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Sliding and rocking response of rigid blocks due to horizontal excitations

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Abstract. To study the dynamic response of a rigid block standing unrestrained on a rigid foundation which shakes horizontally, four modes of motion can be identified, i.e., rest, slide, rock, and slide and rock. The occurrence of each of these four modes and the transition between any two modes depend on the parametric values specified, the initial conditions, and the magnitude of ground acceleration. In this paper, a general two-dimensional theory is presented for dealing with the various modes of a free-standing rigid block, considering in particular the impact occurring during the rocking motion. Through selection of proper values for the system parameters, the occurrence of each of the four modes and the transition between different modes are demonstrated in the numerical examples.

Key words: friction; impact; overturning; rocking; sliding.

1. Introduction

Typical examples of sliding structures include objects or structures resting unrestrained on a flat supporting surface, such as sliding structures, delicate equipment, electronic hardware, precious artifacts, and goods carried or transported by vehicles. Previous researches on the dynamic responses of sliding structures have focused primarily on the sliding behavior or horizontal equilibrium of the structure, with little attention paid to the rocking motion. The results generated from these researches remain strictly valid only for structures with small coefficients of friction or for structures that are relatively short, for which the effect of rocking can be ignored. In most previous works, a sliding structure has been treated as a rigid block, while Coulomb's law was adopted for describing the friction behavior. For a fundamental treatment of the subject of sliding structures, the readers should refer to the works of Westermo and Udwadia (1983) and Mostaghel *et al.* (1983). In the study by Yang *et al.* (1990), the effect of high modes of vibration on the response of sliding structures has been shown to be negligible.

For structures which are not relatively short or for sliding surfaces with relatively large coefficients of friction, it is rocking, rather than sliding, that should be considered. Essential to simulation of rocking is a proper treatment of the impact phenomenon involved. Based on the assumption of no bouncing and sufficient friction to prevent sliding before and after impact, Housner (1963) studied the pure rocking mode of a free-standing rigid block. Such problems were also studied by Yim *et al.* (1980), Aslam *et al.* (1980), and Tso and Wong (1989). In the study by Dimentberg *et al.* (1993)

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and Lin *et al.* (1994), both the horizontal and vertical excitations were considered. Recently, the chaotic nature of the rocking motion of rigid objects has been studied by Lin and Yim (1996a, b) from a probabilistic perspective. In general, to study the dynamic response of a rigid block to ground shaking, various modes of motion, including rest, slide, rock, slide and rock, and jump, should be considered (Ishiyama 1982, Shenton and Jones 1991a, b).

In this paper, a two-dimensional theory will be presented for dealing with the various modes of motion and the transition between any two modes for a rigid block resting unrestrained on a rigid foundation that shakes horizontally. The jump mode is excluded, because the block is assumed to be heavy. No assumption will be made concerning the magnitude of rocking rotations. Newmark's finite difference scheme is adopted for solving the nonlinear response of the rigid block within each mode, along with iterations for removing the unbalanced forces.

2. Conditions for transition of different modes

In this study, both the block and the ground on which the block stands are assumed to be rigid. As shown in Fig. 1, the block is rectangular and symmetrical about the vertical axis passing the center of gravity G. Only horizontal excitations are considered. By Coulomb's law, the maximum static (or dynamic) frictional force f is computed as the product of the reaction N multiplied by the static (or dynamic) coefficient of friction. To analyze the behavior of the rigid block, the conditions for transition between any two of the four modes, i.e., rest, slide, rock, and slide and rock, as well as the equations of motion governing each mode, must be identified first.

2.1. Rest mode

Consider a rectangular rigid block standing freely on the surface of a rigid foundation that is excited horizontally in Fig. 2(a). For the rigid block to be at rest, the frictional force f must be less than the maximum static frictional force, while the (absolute) acceleration of the block $\ddot{x}$ remains equal to the ground acceleration $\ddot{x}_g$, that is

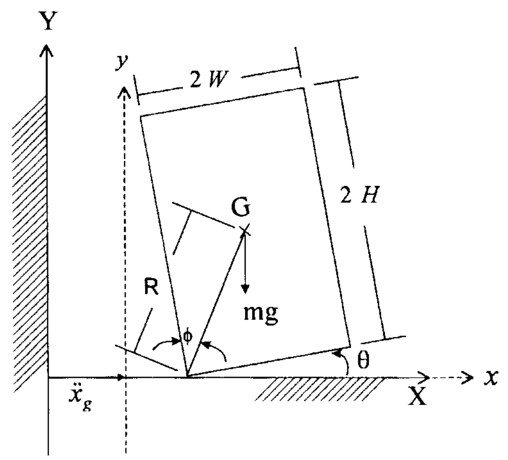
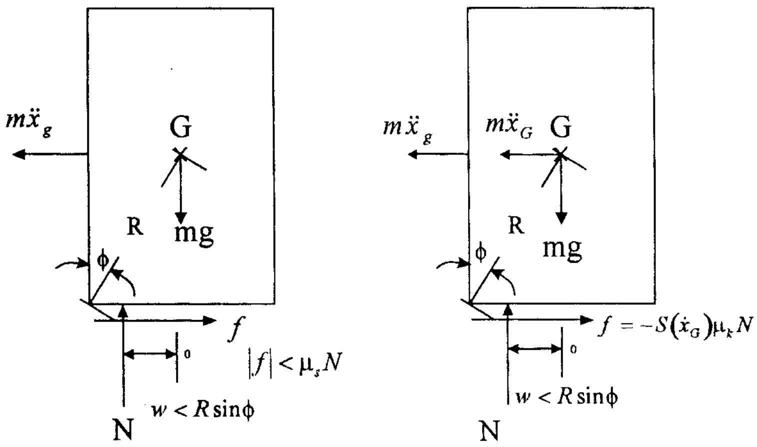


Fig. 1 Model for rigid block



(a) (b)

Fig. 2 Rigid block in: (a) rest mode; (b) slide mode

$$|f|=m|\ddot{x}|=m|\ddot{x}_g|<\mu_s N \tag{1}$$

where μ_s denotes the static coefficient of friction, m the mass of the block, and N the vertical reaction. Noting that $N=mg$, where g is the acceleration of gravity, the condition for the block to remain in the rest mode can be obtained from Eq. (1) as

$$|\ddot{x}_g|<g\mu_s \tag{2}$$

in the absence of vertical ground acceleration.

Consider the rigid block at the instant when it is about to transfer from the rest to the rock mode in Fig. 2(a), in which R denotes the distance from point G to the center of rotation, i.e., one of the two bottom corners, and ϕ the angle between the vertical axis and the diagonal of the block at rest. For this case, the reaction N must act through the center of rotation and the condition for the block to remain in the non-rocking mode is

$$|f|R\cos\phi<NR\sin\phi \tag{3}$$

which reduces to

$$|\ddot{x}_g|<g\tan\phi \tag{4}$$

based on the relations $N=mg$ and $f=m\ddot{x}_g$. Let x_G denote the displacement of the rigid block relative to the ground. The equations of motion for a block at rest are: $\ddot{x}_G=\dot{x}_G=0$ and $\ddot{\theta}=\dot{\theta}=\theta=0$. In general, the (residual) displacement x_G is unequal to zero.

2.2. Slide mode

A block begins to slide, once the following condition is reached:

$$|\ddot{x}_g|=g\mu_s \tag{5}$$

However, for the block to switch first to the slide mode, the condition (4) must also be satisfied. From Eqs. (5) and (4), one observes that for a rigid block with $\mu_s<\tan\phi$, the rest mode will switch

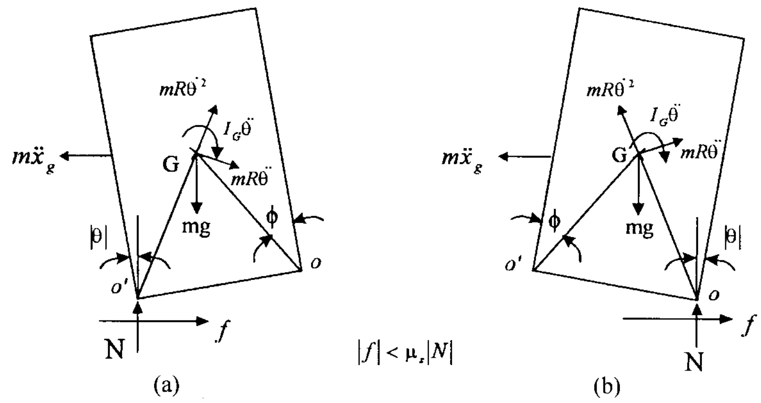


Fig. 3 Rigid block in rock mode: (a) $\theta > 0$; (b) $\theta < 0$

first to the slide mode; for the case with $\mu_s > \tan\phi$, to the rock mode; and for the case with $\mu_s = \tan\phi$, to the slide and rock mode. It should be noted that two parameters μ_s and $\tan\phi$ are determined solely by the material and geometry of the rigid block.

For the rigid block to be in the pure slide mode (Fig. 2b), the following conditions of equilibrium must hold:

$$f = m(\ddot{x}_g + \ddot{x}_G); \qquad N = mg \tag{6a, b}$$

According to Coulomb's law, the (dynamic) frictional force acts in a direction opposite to that of the motion, i.e., $f = -\mathrm{sgn}(\dot{x}_G)\mu_k N$, where $\mathrm{sgn}(\dot{x}_G)$ denotes the sign of $\dot{x}_G$, and μ_k the dynamic coefficient of friction. By substitution, it can be shown that

$$\ddot{x}_g + \ddot{x}_G = -\mathrm{sgn}(\dot{x}_G) g \mu_k \tag{7}$$

This is exactly the equation of motion for the rigid block to slide. For this mode, the angular motion remains identically equal to zero, i.e., $\ddot{\theta} = \dot{\theta} = 0$.

Refer to Fig. 2(b) for the rigid block at the instant to transfer from the slide to rock mode, for which the reaction N must act through one of the corners. Again, for the block to remain in the non-rock mode, the inequality (3) must hold, except that the dynamic frictional force is known as $f = -\mathrm{sgn}(\dot{x}_G)\mu_k N$. It follows that the inequality (3) reduces to

$$\mu_k < \tan\phi \tag{8}$$

This condition is always satisfied, because $\mu_s < \tan\phi$ for the pure slide mode, and μ_k is less than μ_s in general. Evidently, it is impossible for a rigid block to switch from the slide mode to the slide and rock mode. For a rigid block in sliding, the acceleration and velocity relative to the ground will be generally different from zero, i.e., $\ddot{x}_G \neq 0$ or $\dot{x}_G \neq 0$.

2.3. Rock mode

According to the foregoing discussions, the following are the conditions for the rock motion to occur:

$$|\ddot{x}_g| < g\mu_s \quad \text{and} \quad |\ddot{x}_g| < g\tan\phi \tag{9}$$

As was stated in Section 2.2, once the two parameters μ_s and $\tan\phi$ are given, the mode to which the rigid block will transfer from the rest mode can be uniquely determined. The following are the equations of equilibrium for a block in rock mode (Fig. 3):

$$f=m[\ddot{x}_g-R\ddot{\theta}\cos(\phi-|\theta|)-\mathrm{sgn}(\theta)R\dot{\theta}^2\sin(\phi-|\theta|)] \tag{10}$$

$$N-mg=mR[\mathrm{sgn}(\theta)\ddot{\theta}\sin(\phi-|\theta|)-\dot{\theta}^2\cos(\phi-|\theta|)] \tag{11}$$

$$fR\cos(\phi-|\theta|)-\mathrm{sgn}(\theta)NR\sin(\phi-|\theta|)=I_G\ddot{\theta} \tag{12}$$

where I_G denotes the mass moment of inertia of the block with respect to point G, $I_G = 1/3mR^2$. Substituting Eqs. (10) and (11) into (12) yields the equation of motion for the rock motion as

$$\mathrm{sgn}(\theta)mgR\sin(\phi-|\theta|)-mR\ddot{x}_g\cos(\phi-|\theta|)+I_O\ddot{\theta}=0 \tag{13}$$

where I_O denotes the mass moment of inertia with respect to point O (Fig. 3), $I_O = I_G + mR^2$. For this case, the relative displacement between the block and the ground is zero, $\dot{x}_c=0$, where $\dot{x}_c=\dot{x}_O$ for $\theta > 0$ and $\dot{x}_c=\dot{x}_O$ for $\theta < 0$.

For the rigid block to remain in the pure rock (non-slide) mode, it is required that the frictional force f be less than the maximum static frictional force:

$$|f|<\mu_s|N| \tag{14}$$

where f and N have been given in Eqs. (10) and (11). In the meantime, the angular velocity and rotation of the rigid block will be generally different from zero, i.e., $\dot{\theta}\neq 0$ or $\theta\neq 0$.

2.4. Slide and rock mode

This mode can occur either following the rest or rock mode. Depending on the mode from which this mode is converted, two different cases need be considered.

(1) Converted from the rest mode:
The condition for the slide and rock mode to occur is

$$|\ddot{x}_g|=\mu_s g \quad \text{and} \quad |\ddot{x}_g|=g\tan\phi \tag{15}$$

(2) Converted from the rock mode:
According to Eq. (14), the condition for initiation of the slide and rock mode is

$$|f|<\mu_s|N| \tag{16}$$

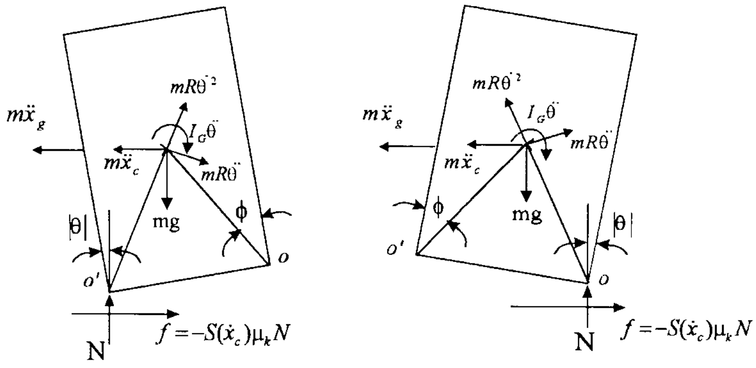
where the forces f and N have been given in Eqs. (10) and (11).

The equation of equilibrium for the rocking motion of the present mode can be obtained from Fig. 4 by taking moments about the center of gravity G as

$$-\mathrm{sgn}(\theta)NR\sin(\phi-|\theta|)+fR\cos(\phi-|\theta|)=I_G\ddot{\theta} \tag{17}$$

Consider the equilibrium of the structure along the horizontal axis. The (dynamic) frictional force f can be given as

$$f=m[\ddot{x}_g+\ddot{x}_c-\mathrm{sgn}(\theta)R\dot{\theta}^2\sin(\phi-|\theta|)-R\ddot{\theta}\cos(\phi-|\theta|)] \tag{18}$$



(a) (b)

Fig. 4 Rigid block in slide and rock mode: (a) $\theta > 0$; (b) $\theta < 0$

Also, consider the equilibrium in the vertical direction. The reaction N can be shown to be identical to the one given in Eq. (11). To ensure the occurrence of sliding in addition to rocking, the following condition should be satisfied:

$$f=-\operatorname{sgn}(\dot{x}_c)\mu_k N \tag{19}$$

By substitution of f and N in Eqs. (18) and (11), the equation of motion as given in Eq. (19) for sliding can be rearranged as follows:

$$\begin{aligned}&-\operatorname{sgn}(\dot{x}_c)\mu_k[g+\operatorname{sgn}(\theta)R\ddot{\theta}\sin(\phi-|\theta|)-R\dot{\theta}^2\cos(\phi-|\theta|)]\\&=\ddot{x}_g+\ddot{x}_c-\operatorname{sgn}(\theta)R\dot{\theta}^2\sin(\phi-|\theta|)-R\ddot{\theta}\cos(\phi-|\theta|)\end{aligned} \tag{20}$$

Moreover, by the use of Eqs. (19) and (11) for f and N, the equation of motion for rocking in Eq. (17) can be rearranged as

$$\begin{aligned}&\{I_G/m+R^2\sin^2(\phi-|\theta|)+1/2\operatorname{sgn}(\dot{x}_c)\operatorname{sgn}(\theta)\mu_k R^2\sin[2(\phi-|\theta|)]\}\ddot{\theta}\\&-\{\operatorname{sgn}(\dot{x}_c)\mu_k R^2\cos^2(\phi-|\theta|)+1/2R^2\operatorname{sgn}(\theta)\sin[2(\phi-|\theta|)]\}\dot{\theta}^2\\&+[\operatorname{sgn}(\dot{x}_c)R\mu_k\cos(\phi-|\theta|)+\operatorname{sgn}(\theta)R\sin(\phi-|\theta|)]g=0\end{aligned} \tag{21}$$

At the instant $\dot{x}_c = 0$, a transition to pure rock mode is possible, depending on the relative values of $\ddot{x}_g$ and μ_s. For example, for the case with $|f| < \mu_s |N|$, the system will transfer to the pure rock mode at the instant $\dot{x}_c = 0$. In the slide and rock mode, it is required that the translational and angular responses of the block remain unequal to zero, i.e., $\ddot{x}_c \neq 0$ or $\dot{x}_c \neq 0$ and $\dot{\theta} \neq 0$ or $\theta \neq 0$.

Throughout all the foregoing derivations no assumption has been made concerning the magnitude of rotations. To enhance comparison with equations existing in the literature, let us consider the special case of slender blocks, for which the angles $(\phi - |\theta|)$ can be assumed to be small (Tso and Wong 1989). It follows that the term containing $\dot{\theta}^2$ can be neglected, $\sin(\phi - |\theta|)$ can be approximated as $(\phi - |\theta|)$ and $\cos(\phi - |\theta|)$ as 1. Consequently, Eqs. (20) and (21) can be linearized as

$$-\operatorname{sgn}(\dot{x}_c)\mu_k[g+\operatorname{sgn}(\theta)R\ddot{\theta}(\phi-|\theta|)]=\ddot{x}_g+\ddot{x}_c-R\ddot{\theta} \tag{22}$$

$$[I_G/m+R^2(\phi-|\theta|)^2+\mathrm{sgn}(\dot{x}_c)\,\mathrm{sgn}(\theta)\mu_k R^2(\phi-|\theta|)]\ddot{\theta}$$
$$+[\mathrm{sgn}(\dot{x}_c)R\mu_k+\mathrm{sgn}(\theta)R(\phi-|\theta|)]g=0 \tag{23}$$

which are identical to those for the special case of small rotations (Shenton and Jones 1991a).

3. Impact between rigid block and foundation

When in the rock motion, a rigid block will experience some impacts with the rigid foundation, and loses its energy in such a process. Following Shenton and Jones (1991a), the classical point-impact theory will be adopted herein, which implies that the rigid block touches the foundation at a point, i.e., at one of its two bottom edges, at the instant of impact. No change in position of the contact point will be considered. Moreover, it is assumed that no bouncing occurs at the point of impact, in the sense that the coefficient of restitution used to relate the normal velocities at the point of impact prior to and after impact is taken as zero. Based on these assumptions, a block is not allowed to impact repeatedly at the same corner. In some cases, a change of the modes prior to and after impact is also possible, depending on the magnitude of the horizontal impulse $\int f_x dt$ relative to the vertical impulse $\mu_s \int f_y dt$, where f_x and f_y denote the impulsive forces acting at the corner of the block along the two directions.

The principle of impulse and momentum, both linear and angular, should be obeyed by impact. As the duration of impact is quite short and the velocities are not continuous, the accelerations and the impulsive forces can be quite large, relative to the weight mg of the block. For this reason, the gravitational effect was usually neglected by the classical impact theory. It should be noted that the impulses $\int f_x dt$ and $\int f_y dt$ remain generally finite in magnitude, as the impulsive forces f_x and f_y, which can be quite large, are integrated only over a short time interval.

3.1. Impact model

Consider the impact diagram in Fig. 5. The following equations can be written based on the principle of impulse and momentum:

(a) x-direction:

$$-mR\dot{\theta}_1\cos\phi+m\dot{x}_{c1}+\int f_{x_i}dt+\int f_{x_r}dt=-mR\dot{\theta}_2\cos\phi+m\dot{x}_{c2} \tag{24}$$

(b) y-direction:

$$-\mathrm{sgn}(\dot{\theta}_1)mR\dot{\theta}_1\sin\phi+\int f_{y_i}dt+\int f_{y_r}dt=\mathrm{sgn}(\dot{\theta}_2)mR\dot{\theta}_2\sin\phi \tag{25}$$

(c) Rotation about center of mass:

$$I_G\dot{\theta}_1+[\int f_{x_i}dt+\int f_{x_r}dt]R\cos\phi+\mathrm{sgn}(\dot{\theta}_1)[\int f_{y_r}dt-\int f_{y_i}dt]R\sin\phi=I_G\dot{\theta}_2 \tag{26}$$

in which the subscripts i and r respectively refer to the "impact" and "rotate" corner.

For the present case, the following are the conditions for the block to remain in the non-slide and slide modes, respectively,

$$\left|\int f_{x_i}dt+\int f_{x_r}dt\right|\le\mu_s\left|\int f_{y_r}dt+\int f_{y_i}dt\right|$$
$$\left|\int f_{x_i}dt+\int f_{x_r}dt\right|=-\mu_k\,\mathrm{sgn}(\dot{x}_{c2})\left|\int f_{y_r}dt+\int f_{y_i}dt\right| \tag{27a, b}$$

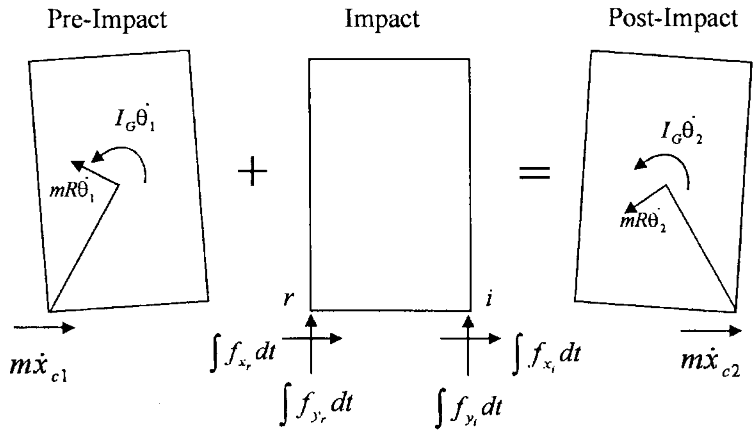


Fig. 5 Impact diagram for $\dot{\theta} < 0$

Let the resultant impulse in the x direction be denoted

$$\int f_x dt = \int f_{x_i} dt + \int f_{x_r} dt \tag{28}$$

Two different conditions can be identified. First, if the rigid block remains in the non-slide mode after impact, the term containing $\dot{x}_{c2}$ in Eq. (24) should be dropped, i.e.,

$$-mR\dot{\theta}_1\cos\phi + m\dot{x}_{c1} + \int f_x\, dt = -mR\dot{\theta}_2\cos\phi \tag{29}$$

For this case, there is a total of four unknowns, i.e., $\int f_{yr}\,dt$, $\int f_{yi}dt$, $\int f_x dt$ and $\dot{\theta}_2$, but only three impact equations, i.e., Eqs. (25), (26), and (29). Second, if the rigid block starts to slide after impact, there will be five unknowns, i.e., $\int f_{yr}dt$, $\int f_{yi}dt$, $\int f_x dt$, $\dot{\theta}_2$, and $\dot{x}_{c2}$, but only three impact equations, i.e., Eqs. (24)-(26), and one condition (27b).

Clearly, one additional condition is needed before the impact problem can be solved. Although the impulsive force is equal in magnitude to the change in momentum, previous impact tests indicated that the larger the initial momentum, the larger the impulsive force is. Further, it can be imagined that the velocity reaches its maximum at the impact corner and equals zero at the rotate corner. Thus, the momentum at the impact corner should be very much greater than that at the rotate corner, and it becomes reasonable to assume $\int f_{yr}dt \ll \int f_{yi}dt$. In other words, the quantity $\int f_{yr}dt$ can be neglected and Eq. (25) reduces to

$$-\mathrm{sgn}(\dot{\theta}_1)mR\dot{\theta}_1\sin\phi + \int f_{y_i}dt = \mathrm{sgn}(\dot{\theta}_2)mR\dot{\theta}_2\sin\phi \tag{30}$$

As the total number of unknowns is equal to the number of equations, the angular and linear velocities after impact can be solved.

3.2. Derivation of impact equations

As was stated previously, a rigid block may change its mode of motion after impact. For this reason, two cases have to be identified:

(1) The block remains non-sliding after impact:

By substituting the relation $\mathrm{sgn}(\dot{\theta}_2) = \mathrm{sgn}(\dot{\theta}_1)$ and Eqs. (29) and (30) into the angular impulse-momentum Eq. (26), one can solve

$$\dot{\theta}_2=\dot{\theta}_1-(2mR^2\dot{\theta}_1\sin^2\phi+mR\dot{x}_{c1}\cos\phi)/I_O \tag{31}$$

Moreover, by the use of Eqs. (29)-(31), one can derive from the non-slide condition (28a) the following:

$$\mu_s \geq |(e_1+e_2)H/W|/|1+e_1| \tag{32}$$

where

$$\begin{aligned} e_2&=\dot{x}_{G1}/(H\dot{\theta}_1);\ \ \dot{x}_{G1}=\dot{x}_{c1}-R\dot{\theta}_1\cos\phi \\ e_1&=1-(3/4)(1+e_2)\cos^2\phi-(3/2)\sin^2\phi \end{aligned} \tag{33}$$

(2) The block starts sliding after impact:
By the use of Eqs. (25) and (28), the slide condition (27b) reduces to

$$\int f_x\,dt=-\mu_k\text{sgn}(\dot{x}_{c2})mR(\dot{\theta}_1+\dot{\theta}_2)\text{sgn}\dot{\theta}_1\sin\phi \tag{34}$$

Next, with the substitution of Eqs. (30) and (34), the angular impulse-momentum Eq. (26) can be rearranged as

$$\dot{\theta}_2=e\dot{\theta}_1 \tag{35}$$

where the coefficient of restitution e for angular velocities is

$$e=\frac{I_G-\text{sgn}(\dot{\theta}_1)\mu_k\text{sgn}(\dot{x}_{c2})mR^2\sin\phi\cos\phi-mR^2\sin^2\phi}{I_G+\text{sgn}(\dot{\theta}_1)\mu_k\text{sgn}(\dot{x}_{c2})mR^2\sin\phi\cos\phi+mR^2\sin^2\phi} \tag{36}$$

Finally, by the use of Eqs. (28), (34), and (35), the linear impulse-momentum equation for the x-direction as given in Eq. (24) can be rearranged as

$$\dot{x}_{c2}=R[(e-1)\cos\phi-(e+1)\text{sgn}(\dot{\theta}_1)\mu_k\text{sgn}(\dot{x}_{c2})\text{sgn}\phi]\dot{\theta}_1+\dot{x}_{c1} \tag{37}$$

As can be seen from Eq. (36), before the e value can be determined, the sign of $\dot{x}_{c2}$ must be known beforehand. However, the value of $\dot{x}_{c2}$ remains unknown at this stage. To circumvent this problem, one may simply assume a sign for $\dot{x}_{c2}$, and then determine e and $\dot{x}_{c2}$ from Eqs. (36) and (37). If the $\dot{x}_{c2}$ value solved bears the same sign as assumed, then the solution is exactly the one sought. Otherwise, reverse the sign of $\dot{x}_{c2}$ and repeat the procedure.

It is true that the pre- and postimpact velocities of a rigid block can be determined from the impact equations, based on the hypothesis that the duration of impact is so short that the position of the block remains unchanged before and after impact. However, to determine the acceleration of the block after impact is not as easy, which remains a problem not very well documented in the literature. Someone might suggest that the postimpact acceleration be determined using integration schemes of the Newmark type, based on information available prior to impact, e.g., the preimpact velocity and acceleration. Such an approach is deemed improper, because the Newmark procedure can only be applied to problems that are continuous in response, but not to problems involving discontinuity in response, as encountered here. To solve this problem, recourse must be Newton's second law, which remains valid for the block at all times, including the instants immediately before and after impact. Namely, based on the hypothesis that the position of the block after impact remains unchanged, the linear and angular accelerations can be determined from the conditions of equilibrium for the free body along the three directions x, y, and θ.

Let us consider a special case of impact by setting the response $\dot{x}_{c1}$ in Eq. (24) equal to zero. This is exactly the case for pure rock to occur. For this case, the angular impulse-momentum Eq. (31) reduces to

$$\dot{\theta}_2 = \dot{\theta}_1 (I_O - 2mWR\sin\phi)/I_O \tag{38}$$

and the condition Eq. (32) becomes

$$\mu_s \geq 3\sin\phi\cos\phi/(1 + 3\cos^2\phi) \tag{39}$$

which is independent of either the linear or angular velocities. As was stated in Sect. 2.2, for the case with $\mu_s > \tan\phi$, a rigid block begins to rock. Further, Eq. (39) indicates that if μ_s is less than $3\sin\phi\cos\phi/(1+3\cos^2\phi)$, the rigid block will switch to the slide and rock mode after impact. However, because of the relation: $3\sin\phi\cos\phi/(1+3\cos^2\phi) = [3\cos^2\phi/(1+3\cos^2\phi)]\tan\phi \leq \tan\phi$, a rigid block will never switch to the slide and rock mode after impact, once it has been in the pure rock mode.

3.3. Transition of modes

The transition between the various modes of a rigid block has been depicted in Fig. 6. As can be seen, except for the rest-slide link and rock-slide and rock link, which are reversible, all the other modes will return to their original modes only after experiencing the impact process. In other words, the velocity change (or energy change) remains continuous only when undergoing the transition from the rest to slide mode or from rock to slide and rock mode, and vice versa; it is not continuous for the other transitions involving impact. It should be noted that due to the discontinuity inherent in the impact process, various forms of energy transformation may occur. For instance, the kinetic energy involved in slide and rock may be converted all to that of the slide mode, of the rock mode, or of the slide and rock mode, depending on the energy level of the rigid block after impact.

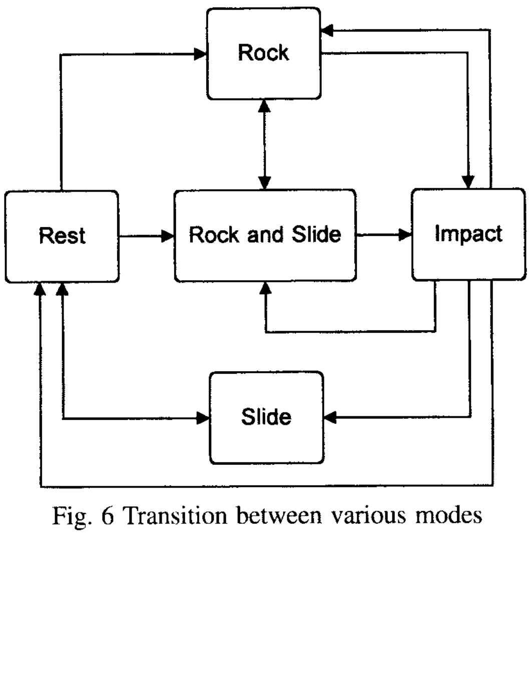


Fig. 6 Transition between various modes

4. Numerical method of integration

Consider an equation of equilibrium of the form $f(\theta_k, \dot{\theta}_k, \ddot{\theta}_k)=0$, where the subscript k denotes the kth time step using an incremental approach. In this study, a trial value will be denoted with a superscript "*". Based on Newmark's β method (1959), the following relations can be written:

$$\begin{aligned} \ddot{\theta}_k^* &= \ddot{\theta}_{k-1} \\ \theta_k^* &= \theta_{k-1}+\Delta t\dot{\theta}_{k-1}+\Delta t^2[(1/2-\beta)\ddot{\theta}_{k-1}+\beta\ddot{\theta}_k^*] \\ \dot{\theta}_k^* &= \dot{\theta}_{k-1}+\Delta t[(1-\gamma)\ddot{\theta}_{k-1}+\gamma\ddot{\theta}_k^*] \end{aligned} \tag{40a-c}$$

where $\beta=1/4$ and $\gamma=1/2$ are adopted, which implies a finite difference expansion of constant average acceleration. The preceding expressions can be substituted into the equation of motion to yield a single-step finite difference equation. Due to the nonlinear nature and the approximation brought by finite differences, the equation of motion cannot be exactly satisfied. The unbalanced force may be given as $f(\theta_k^*, \dot{\theta}_k^*, \ddot{\theta}_k^*)=U(\theta_k^*, \dot{\theta}_k^*, \ddot{\theta}_k^*)$. As will be shown below, the values of θ_k^*, $\dot{\theta}_k^*$, and $\ddot{\theta}_k^*$ can be improved by iteration, until the unbalanced force U becomes smaller than a preset tolerance, in which case the values θ_k^*, $\dot{\theta}_k^*$, and $\ddot{\theta}_k^*$ computed are taken as the solution for the particular time step.

For the present purposes, one may rewrite $U(\theta_k, \dot{\theta}_k, \ddot{\theta}_k)$ as $U(\theta_k^j, \dot{\theta}_k^j, \ddot{\theta}_k^j)$, in which the superscript j denotes the number of iteration. Expanding U^j with respect to U^{j-1} in Taylor's series and taking the first two terms:

$$U_k^j \cong U_k^{j-1} + \left.\frac{\partial U}{\partial \theta_k}\right|_{\theta=\theta_k^{j-1}} \Delta\theta_k \tag{41}$$

in which the derivative $\partial U/\partial\theta$ contains terms such as $\partial\ddot{\theta}/\partial\theta$ and $\partial\dot{\theta}/\partial\theta$. By differentiation, one can derive from Eqs. (40b) and (40c) the following:

$$\frac{\partial\ddot{\theta}}{\partial\theta}=\frac{1}{\Delta t^2\beta}; \quad \frac{\partial\dot{\theta}}{\partial\theta}=\frac{\gamma}{\Delta t\beta} \tag{42}$$

Assume that for the jth iteration, $|U_k^j| < \varepsilon_0$, with ε_0 denoting a preset tolerance. Then,

$$-U_k^{j-1}=\frac{\partial U}{\partial\theta_k}\Delta\theta_k \tag{43}$$

from which the increment $\Delta\theta_k$ can be solved,

$$\Delta\theta_k=-\frac{\partial U_k^{j-1}}{\partial U/\partial\theta_k} \tag{44}$$

It follows that the responses θ_k, $\dot{\theta}_k$, and $\ddot{\theta}_k$ can be updated as

$$\theta_k^j=\theta_k^{j-1}+\Delta\theta_k; \quad \dot{\theta}_k^j=\dot{\theta}_k^{j-1}+\frac{\gamma}{\Delta t\beta}\Delta\theta_k; \quad \ddot{\theta}_k^j=\ddot{\theta}_k^{j-1}+\frac{1}{\Delta t^2\beta}\Delta\theta_k \tag{45}$$

The preceding expressions can be substituted back into Eq. (41) to check if the condition $|U_k^j| < \varepsilon_0$ is satisfied. If yes, it means that a converged solution has been obtained for the kth step. Otherwise, repeat the procedure from Eqs. (41) to (45).

5. Illustrative examples

5.1. Slide mode

Consider the case when a rigid block is subjected to a harmonic ground excitation: $\ddot{x}_g = a_0 \sin \omega t$. For simplicity, the static and dynamic coefficients of friction will be taken to be equal, i.e., $\mu_s = \mu_k = \mu$. Let y denote the absolute displacement of the rigid block. Once the rigid block is in sliding, $\ddot{y}(t) = \pm\mu g$, which can be integrated to yield

$$\dot{y}(t) = \pm\mu g(t - t_0) + \dot{y}(t_0) \tag{46}$$

where t_0 denotes the time at which sliding starts. At the moment when the block starts to slide, $\ddot{x}_g(t_0) = \ddot{y}(t_0)$, i.e.,

$$a_0 \sin \omega t_0 = \pm a_0 \eta \tag{47}$$

where $\eta = \mu g / a_0$. From Eq. (47), it can be solved,

$$t_0 = 1/\omega[\sin^{-1}(\pm\eta) \pm 2m\pi], \quad m = 0, 1, 2, 3, \ldots \tag{48}$$

Let t_f denote the instant when the body ceases to slide. One can write

$$\dot{x}_g(t_f) = \dot{y}(t_f) = \pm\mu g(t_f - t_0) + \dot{y}(t_0) \tag{49}$$

Since $\dot{x}_g(t_f) - \dot{x}_g(t_0) = \dot{x}_g(t_f) - \dot{y}(t_0) = \pm\mu g(t_f - t_0)$ and $\dot{x}_g = -(a/\omega)\cos \omega t$, it can be shown that

$$\cos \omega t_f - \cos \omega t_0 = \mp\eta\,\omega(t_f - t_0) \tag{50}$$

For $t_f > t_0$, the rigid block is to remain at the slide mode. This is the case to be encountered in the slide-reverse-slide mode. According to Westermo and Udwadia (1983), when t_f equals the time at which the next slide starts, it can be proved that $\omega(t_f - t_0) = \pi$. Consequently, one can solve from Eq. (50) that $\eta = 0.53$, which is the limit value for a sliding object to reverse its direction of sliding with no rest.

By letting $a_0 = g$, i.e., $\eta = \mu$, and assuming $\omega = 0.4$, $\tan\phi = 3$, and zero initial conditions, the responses of the block have been plotted in Fig. 7 with the coefficient of friction increasing from 0.4 to 0.7. As can be seen, under the condition $a_0 = g$ or $\eta = \mu$, the block tends to behave in the manner of slide-reverse-slide mode for $\eta \leq 0.53$, and in the manner of slide-rest-slide mode for $\eta > 0.53$. Here, one observes that for the same ground acceleration, how the rigid block transits from one mode to the other is determined by the magnitude of the coefficient of friction.

5.2. Rock mode

Whenever the static coefficient μ_s is greater than $\tan\phi$, where ϕ denotes the angle between the vertical axis and a diagonal of the rectangular block at rest, the rigid block will stay in the pure rock mode. Based on the fact that a system loses its energy by impact, a block in free rocking, i.e., with no ground acceleration, will eventually approach an equilibrium position after a number of cycles of rocking, as can be seen from Fig. 8(a), where the solid and dashed lines respectively denote the case with and with no impact. It is clear that due to impact, the angular response eventually converges to zero. From the phase plot shown in Fig. 8(b), the area enclosed by the solid line diminishes, indicating the gradual loss of energy, while that enclosed by the dashed line remains

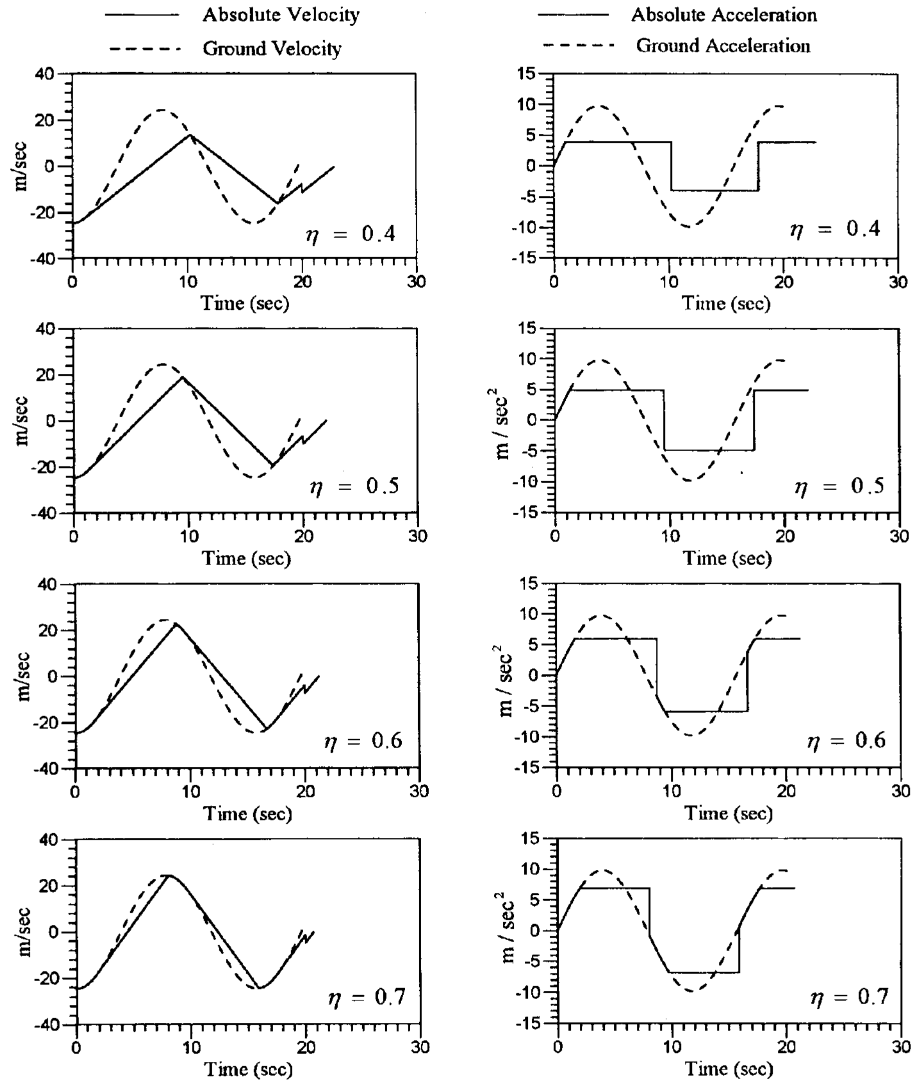


Fig. 7 Response of block response to harmonic excitation

basically constant, indicating that no energy has been lost, as is typical of elastic impact.

Consider the case when a rigid block of width $W = 10$ m and height $H = 40$ m is subjected to a harmonic excitation $\ddot{x}_g = 1g\cos 6t$, assuming $\mu_s = \mu_k = 1.0$ implying that $\mu_s > \tan\phi$. Due to the effect of friction, the angular motion will either exceed the stability limit, resulting in overturn of the block, or remain well bounded; the latter can be observed from Fig. 9. As can be seen from the phase plots, for the case of harmonic ground shaking, the block tends to reach a steady state response after a few cycles of irregular motion, which are stable and bounded by nature. Of interest is the subharmonic phenomenon that occurs around the peak responses of the rigid block, corresponding to which is the appearance of small loops in the phase plots. Finally, the vertical line segments in the phase plots represent in particular the impact process.

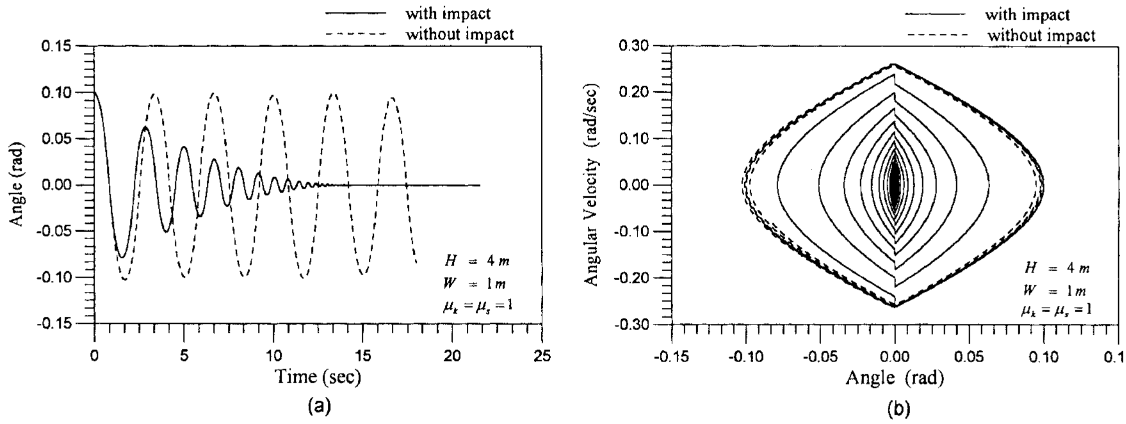


Fig. 8 Response of block in free rocking: (a) time history; (b) phase plot

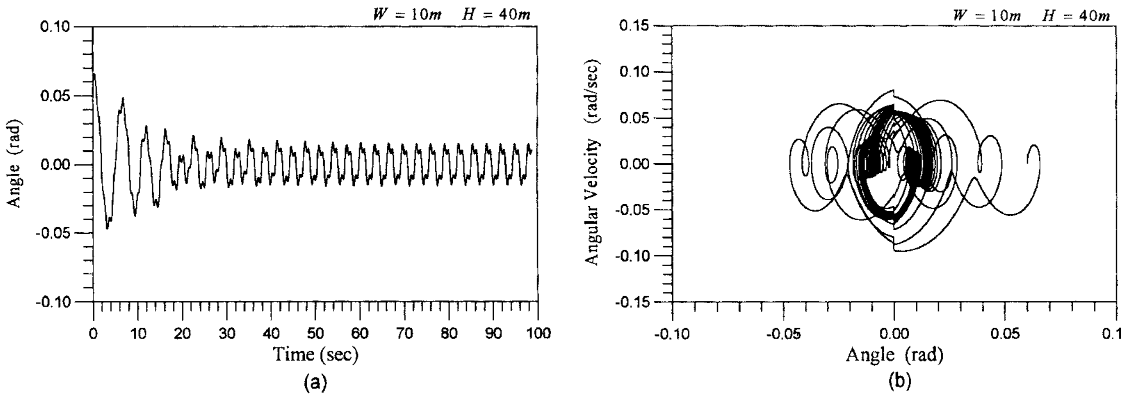


Fig. 9 Rock response of block: (a) time history; (b) phase plot

5.3. Slide and rock mode

For a rigid body under harmonic ground shaking $\ddot{x}_g = 2g\cos 4t$ and with initial rotation $\theta = 0.06$ rad, the rigid block tends to behave in a manner that combines slide and rock modes. As can be seen from Fig. 10(a), the zero values of $\dot{x}_c$ indicate the occurrence of rock mode, while the nonzero values of $\dot{x}_c$ relate to the slide and rock mode. Again, the vertical segments in the phase plots of Fig. 10(b) indicate the occurrence of impact, corresponding to which the deviations of x_G from zero indicates the residual displacements.

6. Concluding remarks

For a two-dimensional rectangular rigid block that is allowed to slide on a smooth rigid foundation, four modes of motion need to be considered, i.e, rest, slide, rock, and slide and rock. Based on the numerical studies presented in this paper, the following concluding remarks can be made. First, for a rigid block starting to move from rest, the second mode of motion is determined by the rela-

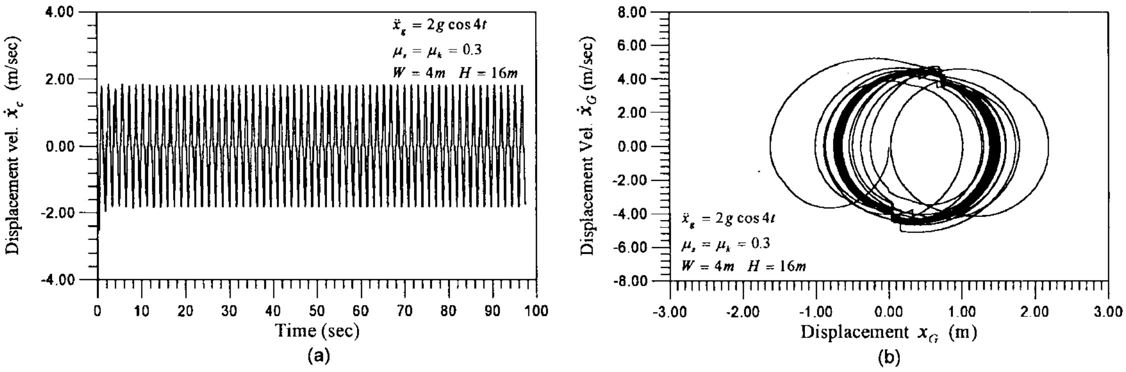


Fig. 10 Rock and slide response of block: (a) time history; (b) phase plot

tive magnitude of the coefficient μ_s of friction and the angle ϕ between the vertical axis and a diagonal of the block. Second, the behavior of a block in rocking is largely affected by impact, which serves as the mechanism for dissipating the energy. Third, for a block in slide mode and with $\mu_k g/a_0 > 0.53$, where μ_k denotes the dynamic coefficient of friction and a_0 the amplitude of harmonic ground shaking, the block will behave in the manner of slide-rest-slide mode. Otherwise, for $\mu_k g/a_0 \leq 0.53$, it will behave in the slide-reverse-slide mode. Fourth, for the case with $\mu_s > \tan\phi$, a rigid block subjected to harmonic ground shaking will reach the steady state of pure rock and stay in rock mode after impact. Finally, for the case where no overturning has occurred, a rigid block with initial rotation and under harmonic excitation, the steady-state response is generally a combination of the slide and/or rock mode, depending on the relative values of $|f|$ and $\mu_s|N|$ at the instant when velocity of the block $\dot{x}_c$ equals zero.

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References

Aslam, M., Godden, W.G. and Scalise, D.T. (1980), "Earthquake rocking response of rigid bodies", *J. Struct. Div., ASCE*, **106**(2), 377-392.

Dimentberg, M.F., Lin, Y.K. and Zhang, R. (1993), "Toppling of computer-type equipment under base isolation", *J. Eng. Mech., ASCE*, **119**(1), 145-160.

Housner, G.W. (1963), "The behavior of inverted pendulum structures during earthquakes", *Bull. Seismological Soc. Amer.*, **53**(2), 403-417.

Ishiyama, Y. (1982), "Motions of rigid bodies and criteria for overturning by earthquake excitations", *Earthquake Eng. Struct. Dyn.*, **10**, 635-650.

Lin, H. and Yim, S.C.S. (1996a), "Nonlinear rocking motions, I: Chaos under noisy periodic excitations", *J. Eng. Mech., ASCE*, **122**(8), 719-727.

Lin, H. and Yim, S.C.S. (1996b), "Nonlinear rocking motions, II: Overturning under random excitations", *J.*

Eng. Mech., ASCE, **122**(8), 728-735.

Lin, Y.K., Dimentberg, M.F., Zhang, R. and Cai, G.Q. (1994), "Sliding motion of anchored rigid block under random base excitations", *Probabil. Eng. Mech.*, **9**, 33-38.

Mostaghel, N., Hejazi, M. and Tanbakuchi, J. (1983), "Response of sliding structures to harmonic support motion", *Earthquake Eng. Struct. Dyn.*, **11**, 355-366.

Newmark, N.M. (1959), "A method of computation for structural dynamics", *J. Eng. Mech., ASCE,* **85**(3), 67-94.

Shenton, H.W. III and Jones, N.P. (1991a), "Base excitation of rigid bodies, II: Formulation", *J. Eng. Mech., ASCE,* **117**(10), 2286-2306.

Shenton, H.W. III and Jones, N.P. (1991b), "Base excitation of rigid bodies, I: Periodic slide-rock response", *J. Eng. Mech., ASCE,* **117**(10), 2307-2328.

Tso, W.K. and Wong, C.M. (1989), "Steady state rocking response of rigid blocks, Part 1: Analysis", *Earthquake Eng. Struct. Dyn.*, **18**, 89-106.

Westermo, B. and Udwadia, F. (1983), "Periodic response of sliding oscillator system to harmonic excitation", *Earthquake Eng. Struct. Dyn.*, **11**, 135-146.

Yang, Y.B., Lee, T.Y. and Tsai, I.C. (1990), "Response of multi-degree-of-freedom structures with sliding supports", *Earthquake Eng. Struct. Dyn.*, **19**, 739-752.

Yim, C.S., Chopra, A.K. and Penzien, J. (1980), "Rocking response of rigid blocks to earthquakes", *Earthquake Eng. Struct. Dyn.*, **8**, 565-587.