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Transverse vibrations of simply supported orthotropic rectangular plates with rectangular and circular cut-outs carrying an elastically mounted concentrated mass

D.R. Avalos† and H.A. Larrondo†

Physics Department, School of Engineering, Universidad Nacional de Mar del Plata, 7600-Mar del Plata, Argentina

P.A.A. Laura†

Department of Engineering, Universidad Nacional del Sur and Institute of Applied Mechanics, (CONICET) 8000-Bahia Blanca, Argentina

Abstract. Practicing a hole or an orifice through a plate or a slab constitutes a very frequent engineering situation due to operational reasons imposed on the structural system. From a designer's viewpoint it is important to know the effect of this modification of the mechanical system upon its elastodynamic characteristics. The present study deals with the determination of the lower natural frequencies of the structural element described in the title of the paper using a variational approach and expressing the displacement amplitude of the plate in terms of the double Fourier series which constitutes the classical, exact solution when the structure is simply supported at its four edges.

Key words: simply supported; orthotropic; rectangular plates; cut-outs; elastically mounted mass; Rayleigh-Ritz; double Fourier series.

1. Introduction

Consider the mechanical system shown in Fig. 1: an orthotropic rectangular plate simply supported at its edges, carrying an elastically supported mass m while k is the spring constant of the foundation of the mass which is assumed to possess lineal characteristics. Obviously the mass represents a motor or engine in the case of a mechanical design, or an electronic element in the case of a printed circuit board (Steinberg 1973). Practicing a hole of dimensions $(a_1 \times b_1)$ (like in Fig. 1) or a circular orifice is common practice in a great variety of situations since the passage of ducts, conduits or electrical cables requires it. The present paper deals with the determination of the lower natural frequencies of the system.

Since an exact solution seems to be out of the question in view of the impossibility of identically satisfying the natural boundary conditions at the free edge, use is made of the classical

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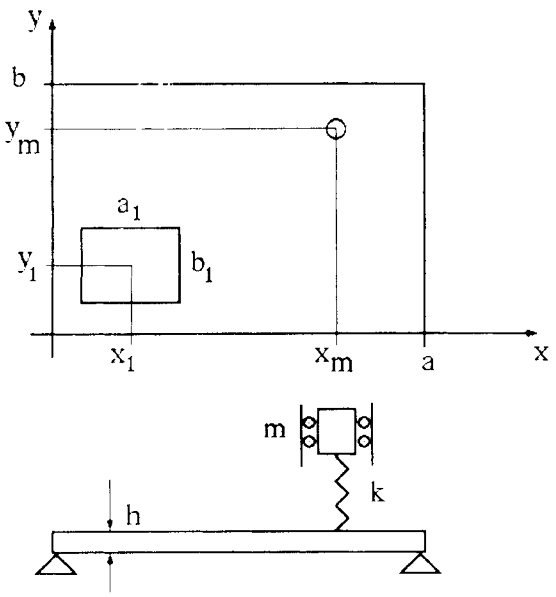

Fig. 1 Vibrating system under consideration

Rayleigh-Ritz method. The displacement amplitude is expressed in terms of a truncated double Fourier series which satisfies identically the governing boundary conditions at the outer edge and which leads to the exact solution of the problem when no holes are present.

The eigenvalues have been computed from (401×401) and (901×901) secular determinants obtaining excellent agreement between the lower roots. It has been previously shown that in the case of an isotropic plate the frequency coefficients agree with those determined independently by means of a very accurate finite element code (Avalos *et al.* 1997, Laura *et al.* 1997a, b).

2. Approximate analitical solution

The Rayleigh-Ritz method requires minimization of the combined functional

$$J[W',z']=J_p[W']+J_m[z'] \tag{1}$$

where $J_p[W']$ is the functional for the displacement amplitude of the plate and $J_m[z']$ is the corresponding functional for the displacement amplitude of the concentrated mass (see Fig. 1). Each functional, in turn, has the general form:

$$J=U-T \tag{2}$$

where U and T are the Maximum Strain Energy and Maximum Kinetic Energy of plate-mass-spring system, respectively.

As is well known, in the case of rectangular plates of orthotropic anisotropy its functional can be written as (Lekhnitskii 1968)

$$J_p=1/2\int_{A_p}\left[D_1\left(\frac{\partial^2 W'}{\partial x'^2}\right)^2+2D_1\nu_2\frac{\partial^2 W'}{\partial x'^2}\frac{\partial^2 W'}{\partial y'^2}+D_2\left(\frac{\partial^2 W'}{\partial y'^2}\right)^2\right.$$

$$+4D_k\left(\frac{\partial^2 W'}{\partial x'\partial y'}\right)^2\Bigg]dx'dy' - \frac{\rho h\,\omega^2}{2}\int_{A_p} W'^2 dx'dy' \tag{3a}$$

where W' is the true displacement amplitude of the plate; the first integral in (3a), taken over the actual doubly connected area of the plate surface A_p, is the (maximum) Strain Energy of the plate and the second integral measures the maximum Kinetic Energy of the plate.

The functional for the concentrated mass-spring system has the form (Avalos *et al.* 1994)

$$J_m = \frac{k_m}{2}z'^2 - \frac{m\,\omega^2}{2}(z' + W_m')^2 \tag{3b}$$

Where z' is the mass displacement amplitude relative to the plate; $(z'+W_m')$ is the total displacement amplitude of the point mass, and W_m' is the displacement amplitude of the plate at the concentrated mass position.

In Eqs. (3a) and (3b) above:

$$D_1 = \frac{E_1 h^3}{12(1-\nu_1\nu_2)};\quad D_2 = \frac{E_2 h^3}{12(1-\nu_1\nu_2)} \quad \text{and} \quad D_k = \frac{Gh^3}{12} \tag{4a}$$

are the well known flexural rigidities of the (orthotropic) plate. For an isotropic plate, Eqs. (4a) take the well known form:

$$D_1 = D_2 = \frac{Eh^3}{12(1-\nu^2)};\quad D_k = \frac{(1-\nu)}{2}\,\frac{Eh^3}{12(1-\nu^2)} \tag{4b}$$

If the length of the sides of the rectangular plate are a and b in the x and y directions, respectively, Eqs. (3a) and (3b) can be cast in a non-dimensional form by introducing:

$$W = W'/a;\quad x = x'/a;\quad y = y'/b \quad \text{and} \quad z = z'/a \tag{5}$$

One obtains, for the functional for the whole system depicted in Fig. 1:

$$J_{nd} = \frac{2J}{D_1} = \int_{A_p}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \frac{2\nu_2}{s_1^2}\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + \frac{D_2}{D_1 s_1^4}\left(\frac{\partial^2 W}{\partial y^2}\right)^2\right.$$

$$\left. + \frac{4D_k}{D_1 s_1^2}\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]dxdy - \Omega^2\int_{A_p} W^2 dxdy + K_m z^2 - M\,\Omega^2(z + W_m)^2 \tag{6}$$

where

$$\Omega^2 = \frac{\rho h\,\omega^2 a^4}{D} \tag{7}$$

is the square of the non-dimensional frequency coefficient;

$$M = \frac{m}{M_p} \tag{8}$$

M_p being the mass of the plate without holes;

$$K_m = \frac{k_m a^2}{D_1 s_1} \tag{9}$$

is the non-dimensional mass-spring constant
and

$$s_1 = \frac{b}{a} \tag{10}$$

is the aspect ratio of the plate.

Expressing the displacement amplitude $W(x, y)$ of the plate in an approximate way by means of a double Fourier series

$$W_a(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{mn} \sin(m\pi x)\sin(n\pi y) \tag{11}$$

one gets as a final expression for the functional the following sum:

$$J_{nd} = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{l=1}^{N} \sum_{k=1}^{M} b_{kl} b_{mn} \left\{ \pi^4 \left[m^2 k^2 + 2\nu_2 \frac{k^2 n^2}{s_1^2} + \frac{D_2}{D_1} \frac{n^2 l^2}{s_1^4} \right] A_{ss} \right.$$
$$\left. + \pi^4 \left[\frac{4D_k}{D_1} \frac{klmn}{s_1^2} \right] A_{cc} - \Omega^2 A_{ss} \right\} + K_m z^2 - M\Omega^2 s_1 (z + W_m)^2 \tag{12}$$

where

$$W_m = W_a(x_m, y_m) = \sum_{n=1}^{N} \sum_{m=1}^{M} b_{nm} \sin(m\pi x_m)\sin(n\pi y_m)$$

is the amplitude displacement of the plate at the mass position, x_m and y_m being the non-dimensional mass position coordinates;

$$A_{ss} = \int_{A_p} \sin(k\pi x)\sin(l\pi y)\sin(m\pi x)\sin(n\pi y)\,dxdy \tag{13}$$

and

$$A_{cc} = \int_{A_p} \cos(k\pi x)\cos(l\pi y)\cos(m\pi x)\cos(n\pi y)\,dxdy \tag{14}$$

In order to minimize the functional one has to take its partial derivatives with respect to the coefficient b_{ij} and z in expression (12) and equal these derivatives to zero.

That is to say

$$\frac{\partial J}{\partial b_{11}} = 0$$
$$\frac{\partial J}{\partial b_{12}} = 0$$
$$\frac{\partial J}{\partial b_{MN}} = 0$$
$$\frac{\partial J}{\partial z} = 0$$

This yields and ($M\times N+1$) homogeneous linear system of equations in the b_{ij}'s and the z. A secular determinant in the natural frequency coefficients of the system results from the non triviality condition.

The present study is concerned with the determination of the frequency coefficients Ω_1 and Ω_2 in the case of orthotropic plates with rectangular and circular holes and carrying elastically mounted concentrated masses. It is important to emphasize the fact that basic, important problems like the case where a slit is practiced in the plate, cannot be analyzed using the present approach.

3. Numerical results

Numerical experiments on vibrating isotropic rectangular plates with simply supported edges and rectangular, free edge holes have recently been reported (Laura *et al.* 1997b). It has been shown that the analytical results obtained using the approach previously explained, using a 901×901 secular determinant, are in excellent agreement with the eigenvalues obtained by means of a very accurate finite element code, the maximum differences being of the order of 0.3%. In some instances four significant digits agreed.

As previously stated, the natural boundary conditions at the hole edge are not satisfied when using the analytical approach. But, on the other hand, the coordinate functions employed constitute a complete set of trial functions and the functional minimization process required by the

Table 1 Values of Ω_1 and Ω_2 in the case of orthotropic square plates with different positions and values of the mass-spring system when the cutout with a_1/a=0.1 moves along the diagonal (Fig. 2)

Mass coord.	m/M_p	Ka^2/D	(A)		(B)		(C)	
			Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
x/a=.50	0.1	1	3.148	20.01	3.149	19.96	3.148	19.77
y/b=.75		10	9.526	20.59	9.534	20.52	9.532	20.32
		∞	17.92	36.90	17.93	36.86	17.81	36.44
(a)	0.3	1	1.818	20.01	1.818	19.96	1.818	19.77
		10	5.529	20.49	5.533	20.42	5.533	20.22
		∞	14.68	32.09	14.75	31.96	14.69	31.54
x/a=.75	0.1	1	3.152	19.98	3.152	19.93	3.152	19.74
y/b=.75		10	9.674	20.27	9.676	20.22	9.676	20.02
		∞	18.86	37.46	18.83	37.44	18.68	37.01
(b)	0.3	1	1.820	19.98	1.820	19.93	1.820	19.74
		10	5.602	20.22	5.602	20.17	5.602	19.97
		∞	16.49	30.43	16.50	30.37	16.42	30.02
x/a=.75	0.1	1	3.149	20.01	3.149	19.96	3.149	19.77
y/b=.50		10	9.549	20.60	9.559	20.53	9.556	20.33
		∞	17.96	42.53	17.98	42.46	17.85	41.99
(c)	0.3	1	1.818	20.01	1.818	19.96	1.818	19.77
		10	5.542	20.49	5.547	20.43	5.547	20.23
		∞	14.83	35.77	14.91	35.63	14.84	35.28

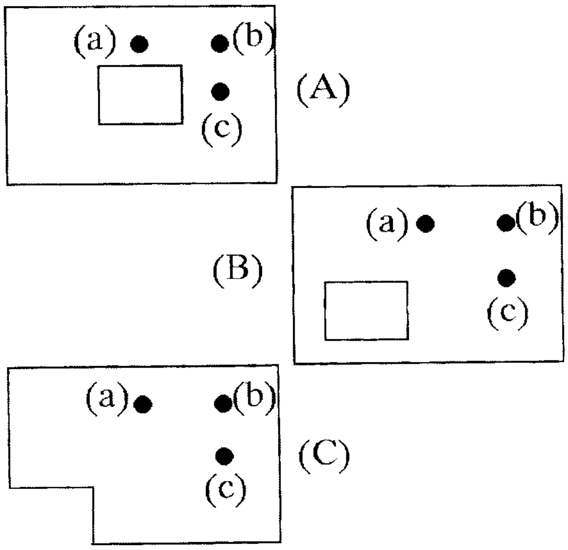

Fig. 2 Orthotropic simply supported rectangular plates with cutouts of the same aspect ratio when the cutout displaces along the diagonal and different positions of the spring-mass system. Positions of the cutout center: (A) $x_1=a/2$, $y_1=b/2$; (B) $x_1=a/4$, $y_1=b/4$; (C) $x_1=a_1/2$, $y_1=b_1/2$

Table 2 Values of Ω_1 and Ω_2 in the case of orthotropic square plates with different positions and values of the mass-spring system when the cutout with a_1/a=0.3 moves along the diagonal (Fig. 2)

Mass coord.	m/M_p	Ka^2/D	(A)		(B)		(C)	
			Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
x/a=.50	0.1	1	3.145	21.05	3.148	19.42	3.148	18.43
y/b=.75		10	9.432	21.79	9.511	20.02	9.503	19.98
		∞	18.06	35.50	17.44	37.01	16.80	35.10
(a)	0.3	1	1.816	21.05	1.817	19.42	1.817	18.43
		10	5.477	21.67	5.523	19.90	5.522	18.87
		∞	14.07	31.91	14.37	32.06	14.09	29.97
x/a=.75	0.1	1	3.151	21.01	3.152	19.39	3.152	18.40
y/b=.75		10	9.638	21.37	9.668	19.68	9.664	18.67
		∞	19.43	36.34	18.34	37.79	17.57	35.63
(b)	0.3	1	1.819	21.01	1.820	19.39	1.820	18.40
		10	5.582	21.31	5.600	19.63	5.599	18.62
		∞	16.39	30.73	16.15	30.34	15.73	28.57
x/a=.75	0.1	1	3.145	21.05	3.149	19.42	3.149	18.43
y/b=.50		10	9.432	21.79	9.533	20.03	9.523	19.01
		∞	18.07	38.60	17.47	42.01	16.79	40.11
(c)	0.3	1	1.816	21.05	1.818	19.42	1.818	18.43
		10	5.477	21.68	5.536	19.91	5.534	18.89
		∞	14.08	33.74	14.50	35.86	14.16	33.73

Table 3 Values of Ω_1 and Ω_2 in the case of orthotropic square plates with different positions and values of the mass-spring system when the cutout with a_1/a=0.5 moves along the diagonal (Fig. 2)

Mass coord.	m/M_p	Ka^2/D	(A) Ω_1	(A) Ω_2	(B) Ω_1	(B) Ω_2
x/a=.50 y/b=.75	0.1	1	3.131	26.67	3.146	18.71
		10	9.103	27.69	9.443	19.42
		∞	18.42	34.24	16.62	36.26
(a)	0.3	1	1.808	26.67	1.816	18.70
		10	5.277	27.60	5.494	19.28
		∞	12.14	32.87	13.60	32.27
x/a=.75 y/b=.75	0.1	1	3.145	26.61	3.152	18.67
		10	9.477	27.19	9.645	19.01
		∞	21.63	35.56	17.62	38.07
(b)	0.3	1	1.816	26.61	1.819	18.67
		10	5.486	27.14	5.590	18.94
		∞	15.48	33.10	15.52	30.96
x/a=.75 y/b=.50	0.1	1	3.131	26.67	3.147	18.71
		10	9.976	27.67	9.454	19.46
		∞	18.31	34.44	16.60	39.18
(c)	0.3	1	1.807	26.66	1.817	18.71
		10	5.216	27.58	5.501	19.31
		∞	11.98	32.90	13.63	35.95

Table 4 Values of Ω_1 and Ω_2 in the case of orthotropic rectangular plates, with b/a=2/3, when the cutout, with equal aspect ratio and a_1/a=0.3 moves along the diagonal (Fig. 2)

Mass coord.	m/M_p	Ka^2/D	(A) Ω_1	(A) Ω_2	(B) Ω_1	(B) Ω_2
x/a=.50 y/b=.75	0.1	1	3.154	31.68	3.156	27.86
		10	9.758	32.13	9.809	28.18
		∞	27.16	58.00	25.37	59.82
(a)	0.3	1	1.821	31.68	1.822	27.85
		10	5.639	32.10	5.669	28.15
		∞	20.97	50.01	21.25	49.84
x/a=.75 y/b=.75	0.1	1	3.157	31.66	3.158	27.84
		10	9.848	31.87	9.861	27.99
		∞	19.25	53.59	26.57	52.25
(b)	0.3	1	1.823	31.66	1.823	27.84
		10	5.688	31.85	5.696	27.97
		∞	24.39	45.02	23.59	41.49
x/a=.75 y/b=.50	0.1	1	3.154	31.68	3.156	27.85
		10	9.758	32.13	9.795	28.16
		∞	27.14	52.25	25.38	51.10
(c)	0.3	1	1.821	31.68	1.822	27.85
		10	5.640	32.10	5.662	28.13
		∞	20.94	46.87	21.09	43.34

Table 5 Values of Ω_1 and Ω_2 in the case of orthotropic rectangular plates, with $b/a=2/3$, when the circular cutout, with $r_o/a=0.15$ moves along the diagonal (Fig. 3)

Mass coord.	m/M_p	Ka^2/D	(A) Ω_1	(A) Ω_2	(B) Ω_1	(B) Ω_2
$x/a=.50$ $y/b=.75$	0.1	1	3.155	33.11	3.156	29.03
		10	9.763	33.56	9.808	29.38
		∞	28.13	57.55	26.14	62.08
(a)	0.3	1	1.821	33.11	1.822	29.03
		10	5.641	33.53	5.669	29.35
		∞	21.43	51.02	21.57	52.23
$x/a=.75$ $y/b=.75$	0.1	1	3.157	33.09	3.157	29.01
		10	9.851	33.30	9.861	29.18
		∞	30.37	51.72	27.45	52.62
(b)	0.3	1	1.822	33.09	1.823	29.01
		10	5.690	33.29	5.696	29.16
		∞	24.98	45.17	24.01	43.90
$x/a=.75$ $y/b=.50$	0.1	1	3.154	33.12	3.155	29.03
		10	9.752	33.57	9.794	29.38
		∞	27.91	49.31	26.06	54.44
(c)	0.3	1	1.821	33.12	1.822	29.03
		10	5.636	33.54	5.660	29.35
		∞	21.07	45.38	21.29	46.64

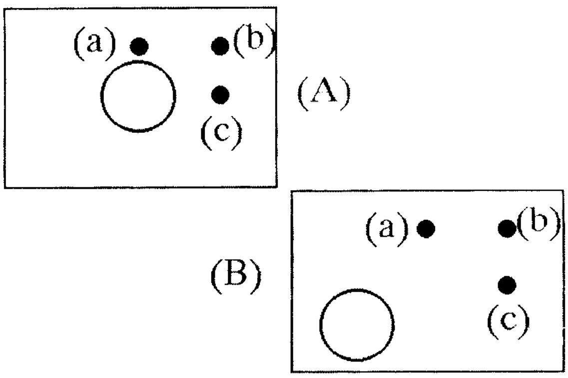

Fig. 3 Orthotropic simply supported rectangular plates with circular cutout of radius $r_o/a=0.15$ when the cutout displaces along the diagonal and different positions of the spring-mass system. Positions of the cutout center: (A) $x_1=a/2$, $y_1=b/2$; (B) $x_1=a/4$, $y_1=a/4$

Rayleigh-Ritz method guarantees that, as the number of coordinate functions used approaches infinity, the generalized force type boundary conditions and the governing partial differential equation of motion will be exactly satisfied (Kantorovich and Krylov 1964, Mikhlin 1964).

Tables 1 through 5 show values for orthotropic rectangular plates with both equal aspect ratio rectangular holes and circular holes. In these tables, $\nu_2=0.3$, $D_2=D_k=D_1/2$ and $M=N=20$.

Tables 1, 2 and 3 depict values of the fundamental and first excited frequency coefficients, Ω_1,

Ω_2, in the case of square orthotropic plates as the center of the cut-out displaces along a diagonal for the following positions: $x=a_1/2$ and $y=b_1/2$; $x=a/4$ and $y=b/4$ and finally $x=a/2$ and $y=b/2$ and for different values of the mass, spring constant and mass-spring coordinates.

Table 4 shows values of Ω_1 and Ω_2 in the case of rectangular plates, with $b/a=2/3$, when an equal aspect ratio cutout, with $a_1/a=0.3$, takes the following two positions along the diagonal: $x=a_1/2$ and $y=b_1/2$ and $x=a/2$ and $y=b/2$.

Finally Table 5 shows values of Ω_1 and Ω_2 in relation to rectangular plates, with $b/a=2/3$, when a circular cutout of non-dimensional ratio $r_o/a=0.15$, takes the following two positions along the diagonal: $x=a/4$ and $y=b/4$ and $x=a/2$ and $y=b/2$.

The numerical results presented in Tables 1 through 5 have been obtained from a 401×401 determinantal equation ($N=M=20$) since from a practical viewpoint the differences between the lower roots obtained from this determinantal equation and those determined from a (901/901) determinantal equation was less than 0.5% in the case of isotropic plates. It is reasonable to expect that as to orthotropic plates the accuracy achieved from a (401×401) determinantal equation will be sufficient from a designer's viewpoint.

Special care was taken to manipulate such large determinants and 80 bit floating point variables (IEEE-standard temporary reals) have been used in order to obtain accurate results. When the spring constant k attains a finite value the frequency coefficient Ω_1 is, obviously, the frequency coefficient corresponding to the spring-mass system "disturbed" by the presence of the plate while Ω_2 is the plate frequency coefficient modified by the existance of the mass-spring system (Laura *et al.* 1997).

It is interesting to point out that in some instances, for the same values of x/a, y/b, m/M_p and Ka^2/D, Ω_2 increases as the hole size increases. This is the case for $x/a=0.5$; $y/b=0.75$; $m/M_p=0.1$ and $Ka^2/D=1$ as a_1/a increases from 0.1 to 0.3 and 0.5, case (A) of Tables 1, 2 and 3. This phenomenon is known as "dynamic stiffening effect" since the actual plate is reduced in its actual weight but its fundamental frequency is higher.

The approach presented in this paper can be extended to the case of plates of non uniform thickness and also to other combinations of boundary conditions using appropriate coordinate functions, for instance the popular beam functions following Young's classical approach (Young 1950). The approach is also applicable in the case of anisotropic plates with elastic restrictions along the edges. Admittedly certain problems are excluded if the present approach is used, e.g., the case of a slit practiced in the plate.

Conclusions

It is shown in this paper that a simple approximate method allows for the solution of a rather complex elastodynamics problem. A subsidiary energy functional corresponding to the cut-out is subtracted from the energy functional of the solid, simply connected plate. The lower natural frequencies are determined in a straightforward fashion for several combinations of governing geometric and mechanical parameters.

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Notations

a, b	sides of the rectangular plate
a_1, b_1	sides of the rectangular hole
h	plate thickness
r_o	radius of the circular hole
s_1	b/a
W'	plate displacement amplitude
W	dimensionless plate displacement amplitude
z'	mass displacement amplitude relative to the plate
z	dimensionless mass displacement amplitude relative to the plate
ρ	mass density of the plate material
Ω	frequency coefficient