Structural Engineering and Mechanics, Vol. 8, No. 3 (1999) 243-256
DOI: http://dx.doi.org/10.12989/sem.1999.8.3.243

Flexural free vibration of cantilevered structures of variable stiffness and mass

Q.S. Li†

Department of Building and Construction,
City University of Hong Kong, Tat Chee Avenue, Hong Kong

Abstract. Using appropriate transformations, the differential equation for flexural free vibration of a cantilever bar with variably distributed mass and stiffness is reduced to a Bessel's equation or an ordinary differential equation with constant coefficients by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass. The general solutions for flexural free vibration of one-step bar with variable cross-section are derived and used to obtain the frequency equation of multi-step cantilever bars. The new exact approach is presented which combines the transfer matrix method and closed form solutions of one step bars. Two numerical examples demonstrate that the calculated natural frequencies and mode shapes of a 27-storey building and a television transmission tower are in good agreement with the corresponding experimental data. It is also shown through the numerical examples that the selected expressions are suitable for describing the distributions of stiffness and mass of typical tall buildings and high-rise structures.

Key words: vibration; natural frequency; mode shape; tall building.

1. Introduction

A broad range of engineering problems involves vibration analysis of non-uniform bars and beams. For example, when analysing free vibrations of tall buildings and high-rise structures, it is possible to regard such structures as a cantilever bar with variable cross-section (e.g., Wang 1978, Li *et al.* 1996, 1998). However, in general, it is not possible or, at least, very difficult to get the exact analytical solutions of differential equations of free vibrations of bars with variably distributed mass and stiffness. These exact bar solutions are available only for certain bar shapes and boundary conditions. Wang (1978) derived the closed-form solutions for the free vibration of a flexural bar with variably distributed stiffness, but uniform mass. Kumar *et al.* (1997) found the exact solutions for the longitudinal vibration of non-uniform rods whose cross-section varies as $A=(a+bx)^n$ and $A=A_0 \sin^2 (ax+b)$. In this paper, the exact solutions of flexural free vibration of a bar with variably distributed stiffness and mass are found by selecting suitable expressions, such as power functions and exponential functions, for the distributions of flexural stiffness and mass of the bar. The flexural free vibration of a multi-step bar is a complex problem, and the exact solution of this problem has not previously been obtained. Use of the exact solution of a one-step bar together with a transfer matrix method is presented in this paper in order to resolve this

† Assistant Professor

problem. It is shown through two numerical examples that the selected expressions are suitable for describing the distributions of flexural stiffness and mass of typical tall structures and high-rise structures, and that the total number of the segments required in the proposed transfer matrix method could be significantly less than that normally used in conventional finite element methods. Thus, the proposed method has practical significance for free vibration analysis of non-uniform cantilevers.

In this paper, an attempt is made to present exact analytical solutions to flexural free vibration of cantilever bars with variably distributed mass and stiffness. In the absence of exact solutions, this problem can be solved using approximated methods (e.g., the Ritz method) or numerical methods (e.g., the finite element method and the finite strip method). However, the present exact solutions can provide adequate insight into the physics of the problem and can be easily implemented even without the availability of a computer. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain exact solutions to such problems.

2. One-step bars

The governing differential equation for flexural vibration of a one-step cantilever bar with variable cross-section considering damping effect under the action of transverse forces (Fig. 1) and neglecting the effect of rotatory inertia and transverse shear deformation can be expressed as follows:

$$\frac{\partial^2}{\partial x^2}(EI_x \frac{\partial^2 y}{\partial x^2}) + \bar{m}_x \frac{\partial^2 y}{\partial t^2} + C_x \frac{\partial y}{\partial t} = P(x,t) \tag{1}$$

In which EI_x, $\bar{m}_x$, C_x and $P(x,t)$ are the flexural stiffness, the mass, the damping coefficient per unit length and the transverse force at section x of the bar, respectively.

If $P(x,t)$=0, then, Eq. (1) becomes the equation of damped flexural free vibration. Setting C_x=0 obtains the equation of undamped flexural free vibration.

All structures dissipate energy when they vibrate. Hence, damping is present to some degree in all structural systems. However, in general, the effect of damping on structural natural frequency and vibration mode shape is neglected in free vibration analysis. Although in the majority of engineering systems this effect is small and may be disregarded, there are cases in which the effect reaches an appreciable magnitude and must be included in the analysis, for example, it is possible that the damping factor of a controlled structure is twenty times or more greater than that of corresponding uncontrolled structures in some cases (Soong 1990). There is, therefore, a need to carry out further research on the evaluation of free vibration of structural systems considering damping effect. In this paper, the damping coefficient of a bar is assumed to be proportional to its mass.

Let

$$C_x = C_0 \bar{m}_x \tag{2}$$

Using the method of separation of variables gives

$$y(x,t) = Y(x)\exp(\lambda t) \tag{3}$$

where $Y(x)$ is the damped mode shape function, λ is a complex value.

Substituting Eq. (2) and Eq. (3) into the equation of damped flexural free vibration gives the equation of $Y(x)$ as follows:

$$\frac{d^2}{dx^2}(EI_x \frac{d^2Y}{dx^2}) + (C_0\lambda + \lambda^2)\bar{m}_x Y = 0 \tag{4}$$

If we set

$$y(x,t) = Y(x)\sin(\omega t + \gamma_0) \tag{5}$$

where $Y(x)$ is the undamped vibration mode function, ω is the undamped circular natural frequency and γ_0 is the initial phase, then the equation of $Y(x)$ becomes

$$\frac{d^2}{dx^2}(EI_x \frac{d^2Y}{dx^2}) - \omega^2\bar{m}_x Y = 0 \tag{6}$$

If we set

$$\omega^2 = -(C_0\lambda + \lambda^2) \tag{7}$$

then, the damped mode shape governed by Eq. (4) is the same as the undamped one governed by Eq. (6). The relationship between the damped natural frequency, ω_d, and the undamped natural frequency, ω, can be found by solving Eq. (7) as follows

$$\lambda = -\frac{C_0}{2} \pm i\omega\sqrt{1 - (\frac{C_0}{\omega})^2} \tag{8}$$

i.e.,

$$\omega_d = k_d\,\omega, \quad k_d = \sqrt{1 - (\frac{C_0}{\omega})^2} \tag{9}$$

It can be seen from Eq. (9) that the effect of damping on natural frequency can be neglected for the case of light damping ($C_0/\omega < 0.1$). Even if the damping coefficient is very large, the effect of damping can also be not considered in free vibration analysis. After the undamped natural frequencies have been found, the damped natural frequencies can be determined from Eq. (9). This suggests that if the distribution of the damping coefficient of a bar is assumed to be proportional to that of the mass ($C_x = C_0\bar{m}_x$), the damped natural frequency is equal to the corresponding undamped natural frequency multiplied by the coefficient, k_d, and the damped mode shape is the same as the corresponding undamped mode shape.

It is difficult to find the general solution of Eq. (6) since the structural parameters in the equation vary with the co-ordinate x. However, the general solution of Eq. (6) may be obtained by making appropriate selections for mass, stiffness and damping distribution functions. As suggested by Wang (1978), Tuma and Cheng (1983) and Li *et al.* (1994, 1995), the functions that can be used to approximate the variation of mass and stiffness are algebraic polynomials, exponential functions, trigonometric series, or their combinations. Three cases are considered as follows:

Case I: Expressions for flexural stiffness and mass per unit length are power functions, which are given as follows

$$EI_x = EI_0(1 - \beta x)^{n+2} \tag{10}$$

$$\bar{m}_x = m_0(1 - \beta x)^n \tag{11}$$

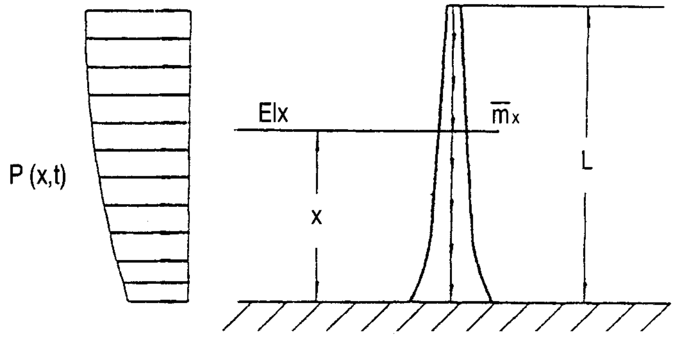

Fig. 1 A bar with variable cross-section

in which EI_0, m_0 are the flexural stiffness and the mass per unit length, respectively, at section x=0; β and n are parameters which can be determined by values of EI_x and $\overline{m}_x$ at x=L/2 and L (Fig. 1) or at other control points.

Substituting Eqs. (10) and (11) into Eq. (6) leads to

$$(1-\beta x)^{n+2}\frac{d^4Y}{dx^4}-2\beta(n+2)(1-\beta x)^{n+1}\frac{d^3Y}{dx^3}+\beta^2(n+2)(n+1)(1-\beta x)^n\frac{d^2Y}{dx^2}$$
$$-(1-\beta x)^n\overline{m}_0\omega^2\frac{Y}{EI_0}=0 \tag{12}$$

If a differential operator is

$$D=\frac{1}{(1-\beta x)^n}\frac{d}{dx}[(1-\beta x)^{n+1}\frac{d}{dx}] \tag{13}$$

Then, Eq. (12) can be written as

$$(D+\Omega^2)(D-\Omega^2)Y=0 \tag{14}$$

Eq. (14) can be divided into two differential equations as follows

$$(D+\Omega^2)Y=0 \tag{15}$$
$$(D-\Omega^2)Y=0 \tag{16}$$

in which

$$\Omega^2=\omega\sqrt{\frac{\overline{m}_0}{EI_0}} \tag{17}$$

Substituting Eq. (13) into Eqs. (15) and (16) gives

$$\frac{d^2Y}{dx^2}-\frac{\beta(n+1)}{1-\beta x}\frac{dY}{dx}+\frac{\Omega^2}{1-\beta x}Y=0 \tag{18}$$

and

$$\frac{d^2Y}{dx^2}-\frac{\beta(n+1)}{1-\beta x}\frac{dY}{dx}-\frac{\Omega^2}{1-\beta x}Y=0 \tag{19}$$

The general solution of Eq. (12) consists of the solution of Eq. (18) and the solution of Eq. (19) and can be expressed as

$$Y(x)=g^{-n}[C_1J_n(g)+C_2Y_n(g)+C_3I_n(g)+C_4K_n(g)] \quad (n = \text{an integer}) \tag{20}$$

or

$$Y(x)=g^{-n}[C_1J_n(g)+C_2J_{-n}(g)+C_3I_n(g)+C_4I_{-n}(g)] \quad (n = \text{a non-integer}) \tag{21}$$

where J_n (g), Y_n (g), I_n (g), K_n (g) are the Bessel functions of the first, second, third and fourth kinds, respectively; $g=\frac{2\Omega}{\beta}\sqrt{1-\beta x}$.

Case II: The distributions of flexural stiffness and mass per unit length are expressed as:

$$EI_x=EI_0(1-\beta x)^{n+4} \tag{22}$$

$$\bar{m}_x=m_0(1-\beta x)^n \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (6) gives

$$(1-\beta x)^4\frac{d^4Y}{dx^4}-2\beta(n+4)(1-\beta x)^3\frac{d^3Y}{dx^3}+\beta^2(n+4)(n+3)(1-\beta x)^2\frac{d^2Y}{dx^2}-\Omega^4Y=0 \tag{24}$$

Let

$$\begin{cases} z=Ln(1-\beta x) \\ E=\dfrac{d}{dz} \end{cases} \tag{25}$$

Eq. (24) then becomes

$$[E^4+2(n+4)E^3+(n+4)(n+3)E^2-\frac{\Omega^4}{\beta^4}]Y=0 \tag{26}$$

The eigenvalue equation of Eq. (26) is as follows

$$r^4+2(n+4)r^3+(n+4)(n+3)r^2-\frac{\Omega^4}{\beta^4}=0 \tag{27}$$

Solving Eq. (27) obtains four roots as follows

$$r_{1,2,3,4}=-\frac{1}{4}\pm\sqrt{\frac{(b_1\pm b_2)^2}{16}-\frac{1}{2}(f\pm\sqrt{f^2-4e})} \tag{28}$$

where

$$\left.\begin{aligned} &f=\sqrt[3]{-\frac{q}{2}\pm\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}}+\sqrt[3]{-\frac{q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}}-\frac{a}{3} \\ &b_1=\frac{4\Omega^4}{\beta^4},\quad b_2=\sqrt{b_1^2-4s_1+4f} \\ &a=-(n+4)(n+3),\quad e=-\frac{\Omega^4}{\beta^4} \\ &s_1=-4(n+4)(n+3)\frac{\Omega^4}{\beta^4}-4(n+4)^3(n+3) \\ &p=b_1-\frac{a^2}{3},\quad q=s_1-\frac{ab_1}{3}-\frac{a^3}{27} \end{aligned}\right\} \tag{29}$$

The general solution of Eq. (24) is

$$Y(x) = C_1(1-\beta x)^{r_1} + C_2(1-\beta x)^{r_2} + C_3(1-\beta x)^{r_3} + C_4(1-\beta x)^{r_4} \tag{30}$$

Case III: The distributions of flexural stiffness and mass per unit length are exponential functions given as:

$$EI_x = EI_0 \exp(-bx) \tag{31}$$

$$m_x = m_0 \exp(-bx) \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (6) leads to a differential equation with constant coefficients as follows

$$\frac{d^4Y}{dx^4} - 2b\frac{d^3Y}{dx_3} + b^2\frac{d^2Y}{dx^2} - \Omega^4 Y = 0 \tag{33}$$

The eigenvalue equation of Eq. (31) is as follows

$$r^4 - 2br^3 + b^2r^2 - \Omega^4 = 0 \tag{34}$$

The roots of the above equation are found as

$$r_{1,2,3,4} = \frac{b}{2} \pm \sqrt{\frac{b^2}{2} \pm \Omega^2} \tag{35}$$

The general solution of Eq. (33) can be expressed as

$$Y(x) = C_1 \exp(r_1 x) + C_2 \exp(r_2 x) + C_3 \exp(r_3 x) + C_4 \exp(r_4 x) \tag{36}$$

3. Multi-step bars

Although the general solutions of the three cases discussed above can be used to determine structural dynamic characteristics of certain structures, there are two problems to be solved. First, some structures consist of several steps or segments (see Fig. 2). Second, the distributions of flexural stiffness and mass per unit length of some structures may not obey the assumed expressions given in the above three cases. Such structures can be divided into several segments (or finite elements) for free vibration analysis. If the segments (finite elements) are divided appropriately, the distributions of flexural stiffness and mass per unit length in each of the segments may match accurately or approximately one of the expressions described in the above three cases. The exact solution of a one-step bar with variable cross-section could be used to derive the eigenvalue equation of a bar with multi-segments by using the transfer matrix method to be described below. One of the advantages of the present method is that the total number of segments required could be much less than that normally used in the conventional finite element methods.

The general solution of mode shape of the i-th segment can be expressed as

$$Y_i(x) = C_{1i}\, W_{1i}(x) + C_{2i}\, W_{2i}(x) + C_{3i}\, W_{3i}(x) + C_{4i}\, W_{4i}(x); \quad (i = 1, 2, \cdots, q) \tag{37}$$

where i denotes the i-th segment and q is the number of segments of the bar divided (Fig. 2).

A transfer matrix method is introduced herein to solve the problem. The mode shape function of displacement $Y_i(x)$, rotation $dY_i(x)/dx$, bending moment $M_i(x)$ and shear force $Q_i(x)$ can be

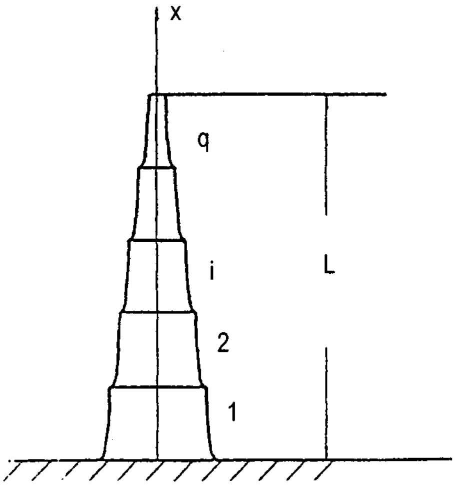

Fig. 2 A multi-step bar

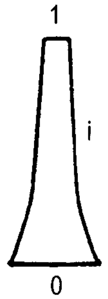

Fig. 3 Definition of the parameters at the two end of the i-th segment

expressed as a matrix

$$[Y_i(x) \ \frac{dY_i(x)}{dx} M_i(x) \ \ Q_i(x)]^T = [W_i(x)] \, [C_1 \, C_2 \, C_3 \, C_4]^T \tag{38}$$

where

$$[W_i(x)] = \begin{bmatrix} W_{1i}(x) & W_{2i}(x) & W_{3i}(x) & W_{4i}(x) \\ \frac{dW_{1i}(x)}{dx} & \frac{dW_{2i}(x)}{dx} & \frac{dW_{3i}(x)}{dx} & \frac{dW_{4i}(x)}{dx} \\ EI_x \frac{d^2W_{1i}(x)}{dx^2} & EI_x \frac{d^2W_{2i}(x)}{dx^2} & EI_x \frac{d^2W_{3i}(x)}{dx^2} & EI_x \frac{d^2W_{4i}(x)}{dx^2} \\ \frac{d}{dx}[EI_x \frac{d^2W_{1i}(x)}{dx^2}] & \frac{d}{dx}[EI_x \frac{d^2W_{2i}(x)}{dx^2}] & \frac{d}{dx}[EI_x \frac{d^2W_{3i}(x)}{dx^2}] & \frac{d}{dx}[EI_x \frac{d^2W_{4i}(x)}{dx^2}] \end{bmatrix} \tag{39}$$

The relationship between the parameters introduced above at the two ends of the i-th segment (Fig. 3) can be expressed as

$$\begin{bmatrix} Y_{i1} \\ \frac{dY_{i1}}{dx} \\ M_{i1} \\ Q_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} Y_{i0} \\ \frac{dY_{i0}}{dx} \\ M_{i0} \\ Q_{i0} \end{bmatrix} \tag{40}$$

in which

$$[T_i] = [W(x_{i1})] \, [W(x_{i0})]^{-1} \tag{41}$$

$$\frac{dY_{i1}}{dx}=\left.\frac{dY_i(x)}{dx}\right|_{x=x_{i1}}\quad \frac{dY_{i0}}{dx}=\left.\frac{dY_i(x)}{dx}\right|_{x=x_{i0}} \tag{42}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end 0 to those at the end 1 of a segment (i-th).

The co-ordinates of the i-th segment can be represented by the co-ordinates of the (i-1)th segment, then, Eq. (38) can be rewritten as

$$\begin{bmatrix} Y_i \\ \dfrac{dY_i}{dx} \\ M_i \\ Q_i \end{bmatrix} = [T_i] \begin{bmatrix} Y_{i-1} \\ \dfrac{dY_{i-1}}{dx} \\ M_{i-1} \\ Q_{i-1} \end{bmatrix} \quad (i=1,2,\cdots,q) \tag{43}$$

The equation for the top segment (Fig. 2) can be established by using Eq. (43) repeatedly:

$$\begin{bmatrix} Y_q \\ \dfrac{dY_q}{dx} \\ M_q \\ Q_q \end{bmatrix} = [T] \begin{bmatrix} Y_0 \\ \dfrac{dY_0}{dx} \\ M_0 \\ Q_0 \end{bmatrix} \tag{44}$$

in which

$$[T]=[T_q]\,[T_{q-1}]\cdots[T_1] \tag{45}$$

If there is a lumped mass, M_i, attached to the i-th segment, then, the transfer matrix $[T_i]$ should be replaced by $[T_{mi}]$:

$$[T_{mi}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega^2 M_i & 0 & 0 & 1 \end{bmatrix} [T_i] \tag{46}$$

The frequency equation can be determined according to boundary conditions. For example, in many cases a high-rise structure can be treated as a cantilever bar (Fig. 2); its boundary conditions in flexural free vibration are

$$Y(0)=0,\ \left.\frac{dY}{dx}\right|_{x=0}=0,\ M(L)=0,\ Q(L)=0 \tag{47}$$

In order to determine the frequency equation, the matrix $[T]$ is written as

$$[T]=\begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{23} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \tag{48}$$

Using Eq. (47) gives the frequency equation as

$$T_{33}T_{44}-T_{34}T_{43}=0 \tag{49}$$

The natural frequencies ω_j (j=1, 2, $\cdots$) can be computed numerically from Eq. (49), then, the mode shape function Y_j (x), rotation $\frac{dY_j}{dx}$, bending moment $M_j(x)$ and shear force Q_j (x) for each segment can be determined by use of Eq. (40) and the general solution, Eq. (37). For other kinds of boundary conditions, the analysis procedure is the same as that for a cantilever bar.

4. Numerical example 1

A typical tall building with 27-storeys located in Guangzhou is used as a numerical example for the present study. This building is a shear-wall structure with variable cross-section. Based on the full-scale measurement of free vibration of this building (Li *et al.* 1994), the building can be treated as a cantilever bar (Fig. 1) in free vibration analysis, and the effect of rotatory inertia and transverse shear deformation can be neglected. The procedure for determining the dynamic characteristics of this tall building is as follows:

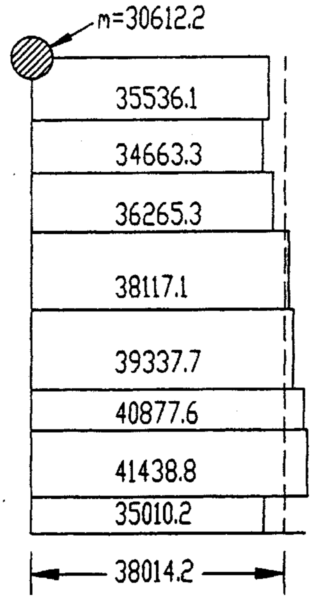

Fig. 4 Mass distribution of the tall building

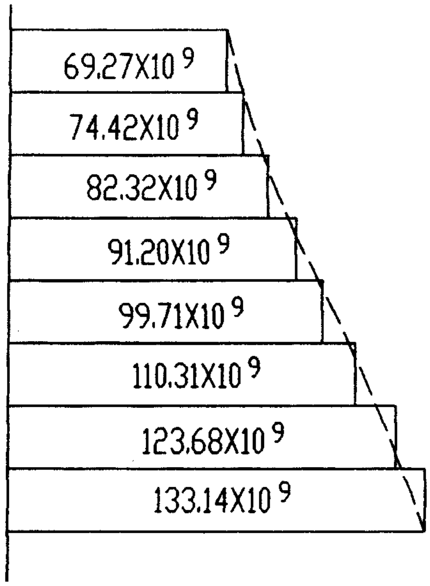

Fig. 5 Stiffness distribution of the tall building

4.1. Determination of the mass per unit length (Fig. 4)

The stiffness and mass per unit length (Fig. 4 and Fig. 5) of this building vary with height. For simplicity, the building is treated as a variable cross-section cantilever bar as shown in Fig. 1. The variation of the mass per unit length is comparatively small, thus, it is reasonable to assume that the mass is uniformly distributed along the height of the building (Fig. 4). The lumped mass attached at the top of the building that is considered in the computation is M=30612.2 kg. The mass per unit length, $\bar{m}$, is found as: $\bar{m}$=38,014.2 kg/m.

4.2. Evaluation of the stiffness, K_x (Fig. 5)

For this example, the distributions of flexural stiffness and mass per unit length along the building height are described as power functions (Eqs. (10) and (11)), which are given as

$$EI_x = EI_0(1-\beta x)^{n+2} \tag{50}$$
$$\bar{m}_x = m_0(1-\beta x)^n \tag{51}$$

Because the mass is considered as uniformly distributed, it is suggested that n=0 in the above two expressions. Then, the stiffness distribution can be expressed as follows

$$EI_x = EI_0(1-\beta x)^2 \tag{52}$$

According to the following information of this building given by Li *et al.* (1994):

$$at\ \ x=0, \quad I_x = 2{,}156.50\,m^4$$
$$at\ \ x=L, \quad I_x = 1{,}099.57\,m^4$$

The constants EI_0 and β are determined as

$$EI_0 = 60.38\times10^{10}\ \mathrm{kN{\cdot}m{\cdot}s^2}, \qquad \beta = 3.796\times10^{-3}$$

The evaluated distribution of stiffness by Eq. (52) is shown in Fig. 5 (dotted line and the values in parentheses).

4.3. Evaluation of the fundamental natural frequency

According to Fig.1, the boundary conditions are

$$x=0, \quad Y=Y'=0 \tag{53}$$
$$x=L, \quad Y''=0, \quad (EIY'')' = -M\omega^2 Y \tag{54}$$

Substituting Eq. (53) and Eq. (54) into Eq. (20) gives the frequency equation as follows:

$$\bar{Y}_3''(L)[EI(L)\bar{Y}_4'''(L)+M\omega^2\bar{Y}_4(L)] = \bar{Y}_4''(L)[EI(L)\bar{Y}_3'''(L)+M\omega^2\bar{Y}_3(L)] \tag{55}$$

in which

$$\bar{Y}_3 = \frac{\pi}{2\beta}\{Y_1(\theta)Y_2 - Y_2(\theta)Y_1 + \frac{2}{\pi}[Y_3(\theta)Y_4 - Y_4(\theta)Y_3]\} \tag{56}$$
$$\bar{Y}_4 = \frac{\pi}{2\beta}\{Y_2'(\theta)Y_1 - Y_1'(\theta)Y_2 - \frac{2}{\pi}[Y_3'(\theta)Y_4 - Y_4'(\theta)Y_3]\} \tag{57}$$
$$\theta = \frac{2\omega}{\beta} \tag{58}$$

Table 1 Fundamental mode shape of the building

Storey Level	1	2	5	8	11	14	17	20	24
x/H	0	0.0704	0.2007	0.3230	0.4454	0.5678	0.6976	0.8125	1
$Y_1(x/H)$ measured	0	0.005	0.070	0.160	0.290	0.390	0.540	0.730	1
$Y_1(x/H)$ calculated	0	0.0068	0.0527	0.1414	0.2644	0.4049	0.5599	0.7336	1

$$Y_1 = g^{-n}J_n(g),\quad Y_2 = g^{-n}Y_n(g),\quad Y_3 = g^{-n}I_n(g),\quad Y_4 = g^{-n}K_n(g) \tag{59}$$

For this tall building, n=0.

Solving the frequency equation, one obtains the natural frequencies. The calculated fundamental natural frequency (when the damping term is omitted) is 6.858rad/sec and the fundamental period T is 0.9162sec. The measured fundamental period T is 0.97 sec (Li *et al.* 1994). It is evident that the computed result based on the procedures proposed in this paper approaches the measured value, suggesting that the proposed methods are applicable to engineering application and practice.

4.4. Calculation of vibration mode shape

After the first natural frequency ω_1 is calculated, the first mode shape, Y_1 (x), can be determined from Eq. (20). The calculated results are listed in Table 1. The measured fundamental mode shape is also presented in Table 1 for the purposes of comparison. It can be seen from Table 1 that the calculated fundamental mode shapes show good agreement with the measured mode shape. Using the aforementioned procedure, the higher natural frequencies and corresponding mode shapes can also be determined.

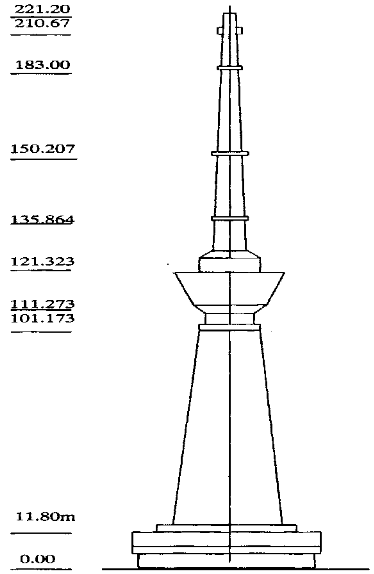

Fig. 6 Sketch of Wuhan T.V. Tower

Table 2 The Structural parameters of Wuhan T.V. Tower

No	Height m	Diameter m	Mass m_i Kg	Stiffness EI_i 10^7 KN·m·sec^2	β_i m^{-1}
1	4.975	15	72,571	28,440	0.01305
2	17.412	12.5	66,739	26,320	0.01300
3	22.387	11.5	60,849	24,157	0.01297
4	27.377	10.875	54,192	17,064	0.00906
5	29.876	10.75	59,317	22,388	0.01144
6	32.375	7.0	41,658	9,582	0.00424
7	34.874	7.0	57,256	24,844	0.00115
8	34.843	7.6	37,208	8,622	0.00422
9	108.853	7.0	20,735	1,426	0.0
10	110.803	7.6	20,735	1,428	0.0
11	114.253	7.0	37,935	2,874	0.0
12	133.653	7.0	20,735	1,428	0.0
13	138.622	5.88	20,735	1,428	0.00613
14	139.864	5.6	20,281	837	−0.03219
15	142.303	4.5	14,314	374	0.00666
16	154.203	4.5	14,314	374	0.0
17	156.685	3.9	9,759	196	0.00599
18	187	3.9	9,759	196	0.0

5. Numerical example 2

Wuhan T.V. Tower is a reinforced concrete tube structure, its geometric configuration is shown in Fig. 6, and its geometric dimension, mass and stiffness distribution are listed in Table 2 which were obtained by Li *et al.* (1995). The top of the tower is of 221 meters. The height of the main tower body is 187 meters. The structural dynamic characteristics of the main tower body were measured by Li *et al.* (1995). This provided an opportunity for comparing the results calculated by the proposed method in this paper with the measured natural frequencies and mode shapes of the main tower body.

The main tower body is treated as a cantilever bar multi-step with variable cross-section as shown in Fig. 2 for free vibration analysis. This structure is divided into 18 segments for computation. The mass and stiffness distribution of the *i*-th segment are assumed as

$$EI_i(x)=EI_i(1-\beta_i\, x)^3 \qquad i=1,2,...,18 \tag{60}$$

$$\bar{m}_x=m_i(1-\beta_i\, x) \tag{61}$$

where EI_i, m_i, β_i are listed in Table 2.

The natural frequencies and the fundamental mode shape of the main tower body which are obtained by use of the methods proposed in this paper are presented in Table 3 and shown in Fig. 7, respectively. In order to examine the accuracy of the methods proposed in this paper, the lumped mass (finite element) method is also employed to calculate the structural dynamic characteristics of the main tower body. This high-rise structure was divided into 40 elements for computation by using the finite element method (FEM) and the results calculated by FEM are also presented in Table 3 and Fig. 7 for comparison purposes. The field measured fundamental natural frequency and mode shape of the main tower body obtained by Li *et al.* (1995) are also shown in

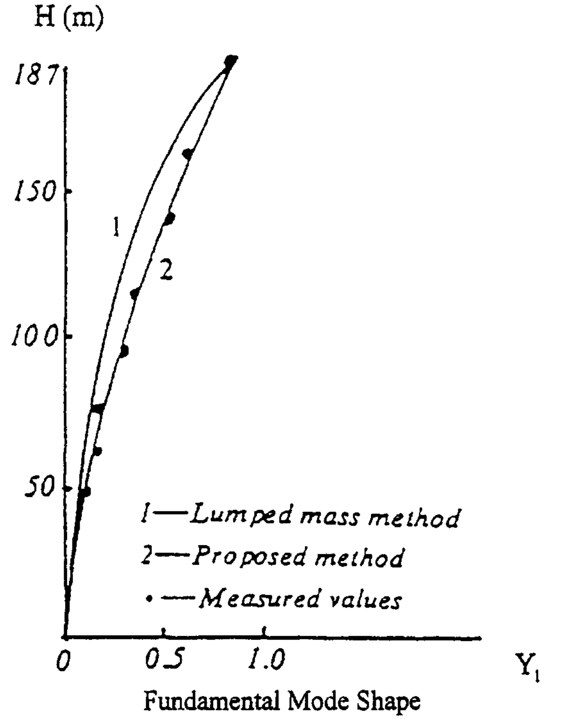

Fig. 7 Fundamental mode shape of the main tower body of Wuhan T.V. Tower

Table 3 The natural frequencies of the main tower body of Wuhan T.V. Tower

ω_i (rad/s)	Proposed method	Finite element method	Measured value
ω_1	1.9899	2.0099	1.99
ω_2	6.4997	6.7797	
ω_3	14.0093	14.9692	

Table 3 and Fig. 7, respectively.

It can be seen from Table 3 and Fig. 7 that the calculated natural frequencies and the fundamental mode shape by the use of the proposed methods are in good agreement with the corresponding field measured data. The assumptions of the variations of flexural stiffness and mass are thus justified. It should be noted that the fundamental mode shape computed by the proposed methods is closer to the measured one than that obtained by the finite element (lumped mass) method. This numerical example also shows that one of the advantages of the present method is that the total number of the segments required in the proposed method could be much less than that normally used in conventional finite element methods. Therefore, the proposed method has practical significance for free vibration analysis.

5. Conclusions

In this paper, tall buildings and high-rise structures are treated as cantilever bars with variable cross-section in free vibration analysis. In general, it is not possible or, at least, very difficult to get the exact solutions of differential equations for free vibration of bars with variably distributed mass and stiffness. In this paper, the general solutions for free vibration of bars with variable cross-section are obtained by selecting suitable expressions, such as power functions and exponential functions, for the distribution of flexural stiffness and mass of the bars. It is found that if the distribution of damping coefficient is assumed to be proportional to that of mass ($C_x=C_0 \overline{m}_x$), there is no effect of damping on the mode shapes and the damped natural frequency is equal to the corresponding undamped natural frequency multiplied by the coefficient, k_d. The numerical examples show that the calculated natural frequencies and fundamental mode shapes of a 27-storey building and a television transmission tower are close to the corresponding measured field data, illustrating that the calculation methods proposed in this paper are applicable to engineering application and practice. The assumptions of the variations of flexural stiffness and mass for typical tall buildings and high-rise structures are justified through the two numerical examples. Thus, the formulae proposed in this paper can be used to determine the natural frequencies and the mode shapes of many tall buildings and high-rise structures. If the distributions of stiffness and mass of some structures do not match any distribution of the three cases discussed, these structures can be divided into several segments until the distributions in each segment accurately or approximately obey one of the types of distribution as described in this paper. When the distributions of flexural stiffness and mass of one-step and multi-step bars are very complex, the accuracy of the method proposed in this paper will also be satisfied by increasing the number of segments. The numerical example (Wuhan T.V. Tower) shows that one of the advantages of the present method is that the total number of the segments required in the proposed method is much less than that normally used in conventional finite element methods. Therefore, the proposed method has practical significance for free vibration analysis of cantilevered structures.

References

B.M. Kumar and R.I. Sujith (1997), "Exact solutions for the longitudinal vibration of non-uniform rods", *Journal of Sound and Vibration*, **207**(5), 721-729.

Q.S. Li, H. Cao and G. Li (1994), "Analysis of free vibrations of tall buildings", *ASCE, Journal of Engineering Mechanics*, **120**(9), 1861-1876.

Q.S. Li, H. Cao and G. Li (1995), "Calculation of free vibration of high-rise structures", *Asian Journal of Structural Engineering*, **1**(1), 17-25.

Q.S. Li, H. Cao and G. Li (1996), "Static and dynamic analysis of straight bars with variable cross-section", *International Journal of Computers & Structures*, **59**(6), 1185-1191.

Q.S. Li, J.Q. Fang and A.P. Jeary (1998), "Free longitudidal vibration analysis of tall buildings and high-rise", *The Structural Design of Tall Buildings*, **1**, 167-176.

T.T. Soong (1990), "Active structural control, theory and practice", *Longman Scientific & Technical*, 18-43.

J.J. Tuma and F.Y. Cheng (1983), *Dynamic Structural Analysis*, McGraw-Hill Book Company.

G.Y. Wang (1978), *Vibration of Building and Structures*, Science and Technology Press, 168-178.