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New phenomena associated with the nonlinear dynamics and stability of autonomous damped systems under various types of loading

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Abstract. The present study deals with the nonlinear dynamics and stability of autonomous dissipative either imperfect potential (limit point) systems or perfect (bifurcational) non-potential ones. Through a fully nonlinear dynamic analysis, performed on two simple 2-DOF models corresponding to the classes of systems mentioned above, and with the aid of basic definitions of the theory of nonlinear dynamical systems, new important phenomena are revealed. For the first class of systems a third possibility of postbuckling dynamic response is offered, associated with a point attractor on the prebuckling primary path, while for the second one the new findings are chaos-like (most likely chaotic) motions, consecutive regions of point and periodic attractors, series of global bifurcations and point attractor response of always existing complementary equilibrium configurations, regardless of the value of the nonconservativeness parameter.

Key words: autonomous systems; dynamic buckling; point attractors; static and dynamic bifurcations; chaotic transients.

1. Introduction

The theory of nonlinear dynamical systems (with all its methods and variations) is undoubtedly the most powerful tool for the qualitative study of local as well as of global dynamics, associated with various classes of problems arising in numerous branches of both theoretical and applied sciences and engineering.

It is only in recent years however, that the application of the new concepts of nonlinear dynamics has gained a growing impact on the nonlinear dynamic response and global stability aspects of discrete structural systems; this is a branch of major importance, since quite often in engineering practice systems with a few degrees of freedom (DOF) are comprehensively and successfully used as simulations (models) of actual continuous structures, capturing their salient features and most dominant characteristics.

More specifically, the dynamics and stability of autonomous conservative (potential, symmetric) and non-conservative (non-self adjoint, asymmetric) systems, either perfect (bifurcational) or imperfect (limit point ones), have been the subject of numerous significant studies reported the last 15 years or so in the technical literature. Following a rather limited number of works, dealing mainly with the linearized (local) dynamics of single-degree-of-freedom systems, Kounadis and his

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associates (1983-1999), employing a fully nonlinear approach and introducing readily applicable qualitative and quantitative analyses, have developed various pertinent criteria and enriched the knowledge concerning, among others, the dynamic buckling mechanism of conservative multi-DOF systems under step (constant directional) loading and the overall (global) dynamic response of non-potential ones acted upon by path-dependent (especially partial follower) forces.

According to these studies, dealing with the nonlinear dynamic stability of autonomous dissipative/ nondissipative potential structural systems (Kounadis 1993, 1996, Gantes and Kounadis 1995 and many other citations not mentioned herein), based on topological, geometrical and energy considerations, dynamic buckling is defined as that state, for which an escaped motion is initiated, due to an infinitesimal increase of the main control parameter (i.e., of the loading); this phenomenon takes place through a particular equilibrium point (or its neighborhood), being at the same time a saddle of both the total potential energy surface (in the V-displacement space) and of the loading surface (in the load-displacement space). It has been also shown and evidently reported that this saddle (with negative total potential energy) may belong either to an unstable branch of the natural postbuckling equilibrium path (for limit point systems) or to unstable complementary paths (for statically stable or imperfection sensitive systems). In addition to the above, when the boundary of the basin of attraction of a stable (prebuckling) fixed pint or the amplitude of motion touches one of these unstable paths (with $V < 0$), then all the conditions for the starting of the dynamic buckling mechanism are fulfilled and in the sequel the motion of the system escapes through the saddle or its vicinity.

After the instant of dynamic buckling the relevant literature has reported till now *two types* of possible dynamic response. The one is associated with a point attractor in the large, i.e., on a remote postbuckling stable equilibrium (natural or sometimes complementary) path, with the motion being unable to return backwards through the saddle - following an opposite direction and the other with an unbounded motion (overflow in a computational sense). The latter may occur when no postbuckling stable equilibrium configurations exist or the ones existing are not “strong” enough to eventually capture the motion.

This work, extending the previous significant findings, as its first goal, reveals a third possibility of dynamic postbuckling response (Sophianopoulos 1999); this is associated with the new phenomenon of a point attractor response on the prebuckling natural equilibrium path after dynamic buckling. On the basis of the geometrical point of view and some widely accepted definitions of the theory of nonlinear dynamical systems and focusing on the geometrical and energy properties of all available stable fixed points, corresponding to loads equal or higher than the dynamic buckling load, as well as to special cases of limit-point like systems (with always negative total potential along the physical equilibrium path), the aforementioned new finding will be shown and validated. This will be done through a straightforward fully nonlinear dynamic stability analysis of a 2-DOF autonomous dissipative system under step conservative loading of infinite duration, simulating an asymmetrical suspended roof (Michaltsos and Sophianopoulos 1998).

On the other hand, the second goal of the present study is to re-examine in detail non-potential perfect damped systems under partial follower loading in regions of divergence and beyond these. For this type of systems, the effect of damping, geometric nonlinearities and other parameters have been thoroughly discussed (Kounadis 1993, 1997), various findings contradicting existing and widely accepted results have been obtained (Kounadis *et al*. 1992, Kounadis and Sophianopoulos 1999), while criteria for the existence of flutter (dynamic) instability prior to divergence (static instability) have been properly and comprehensively presented. Aiming to add some new important results on the global dynamic response of the systems just mentioned, the main objectives of the second goal of this investigation are as follows:

- To establish criteria for the occurrence of flutter (dynamic) instability through the birth of codimension one Poincare-Andronov-Hopf bifurcations, as well as through double-zero eigenvalue critical state (codimention two bifurcation) leading to a stable limit cycle response.
- To seek out whether it is possible, by varying only the level of the loading (being for this case also the principle control parameter), to obtain different types of dynamic bifurcations, even for systems exhibiting independent postbuckling equilibrium paths (similar to the ones of symmetric, conservative systems).
- To reveal regions of non-existence of static equilibria and obtain the corresponding nonlinear dynamic response, related mostly to chaotic transients (chaos-like motions) within the whole autonomous formulation.
- To discuss whether trivial states, although locally stable, can be globally unstable.
- To determine the role of physically not accepted complementary equilibrium configurations on the overall response, in case of high level loads.

The aforementioned objectives, based on a fully nonlinear static and dynamic analysis, will be sought through the study of the well known 2-DOF perfect Hookean dissipative model of Ziegler in its most general case. New phenomena will be revealed, such as series of dynamic bifurcations, consecutive regions of point and periodic attractor response, possible chaotic behavior etc., reported for the first time for the systems under consideration.

2. Mathematical formulation and general observations

Let us consider the most general case of an n-DOF, n-mass, initially imperfect dissipative system under partial follower loading λ, associated with a non-conservativeness parameter η. The differential equations of Lagrange governing the motion of this autonomous generally non-potential system (in terms of generalized displacements q_i and velocities $\dot{q}_i = dq_i/dt$) are given by the relation:

$$\frac{d}{dt}\left\{\frac{\partial K}{\partial \dot{q}_i}\right\} - \frac{\partial K}{\partial \dot{q}_i} + \frac{\partial F}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} - Q_i = 0, \quad i = 1 \div n \tag{1}$$

in which the energy components K, F, U and their characteristics can be found in the references listed herein, while $Q_i = \lambda \boldsymbol{Q}(q_i, \eta)$ are generalized non-potential forces, being non-linear functions of q_i and η.

For a system initially at rest, and for this type of (step) loading, the following initial conditions may be assumed:

$$q_i(0) = q_i^0, \ \dot{q}_i(0) = 0 \ (i = 1, 2, .., n) \tag{2}$$

After setting $y_i = q_i$, $y_{n+i} = dq_i/dt$, $i = 1, 2, \ldots, n$ the highly nonlinear initial-value problem defined by Eqs. (1) and (2) can be written in a matrix-vector form as:

$$\begin{aligned} &\dot{\boldsymbol{y}} = \boldsymbol{Y}(\boldsymbol{y}; \boldsymbol{k}; \boldsymbol{c}; \boldsymbol{m}; \varepsilon; \lambda; \eta), \quad \boldsymbol{y} \in E^{2n}, \quad \varepsilon, \boldsymbol{k} \in E^n, \quad \lambda, \eta \in E \\ &\dot{\boldsymbol{y}}(t = 0) = \boldsymbol{y}^0 \end{aligned} \tag{3}$$

with λ, η being the main control parameters.

This can be treated as the state equation of a 2-nth order autonomous nonlinear dynamical system, with vector field $\boldsymbol{y}: E^{2n} \rightarrow E^{2n}$. Such a system can be comprehensively classified in terms of its steady-state solutions and limit sets (Parker and Chua 1987, Wiggins 1989). A point $\boldsymbol{x}$ is a limit point of $\boldsymbol{y}$ if, for every neighborhood U of $\boldsymbol{x}$, the trajectory $\varphi_t(\boldsymbol{y})$ repeatedly enters U as $t \rightarrow \infty$; furthermore, the set of all limit points of $\boldsymbol{y}$ is the (closed and invariant under φ_t) limit set $\boldsymbol{L}(\boldsymbol{y})$. Adopting these basic definitions of the theory of nonlinear dynamical systems, one may proceed by defining $\boldsymbol{L}$ as attracting, if there exists an open neighborhood U of $\boldsymbol{L}$, such that $\boldsymbol{L}(\boldsymbol{y})=\boldsymbol{L}\ \forall y \in U$, and consequently the domain $\boldsymbol{B}(\boldsymbol{L})$ (basin of attraction) of $\boldsymbol{L}$ is the union of all open neighborhoods U of $\boldsymbol{L}$, for which $\boldsymbol{L}(\boldsymbol{y})=\boldsymbol{L}$, $\forall y \in U$. Every trajectory starting within $\boldsymbol{B}(\boldsymbol{L})$ will eventually tend towards $\boldsymbol{L}$ as $t \rightarrow \infty$. The latter stands as a first approach to the concept of attractors, which are precisely defined (Wiggins 1989) as topologically transitive attracting sets. Moreover, for the type of loading under consideration and the whole autonomous formulation, the most prominent steady-states (after the transients decay to zero at $t \rightarrow \infty$) are equilibria (stationary solutions) and periodic motions, while also unbounded motions may occur (due to dynamic buckling or unstable limit cycles). However, an autonomous system can also exhibit quasi-periodic motions, with more than one basic frequency (and limit sets diffeomorphic to tori) and chaotic motions, depending on the structure of the vector field.

Starting thereafter from the simplest limit set case (in the state-space domain), the equilibrium (fixed) point $\boldsymbol{y}_E$ of system (3) is given by

$$\boldsymbol{Y}(\boldsymbol{y}_E) = 0 \tag{4}$$

Eq. (1) then yields

$$\frac{\partial V}{\partial q_i} = \frac{\partial U}{\partial q_i} - \lambda\, \boldsymbol{Q}_i = 0, \quad (i = 1, 2, ..., n) \tag{5}$$

constituting a necessary and sufficient criterion for static equilibrium.

The present work focuses on the study of two characteristic classes of systems, among the many described by Eq. (3). The one deals with imperfect potential systems, related thus to a nonlinear equilibrium path with a maximum (limit point) or a monotonically rising path and having only one main control parameter, the loading λ; the second refers to perfect non-potential systems with trivial fundamental equilibrium paths (perfect bifurcational systems).

2.1. Imperfect (limit point) potential systems

For this first type of autonomous Hamiltonian systems, the only type of steady-state behavior is the equilibrium point, and since such points are not affected by damping, all orbits (the system's motion) are restricted to lie in ($2n$-1) dimensional energy "hypersurfaces" given by the level sets of the Hamiltonian (being the sum of V and K, as defined previously). These surfaces are the union of all attracting limit sets, i.e., in this case of the domains of all equilibria, corresponding to all acceptable levels of V (≤ 0); inside these lie all the fixed points resulting from Eq. (5). Hence, stable equilibria are relative minima (wells) of the aforementioned surfaces and saddles (in case they exist) are relative maxima (hilltops). These saddles are in fact non-stable, remaining that way if the flow is reversed, contrary to unstable points, becoming stable in reverse time (sources $\leftrightarrow$ sinks).

Considering a (rather simple) typical limit point system with snapping as its main feature, at the instant and after dynamic buckling, the energy surface is made of two domains interconnected in the neighborhood of a saddle. The one domain belongs to the natural stable prebuckling equilibrium and the other to a remote also stable postbuckling equilibrium configuration. Using the characteristics of saddles (defined above) and from a purely qualitative viewpoint, it is evident that the motion may as well find its way from through the vicinity of the saddle from one basin of attraction to the other (guaranteed by the presence of inset and outset manifolds), from both available directions. Consequently, which stable equilibrium will finally act as an attractor depends not only on the geometry of the first (prebuckling, departure) domain, but also on the overall surface structure of the second (postbuckling) domain; another critical parameter is indeed the number of degrees of freedom (which restricts the motion if increased) and finally the amount of dissipation accounted for.

On the other hand, if one deals with a far more complicated case, associated with multiple domains (i.e., with multiple stable fixed points for the same amount of loading, either physical or/and complementary) and saddles, then many different choices are offered to the motion and in order to eventually determine where (for non zero damping, realistic case) the system will rest at $t \rightarrow \infty$, one must take into account the dimensions and surface structure of all basins of attraction. In addition to this pertinent conclusion, if all the above are combined with always negative total potential (implying motion), the valid and well accepted zero V_T criterion (Kounadis 1993) cannot be readily applied; hence neither lower bound dynamic buckling estimates nor the direction of the motion and its global response can be established on the basis of an essentially energy approach. Apparently, only a straightforward nonlinear dynamic analysis can provide accurate information about global dynamic stability; this is so especially because the most governing nonlinear terms in the vector field are not a priori known and seeking a simpler normal form (Wiggins 1989, 1993, Guckenheimer and Holmes 1983) would rather complicate than simplify the whole investigation.

2.2. Perfect bifurcational non-potential systems

For such systems, the steady-state solutions (mainly equilibria and limit cycles) can be associated with:

a. Local bifurcations (static and dynamic) occurring when a hyperbolic equilibrium point transforms into a non-hyperbolic one, related to:

- A zero or a double zero Jacobian eigenvalue (limit cycles, static bifurcation)
- At least a pair of pure imaginary eigenvalues (Hopf dynamic bifurcations)

b. Global bifurcations (homoclinic or saddle connection).

Furthermore, one should also consider cases, where the corresponding static equilibrium equations have no solutions in R^{2n}, within certain sets of values of the loading parameter λ. If such sets exist, the question immediately arising is which would be the type of the consequent dynamic response and how – if possible – could one classify and study the resulting motions. One more quite interesting task would be to determine and discuss the dynamics of the system in the presence of more than one stable equilibrium configurations (for the same loading), as well as the effect of sensitivity to initial conditions on the long term response. To the knowledge of the author, although a large number of works have been published, dealing with non-potential perfect dissipative systems, there seems to be a lack of pertinent results and information concerning the aforementioned issues; only a limited number of studies (Kounadis *et al.* 1992, 1993, Kounadis 1997) have offered, only as by-products, some answers to the previously set questions. The present study aims to provide evident new findings related to the remarks stated above and to cast new light on the global dynamic response of the systems considered, paying special attention on those areas, which have not been tackled yet by previous analyses.

3. Four dimensional (2-DOF statical) systems

The whole analysis can be substantially simplified by reducing the dimensionality of higher order systems into fewer dimensions, the number of which depends on the multiplicity of critical Jacobian eigenvalues of the original system. This can be achieved via Lyapunov-Schmidt technique as well as the local techniques of the center manifold and normal forms (Wiggins 1988) or the splitting lemma of the catastrophe theory. In doing this and for visualization and comprehensibility reasons, the foregoing theoretical aspects will be subsequently discussed through the analysis of two four dimensional (2-DOF Lagrangian) systems-models of structural engineering importance.

3.1. Imperfect (limit point) conservative systems

The first simple model adopted herein for the analysis of limit point conservative system to follow, is the one depicted in Fig. 1, presented recently for the simulation of asymmetrical suspended roofs and the study of their nonlinear stability aspects (Michaltsos and Sophianopoulos 1998, Sophianopoulos 1999). This particular model consists of three linear springs K_i with corresponding dashpots C_i and a concentrated mass m, being initially at rest at the deformed configuration $AB\Gamma'\Delta'$, where all springs are considered unstressed; A, B are immovable hinges, while support Δ slides freely along horizontal tracks, leaving therefor spring 1 always vertical. Using υ and w as generalized coordinates, introducing the following dimensionless quantities

dimensionless time $\tau=\sqrt{\dfrac{K_1}{m}}t$, $q_1=\upsilon/l_a$, $q_2=w/l_a$, $k_2=K_2/K_1$, $k_3=K_3/K_1$, $c_2=C_2/\sqrt{K_1 m}$, $c_3=C_3/\sqrt{K_1 m}$,

$$\lambda = P/l_a K_1,\ q_{10}=\upsilon_o/l_a,\ q_{20}=w_0/l_a,\ \gamma=l_b/l_a \tag{6}$$

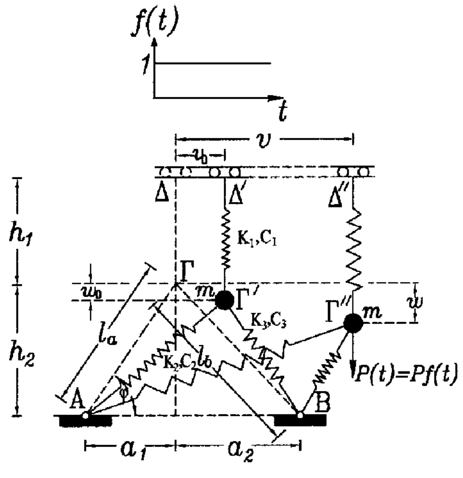

Fig. 1 Two degrees of freedom imperfect model, simulating an asymmetrical suspended roof

we reach to a set of two strongly nonlinear Lagrange differential equations of motion (resulting from Eq. 1) that can be found in the papers listed above, subject to initial conditions

$$q_1(0)=q_{10},\ q_2(0)=q_{20},\ \dot{q}_1(0)=\dot{q}_2(0)=0 \tag{7}$$

In these citations one can also find the nondimensionalized static equilibrium equations and the expression of the total potential energy involved.

Due to the absence of an exact solution of the O.D.Es of motion and the fact that the most governing nonlinear terms inherent are not a priori known, it is preferable to perform a direct numerical integration. Thus, choosing an appropriate numerical scheme, a straightforward fully nonlinear dynamic analysis is employed for the study of motion itself, as well as of the stability of steady-states.

In doing this, we examine as a first application an asymmetrical roof model with $q_{10}=q_{20}=0.01$, $\gamma=1.50$, $\varphi=20^0$, $k_2=20$, $k_3=18$. Numerical solution of the static equilibrium equations yield the physical equilibrium paths (λ vs q_i) of Fig. 2, revealing a limit point instability, while complementary fixed points do not exist. The pertinent zero total potential energy criterion (Kounadis 1993) produces the value of the approximate dynamic buckling load $\tilde{\lambda}_D$, being a lower bound of the exact one λ_D (for zero damping); the latter is obtained using the criterion requiring that the motion should touch the postbuckling "unstable" path. For $\lambda=\tilde{\lambda}_D$ as expected, $V_T=0$ constitutes a repeller barrier and hence no escape passage exists, while the domains of the two stable equilibria corresponding to $\tilde{\lambda}_D$ are simply tangent at point *D*. Futhermore, for $\lambda=\lambda_D$ dynamic buckling occurs in the vicinity of a saddle and the motion is lead from one basin to the other (two-well potential). Finally, if damping is accounted for (introducing $c_1=0.02$, $c_2=0.05$, $c_3=0.04$) the exact dynamic buckling load λ_{DD} is computed (in the same manner as λ_D), while the corresponding motion crosses the escape passage near the saddle DD to be attracted by a remote physical stable equilibrium (in the large). For $\lambda>\lambda_{DD}$, according to previous analyses, one would expect a similar response. But this is not the case for the model considered, since for $\lambda=0.32$ and although the phenomenon of dynamic buckling has already taken place, the motion comes back and forth from one domain to the other, to be finally attracted by the prebuckling stable fixed point. On the contrary, for $\lambda=0.33$ the system moves in a similar way, exhibiting a point attractor response in the large. All the above are clearly

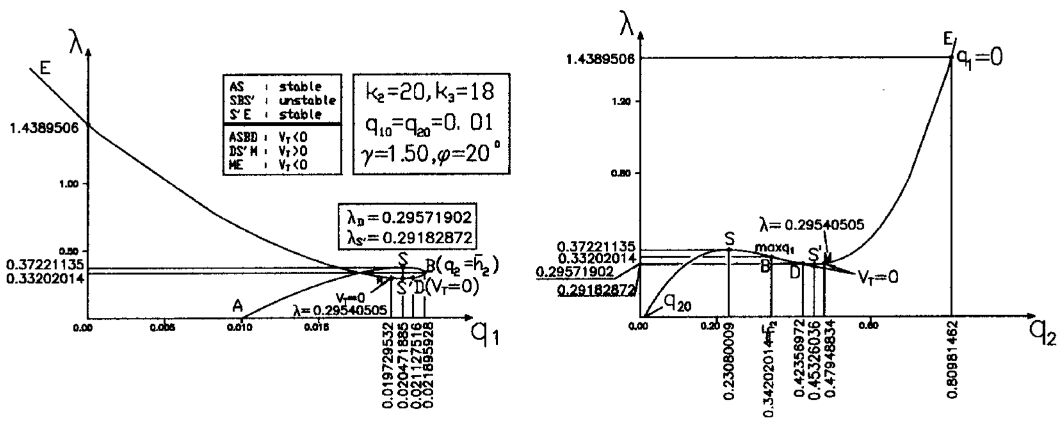

Fig. 2 Physical equilibrium paths (λ vs q_i, i=1, 2) of a roof model with $q_{10}=q_{20}=0.01$, $\gamma=1.50$, $\varphi=20^0$, $k_2=20$, $k_3=18$, exhibiting a limit point instability

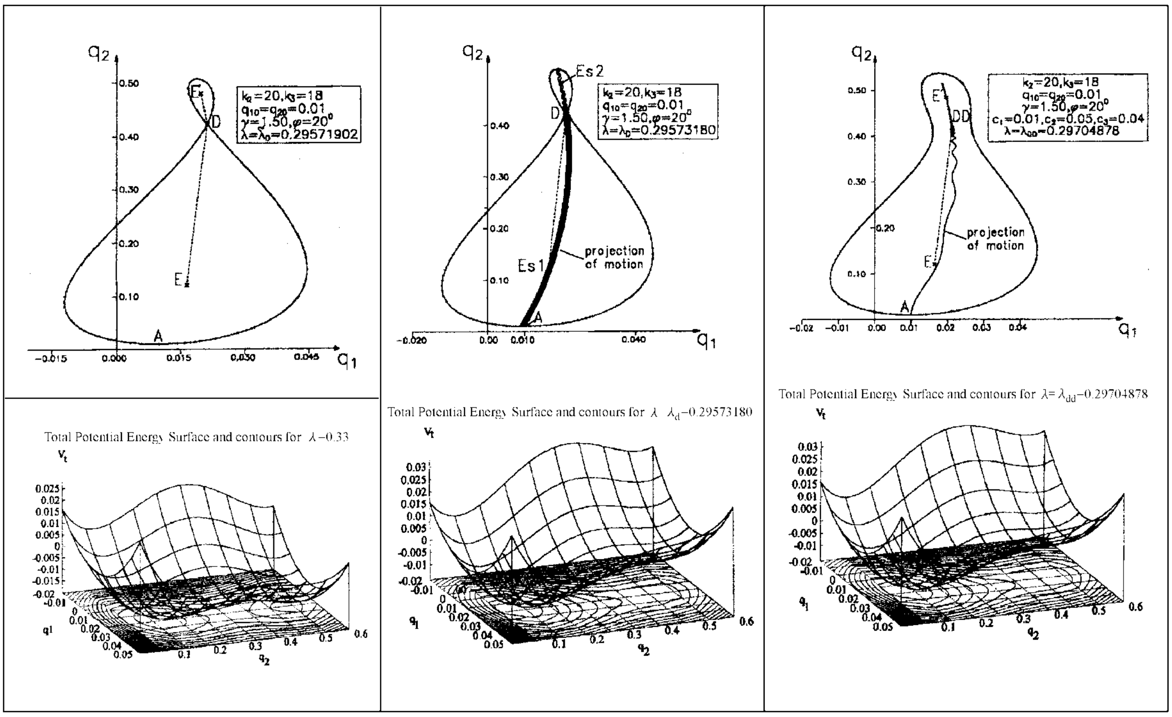

Fig. 3 Contours V_T=0, motion projection on the q_1q_2 plane and total potential energy surfaces for a roof model with q_{10}=q_{20}=0.01, γ=1.50, φ=20^0, k_2=20, k_3=18 and three characteristic values of the loading λ (λ_D, λ_{DD}, 0.33)

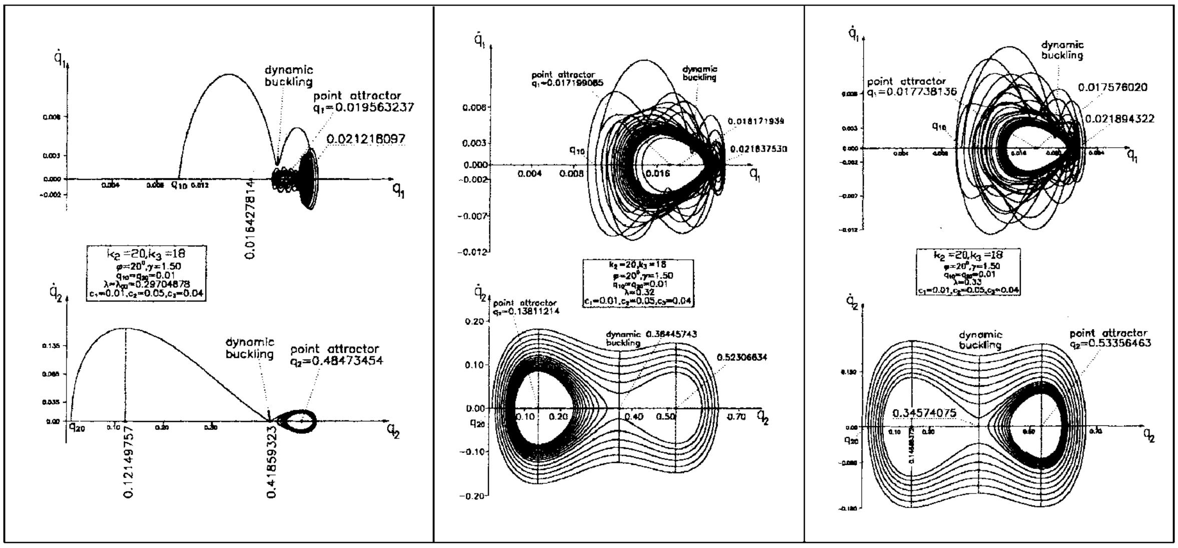

Fig. 4 Phase plane portraits $[q_i(\tau),\ \dot{q}_i(\tau)]$, i=1, 2 for a roof model with q_{10}=q_{20}=0.01, γ=1.50, φ=20^0, k_2=20, k_3=18, c_1=0.01, c_2=0.05, c_3=0.04 and three values of the loading λ (λ_{DD}, 0.32, 0.33), implying two different kinds of postbuckling point attractor response

depicted in Figs. 3 and 4, where one can perceive the difference in dimensions between domains and its evolution, as λ increases, and the two different kinds of postbuckling point attractor response.

This new phenomenon, as described in detail above, can be reached only via a fully nonlinear (global) dynamic analysis, specifying that after dynamic buckling (i.e., for $\lambda > \lambda_{DD}$) some but not all statically stable equilibria of the natural primary (prebuckling) path, are for limit point systems dynamically locally stable and globally unstable, while others are both locally and globally asymptotically stable, like the ones corresponding to $\lambda < \lambda_{DD}$ (before dynamic buckling).

A case of equal interest and importance is associated with a roof model with the following properties: $q_{10}=q_{20}=0.01$, $k_2=1.00$, $k_3=0.80$, $\varphi=60^0$. The physical equilibrium paths for $\gamma=1$, 1.5, and 2 are presented graphically in the L.H.S of Fig. 5 and posses the special attribute of having negative definite total potential all along; the corresponding complementary paths of the limit-point like symmetric model ($\gamma=1.00$) are shown in the R.H.S. of the same illustration. Thus, for $\lambda_S > \lambda > \lambda_{S'}$ evidently multiple stable equilibrium configurations, very close to each other, correspond to the same amount of loading. Therefore, their domains may as well form a highly irregular energy surface with numerous local maxima and minima, offering to the undamped motion more than one possible (and available) directions and escape passages. In this manner the system could move in a

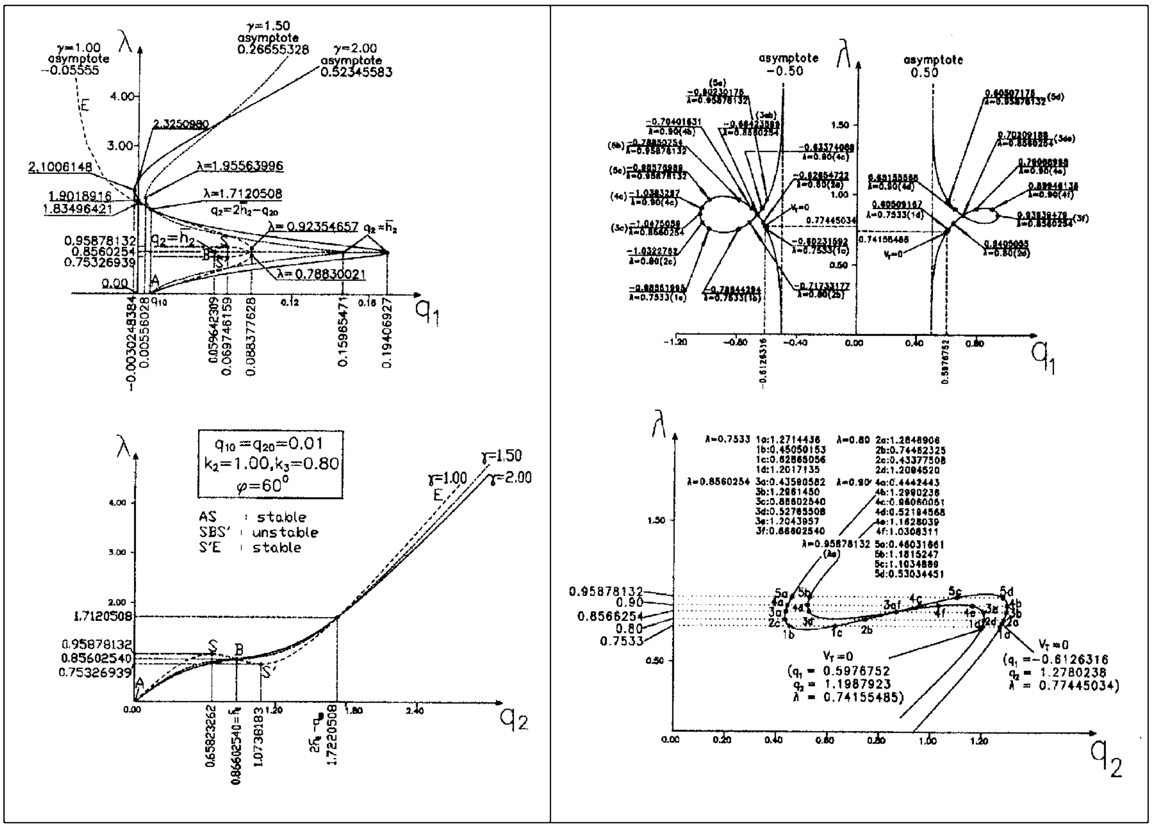

Fig. 5 Physical equilibrium paths (λ vs q_i, i=1, 2) of a roof model with $q_{10}=q_{20}=0.01$, $\varphi=60^0$, $k_2=1.00$, $k_3=0.8$, for $\gamma=1.00$ (symmetric), $\gamma=1.50$, 2.00 (asymmetric) and corresponding complementary ones for the symmetric case

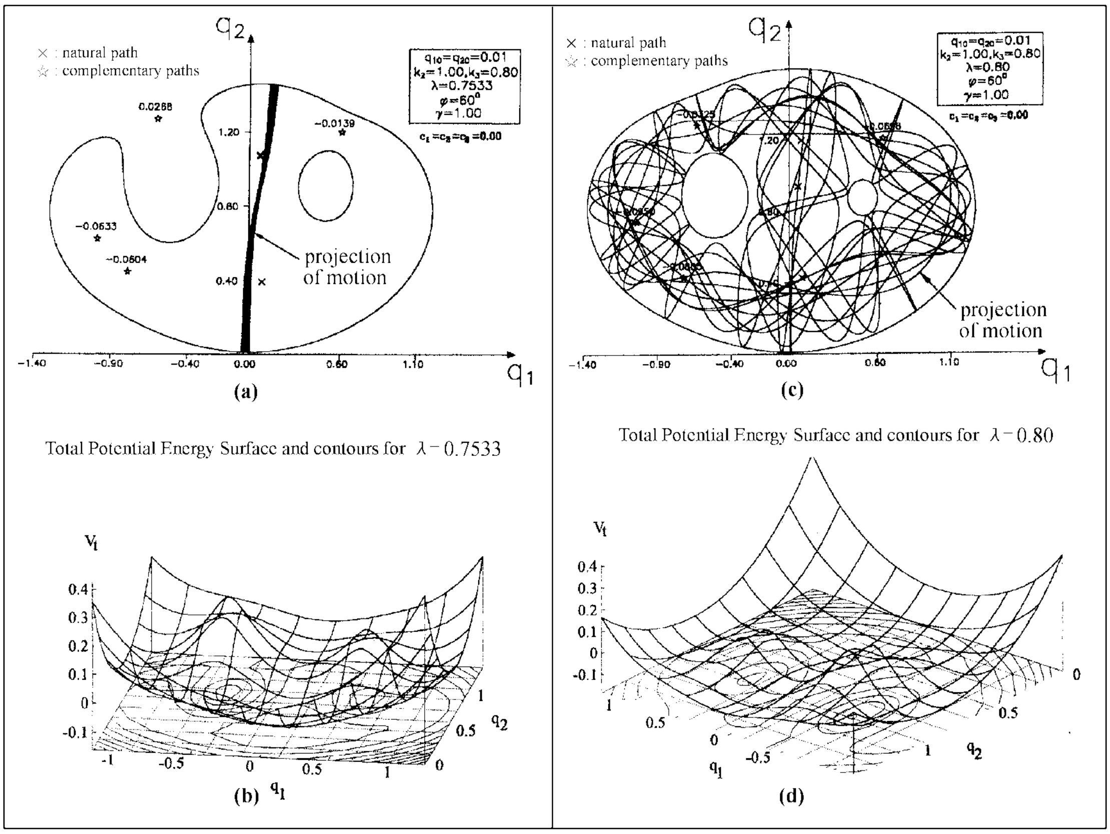

Fig. 6 Contours $V_T = 0$ and undamped motion projection on the q_1q_2 plane (a,c) and corresponding total potential energy surfaces (b,d) for a symmetric roof model with q_{10}=q_{20}=0.01, φ =60^0, k_2=1.00, k_3=0.8 at λ = 0.7533 and λ = 0.80 respectively

complicated (almost arbitrary) pattern, from one domain to the other. This indeed happens in the case under study, shown in the zero potential contours, motion projections and corresponding energy surfaces of Fig. 6.

The questions immediately arising are concerned with the nature of the system's dynamic behavior and the effect of the amount of dissipation on the corresponding global dynamics. These are clarified examining the response of the foregoing model for λ = 0.80, c_1=c_2=0.01, c_3=0.008. It is found that the validity of the dynamic buckling mechanism is initially verified and as predicted by relevant studies a large amplitude motion is initiated, lead through the escape channel back and forth (from domain to domain) and is at last attracted by the prebuckling natural fixed point. If again damping is doubled, the system exhibits a point attractor response in the large. In this case there also exist four (4) complementary equilibria, not crucially affecting the damped motion, as clearly shown in Fig. 7, which nevertheless is more "wondering" than in usual cases. For higher levels of the loading, the system experiences a dynamic response of a different pattern as for instance in the case of λ = 0.90, associated with 6 complementary fixed points. The motion, depending on the amount of dissipation, may or may not "visit" the domain of all possible attractors, as presented in the phase planes of Fig. 8.

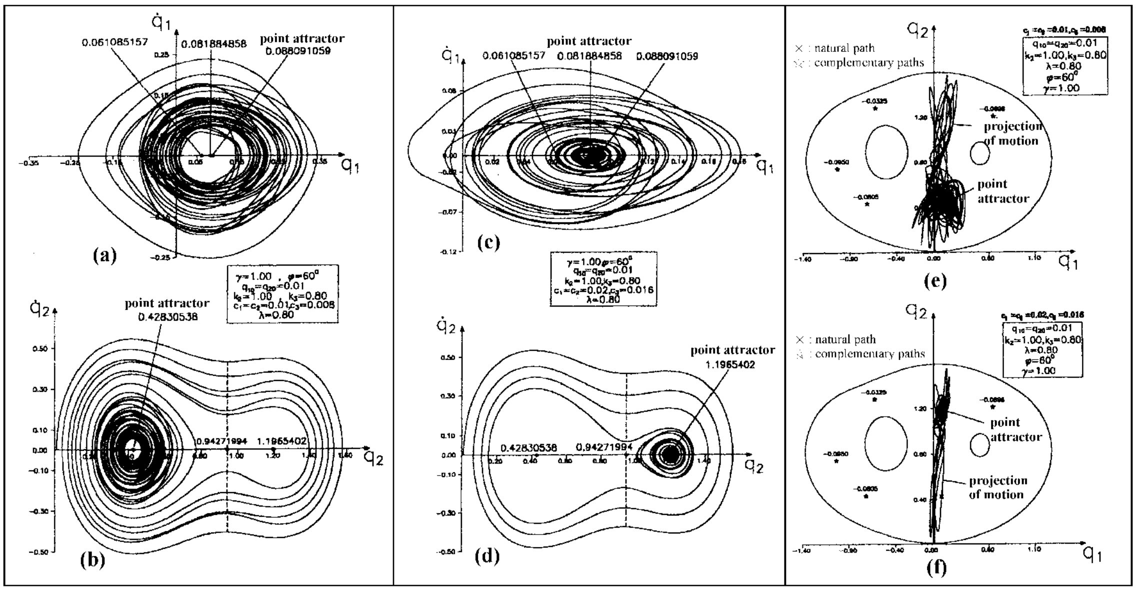

Fig. 7 Phase plane portraits $[q_i(\tau),\ \dot{q}_i(\tau)]$, $i=1$, 2 and corresponding motion projection on the q_1q_2 plane for a symmetric roof model with $q_{10}=q_{20}=0.01$, $\varphi=60^0$, $k_2=1$, $k_3=0.80$ at $\lambda=0.80$, for $c_1=c_2=0.01$, $c_3=0.008$ (a,b,e) and $c_1=c_2=0.02$, $c_3=0.016$ (c,d,f), implying different point attractor responses

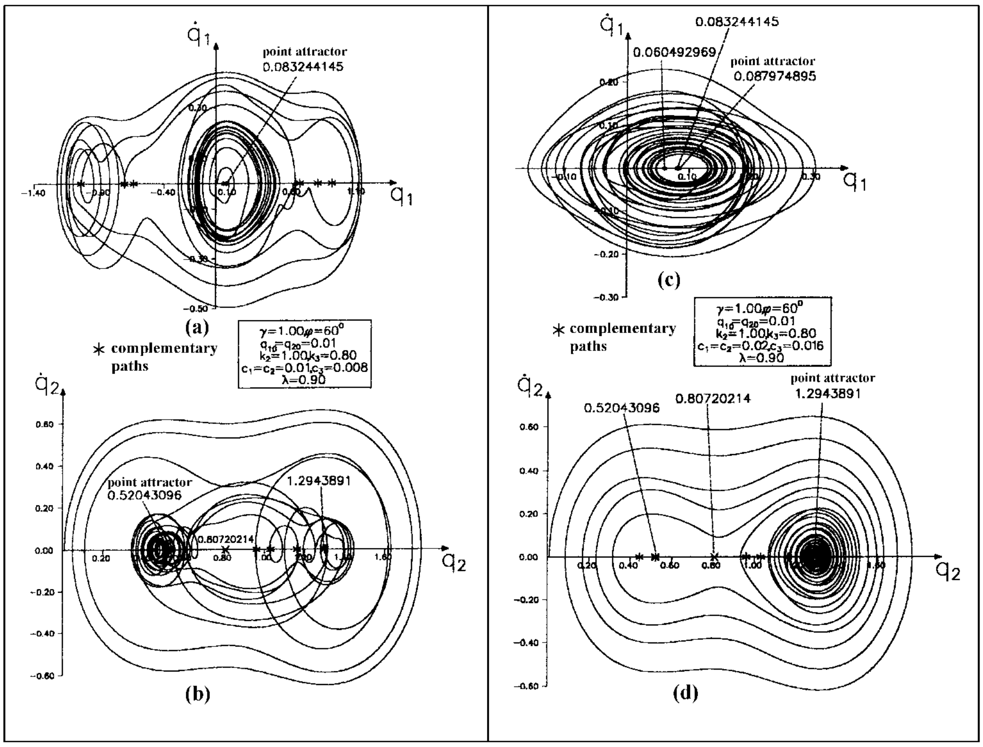

Fig. 8 Phase plane portraits $[q_i(\tau),\ \dot{q}_i(\tau)]$, $i=1$, 2 for the symmetric roof model of Fig. 7 at $\lambda=0.90$, implying different point attractor responses

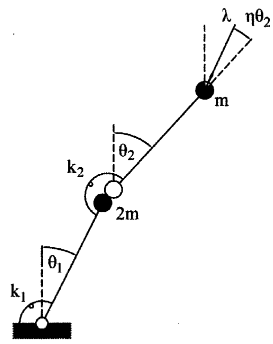

Fig. 9 Perfect 2-DOF Ziegler's model under partial follower loading

3.2. Perfect (bifurcational) non-potential systems

The second model dealt with herein is the classical model of Ziegler shown in Fig. 9, for which a large amount of numerical results is available, sometimes coming from totally different scientific approaches (see Jin and Matsuzaki 1988). In contrast with previous investigations, in the present paper $k_1 \neq k_2$, while the Rayleigh type viscous damping coefficients are β_1 and β_2 respectively. Lagrange equations of motion in dimensionless form are given by (Sophianopoulos 1996):

$$(1+m)\ddot{\theta}_1+\cos(\theta_1-\theta_2)\ddot{\theta}_2+\dot{\theta}_2^2\sin(\theta_1-\theta_2)+(\beta_1+\beta_2)\dot{\theta}_1-\beta_2\dot{\theta}_2+\frac{\partial V_T}{\partial \theta_1}=0$$

$$\cos(\theta_1-\theta_2)\ddot{\theta}_1+\ddot{\theta}_2-\dot{\theta}_1^2\sin(\theta_1-\theta_2)+\beta_2\dot{\theta}_2-\beta_2\dot{\theta}_1+\frac{\partial V_T}{\partial \theta_1}=0 \tag{8}$$

where m=m_1/m_2 (usually set equal to 2) and

$$\frac{\partial V_T}{\partial \theta_1}=(1+k)\theta_1-\theta_2-\lambda\sin[\theta_1+(\eta-1)\theta_2]$$

$$\frac{\partial V_T}{\partial \theta_2}=(-\theta_1+\theta_2)-\lambda\sin\eta\theta_2 \tag{9}$$

with $\lambda = Pl/k_2$, $k = k_1/k_2$ and dimensionless time $\tau = t(k_1/m_2 l^2)^{1/2}$

The characteristic equation of the corresponding linearized system can be written in the form:

$$\rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4 = 0 \tag{10}$$

where

$$a_1=\frac{1}{m}[\beta_1+(m+4)\beta_2],\ a_2=\frac{1}{m}[\beta_1\beta_2+k+m+4-\lambda(2+m\eta)]$$

$$a_3=\frac{1}{m}[\beta_1(1-\eta\lambda)+\beta_2(k-2\eta\lambda)],\ a_4=\frac{1}{m}[\eta\lambda^2-\eta\lambda(k+2)+k] \tag{11}$$

The boundary between divergence (static) and flutter (dynamic) instability is determined by solving the systems of equations a_4=$da_4/d\lambda$=0, yielding the double (compound) branching point (η_0, λ_0^c)

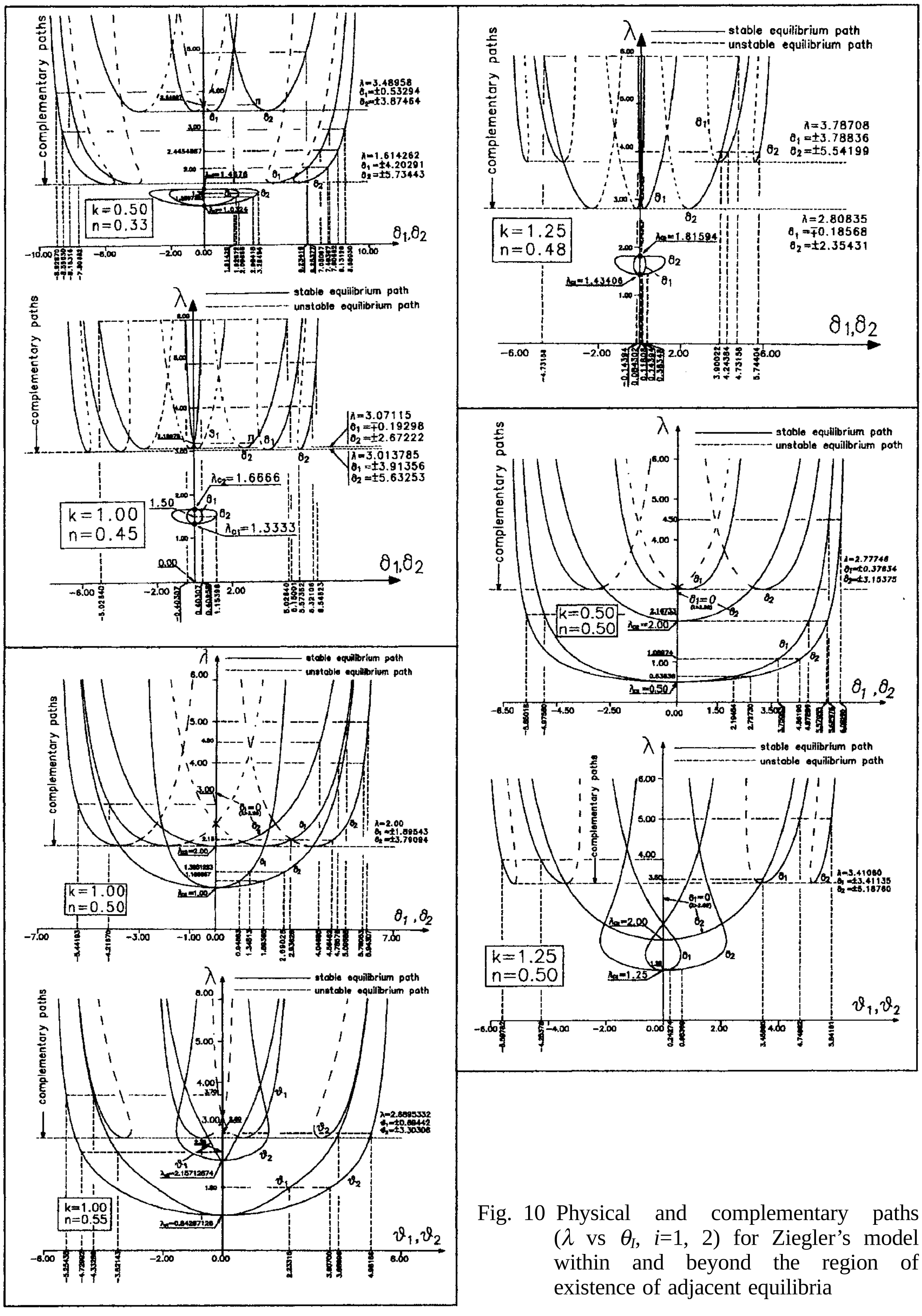

Fig. 10 Physical and complementary paths (λ vs θ_I, i=1, 2) for Ziegler's model within and beyond the region of existence of adjacent equilibria

given by:

$$\eta_0 = \frac{4k}{(k+2)^2}, \quad \lambda_0^c = \frac{k+2}{2} \tag{12}$$

Thus, the region of divergence (of existence of adjacent equilibria) is defined by $4/9 \le \eta_0 < 0.50$, within which the foregoing model exhibits one postbuckling path connecting the 1st (stable) branching point $\lambda^c_{(1)}$ with the second (unstable) one $\lambda^c_{(2)}$; on the other hand, the 1st and 2nd buckling modes for $\eta \ge 0.50$ are independent. This is depicted in Fig. 10, from which it is evident that regardless of the value of η, there exist more than one complementary (physically not accepted)

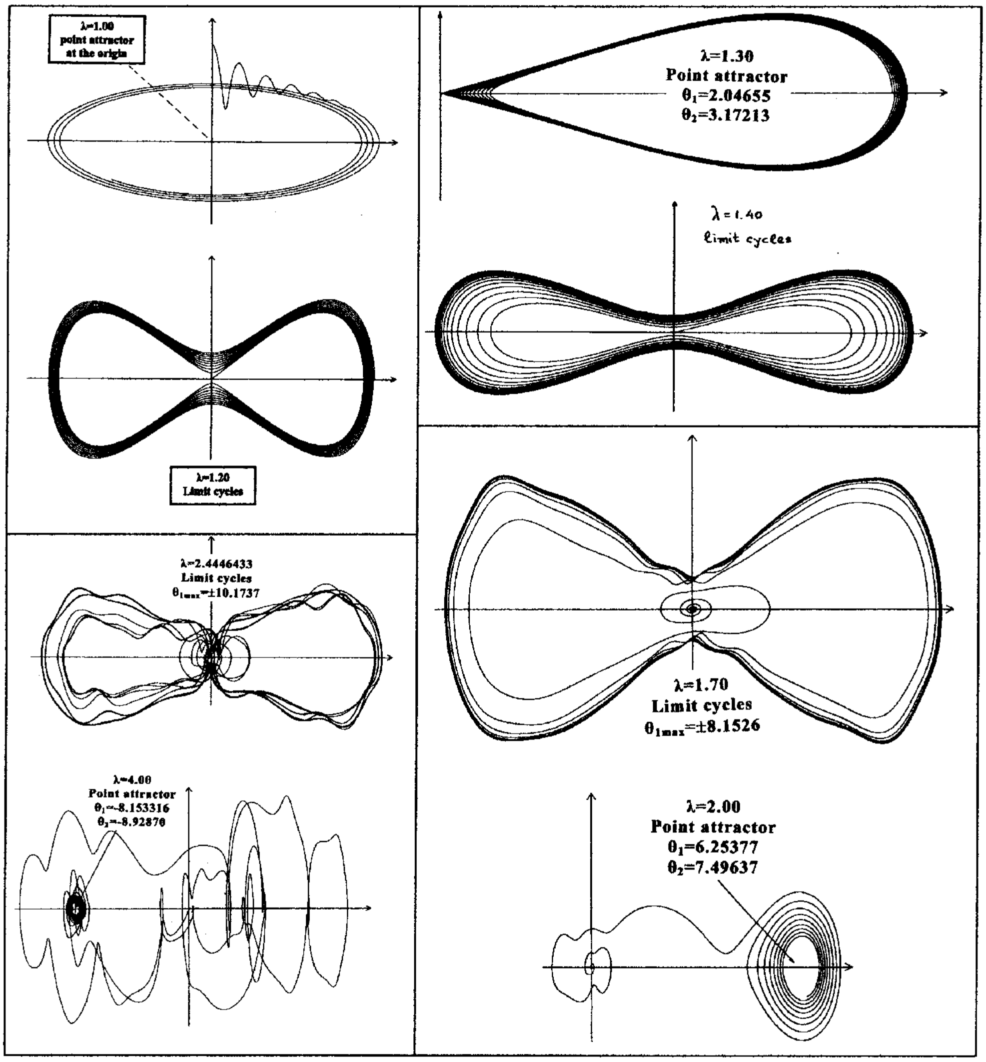

Fig. 11 Phase plane portraits $[\theta_1(\tau), \dot{\theta}_1(\tau)]$ of Zieglers model with k=0.50, η=0.33, $\beta_1=\beta_0/10$, β_2=0.10 (β_0=0.27512) for various values of λ, implying consecutive regions of point and periodic attractor response

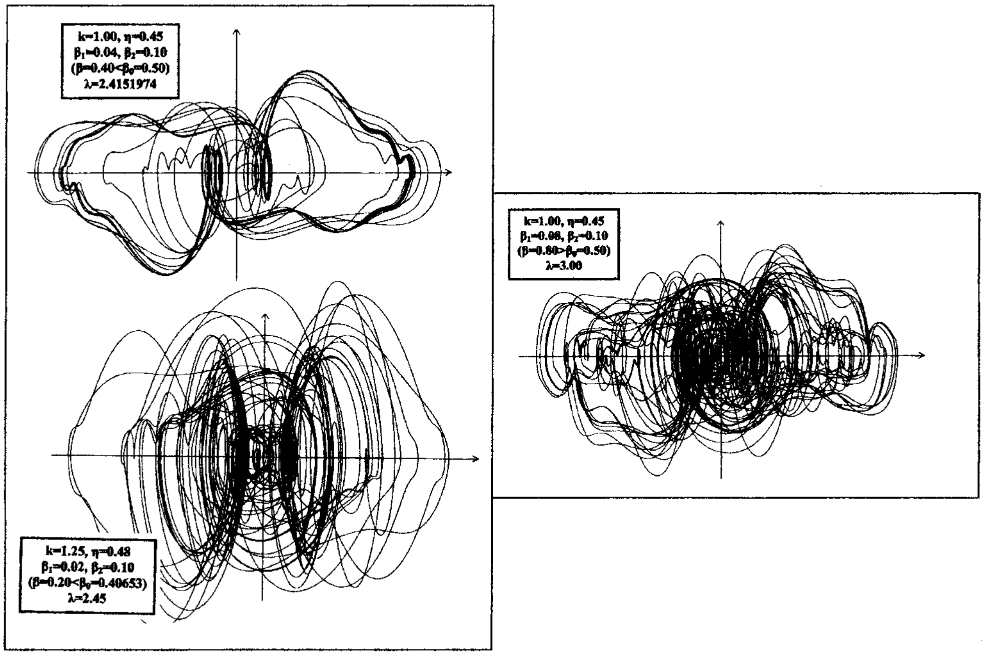

Fig. 12 Phase plane portraits $[\theta_1(\tau), \dot{\theta}_1(\tau)]$ of Ziegler's model for three characteristic cases exhibiting complicated chaos-like motions

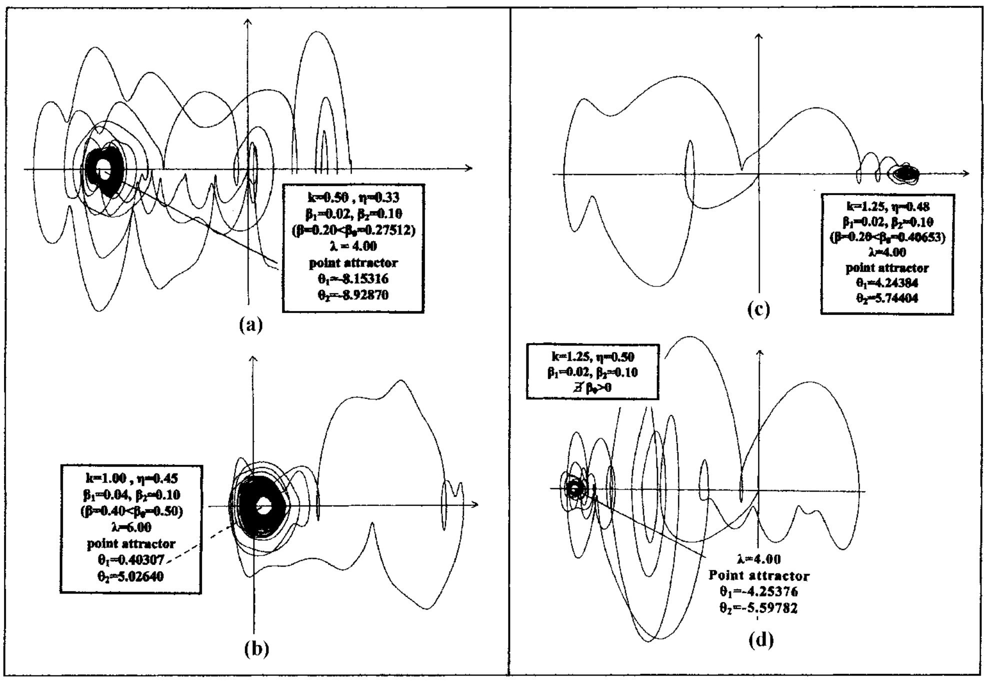

Fig. 13 Phase plane portraits $[\theta_1(\tau), \dot{\theta}_1(\tau)]$ of Ziegler's model for four characteristic cases of high level loading, exhibiting a point attractor on complementary stable equilibria

paths, for $\lambda \geq \lambda^c_{(2)}$. From these drawings however, a new phenomenon is revealed. It is the existence of an open set of values of λ, within the region of divergence, between $\lambda^c_{(2)}$ and the minimum of the complementary paths, with the unique characteristic that there are no solutions in R^2 of the static equilibrium equations. The dynamics of the system inside this set will be discussed at the end of this subtopic. Moreover, it can be shown (Sophianopoulos 1996) that in order the Routh-Hurwitz determinant Δ_3 to have a positive real root smaller than $\lambda^c_{(1)}$ and thus flutter to occur prior to divergence, the damping ratio $\beta = \beta_1/\beta_2$ must be less than β_0, given by

$$\beta_0 = \frac{2\eta\lambda^c_{(1)}}{1 - \eta\lambda^c_{(1)}} \tag{13}$$

At $\beta = \beta_0$, the corresponding (dynamic) Hopf bifurcation is degenerated at $\lambda^c_{(1)}$ to a double zero eigenvalue critical state, where it is also static, since $\Delta_3 = a_3 = a_4 = 0$, and the system's motion is associated with stable limit cycles.

Choosing k=0.50, η=0.33, $\beta_1 = \beta_0/10$, β_2=0.10 (β_0=0.27512) and varying smoothly control parameter λ, consecutive regions of point and periodic attractor response are revealed, as shown in Fig. 11, while within the region of absence of static equilibria the majority of cases is associated with complicated chaotic transients, shown in Fig. 12. Considering thereafter the effect of complimentary equilibrium configurations on the global response of the system, it is found that its motion is always associated with a point attractor in case of high loading values, regardless of the value of k, as illustrated in Fig. 13. Finally, paying significant attention at the systems dynamics at

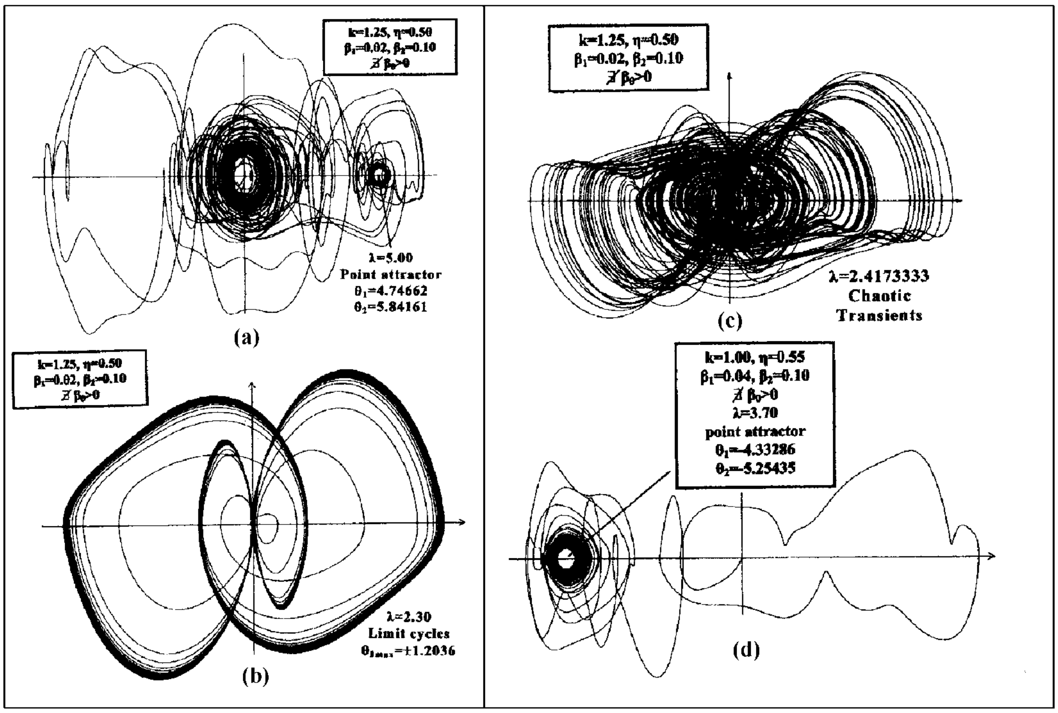

Fig. 14 Phase plane portraits $[\theta_1(\tau), \dot{\theta}_1(\tau)]$ of Ziegler's model at $\eta = 0.50$, exhibiting four different types of dynamic behavior

$\eta = \max\eta_0 = 0.50$, which is the boundary between the region of existence and non-existence of adjacent equilibria, the interaction between buckling modes gives birth to new phenomena, such as competitive equilibria (Fig. 14a), global bifurcations (Fig. 14b), large amplitude motion before attraction (Fig. 14d) and complicated chaotic transients (Fig. 14c).

3.3. Discussion of the region of absence of static equilibria

The motions of the system for λ inside this particular and quite challenging area, cannot be easily classified, but are neither periodic nor associated with a "nonstrange" attractor (Parker and Chua 1988). Consequently, useful information may be gained based on the corresponding time waveforms (time series). Reexamining for instance, the motion of a model with k =1.25, η =0.50, β_1=0.02, β_2=0.1 at λ =2.417333 and λ =2.80 and carefully observing the time series (τ, θ_2) shown at the L.H.S. of Fig. 15, it is readily perceivable that their shape is almost identical to the ones used in pioneer literature (Bolotin 1956) to describe "mild" and "violent" chaos. Except this, both phase plane portraits $(\theta_2, \dot{\theta}_2)$ of the R.H.S of Fig. 15 are strong reminders of chaotic trajectories belonging to a third-order autonomous system (Double Scroll equation). Beyond these qualitative

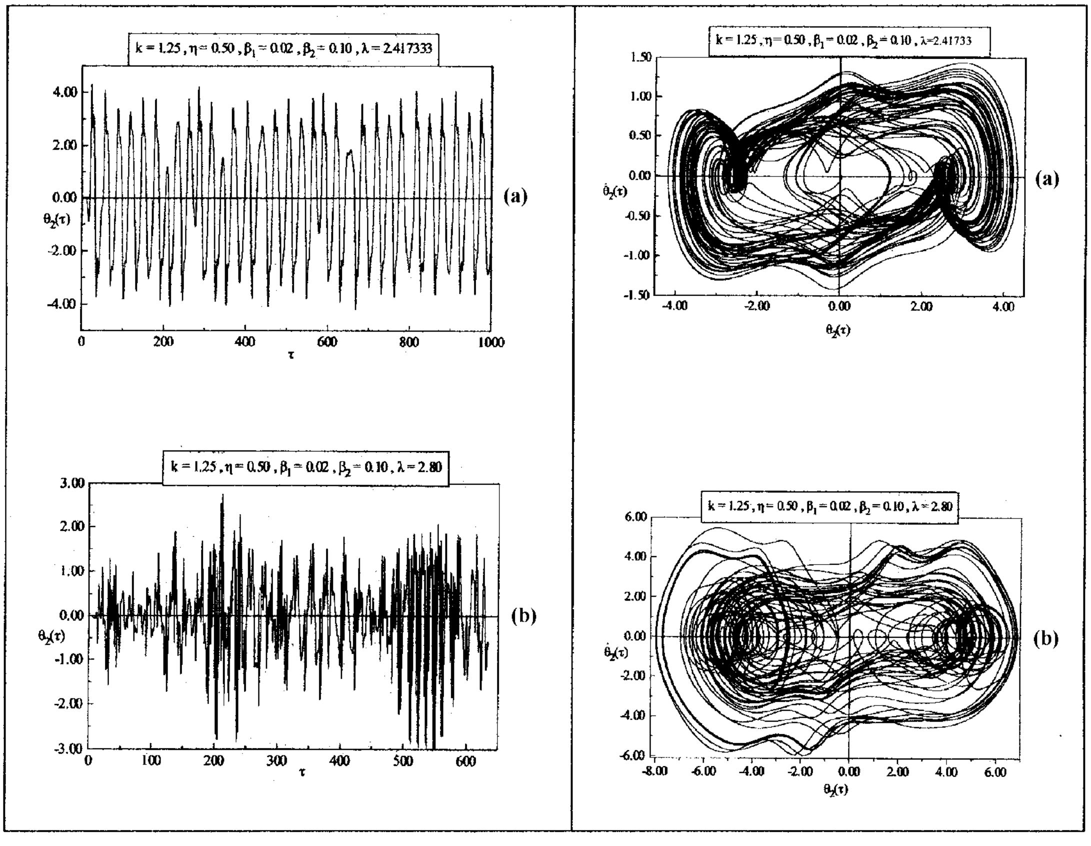

Fig. 15 Time series [τ, $\theta_2(\tau)$] and corresponding phase plane portraits $[\theta_2(\tau), \dot{\theta}_2(\tau)]$ of Ziegler's model with k=1.25, η =0.50, β_1=0.02, β_2=0.10 for $\lambda = 2.41733$ and $\lambda = 2.80$, implying possible chaotic behavior

remarks, the exact classification of these motion, lying beyond the scopes of this paper, is without doubt a very intriguing subject for future investigation.

4. Conclusions

The most important conclusions drawn from the present study are as follows:

4.1. Imperfect conservative (limit point) systems

• Dynamic buckling of autonomous potential dissipative systems of a classical limit point type is not always associated either with a point attractor response on a remote stable equilibrium (in the large) or with an unbounded motion. Under certain conditions and although dynamic buckling has already taken place, the escaped motion returns through the vicinity of the saddle and is finally attracted by the stable fixed point belonging to the physical prebuckling stable (primary) equilibrium path. This new phenomenon can be captured only via a straightforward fully nonlinear (global) dynamic analysis, revealing for some fixed points local and global dynamic stability, after dynamic buckling.

• This third possibility of postbuckling dynamic response leads to the significant remark, that from a qualitative viewpoint the geometry and surface structure of both pre- and postbuckling domains must be accounted for, in conjunction with other parameters.

• The phenomena described above may also occur for limit-point like systems, whose total potential is always negative along their physical equilibrium paths. Such systems may exhibit also various complementary equilibrium configurations, and for loads higher than a certain level the total potential energy surface contains numerous domains, becoming quite irregular and complicated. Depending on the number of stable complementary equilibria corresponding to the same amount of loading, the systems motion may enter these domains, to be finally attracted by always physical equilibria, either pre- or postbuckling. Hence, the surface structure of all basins of attraction must be considered, while the only means to establish the overall response is - once again- to perform a fully nonlinear dynamic analysis, since energy criteria are no longer valid, due to negative potential.

4.2. Perfect (bifurcational) non-potential systems

Within the region of existence of adjacent equilibria (divergence) and beyond this, the system’s dynamics may be associated with consecutive regions of point and periodic attractor response, while the case of a double zero eigenvalue leads to a degenerate Hopf bifurcation (both static and dynamic) for a specific value of the damping ratio.

• Regardless of the values of the stiffness coefficients k_i and the nonconservativeness parameter η, there exist more than one complementary equilibrium paths (physically not accepted), for loading levels equal or higher than the load of the 2nd branching point. The stable fixed points of these paths affect the systems dynamic behavior, by always acting as point attractors.

• Except the above new phenomena, the foregoing systems may under certain combinations of the parameters involved exhibit at the invariant point η_0=0.50 series of global (dynamic) bifurcations – stable limit cycles, competition (interaction) of modes and large amplitude motions before final attraction.

• In the region of divergence, there is an open set of values of the loading (main control parameter), higher than the 2nd branching point and lower than the minimum complementary fixed point, for which no equilibria exist. The corresponding motion is found to be chaos-like and quite similar to chaotic, a fact that requires further in depth investigation, employing advanced methods of the theory of nonlinear dynamical systems.

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