
C-S-C-S	66.521	64.159	−3.55	67.130	+0.92
C-C-C-C	99.387	84.504	−14.97	97.672	−1.72

Table 2 Buckling load for varied loading conditions

	Exact solution	5×5 stations		7×7 stations	
	$\sigma_{cr}\left(\frac{D}{b^2h}\right)$	$\sigma_{cr}\left(\frac{D}{b^2h}\right)$	Error%	$\sigma_{cr}\left(\frac{D}{b^2h}\right)$	Error%
Triangular load	76.983	77.534	+0.72	77.123	+0.18
Biaxial load	19.739	19.754	+0.08	19.741	0.00

Table 3 Buckling load for the clamped plate subjected to axial force and shear force

$\frac{\sigma}{\tau}$	Exact solution	5×5 stations		7×7 stations	
	$\tau_{cr}\left(\frac{D}{b^2h}\right)$	$\tau_{cr}\left(\frac{D}{b^2h}\right)$	Error%	$\tau_{cr}\left(\frac{D}{b^2h}\right)$	Error%
0	145.182	161.597	+11.30	146.973	+1.23
0.5	69.975	67.484	− 3.56	70.890	+1.30
1.0	44.413	39.273	−11.57	44.740	+0.73
1.5	31.978	27.206	−14.92	35.152	+0.54
2.0	24.773	20.707	−16.41	24.914	+0.57

Table 4 Maximum % errors of simulation experimentations

Experimental error range%	Boundary condition					
	S-F-S-F	S-S-S-F	S-C-S-F	S-S-S-S	C-S-C-S	C-C-C-C
2	6	5	5	5	6	6
4	11	10	10	10	11	11
6	17	16	17	16	17	17

by factors such as experimental apparatus error, operator error, etc. To assess the possible errors for the buckling load determination, a simulation experimentation was designed (Hillier 1974) and carried out as follows:

(1) calculation of theoretical vibration parameters $[\overline{\phi}_n]$ and $\overline{\omega}_n$;
(2) generation of experimental errors e within an error range using the Monte-Carlo method (Hillier 1974);
(3) simulation of experimental vibration parameters $[\phi_n]$ and ω_n using formulas $\phi_i = \overline{\phi}_i \times (1+e_i)$ and $\omega_n = \overline{\omega}_n \times (1+e_n)$;
(4) determination of $[G]$ using Eq. (23);
(5) determination of $[B]$ using Eq. (7);
(6) identification of N_{cr} using Eq. (11).

In the analysis, three different cases, with error ranges 2%, 4% and 6% were taken into consideration. With 10,000 simulation experimentations in each case, the maximum error of N_{cr} for different boundary conditions is shown in Table 4. It may be inferred that the accuracy of the buckling load identification is proportional to the measurement errors.

Since the model testing technique is well established, an identification error within 3% can easily be reached. This means, from Table 4, that the proposed approach can provide an effective way to determine the buckling load.

6. Conclusions

An analysis model for determining the buckling load of a flat plate utilizing dynamic parameters determined experimentally was proposed in this paper. It was concluded that the proposed analysis model is a valid method for determining the buckling load. The main advantages of applying the proposed analysis model may be stated as follows:

(1) It requires only dynamic parameters, i.e. natural frequencies and the corresponding mode shapes, for buckling load determination.
(2) It is suitable for all kinds of boundary conditions.
(3) There is no axial force required in the process.
(4) It is suitable for various loading situations.
(5) It does not require a detailed knowledge of the material properties.

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