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Plate on non-homogeneous elastic half-space analysed by FEM

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Abstract. The isoparametric element method is used for a plate on non-homogenous foundation. The surface displacement due to a point force acting on the non-homogeneous foundation is the fundamental solution. Based on this analysis, the interaction between the foundation and plate can be determined and the reaction of the foundation can be treated as the external force to the plate. Therefore, only the plate needs to be divided into some elements. The method presented in this paper can be used in cases such as thin or thick plate, different plate shapes, various loading, homogenous and non-homogenous foundations. The examples in this paper show that this method is versatile, efficient and highly accurate.

Key words: plate on foundation; elastic half-space; non-homogeneous elastic half-space; fundamental solution.

1. Introduction

The plate on elastic foundation serves as an important type of construction in civil engineering. In the analysis, it is necessary to choose a suitable soil model for the foundation. Even in the elastic range, there are different soil models. The simplest among them is the Winkler's model, in which the deformation of a surface point is directly proportional to the intensity of the vertical stress at the point, resulting in only one material parameter in the model equation. Although the Winkler's model is very simple and convenient in applications, the simulated result to the practice is not good. Another idealization assumes continuum behaviour of the soil, and the soil medium is thus represented by an elastic half-space. The basic solution for the continuum representation of soil media is from the work of Boussinesq, who analysed the problem of a semi-infinite homogeneous isotropic linearly elastic solid subjected to a concentrated force that acts normal to the plane boundary. Except for the above two kinds of soil models, there are the two-parameter models of idealized soil behaviour. In one category of two-parameter soil models mechanical interaction is introduced between the spring elements of the Winkler medium. In the second category restrictions

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are imposed on the types of displacement and stress distribution which can be occur in the elastic half-space model (Vlazov and Leontiev 1966). A review of the three basic models of soil behaviour has been given by Selvadurai (1979).

In national soil deposits the stiffness usually increases with depth due to the increasing overburden pressure, which results in the contradiction to the assumption of homogeneity. Gibson (1967), Brown and Gibson (1972) have studied the half-space exhibiting this kind of non-homogeneity. They have considered the behaviour of the soil whose shear modulus (or Young's modulus) increases linearly with depth. Booker *et al*. (1985a, 1985b) have studied this problem in a more general form, in which the Young's modulus is in power variation with depth.

In current engineering practice, the finite element method (FEM) is the most powerful method for the practical problems. There are two ways of using FEM analysis for the plate on foundation. The first one is to divide both the plate and the foundation into some elements. Because the foundation is a half-space, it needs to take a large chunk of the foundation and divide it with a large number of elements. This will lead to a lot of work in preparation and computing. The second method was proposed first by Cheung and Zienkiewicz (1965). The elements only need to be arranged in the plate. The actions from the foundation can be treated as the external forces to the plate. According to the Boussinesq's solution, the acting forces from the foundation can be represented by the displacements as those given by Cheung (1977). Using this method and rectangular elements, they evaluated the problem of a rectangular plate on the elastic isotropic foundation. Furthermore, Wang *et al*. (1996, 1998) used triangular elements and isoparametric elements for the plates with different shapes on the homogeneous foundations.

Based on the results published by Booker *et al.* (1985a, 1985b), Stark *et al.* (1997a, 1997b) have described a numerical procedure to determine the surface displacements on a non-homogenous soil mass subjected to uniformly distributed surface traction.

In this paper, the interaction of a plate on a non-homogeneous half-space is analysed. At first, the result of the surface deflection for the case of a concentrated loading on the non-homogeneous foundation is taken as a fundamental solution. This is equivalent to the Boussinesq's solution for the homogeneous half-space. Therefore, the interaction of the plate and foundation can be determined. The reaction from the foundation can be treated as the external force to the plate.

Secondly, isoparametric elements are used for the plate on the foundation. The Reissner-Mindlin plate formulae are introduced. The rotations of a point in the plate no longer depend on the deflection and they are chosen as independent variables. After this treatment, the shear deformations can be considered at the same time. Therefore, the present method can be used for both thick and thin plates with different shapes. A series of examples have been given to show the characteristics of this method.

2. Fundamental solution

In the following, it is assumed that the soil mass can be idealised as an isotropic but non-homogeneous medium. The Poisson's ratio is a constant, and Young's modulus is given by (Stark and Booker 1997):

$$E(z) = m_E z^{\alpha} \qquad (0 \le \alpha \le 1) \tag{1}$$

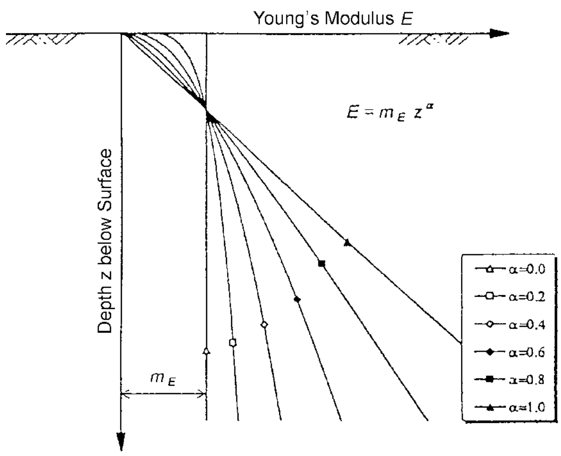

Fig. 1 Variations of Young's modulus with depth for different degrees of non-homogeneity

where m_E is a constant, which determines Young's modulus when z=1, α is referred to as the non-homogeneity parameter. Fig. 1 shows schematically the variation of E with z for some values of α. When α=0, this is the homogenous half-space case. When α=1, this is the Gibson's soil. This case has been investigated by Holl (1940) for the particular case in which Poisson's ratio (v_s) is linked to the exponent α by the relationship v_s=1/(α+2), so that Holl's solutions are relevant to an incompressible homogeneous soil and a Gibson soil (α=1) with a Poisson's ratio v_s=1/3.

Let us suppose that a normal point load of magnitude P is acting on the surface of the non-homogeneous half-space. The vertical component of the surface deflection, w, at a distance r from the load, is given by (Booker *et al*. 1985a, 1985b):

$$w = \frac{PB}{m_E r^{1+\alpha}} \tag{2}$$

where

$$\begin{cases} B = \dfrac{b\Gamma\left(\dfrac{\alpha+1}{2}\right)}{\Gamma\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{\alpha}{2}\right)} \\ b = \dfrac{1-v_s^2}{\alpha}\sin\left(\dfrac{\beta\pi}{2}\right)\dfrac{\beta}{\alpha+1}F_{\alpha\beta} \\ \beta = \sqrt{(1+\alpha)\left(1-\dfrac{\alpha v_s}{1-v_s}\right)} \\ F_{\alpha\beta} = \dfrac{2^{\alpha+1}}{\pi}(\alpha+2)\dfrac{\Gamma\left(\dfrac{3+\alpha+\beta}{2}\right)\Gamma\left(\dfrac{3+\alpha-\beta}{2}\right)}{\Gamma(3+\alpha)} \end{cases} \tag{3}$$

In the above equations Γ denotes the gamma function. It may be worthwhile to point out the limiting cases:

(1) For homogeneous half-space, α=0, we find that β=1, $F_{\alpha\beta}$=2/π, B= (1–v_s^2)/π, m_E=E_s, then Eq. (2) becomes:

$$w = \frac{P(1-v_s^2)}{\pi E_s r} \tag{4}$$

This is the Boussinesq's formula for the homogeneous case.

(2) For the incompressible Gibson soil, α=1 and v_s=0.5, we find that β=0, $F_{\alpha\beta}$=2/π, B=(1–v_s^2)/π, then Eq. (2) becomes:

$$w = 0 \tag{5}$$

This is a Winkler material model.

3. Formulae of bending plate

For a Mindlin plate the displacement field can be uniquely specified by an independent variation of the deflection w and two independent variations of angles defining the direction of the line originally normal to the midsurface of the plate, θ_x and θ_y. At point i, the vector of displacements is (Zienkiewicz *et al.* 1991, Hinton *et al.* 1977):

$$\{\delta\}_i = \{w_i,\ \theta_{xi},\ \theta_{yi}\}^T \tag{6}$$

Here θ_x and θ_y are independent of the deflection w.

The loading vector of a point can be written as

$$\{F\}_i = \{P_i,\ M_{xi},\ M_{yi}\}^T \tag{7}$$

In the theory of FEM for plate bending problem, the final equilibrium equations can be written as

$$\boldsymbol{K\delta} = \boldsymbol{F} \tag{8}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{\delta}$ is the vector of global displacements of nodes, $\boldsymbol{F}$ is the vector of global forces at the nodes.

In the common plate bending problems, the applied forces and the boundary conditions are known. Therefore, the above equations can be solved, and the displacement $\boldsymbol{\delta}$ can be obtained. Then according to the related formulae, the nodal force $\boldsymbol{F}$, stress $\boldsymbol{\sigma}$, strain $\boldsymbol{\varepsilon}$, can all be obtained.

For the plate on foundation, it is acted at the same time by the forces from the upper structure and the foundation force. If we note $\boldsymbol{F}$ is the nodal force vector acting from the upper structure, and $\boldsymbol{Q}$ is that from the foundation, then Eq. (8) can be written as

$$\boldsymbol{K\delta} = \boldsymbol{F} - \boldsymbol{Q} \tag{9}$$

If we know the relation between $\boldsymbol{Q}$ and $\boldsymbol{\delta}$, i.e.,

$$\boldsymbol{Q} = \boldsymbol{K_f}\,\boldsymbol{\delta} \tag{10}$$

then Eq. (9) can be written as

$$(\boldsymbol{K} + \boldsymbol{K_f})\, \boldsymbol{\delta} = \boldsymbol{F} \tag{11}$$

If $\boldsymbol{K_f}$ is determined, then the deflection $\boldsymbol{\delta}$ can be calculated and the interaction of the plate and the foundation can be obtained as well.

4. Interaction between the plate and foundation

Apart from the loading from the upper structure, the plate has the action from the foundation. Before determining the deflection, the action from the foundation is unknown. On the other hand, when the action is unknown, the deflection of the plate cannot be obtained by FEM. For the Winkler's plate, the relationship between the deflection of a point and its force is obvious. For the plate on the half-space foundation, how to determine that relationship is the key step of the problem.

For the 8-node parametric elements, the reaction of the foundation cannot be calculated as previously in which the element area is divided into several parts by the direct proportion. Even when a uniform loading acts on a square element with a unit area, the values of the nodal forces are different. Furthermore, their signs are not the same for the corner nodes and the midside nodes. As a result when the 8-node elements are used, it is necessary to analyse the relationship among the deflections, nodal forces and the action of the foundation further.

The relationship between the loading action $p(x, y)$ on one element and the nodal loading value $\boldsymbol{P^e}$ is given by

$$p = \boldsymbol{N}\boldsymbol{P^e} \tag{12}$$

where $\boldsymbol{N}$ is the shape function.

The force vector of nodes is

$$\boldsymbol{Q^e} = \iint \boldsymbol{N}^T p\,dxdy \tag{13}$$

Inserting Eq. (12) into the above equation and considering that $\boldsymbol{P^e}$ is constant and can therefore be taken out, then

$$\boldsymbol{Q^e} = \boldsymbol{C}\boldsymbol{P^e} \tag{14}$$

where

$$\boldsymbol{C} = \iint \boldsymbol{N}^T \boldsymbol{N}\,dxdy \tag{15}$$

The displacement at a point due to the element area loaded to intensity $p(x, y)$ at the surface can be obtained by dividing the element area into many small elements, each with an area of $dxdy$ and a small concentrated load at its center equal to

$$dP = p(x, y)dxdy$$

According to Eq. (2), the deflection of a point, i, in the element due to dP is

$$dw_{ii} = \frac{pdxdyB}{m_E r^{1+\alpha}}$$

The deflection w_{ii} can be found by integrating the above equation over the element area. It can be written as

$$w_{ii}=\frac{B}{m_E}\iint\frac{p}{r_i^{1+\alpha}}dxdy \tag{16}$$

where r_i is the distance from the integral point to the fixed node i.

Inserting Eq. (12) into the above equation, we have

$$w_{ii}=\frac{B}{m_E}\iint\frac{\boldsymbol{N}\boldsymbol{P}^{\boldsymbol{e}}}{r_i^{1+\alpha}}dxdy \tag{17}$$

Obviously, $\boldsymbol{P}^{e}$ can also be taken out from the integral. At the same time, from Eqs. (14) and (17), we can obtain

$$w_{ii}=\boldsymbol{E}^{\boldsymbol{e}}_{\boldsymbol{1}\times\boldsymbol{j}}\boldsymbol{C}^{-1}_{j\times j}\boldsymbol{Q}^{\boldsymbol{e}}_{\boldsymbol{j}\times\boldsymbol{1}} \tag{18}$$

where $\boldsymbol{C}^{-1}$ is the inverse matrix of $\boldsymbol{C}$ defined in Eq. (15), j is the number of the terms of the shape function, and

$$\boldsymbol{E}^{e}=\frac{B}{m_E}\iint\frac{\boldsymbol{N}}{r_i^{1+\alpha}}dxdy \tag{19}$$

For the 8-node element, j in Eq. (18) should be 8. Also, i can be taken from 1 to 8, then

$$\boldsymbol{w}^{\boldsymbol{e}}_{8\times 1}=\boldsymbol{G}_{8\times 8}\boldsymbol{Q}^{e}_{8\times 1} \tag{20}$$

where $\boldsymbol{G}_{8\times8}$ can be obtained by combining $\boldsymbol{E}^{\boldsymbol{e}}_{1\times8}$ and $\boldsymbol{C}^{-1}_{8\times8}$.

The deflection w_{ij}, at an arbitrary nodal point i that is not in the same element of j, can be obtained by performing integration similar to Eq. (17). However, a reasonable approximation to this result may be obtained by using Eq. (2). For α=1 i.e., homogeneous case, the exact integration result and the corresponding Boussinesq's solution were compared by Cheung (1977). It is evident that the error is small for the two points that are not in the same element, and it decreases rapidly as r_{ij} becomes large. This is to be expected by virtue of St. Venant's principle. It is in reasonable agreement with the cases of non-homogeneous media.

According to the analysis above, the relation among the deflections of all nodes, $\boldsymbol{\delta}$, and the all nodal forces, $\boldsymbol{Q}$, can be obtained as:

$$\boldsymbol{\delta}_{m\times1} = \boldsymbol{H}_{m\times m}\boldsymbol{Q}_{m\times1} \tag{21}$$

where m is the number of total nodes of the plate elements.

In the above analysis, there is an assumption that the reaction of the foundation only affects the deflection of the plate. If δ also includes the rotation angles, then Eq. (21) needs to be changed only by adding some zeros. So it can be rewritten as

$$\boldsymbol{Q} = \boldsymbol{H}^{-1}\boldsymbol{\delta} \tag{22}$$

Comparing the above equation with Eq. (10), we have

$$\boldsymbol{K_f} = \boldsymbol{H}^{-1} \tag{23}$$

Therefore, $\boldsymbol{K}_f$ can be obtained by the above equations, and then the deflection solution can be given by Eq. (11). The interaction of the plate and the foundation can be determined further.

In the integration calculation, a numerical method of reduced integration is used. It can eliminate the fictitious shear deformation, increase the accuracy, reduce the computing time and be suitable for both thick and thin plates.

5. Plate on homogeneous foundation

In order to verify the effectiveness of the proposed isoparametric finite element method, a square plate on an isotropic and homogeneous elastic foundation evaluated by Chen *et al.* (1980) is shown as the first example. The relevant data are:

The dimensions of the plate:
Length $L = 4$ m
Thickness $h = 0.2$ m
The material parameters of the plate:
Young's modulus $E = 0.343\times10^5$ MPa
Poisson's ratio $\nu = 0.167$
The material parameters of the foundation:
$m_E = 0.343\times10^3$ MPa
$\alpha = 0$
(the above m_E and α mean: Young's modulus $E_S = 0.343\times10^3$ MPa)
Poisson's ratio $\nu_S = 0.4$
The magnitude of the uniform loading:
$q = 0.98$ MPa

For this symmetric problem, a quarter of the plate is divided into equal elements. By using different meshes, the computed deflections at the plate centre are listed in Table 1, and compared in which the spline method and displacement method are used by Chen *et al.* (1980). From the table it can be seen that the results obtained by different methods agree well generally. Even for the 2×2 mesh, the results of the 8-node element method are still good. The results tend to converge to stable values for increasing number of elements.

For the symmetric problem, the deflection at the plate center is maximum, however the variation from the center to the edge is only moderate. Based on results of the 6×6 element mesh, Fig. 2 shows a 3-dimensional deformation of the plate. It should be noted that the observed deformation in the figure is highly magnified. In fact, the unit of the plate dimensions is m, while that of the deflection is 10^{-3} m. The actual deflection is not really large.

Table 1 The deflection (m) at the plate centre

Mesh	Present method	Spline method*	Displacement method*
2×2	0.01077	/	/
4×4	0.01068	0.01054	0.01054
6×6	0.01063	0.01059	0.01059
8×8	0.01061	0.01062	0.01062

*Results taken from Chen *et al.* (1980)

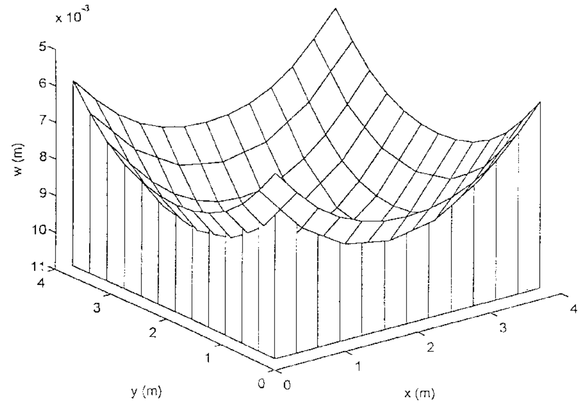

Fig. 2 The deformation of a plate on elastic foundation

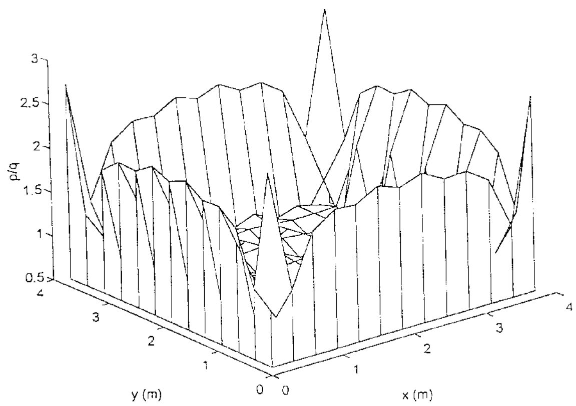

Fig. 3 Contact pressure distribution

Fig. 3 is a 3-D figure of the contact pressure of the plate shown by a non-dimensional parameter, *p/q*. It can be seen that at the boundary of the plate, there is some stress concentration, and the concentration is most serious at the corners of the plate. For the internal points of the plate, however, *p/q* is almost equal to unity. This means that there is no concentration in the internal part of the plate.

6. Plate on non-homogeneous foundation

6.1. Influence of α

In order to obtain variation trends of plate on non-homogenous media, consider the same plate on foundation. Except for the material parameters of the soil, the other data is the same as in the last section. Let m_E=0.343×10^3 (MPa /m$^\alpha$), and α is changed from 0 to 1.

Fig. 4 shows the variation of w_{max} (the deflection at the plate centre) for various values of α. It can be observed that w_{max} decreases as long as α increases at first (α<0.4). When α>0.4, w_{max} increases as long as α increases.

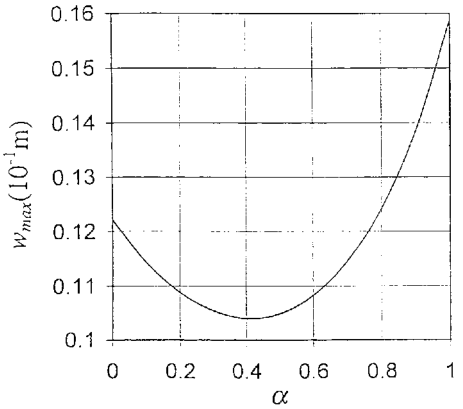

Fig. 4 $w_{\max}$ for different α

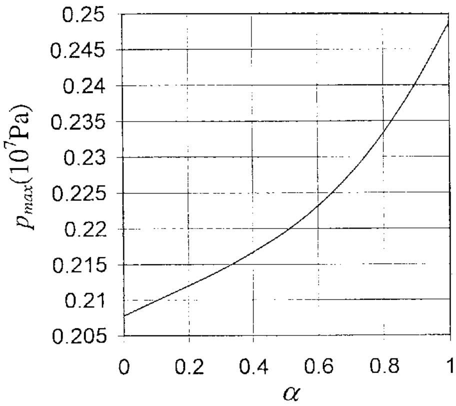

Fig. 5 $p_{\max}$ for different α

Fig. 5 shows the variation of $p_{\max}$ (the contact pressure at the plate corners) for several α values. When α increases, $p_{\max}$ also increases monotonously. When α=1, $p_{\max}$ reaches the maximum value.

From Figs. 4 and 5 it can be seen that the variation trends of $w_{\max}$ and $p_{\max}$ are different. Fig. 4 shows that in the shallow layer of soil (z<1m), when α is bigger, E_s value is smaller, and the settlement in this part will be larger. For deeper layers, (z>1m), the variation turns to be in the opposite way. These two parts sum up to become the total settlement. So that the variation of $w_{\max}$ with α in Fig. 4 decreases first. After α>0.5, the second part will have more influence, and then the curve will rise.

For the interaction, the influence of the soil may be limited mainly to the shallow part. In this part, E_s becomes smaller for a larger α value. Therefore, when α increases, E_s will decrease, $p_{\max}$ value will increase monotonously at the same time.

6.2. Soft plate

In order to study the relative stiffness of the plate on the foundation, the Young's modulus E of the plate is taken to be different values. When E = 0.343 kPa, it means that the plate is soft.

Assume that the dimension and Poisson's ratio of the plate, m_E of the foundation, and the loading value are the same as the previous example. For α=10^{-5} (it tends to 0), 0.5, 1.0, the computed $w_{\max}$ curves for various Poisson's ratio of the soil, v_s, are shown in Fig. 6. It can be seen that when v_s increases, $w_{\max}$ for different α values decrease monotonously. For a fixed v_s value, for example, v_s= 0.2, the magnitudes of $w_{\max}$ for different α values are not ordinal. The reason of the variation has been explained in the above section and Fig. 4.

From Fig. 6 it can be seen that the decrease rate for various α values is different. When v_s is small, the curve for α=1.0 is the highest. As long as v_s increases, the curve decreases rapidly. When v_s>0.33, the curve of α=1.0 is lower than that of α=10^{-5}.

Fig. 7 shows the variations of $w_{\max}$ (at the plate center) and w_c (at the plate corner) with v_s for α=0.5. The two curves have the similar tendency of variation.

For the soft plate, the interaction of the plate and the foundation is uniform. The contact pressure at all points of the plate is equal to the applied uniform force. This is reasonable for the soft plate on the foundation.

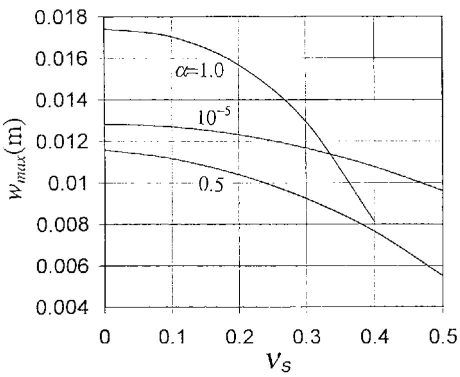

Fig. 6 $w_{\max}$ of soft plate for different α and v_s

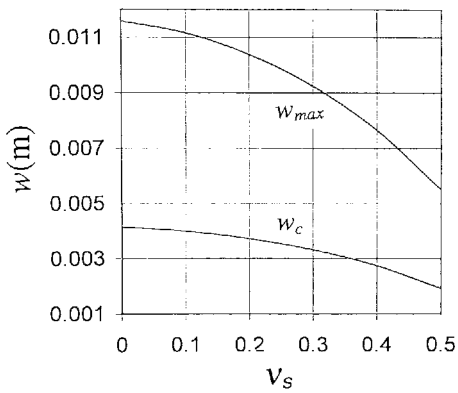

Fig. 7 w for the center and corners of soft plate for different v_s (α=0.5)

6.3. Plate with medium stiffness

Let the Young's modulus of the plate be: E = 0.343 GPa. The $w_{\max}$ curves for different α values and v_s values are shown in Fig. 8. The variation trend of the curves is almost the same as those for the soft plate. Fig. 9 shows the variation of $w_{\max}$ (at the plate center) and w_c (at the plate corner) with v_s. Fig. 7 shows the similar results of the soft plate.

Fig. 10 shows $p_{\max}$ variation for different α values and v_s values. From the figure it can be seen that when v_s increases, $p_{\max}$ curves decrease. For the small v_s values, the variation of $p_{\max}$ with v_s has the same trend for different α values. As long as v_s becomes rather large, for example, v_s>0.4, the variation of these curves with different α values may be not ordinal.

6.4. Rigid plate

Let the Young's modulus of the plate be $E = 0.343\times10^8$ GPa. Since the plate is nearly rigid, there should not be any deformation in the plate. This means that the settlements at all plate points are the same. Fig. 11 shows the w curves for 3 values of α and various v_s values. From Figs. 6, 8, 11, it can be observed that there is little difference and the general variation of $w_{\max}$ for the plates with

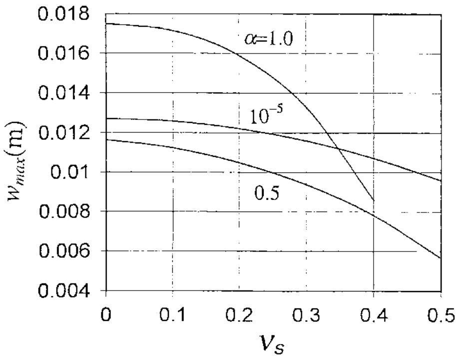

Fig. 8 $w_{\max}$ of medium rigidity plate for different α and v_s

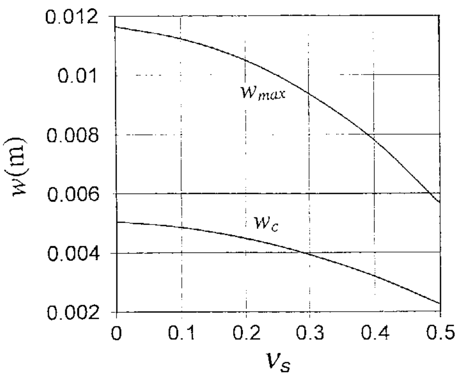

Fig. 9 w for the center and corners of medium rigidity plate for different v_s (α =0.5)

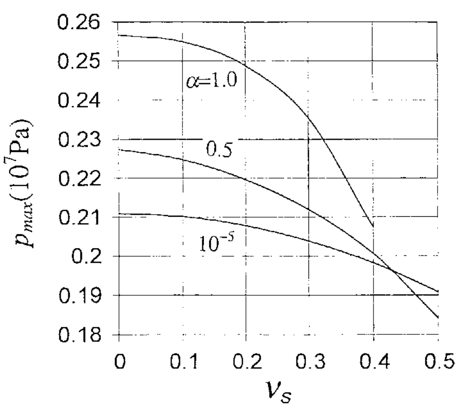

Fig. 10 $p_{\max}$ of medium rigidity plate for different α and v_s

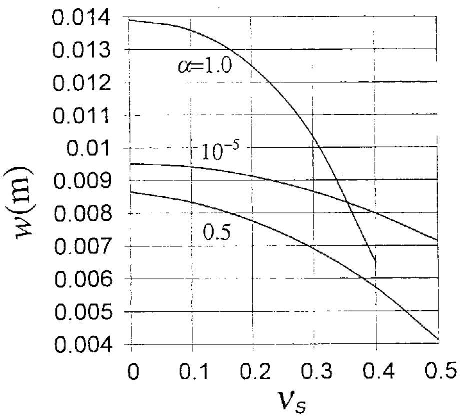

Fig. 11 $w_{\max}$ of rigid plate for different α and v_s

different rigidities is similar.

Fig. 12 shows the $p_{\max}$ - v_s relations for 3 values of α and which are all horizontal lines. This means that $p_{\max}$ is independent of v_s of the rigid plate. On the other hand, for different α values, $p_{\max}$ values are different. When α is smaller, the $p_{\max}$ value is larger.

Comparing the soft plate with the medium and rigid plates, there is a lot of difference for $p_{\max}$. For the soft plate with a uniform loading, the contact forces are the same at all points for all α and v_s values. The magnitude is equal to the external loading. For the plate with medium rigidity, $p_{\max}$ occurs at the plate corners. The magnitude is different for various α and v_s values. For the rigid plate, $p_{\max}$ also appears at the plate corners and there has serious stress concentration. Moreover, $p_{\max}$ values are independent of v_s, which are represented by horizontal lines in the $p_{\max}$ - v_s figure.

7. Conclusions

The problem of a plate on elastic foundation has practical significance in engineering. In order to

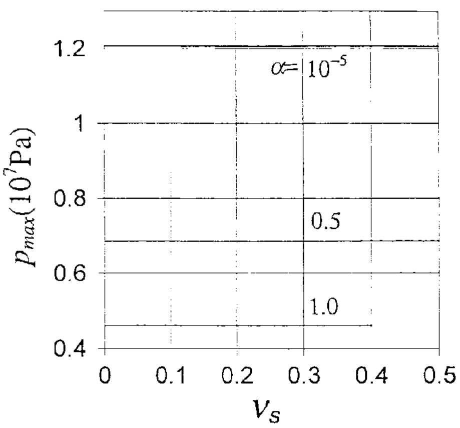

Fig. 12 $p_{\max}$ of rigid plate for different α and v_s

reduce the preparation and computing work, a semi-analytical and semi-numerical method is used. Several fundamental solutions are available. For isotropic soil, Boussinesq's solution is used, while for non-homogeneous soil, Booker's formulae are introduced. Therefore, the reaction from the foundation can be treated as the external force. In the current method, the finite elements are only needed to be arranged in the plate. This will reduce the computing time and storage space of the computer.

The interaction between the plate and the foundation is studied numerically by the isoparametric element method. This method also lifts the restriction that the contact pressure is assumed to be uniformly distributed around each node point. The present method is more reasonable and can have more accurate results. Furthermore, the 8-node elements can be used for the plate with different boundaries. Thus, the proposed method can be applied to general cases.

Since the rotation angles of the plate are assumed to be independent of the deflection and the reduced integration method is used, the current method can be applied for both thick and thin plates. This is especially suitable for the civil engineering practice.

A number of examples for plates on non-homogenous foundation have been studied. Among them, the plates have different rigidities (depending on the plate thickness, Young's modulus and Poisson's ratio) and the foundations have different materials parameters, such as α and v_s. The variation trends of the deformation and contact pressure have been obtained.

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