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Sub-degrees of freedom method with perturbation procedure for reduction of eigenvalue computation

Xiao-Lin Liu†

Laboratório Nacional de Engenharia Civil, Avenida do Brasil, 101. 1799 Lisboa Codex, Portugal

Abstract. A new way to reduce the eigenvalue computation effort in structural dynamics is presented in this paper. The degrees of freedom of a structure may be classified into groups that are termed as sub-degrees of freedom. The eigenvalue analysis is performed with each of sub-degrees of freedom so that the computing time is much shortened. Since the dynamic coupling between sub-degrees of freedom is selected to be small and it may be considered as a perturbation, the perturbation algorithm is used to obtain an accuratae result. The accuracy of perturbation depends on the coupling between sub-degrees of freedom. The weaker the coupling is, the more accurate the result is. The procedure can be used to simplify a problem of three dimensions to that of two dimensions or from two dimensions to one dimension. The application to a truss and a space frame is shown in the paper.

Key words: structural dynamics; eigenvalue computation; reduction; sub-degrees of freedom; perturbation.

1. Introduction

A structure is discretized into the degrees of freedom(d.o.f.) as variables by using the finite element method, which has been getting more and more popular in the structural analysis. It is observed that the accuracy of analysis relies on the number of d.o.f. The more d.o.f. are selected, the more accurate structural behavior is obtained. It is common that up to thousands of d.o.f. are used in a structural analysis. However, due to the efficiency of analysis and the capacity of a computer, to reduce the number of d.o.f. is desirable. This is even more important for the structural dynamics.

Usually, the earthquake or dynamic analysis focuses on an eigenvalue problem to obtain the dynamic behavior of the sturcture. The lower dynamic modes are excited under an earthquake and they dominate the response. It is known that the eigenvalue analysis takes much computing time. Various ways have been developed to reduce the computation. To reduce the size of stiffness matrix as well as mass matrix with fewer d.o.f. is the commonly used approach.

Substructuring is a normal technique to reduce the structural analysis (Cook 1981). The structure is partitioned into substructures and into even multi-level substructures. The output d.o.f. are formed in a substructure and the rest of d.o.f. are condensed. By this way, the number of d.o.f. is reduced and the structural matrix becomes much small. The modal synthesis of substructures is

† Research Fellow

used to reduce eigenvalue analysis (Hurty 1965, Bathe and Wilson 1976, Craig 1977). The eigensolution of the structure is from the dynamic modes of substructures. For an accurate eigensolution, the weak connectors between the substructures must be found. However in many cases, it is rather difficult to find the weak connectors and the structure is unable to be partitioned into substructures. This can be observed in a dam structure or a frame structure.

The d.o.f. of a structure are able to be classified into groups, for example, the d.o.f. in any one of two horizontal directions for a frame structure. The group of d.o.f. may be termed as sub-d.o.f. The weak coupling between the groups may be found so that the submatrices of the structural matrices are small. Without considering the sub-matrices, the stiffness matrix and mass matrix can be divided into two or more parts related to sub-d.o.f. By this way, the number of d.o.f. is reduced into several parts of sub-d.o.f.

The perturbation method is an approximate way to obtain the solution in the form of an asymptotic expansion from the zeroth-order result. It is useful for the structural analysis with small changes of properties and for the modal sensitive analysis. The reduction with the perturbation method has been studied. The perturbation method is used in the substructuring for structures with weak connectors (Chen and Liu 1993). A reduction algorithm is applied to the structures with large stiffnesses and small masses(Liu 1995, Liu 1997).

In this paper, a new way to reduce the eigenvalue computation in structural dynamics is presented. The term of sub-degrees of freedom, sub-d.o.f., is introduced. The perturbation algorithm is employed to obtain an accurate eigensolution.

For a structure being partitioned into sub-d.o.f., the eigenvalue analysis is performed for the separate computation of each sub-d.o.f. As the computing time takes only a small portion, the total time is much less than that for the computation with full d.o.f. Since the weak coupling between sub-d.o.f., the eigensolution of sub-d.o.f. is an approximation for the full d.o.f. The small sub-matrices, which are not included for sub-d.o.f. are considered to be a perturbation. Taking the eigensolution to be the zeroth-order result, the perturbation procedure can be used. The perturbed result is assumed to be a sequence of small parameter. Using the standard perturbation procedure, the simple algorithm is derived and is given in the paper. The perturbed eigenvector is obtained from the first-order or higher-order perturbation, while the perturbed eigenvalue is obtained from the second-order perturbation. For closely spaced eigenvalues, a specific perturbation must be used and an eigenvalue equation needs to be solved.

The application of the method is shown in the numerical examples, a plane truss with 5 bars and a spatial frame of one-storey. The perturbed result is obtained from the second order perturbation. The relative errors with the increase of the coupling between sub-d.o.f. are examined.

2. Sub-d.o.f.

In the structural dynamics, the eigenvalue equation with the symmetrical stiffness matrix $\boldsymbol{K}$ and mass matrix $\boldsymbol{M}$ is given by

$$(\boldsymbol{K} - \lambda \boldsymbol{M})\phi = \boldsymbol{0} \tag{1}$$

where λ and ϕ are eigenvalue and eigenvector respectively. The structure is discretized into N d.o.f. through the finite element method.

Suppose eigenvector ϕ may be classified into two or even more groups. The d.o.f. for the

groups could be termed as sub-d.o.f. Hence Eq. (1) is rewritten to

$$\left(\begin{bmatrix} \boldsymbol{K}_{aa} & \boldsymbol{K}_{ab} \\ \boldsymbol{K}_{ba} & \boldsymbol{K}_{bb} \end{bmatrix} - \lambda \begin{bmatrix} \boldsymbol{M}_{aa} & \boldsymbol{M}_{ab} \\ \boldsymbol{M}_{ba} & \boldsymbol{M}_{bb} \end{bmatrix} \right) \begin{Bmatrix} \phi_a \\ \phi_b \end{Bmatrix} = \mathbf{0} \tag{2}$$

where a and b denote two sub-d.o.f.

For each of two sub-d.o.f. a and b, an eigenvalue problem is set up and is sloved separately, that is

$$(\boldsymbol{K}_{aa} - \lambda_a \boldsymbol{M}_{aa})\phi_a = \mathbf{0} \tag{3}$$

$$(\boldsymbol{K}_{bb} - \lambda_b \boldsymbol{M}_{bb})\phi_b = \mathbf{0} \tag{4}$$

The computation of equations yields the eigensolutions, which may be considered to be from Eq. (1) or Eq. (2) without the sub-matrices for the coupling between two sub-d.o.f. a and b. Since the numbers of d.o.f. in Eq. (3) and Eq. (4) are less than that for Eq. (2), the matrices $\boldsymbol{K}_{aa}$, $\boldsymbol{K}_{bb}$, $\boldsymbol{M}_{aa}$ and $\boldsymbol{M}_{bb}$ become much smaller. This makes the computations of two eigenvalue equations taking less time. Eventually, the whole computing time is greatly reduced.

If the coupling between the sub-d.o.f. is weak, the terms of stiffness and mass in sub-matrices $\boldsymbol{K}_{ab}$, $\boldsymbol{K}_{ba}$, $\boldsymbol{M}_{ab}$ and $\boldsymbol{M}_{ba}$ are small. Therefore, the results obtained from the computation of each sub-d.o.f. are possible to be an approximation of eigensolution of Eq. (1). However, to achieve a better result, the computational procedure needs to be explored.

Similar reduction techniques are widely used in the structural analsis. In many cases, to analyze a complex structure in three dimensions is not feasible. A section of a structure may be taken into consideration in an anlaytical model and it is simplified into a plane structure in two dimensions. The third coordinate of the structure is neglected. By this way, the structure is analyzed and the behavior is obtained. The plane frame is a usual analytical model for a frame structure. Therefore, each frame in any one of two horizontal directions is separately computed. However, this is only a simple way and may result in a large error. The eigensolution obtained with this simplification is only for a lateral vector and the vertical vector. The torsional mode, which can be very important in some cases, is excluded. For this reason, the result lacks the character of three dimensions.

3. Perturbation equations

If sub-d.o.f. are carefully selected, the sub-matrices of stiffness and mass for the coupling between sub-d.o.f., i.e., $\boldsymbol{K}_{ab}$, $\boldsymbol{K}_{ba}$, $\boldsymbol{M}_{ab}$ and $\boldsymbol{M}_{ba}$, are in the small order compared to the diagonal matrices $\boldsymbol{K}_{aa}$, $\boldsymbol{K}_{bb}$, $\boldsymbol{M}_{aa}$ and $\boldsymbol{M}_{bb}$. According to the perturbation theory, the first-order perturbation matrices may be expressed as

$$\boldsymbol{K}^{(1)} = \begin{bmatrix} 0 & \boldsymbol{K}_{ab}^{(1)} \\ \boldsymbol{K}_{ba}^{(1)} & 0 \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} 0 & \boldsymbol{K}_{ab} \\ \boldsymbol{K}_{ba} & 0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{M}^{(1)} = \begin{bmatrix} 0 & \boldsymbol{M}_{ab}^{(1)} \\ \boldsymbol{M}_{ba}^{(1)} & \mathbf{0} \end{bmatrix} = \frac{1}{\varepsilon} \begin{bmatrix} 0 & \boldsymbol{M}_{ab} \\ \boldsymbol{M}_{ba} & \mathbf{0} \end{bmatrix} \tag{6}$$

where ε is a small positive parameter, that is $0 < \varepsilon << 1$. Then the structural stiffness and mass matrices become

$$\boldsymbol{K} = \boldsymbol{K}^{(0)} + \varepsilon \boldsymbol{K}^{(1)} \tag{7}$$

$$\boldsymbol{M} = \boldsymbol{M}^{(0)} + \varepsilon \boldsymbol{M}^{(1)} \tag{8}$$

where superscript (0) denotes the zeroth-order. Therefore, the zeroth-order stiffness matrix $\boldsymbol{K}^{(0)}$ and the zeroth-order mass matrix $\boldsymbol{M}^{(0)}$ are given by

$$\boldsymbol{K}^{(0)} = \begin{bmatrix} \boldsymbol{K}_{aa} & \mathbf{0} \\ 0 & \boldsymbol{K}_{bb} \end{bmatrix} \quad \boldsymbol{M}^{(0)} = \begin{bmatrix} \boldsymbol{M}_{aa} & \mathbf{0} \\ 0 & \boldsymbol{M}_{bb} \end{bmatrix}$$

Considering the zeroth-order eigenvalue problem from Eq. (1) of the perturbed problem, the eigenvalue equation is

$$(\boldsymbol{K}^{(0)} - \lambda^{(0)} \boldsymbol{M}^{(0)}) \phi^{(0)} = \mathbf{0} \tag{9}$$

This is as same as Eq. (3) and Eq. (4) for sub-d.o.f. a and b. The eigensolution of the zeroth-order is formed from the results of Eq. (3) and Eq. (4)

$$\Lambda^{(0)} = (\Lambda_a \quad \Lambda_b) \tag{10}$$

$$\Phi^{(0)} = \begin{bmatrix} \Phi_a & 0 \\ 0 & \Phi_b \end{bmatrix} \tag{11}$$

which are $N \times n$ matrices and n is the number of modes. Suppose that the numbers of eigenvalues computed from Eq. (3) and Eq. (4) are n_a and n_b for two sub-d.o.f. a and b respectively, then

$$n = n_a + n_b \tag{12}$$

It is noted that the computed number of modes n may be smaller than the number of d.o.f. N.

3.1. Perturbation formulas for a distinct eigenvalue

According to the perturbation theory, the eigensolution of Eq. (1) is assumed to take an asymptotic sequence of the small parameter ε

$$\lambda_i = \lambda_i^{(0)} + \varepsilon \lambda_i^{(1)} + \varepsilon^2 \lambda_i^{(2)} + \cdots \tag{13}$$

$$\phi_i = \phi_i^{(0)} + \varepsilon \phi_i^{(1)} + \varepsilon^2 \phi_i^{(2)} + \cdots \tag{14}$$

where $i \in R = R_1 + R_2$, and $R_1 = [1, \ldots, n_a]$, $R_2 = [n_a + 1, \ldots, n]$.

Using the standard perturbation procedure, the solution of Eq. (13) and Eq. (14) is obtained. The simple formulas can be given as follows:

The first-order pertubation

$$\lambda_i^{(1)} = 0 \tag{15}$$

for $i \in R_1$

$$\phi_i^{(1)} = \sum_{j=n_a+1}^{n} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T} (\boldsymbol{K}^{(1)} - \lambda_i^{(0)} \boldsymbol{M}^{(1)}) \phi_i^{(0)} \phi_j^{(0)} \tag{16}$$

and for $i \in R_2$

$$\phi_i^{(1)} = \sum_{j=1}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T} (\boldsymbol{K}^{(1)} - \lambda_i^{(0)} \boldsymbol{M}^{(1)}) \phi_i^{(0)} \phi_j^{(0)} \tag{17}$$

The second-order perturbation

$$\lambda_i^{(2)} = \phi_i^{(0)T} (\boldsymbol{K}^{(1)} - \lambda_i^{(0)} \boldsymbol{M}^{(1)}) \phi_i^{(1)} \tag{18}$$

for $i \in R_1$

$$\phi_i^{(2)} = \sum_{\substack{j=1 \\ j \neq i}}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T} (\boldsymbol{K}^{(1)} - \lambda_i^{(0)} \boldsymbol{M}^{(1)}) \phi_i^{(1)} \phi_j^{(0)} - \frac{1}{2} (\phi_i^{(1)T} \boldsymbol{M}^{(0)} \phi_i^{(1)} + 2\phi_i^{(0)T} \boldsymbol{M}^{(1)} \phi_i^{(1)}) \phi_i^{(0)} \tag{19}$$

and for $i \in R_2$

$$\phi_i^{(2)} = \sum_{\substack{j=n_a+1 \\ j \neq i}}^{n_a} \frac{1}{\lambda_1^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T} (\boldsymbol{K}^{(1)} - \lambda_i^{(0)} \boldsymbol{M}^{(1)}) \phi_i^{(1)} \phi_j^{(0)} - \frac{1}{2} (\phi_i^{(1)T} \boldsymbol{M}^{(0)} \phi_i^{(1)} + 2\phi_i^{(0)T} \boldsymbol{M}^{(1)} \phi_i^{(1)}) \phi_i^{(0)} \tag{20}$$

Note that the first-order perturbation of eigenvalue is unable to be obtained.

3.2. Perturbation formulas for closely spaced eigenvalues

For closely spaced eigenvalues, the perturbation is different from that of a distinct eigenvalue and a specific procedure must be performed (Liu 1995, Liu 1998).

It is supposed that $\mu_i^{(0)}$, where $i \in V=V_1+V_2$, where $V_1 \subset R_1$ and $V_2 \subset R_2$, are p closely spaced zeroth-order eigenvalues and Γ are the respective eigenvectors. The zeroth-order eigensolution in Eq. (13) and Eq. (14) is the perturbed result and is given by

$$\lambda_i^{(0)} = \mu_h \tag{21}$$

$$\phi_i^{(0)} = \Gamma \beta_h \tag{22}$$

where μ_h and β_h, for $h = 1, 2, ..., p$, are obtained from the eigenvalue equation

$$\Gamma^T (\boldsymbol{K} - \mu \boldsymbol{M}) \Gamma \beta = \boldsymbol{0} \tag{23}$$

The first order perturbation is obtained as

$$\lambda_i^{(1)} = 0 \tag{24}$$

for $i \in V_1$

$$\phi_i^{(1)} = \sum_{\substack{j=n_a+1 \\ j\notin V_2}}^{n} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T}(\boldsymbol{K}^{(1)} - \lambda_i^{(0)}\boldsymbol{M}^{(1)})\phi_i^{(0)}\phi_j^{(0)} \tag{25}$$

and for $i \in V_2$

$$\phi_i^{(1)} = \sum_{\substack{j=1 \\ j\notin V_1}}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T}(\boldsymbol{K}^{(1)} - \lambda_i^{(0)}\boldsymbol{M}^{(1)})\phi_i^{(0)}\phi_j^{(0)} \tag{26}$$

while the second-order perturbation is given by

$$\lambda_i^{(2)} = \phi_i^{(0)T}(\boldsymbol{K}^{(1)} - \lambda_i^{(0)}\boldsymbol{M}^{(1)})\phi_i^{(1)} \tag{27}$$

for $i \in V_1$

$$\phi_i^{(2)} = \sum_{\substack{j=1 \\ j\notin V_1}}^{n_a} \frac{1}{\lambda_i^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T}(\boldsymbol{K}^{(1)} - \lambda_i^{(0)}\boldsymbol{M}^{(1)})\phi_i^{(1)}\phi_j^{(0)} - \frac{1}{2}(\phi_i^{(1)T}\boldsymbol{M}^{(0)}\phi_i^{(1)} + 2\phi_i^{(0)T}\boldsymbol{M}^{(1)}\phi_i^{(1)})\phi_i^{(0)} \tag{28}$$

and for $i \in V_2$

$$\phi_i^{(2)} = \sum_{\substack{j=n_a+1 \\ j\notin V_2}}^{n} \frac{1}{\lambda_1^{(0)} - \lambda_j^{(0)}} \phi_j^{(0)T}(\boldsymbol{K}^{(1)} - \lambda_i^{(0)}\boldsymbol{M}^{(1)})\phi_i^{(1)}\phi_j^{(0)} - \frac{1}{2}(\phi_i^{(1)T}\boldsymbol{M}^{(0)}\phi_i^{(1)} + 2\phi_i^{(0)T}\boldsymbol{M}^{(1)}\phi_i^{(1)})\phi_i^{(0)} \tag{29}$$

Though only the first-order perturbation and the second-order perturbation are given, the perturbation may be further computed in the straightfoward way. However, the first two order perturbations are usually enough for a good perturbed result and the higher order perturbation is not needed. The equations are only for two sub-d.o.f., while for more sub-d.o.f. the computation can be derived with the same formulas.

As sub-d.o.f. are employed to make the stiffness and mass matrices into smaller ones, less computing time for the zeroth-order result is required in the analysis for sub-d.o.f. It has been demonstrated that of the perturbation takes only very a small part(Liu 1997). Consequently, the total time is reduced.

4. Numerical examples

4.1. A plane truss

For a plane truss, the d.o.f. are considered in two directions of coordinates, i.e., in X and Y. $\boldsymbol{x}$ and $\boldsymbol{y}$ are assumed to be the eigenvectors and are taken as two sub-d.o.f. respectively, so that

$$\phi_a = \boldsymbol{x}, \quad \phi_b = \boldsymbol{y}$$

Fig. 1 shows a 5-elements plane truss. The eigenvector is $\phi=\{\phi_a\ \phi_b\}^T=\{\boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{y}_1\ \boldsymbol{y}_2\}^T$.

The sectional stiffness of bars and the nodal masses are given by

$$k_1 = k_2 = 1.5; \quad k_3 = 1.0; \quad k_4 = k_5 = 0.2$$
$$m_1 = m_2 = 1.0$$

The stiffness matrix and the mass matrix can be written out as

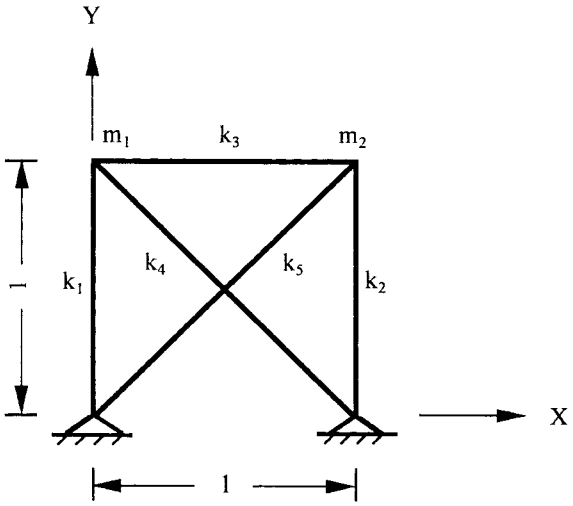


Fig. 1 A plane truss with 5 bars and 4 d.o.f.

$$\boldsymbol{K} = \begin{bmatrix} 1+\frac{\sqrt{2}}{20} & -1 & -\frac{\sqrt{2}}{20} & 0 \\ -1 & 1+\frac{\sqrt{2}}{20} & 0 & \frac{\sqrt{2}}{20} \\ -\frac{\sqrt{2}}{20} & 0 & 1.5+\frac{\sqrt{2}}{20} & 0 \\ 0 & \frac{\sqrt{2}}{20} & 0 & 1.5+\frac{\sqrt{2}}{20} \end{bmatrix}$$

$$\boldsymbol{M} = \mathrm{diag}[\,1 \quad 1 \quad 1 \quad 1\,]$$

The stiffness and mass matrices for two sub-d.o.f. are taken to be

$$\boldsymbol{K}_{aa} = \begin{bmatrix} 1+\frac{\sqrt{2}}{20} & -1 \\ -1 & 1+\frac{\sqrt{2}}{20} \end{bmatrix} \qquad \boldsymbol{M}_{aa} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{K}_{bb} = \begin{bmatrix} 1.5+\frac{\sqrt{2}}{20} & 0 \\ 0 & 1.5+\frac{\sqrt{2}}{20} \end{bmatrix} \qquad \boldsymbol{M}_{bb} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then the first-order perturbation matrices are

$$\boldsymbol{K}^{(1)}=\frac{1}{\varepsilon}\begin{bmatrix}0 & \boldsymbol{K}_{ab}\\ \boldsymbol{K}_{ba} & 0\end{bmatrix}=\frac{1}{\varepsilon}\begin{bmatrix}0 & 0 & -\frac{\sqrt{2}}{20} & 0\\ 0 & 0 & 0 & \frac{\sqrt{2}}{20}\\ -\frac{\sqrt{2}}{20} & 0 & 0 & 0\\ 0 & \frac{\sqrt{2}}{20} & 0 & 0\end{bmatrix}$$

$$\boldsymbol{M}^{(1)}=\boldsymbol{0}$$

Because the lumped nodal mass matrix is used, there is no coupling between sub-d.o.f. and the first-order mass matrix is the zero matrix.

The eigenvalues and eigenvectors for sub-d.o.f. and perturbation are computed. The various eigensolutions with four dynamic modes are compared in Table 1. The exact one is the rusult wih

Table 1 Comparison of the exact, sub-d.o.f. and perturbed eigensolutions

Mode		Exact	Sub-d.o.f.	Perturbed
1	Eigenvalue	0.067384	0.070711	0.067377
	Relative Error (%)		4.9363	-0.0104
	Eigenvector	0.706326	0.707107	0.706321
		0.706326	0.707107	0.706321
		0.033223	0.000000	0.033333
		-0.033223	0.000000	-0.033333
2	Eigenvalue	1.560903	1.570711	1.560711
	Relative Error (%)		0.6284	-0.0123
	Eigenvector	0.097146	0.000000	0.100000
		-0.097147	0.000000	-0.100000
		0.700392	0.707107	0.700036
		0.700412	0.707107	0.700036
3	Eigenavlue	1.574037	1.570711	1.574044
	Relative Error (%)		-0.2113	0.0005
	Eigenvector	-0.033221	0.000000	-0.033333
		-0.033224	0.000000	-0.033333
		0.706336	0.707107	0.706321
		-0.706316	-0.707107	-0.706321
4	Eigenvalue	2.080518	2.070711	2.080711
	Relative Error (%)		-0.4714	0.0092
	Eigenvector	-0.700402	-0.707107	-0.700036
		0.700402	0.707107	0.700036
		0.097147	0.000000	0.100000
		0.097146	0.000000	0.100000

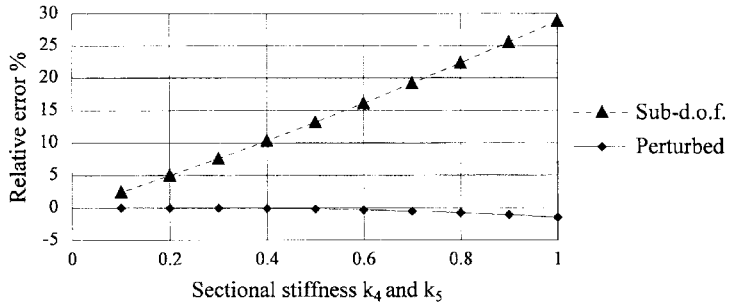


Fig. 2 Relative errors of eigenvalue for the first modes vs. sectional stiffness of bar 4 and bar 5

full d.o.f. The errors of the eigenvalues are the relative errors in percentage to the exact ones. It can be seen that the perturbed eigensolution is improved greatly from the result of sub-d.o.f. The errors of eigenvalues become smaller with the increase of dynamic modes. The big improvement is found for the first mode, with the relative error from 4.9363% to -0.0104%.

It is noted that sub-matrices K_{ab} and K_{ba} for the coupling between two sub-d.o.f. only depend on the sectional stiffness k_4 and k_5 of bar 4 and 5. In order to compare the perturbed eigensolution with the change of the coupling between sub-d.o.f., the sectional stiffness k_4 and k_5 are considered to be a variable and take changes from 0.1 to 1.0. The eigensolutions with different stiffness are obtained. The relative error of the perturbed eigenvalue for the first dynamic mode is plotted in Fig. 2 and is compared with the relative error for sub-d.o.f.

It is noticed in Fig. 2 that the relative error of the eigenvalue for the first mode increases with the change of the sectional stiffness k_4 and k_5 from 0.1. to 1.0. The relative error of the perturbed eigenvalue goes from -0.0008% to -1.5222%. It is much smaller than the error for sub-d.o.f., which is from 2.4131% to 28.8473%. The perturbed eigenvalue is smaller than the exact one, while the zeroth-order result is larger. The error of the perturbed eigenvalue is so small that the result becomes acceptable.

4.2. A spatial frame

A spatial frame of one-storey is shown in Fig. 3. The sectional properties for beam elements,

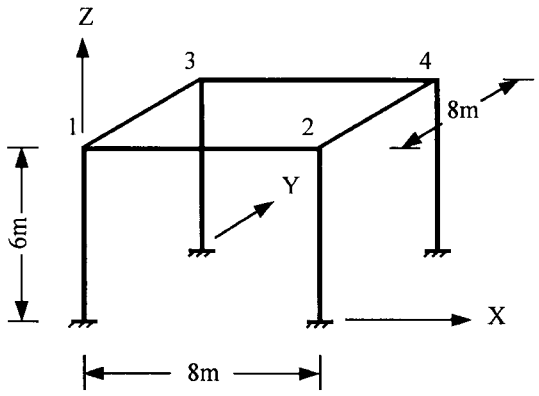


Fig. 3 A spatial frame of one-storey

Table 2 Comparison of eigenvalues

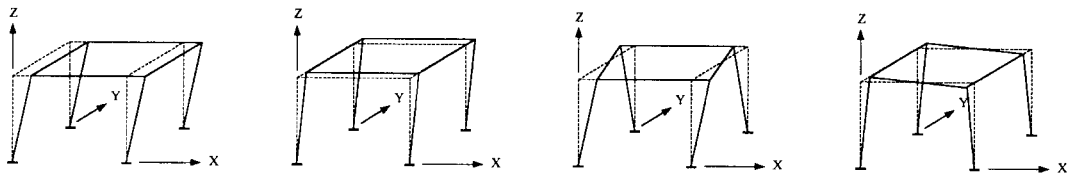
Mode		Exact	Sub-d.o.f.	Perturbed
1	Value	570.9096	572.0888	570.9165
	Relative Error (%)		0.2065	0.0012
2	Value	570.9096	572.0888	570.9165
	Relative Error (%)		0.2065	0.0012
3	Value	719.1937	1036.059	719.1944
	Relative Error (%)		44.0584	0.0001
4	Value	1034.499	1036.059	1034.511
	Relative Error (%)		0.1508	0.0012

column elements, and nodal masses are given by $EA=4.707\times10^6$ kN, $EI=1.412\times10^5$ kN.m^2; $EA=4.903\times10^6$ kN, $EI=1.020\times10^5$ kN.m^2; $m=7000$ kg. There are 4 nodes and 24 d.o.f.. Twenty d.o.f. are divided into three sub-d.o.f. as

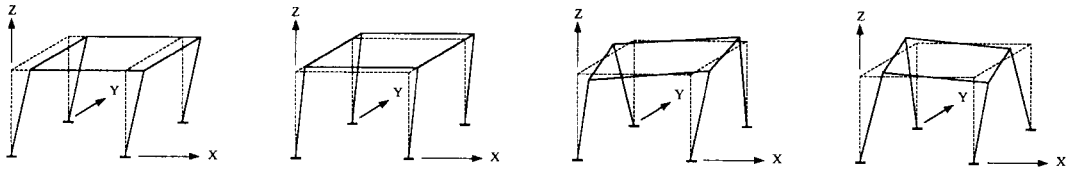
$$\phi_a = \begin{Bmatrix} x \\ \theta_y \end{Bmatrix},\ \phi_b = \begin{Bmatrix} y \\ \theta_x \end{Bmatrix},\ \phi_c = z$$

The eigenvalue analysis is performed with each of three sub-d.o.f. For the frame, the torsional mode is specially important, however it can not be reflected in the eigensolution of sub-d.o.f. As the rotational d.o.f. θ_z are not included in the sub-d.o.f., it needs to be obtained in the perturbation procedure. Considering θ_z as d.o.f. with large stiffnesses, a perturbation algorithm is used (Liu 1995, Liu 1997). The perturbed eigenvalues for the first four modes are given in Table 2 and are also compared with the result of sub-d.o.f. and the exact one.

For the eigensolution of sub-d.o.f. in Table 2, the eigenvalues of the first and third modes are the result of sub-d.o.f. ϕ_a in X direction while the second and forth ones are of ϕ_b in Y direction. However, the third and forth modes appear to be torsional modes, which are restored after the perturbation and are improved greatly. The third eigenvalue changes from 1036.056 to 719.1944,



(a) The shapes of sub-d.o.f.



(b) The perturbed and exact shapes

Fig. 4 The modal shapes of first four modes

which is in the good agreement with the exact one 719.1937.

The first four modal shapes, for sub-d.o.f., perturbed result, and exact one are illustrated in Fig. 4.

5. Conclusions

The sub-d.o.f. is introduced in the paper and is used to divide the stiffness matrix as well as the mass matrix into several parts. The computation is performed individually to the eigenvalue equation for each sub-d.o.f. with a smaller number of d.o.f. The perturbation algorithm is used to obtain the eigensolution with the full d.o.f. Consequently, the computing time is reduced. The method is effective in the reduction and is one way for the efficient computation of eigenvalue problem in the structural dynamics.

The numerical examples show the effectiveness of the method. The eigensolution obtained from the second-order perturbation is greatly improved from the result for sub-d.o.f. and is in the good agreement with the result of full d.o.f. The error of perturbation depends on the sub-matrices of structural stiffness and mass for the coupling between sub-d.o.f. The weaker coupling yields the better perturbed result. The sub-d.o.f., between which the weak coupling stiffness matrices are found, should preferably be selected.

A structure of three dimensions may be divided into sub-d.o.f. with two dimensions and hence is analyzed. However, the torsional dynamic mode must be obtained from the perturbation, as the result of sub-d.o.f. is only for the lateral and vertical modes. If the earthquake loading is considered, the idea of sub-d.o.f. may be extended to obtain the structural behavior.

It must be mentioned that the perturbation algorithm given in the paper can be applied to the perturbation analysis with the reduction method of substructuring. Actually, substructuring may be considered to be one particular case of sub-d.o.f. technique.

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