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A new quasi-3D higher shear deformation theory for vibration of functionally graded carbon nanotube-reinforced composite beams resting on elastic foundation

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Abstract. This study deals with free vibrations analysis with stretching effect of nanocomposite beams reinforced by single-walled carbon nanotubes (SWCNTs) resting on an elastic foundation. Four different carbon nanotubes (CNTs) distributions including uniform and three types of functionally graded distributions of CNTs through the thickness are considered. The rule of mixture is used to describe the effective material properties of the nanocomposite beams. The significant feature of this model is that, in addition to including the shear deformation effect and stretching effect it deals with only 4 unknowns without including a shear correction factor. The governing equations are derived through using Hamilton's principle. Natural frequencies are obtained for nanocomposite beams. The mathematical models provided in this paper are numerically validated by comparison with some available results. New results of free vibration analyses of CNTRC beams based on the present theory with stretching effect is presented and discussed in details. The effects of different parameters of the beam on the vibration responses of CNTRC beam are discussed.

Keywords: free vibration; stretching effect; CNTRC beams; elastic foundation

1. Introduction

Carbon nanotubes (CNTs) have been accepted as an excellent candidate for the reinforcement of polymer composites due to their high elastic modulus, tensile strength and low density. The potential applications of polymer/CNTs are found in the field of reinforcing composites, high performance structural and multifunctional composites (Thostenson *et al*. 2001). To enhance multiple properties of materials, the CNTs can be potentially incorporated into existing aerospace structural composites (Yamamoto *et al.* 2012). The critical challenge of producing polymer/CNTs composites is how to enhance dispersion and alignment of CNTs in a polymer matrix. Xie (2005) reviewed the available techniques and recent progress on dispersion and alignment of CNTs in the polymer matrix using ex situ technique, force and magnetic fields, electro-spinning and liquid crystalline phase induced methods. Material properties of carbon nanotube-reinforced composites (CNTRCs) have been examined by many investigators. The elastic properties in macro scale of CNTRCs through analyzing the elastic deformation of a representative volume element subjected to different loading conditions were presented by Hu and Fukunaga (2005). In actual applications, the CNTRCs can be incorporated in the structural elements such as beams, plates and shells. There are a few studies on the mechanical behavior of the CNTRC beams in the open literature. For example, Yas and Heshmati (2012) presented the dynamic response of the nanocomposite beams with randomly oriented carbon nanotubes under moving load. Baltacıoglu *et al.* (2010) studied the nonlinear static response of laminated composite plates by discrete singular convolution method. Li *et al.* (2010) used a higher-order theory for static and dynamic analysis of functionally graded beams. For mechanical problems of the CNTRC plates, there are some previous reports available in the open literature (Ping *et al.* 2012). Wattanasakulpong and Ungbhakorn (2013) studied the bending, buckling and vibration behaviors of carbon nanotube-reinforced composite (CNTRC) beams where several higher-order shear deformation theories are presented and discussed in details. Tounsi *et al.* (2013) use a refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates. Akgoz *et al.* (2013) developed the buckling analysis of linearly tapered micro-columns based on strain gradient elasticity. Bouderba *et al.* (2013) analyze the thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations. Wan *et al.* (2013) investigated a size-dependent free vibration analysis of composite laminated Timoshenko beam based on new modified couple stress theory. Zidi *et al*. (2014) analyse the bending of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory. Bousahla *et al*. (2014) investigated a novel higher order shear and normal deformation theory based on neutral surface

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position for bending analysis of advanced composite plates. Hebali *et al.* (2014) studied the static and free vibration analysis of functionally graded plates using a new quasi-3D hyperbolic shear deformation theory. Mahi *et al.* (2015), used a new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. Hamidi *et al*. (2015) proposed a sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates. Zemri *et al*. (2015) proposed an assessment of a refined nonlocal shear deformation theory beam theory for a mechanical response of functionally graded nanoscale beam theory. Larbi Chaht *et al.* (2015) studied the bending and buckling of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect. Belabed *et al*. (2014) used an efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates. Ait Yahia *et al.* (2015) analyzed the wave propagation in functionally graded plates with porosities. Bounouara *et al.* (2016) used a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Mercan *et al.* (2016) used the DSC method for buckling analysis of boron nitride nanotube (BNNT) surrounded by an elastic matrix. Demir *et al.* (2016) studied the determination of critical buckling loads of isotropic, FGM and laminated truncated conical panel. Bennoun *et al.* (2016) analyzed the vibration of functionally graded sandwich plates using a novel five variable refined plate theory. Draiche *et al.* (2016) used a refined theory with stretching effect for the flexure analysis of laminated composite plates. Boukhari *et al*. (2016) used an efficient shear deformation theory for wave propagation of functionally graded material plates. Akgoz *et al*. (2016) studied the bending analysis of embedded carbon nanotubes resting on an elastic foundation using strain gradient theory. Ahouel *et al.* (2016) investigated a size-dependent mechanical behavior of functionally graded trigonometric shear deformable nanobeams including neutral surface position concept. Chikh *et al.* (2017) studied the thermal buckling analysis of cross-ply laminated plates using a simplified HSDT. Klouche *et al.* (2017), investigated an original single variable shear deformation theory for buckling analysis of thick isotropic plates. Shokravi and Jalili (2017) analyze the vibration and stability of embedded cylindrical shell conveying fluid mixed by nanoparticles subjected to harmonic temperature distribution. Abdelhak and Hadji (2016) used using a refined shear deformation theory for buckling response of functionally graded sandwich plates. Ait Amar Meziane *et al.* (2014) proposed an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Attia *et al.* (2015) developed the free vibration analysis of functionally graded plates with temperature-dependent properties using various four variable refined plate theories. Al-Basyouni *et al.* (2015) investigated size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position. Bourada *et al.* (2015) used a new simple shear and normal deformations theory for functionally graded beams. Bousahla *et al.* (2016) investigated the thermal stability of plates with functionally graded coefficient of thermal expansion. Houari *et al.* (2016) developed a new simple three-unknown sinusoidal shear deformation theory for functionally graded plates. Civalek *et al.* (2016) used a simple mathematical model of microtubules surrounded by an elastic matrix by nonlocal finite element method. Beldjelili *et al*. (2016) studied the hygro-thermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four-variable trigonometric plate theory. Bouderba *et al.* (2016) studied the thermal stability of functionally graded sandwich plates using a simple shear deformation theory. Abdelaziz *et al*. (2017) studied the bending, buckling and free vibration of FGM sandwich plates with various boundary conditions using an efficient hyperbolic shear deformation theory. Bellifa *et al.* (2016) studied the bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position. Bellifa *et al.* (2017a) used a nonlocal zeroth-order shear deformation theory for nonlinear postbuckling of nanobeams. Bellifa *et al*. (2017b) used an efficient and simple four variable refined plate theory for buckling analysis of functionally graded plates. El-Haina *et al.* (2017) used a simple analytical approach for thermal buckling of thick functionally graded sandwich plates. Menasria *et al.* (2017) analyze the thermal stability of FG sandwich plates using a new and simple HSDT. Belkorissat *et al.* (2015) developed a new nonlocal refined four variable model for the vibration properties of functionally graded nano-plate. Khetir *et al.* (2017) developed a new nonlocal trigonometric shear deformation theory for thermal buckling analysis of embedded nanosize FG plates. Besseghier *et al.* (2017) studied the free vibration analysis of embedded nanosize FG plates using a new nonlocal trigonometric shear deformation theory. Mouffoki *et al.* (2017) studied the vibration analysis of nonlocal advanced nanobeams in hygro-thermal environment using a new two-unknown trigonometric shear deformation beam theory. Zidi *et al.* (2017) used a novel simple two-unknown hyperbolic shear deformation theory for functionally graded beams. Yazid *et al.* (2018) used a novel nonlocal refined plate theory for stability response of orthotropic single-layer graphene sheet resting on elastic medium. Youcef *et al.* (2018) analysis the dynamic of nanoscale beams including surface stress effects. Fang *et al.* (2018) developed the size-dependent three-dimensional free vibration of rotating functionally graded microbeams based on a modified couple stress theory. Attia *et al.* (2018) analyze the thermoelastic of FGM plates resting on variable elastic foundations using a refined four variable plate theory. Zine *et al.* (2018) developed a novel higher-order shear deformation theory for bending and free vibration analysis of isotropic and multilayered plates and shells. Abualnour *et al*. (2018) analyze the free vibration of advanced composite plates using a novel quasi-3D trigonometric plate theory. Benchohra *et al.* (2018) used a new quasi-3D sinusoidal shear deformation theory for functionally graded plates.

Bouafia *et al.* (2017) used a nonlocal quasi-3D theory for bending and free flexural vibration behaviors of functionally graded nanobeams. Lee *et al.* (2017) studied the free vibration analysis of functionally graded Bernoulli-Euler beams using an exact transfer matrix expression. Tohidi and Hosseini (2017) studied the dynamic stability of FG-CNT-reinforced viscoelastic micro cylindrical shells resting on nonhomogeneous orthotropic viscoelastic medium subjected to harmonic temperature distribution and 2D magnetic field. Altekin (2017) developed the free transverse vibration of shear deformable super-elliptical plates. Pradhan and Chakraverty (2015) studied the free vibration of functionally graded thin elliptic plates with various edge supports. Tagrara *et al.* (2015) studied the bending, buckling and vibration responses of functionally graded carbon nanotube-reinforced composite beams. Ghaitani *et al.* (2017) analyze the frequency and critical fluid velocity of pipes reinforced with FG-CNTs conveying internal flows. Tounsi *et al.* (2016) used an efficient and simple shear deformation theory for free vibration of functionally graded rectangular plates on Winkler-Pasternak elastic foundations. Kheroubi *et al.* (2016) used a new refined nonlocal beam theory accounting for effect of thickness stretching in nanoscale beams. Odunayo *et al.* (2016) analyze the dynamic of a transversely isotropic non-classical thin plate. Recently, Fahsi *et al.* (2017) used a new quasi-3D HSDT for buckling and vibration of FG plate. Mohammadimehr *et al.* (2018) studied the buckling and vibration analyses of MGSGT double-bonded micro composite sandwich SSDT plates reinforced by CNTs and BNNTs with isotropic foam & flexible transversely orthotropic cores. Dihaj *et al.* (2018) analyze the free vibration analysis of chiral double-walled carbon nanotube embedded in an elastic medium using non-local elasticity theory and Euler Bernoulli beam model.

This study deals with free vibrations analysis with stretching effect of nanocomposite beams reinforced by single-walled carbon nanotubes (SWCNTs) resting on an elastic foundation. The single-walled carbon nanotubes (SWCNTs) are aligned and distributed in polymeric matrix with different patterns of reinforcement. The material properties of the CNTRC beams are estimated by using the rule of mixture. The significant feature of this model is that, in addition to including the shear deformation effect and stretching effect it deals with only 4 unknowns without including a shear correction factor. The governing equations are derived through using Hamilton's principle and then solved by using the Navier solution. New solutions of frequencies based on the present shear deformation theory with stretching effect are presented and discussed in details. Several aspects of spring constants, thickness ratios, stretching effect, CNT volume fractions, types of CNT distribution, etc., which have considerable impact on the analytical solutions, are also investigated.

2. Material properties of CNTRC beams

The uniform distribution (UD) and functionally graded distributions (FG-V, FG-O> and FG-X) of carbon nanotubes in the thickness direction of the composite beams (z axis direction) are shown in Fig. 1. In this figure the density of CNTs within the area is constant and the volume fraction varies through the thickness of the beam. We used an embedded carbon nanotube in a polymer matrix. Thus there is no abrupt interface between the CNT and polymer matrix in the entire region of the beam. It is assumed the CNTRC beams are made of a mixture of SWCNTs and an isotropic matrix. The rule of mixture is employed to estimate the effective material properties of CNTRC beams. According to rule of mixture model the effective Young's modulus and shear modulus of CNTRC beams can be expressed as Shen (2009).

$$E_{11} = \eta_1 V_{cnt} E_{11}^{cnt} + V_p E_p \tag{1a}$$

$$\frac{\eta_2}{E_{22}} = \frac{V_{cnt}}{E_{22}^{cnt}} + \frac{V_p}{E_p} \tag{1b}$$

$$\frac{\eta_3}{G_{22}} = \frac{V_{cnt}}{G_{12}^{cnt}} + \frac{V_p}{G_p} \tag{1c}$$

where E_{11}^{cnt}; E_{22}^{cnt} and G_{12}^{cnt} are the Young's modulus and shear modulus of SWCNT, respectively and E_p and G_p are the corresponding material properties of the polymer matrix. Also, V_{cnt} and V_p are the volume fractions for carbon nanotube and the polymer matrix, respectively, with the relation of $V_{cnt} + V_P = 1$. To introduce the size-dependent material properties of SWCNT, the CNT efficiency parameters, η_i (i = 1, 2, 3), are considered. They can be obtained from matching the elastic moduli of CNTRCs estimated by the MD simulation with the numerical results determined by the rule of mixture (Han and Elliott 2007). By employing the same rule, Poisson's ratio (ν) and mass density (ρ) of the CNTRC beams are expressed as

$$\nu = V_{cnt}\nu^{cnt} + V_p \nu^p, \quad \rho = V_{cnt}\rho^{cnt} + V_p \rho^p \tag{2}$$

where ν^{cnt}, ν^p and ρ^{cnt}, ρ^p are the Poisson's ratios and densities of the CNT and polymer matrix respectively. For different patterns of carbon nanotube reinforcement distributed within the cross sections of the beams as shown in Fig. 1(b), the continuous mathematical functions employing for introducing the distributions of material constituents are expressed below

UD-Beam

$$V_{cnt} = V_{cnt}^* \tag{3a}$$

O-Beam

$$V_{cnt} = 2\left(1 - 2\frac{|z|}{h}\right)V_{cnt}^* \tag{3b}$$

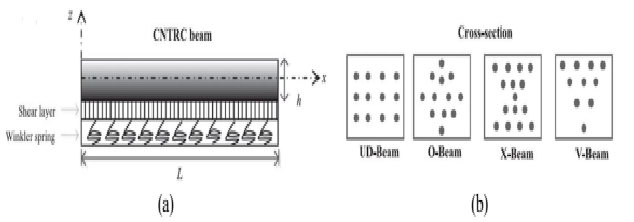

Fig. 1 Geometry of a CNTRC beam on elastic foundation (a); and cross sections of different patterns of reinforcement (b)

X-Beam

$$V_{cnt} = 4\frac{|z|}{h}V_{cnt}^* \tag{3c}$$

V-Beam

$$V_{cnt} = \left(1+2\frac{z}{h}\right)V_{cnt}^* \tag{3d}$$

where V_{cnt}^* is the considered volume fraction of CNTs, which can be determined from the following equation

$$V_{cnt}^* = \frac{W_{cnt}}{W_{cnt} + \left(\rho^{cnt}/\rho^m\right)\left(1-W_{cnt}\right)} \tag{4}$$

where W_{cnt} is the mass fraction of CNTs. From Eq. (3), it can be seen that the O-, X-and V-Beams are some types of functionally graded beams in which their material constituents are varied continuously within their thicknesses; while, the UD-Beam has uniformly distributed CNT reinforcement. In this work, the CNT efficiency parameters (η_i) associated with the considered volume fraction V_{cnt}^* are : $\eta_1 = 1.2833$ and $\eta_2 = \eta_3 = 1.0556$ for the case of $V_{cnt}^* = 0.12$; $\eta_1 = 1.3414$ and $\eta_2 = \eta_3 = 1.7101$ for the case of $V_{cnt}^* = 0.17$; $\eta_1 = 1.3228$ and $\eta_2 = \eta_3 = 1.7380$ for the case of $V_{cnt}^* = 0.28$ (Yas and Samadi 2012).

3. Equations of motion

3.1 Kinematics and constitutive equations

Consider a shear deformation beam theory, the displacement field consisting of the axial displacement, u, and the transverse displacement, w, can be written in the following forms

$$u(x,z,t) = u_0(x,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{5a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) + g(z)\varphi_z(x,t) \tag{5b}$$

where u_0 is the axial displacement, w_b and w_s are the bending and shear components of transverse displacement along the mid-plane of the beam. The additional displacement φ_z accounts for the effect of normal stress is included and $g(z)$ is given as follows

$$g(z) = 1 - f'(z) \tag{6a}$$

In this work, the shape function $f(z)$ is chosen based on a trigonometric function as (Tounsi *et al*. 2013)

$$f(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi}{h}z\right) \tag{6b}$$

Clearly, the displacement field in Eq. (5) contains only four unknowns $(u_0, w_b, w_s, \varphi_z)$. The strains associated with the displacements in Eq. (5) are

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{7a}$$

$$\varepsilon_z = g'(z)\varphi_z \tag{7b}$$

$$\gamma_{xz} = g(z)\left(\frac{\partial w_s}{\partial x} + \frac{\partial \varphi_z}{\partial x}\right) \tag{7c}$$

It can be seen from Eq. (7c) that the transverse shears strain γ_{xz} is equal to zero at the top $(z = h/2)$ and bottom $(z = -h/2)$ surfaces of the beam, thus satisfying the zero transverse shear stress conditions.

By assuming that the material of CNTRC beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x = Q_{11}(z)\,\varepsilon_x + Q_{13}(z)\,\varepsilon_z \tag{8a}$$

$$\tau_{xz} = Q_{55}(z)\,\gamma_{xz} \tag{8b}$$

$$\sigma_z = Q_{13}(z)\,\varepsilon_x + Q_{33}(z)\,\varepsilon_z \tag{8c}$$

where

$$Q_{11}(z) = Q_{33}(z) = \frac{E_{11}(z)}{1-\nu^2}$$
$$Q_{13}(z) = \nu Q_{11}(z) \text{ and } Q_{55}(z) = G_{12}(z) \tag{8d}$$

3.2 Governing equations

Hamilton's principle is employed herein to determine the equations of motion as follows

$$\int_{t_1}^{t_2}\left(\delta U+\delta U_{ef}-\delta T\right)dt=0 \tag{9}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; δU_{ef} is the potential energy of the foundation; and δT is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U &= \int_0^L\int_{-h/2}^{h/2}\left[\sigma_x\delta\varepsilon_x+\sigma_z\delta\varepsilon_z+\tau_{xz}\gamma_{xz}\right]dzdx\\ &=\int_0^L(N_x\frac{d\delta u_0}{dx}N_z\delta\varphi-M_b\frac{d^2\delta w_b}{dx^2}\\ &-M_s\frac{d^2\delta w_s}{dx^2}+Q_{xz}[\frac{d\delta w_s}{dx}+\frac{d\delta\varphi}{dx}])dx\end{aligned} \tag{10}$$

where N_x, M_x^b, M_x^s and Q_{xz} are the stress resultants defined by

$$\begin{aligned}&(N_x,M_x^b,M_x^s)=\int_{-\frac{h}{2}}^{\frac{h}{2}}(1,z,f)\,\sigma_x dz\\ &Q_{xz}=\int_{-\frac{h}{2}}^{\frac{h}{2}}g\,\tau_{xz}dz \quad\text{and}\quad R_z=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_z g'(z)dz\end{aligned} \tag{11}$$

Also, the potential energy of the foundation is expressed as

$$\delta U_{ef}=\int_0^L\left[K_w(w_b+w_s)\delta(w_b+w_s)-K_s\frac{\partial^2(w_b+w_s)}{\partial x^2}\delta(w_b+w_s)\right]dx \tag{12}$$

where K_w and K_s are the Winkler and shearing layer spring constants which can be determined from $K_w=\beta_w A_{110}/L^2$ and $K_s=\beta_s A_{110}$ in which β_w and β_s are the corresponding spring constant factors. It is also defined that A_{110} is the extension stiffness or the value of A_{11} of a homogeneous beam made of pure matrix material. If the foundation is modelled as the linear Winkler foundation, the coefficient K_s in Eq. (12) is zero.

The variation of the kinetic energy can be expressed as

$$\delta T=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)[\dot{u}\,\delta\dot{u}+\dot{w}\,\delta\dot{w}]\,dzdx \tag{13}$$

$$\begin{aligned}&=\int_0^L\begin{cases}I_0[\dot{u}_0\,\delta\dot{u}_0+(\dot{w}_b+\dot{w}_s)(\delta\dot{w}_b+\delta\dot{w}_b)]\\ +J_0[(\dot{w}_b+\dot{w}_s)\delta\dot{\varphi}+\dot{\varphi}\delta(\dot{w}_b+\dot{w}_s)]\end{cases}\\ &+J_0[(\dot{w}_b+\dot{w}_s)\delta\dot{\varphi}+\dot{\varphi}\,\delta(\delta\dot{w}_b+\delta\dot{w}_b)]\\ &-I_1(\dot{u}_0\frac{d\delta\dot{w}_b}{dx}+\frac{d\delta\dot{w}_b}{dx}\delta\dot{u}_0)+I_2(\frac{d\delta\dot{w}_b}{dx}\frac{d\delta\dot{w}_b}{dx})\\ &-J_1(\dot{u}_0\frac{d\delta\dot{w}_s}{dx}+\frac{d\delta\dot{w}_s}{dx}\delta\dot{u}_0)+K_2(\frac{d\delta\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx})\\ &\left.+J_2(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx})+K_0\,\dot{\varphi}\,\delta\dot{\varphi}\right\}dx\end{aligned}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; and (I_i, J_i, K_i) are mass inertias defined as

$$\left(I_0,I_1,I_2\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1,z,z^2\right)\rho(z)dz \tag{14a}$$

$$\left(J_0,J_1,I_2\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(g,f,zf\right)\rho(z)dz \tag{14b}$$

$$\left(K_0,K_2\right)=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(g^2,f^2\right)\rho(z)dz \tag{14c}$$

Substituting the expressions for δU, δU_{ef}, and δT from Eqs. (10), (12), (24), and (13) into Eq. (9) and integrating by parts, and collecting the coefficients of δu_0, δw_b, δw_s and $\delta\varphi_z$, the following equations of motion of the FG beam are obtained

$$\delta u_0:\frac{\partial N_x}{\partial x}=I_0\frac{d^2u_0}{dt^2}-I_1\frac{d^3w_b}{dxdt^2}-J_1\frac{d^3w_s}{dxdt^2} \tag{15a}$$

$$\begin{aligned}&\delta w_b:\frac{\partial^2M_x^b}{\partial x^2}+K_s\left(\frac{\partial^2(w_b+w_s)}{\partial x^2}\right)-K_w(w_b+w_s)\\ &=I_0\left(\frac{\partial^2w_b}{\partial t^2}+\frac{\partial^2w_s}{\partial t^2}\right)+J_0\frac{\partial^2\varphi_z}{\partial t^2}+I_1\frac{\partial^3u_0}{\partial xdt^2}-I_2\frac{\partial^4w_b}{\partial x^2dt^2}-J_2\frac{\partial^4w_s}{\partial x^2\partial t^2}\end{aligned} \tag{15b}$$

$$\delta w_s : \frac{\partial^2 M_x^s}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + K_s\left(\frac{\partial^2 (w_b + w_s)}{\partial x^2}\right) - K_w (w_b + w_s)$$
$$I_0\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + J_0 \frac{\partial^2 \varphi_z}{\partial t^2} + J_1 \frac{\partial^3 u_0}{\partial t^2} - J_2 \frac{\partial^4 w_b}{\partial x^2 \partial t^2} - K_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2} \tag{15c}$$

$$\delta \varphi_z : -N_z + \frac{\partial Q_{xz}}{\partial x} = J_0\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + K_0 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{15d}$$

Eq. (12) can be expressed in terms of displacements (u_0, w_b, w_s, φ_z) by using Eqs. (5), (7), (8) and (11) as follows

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} + X_{13}\frac{\partial \varphi_z}{\partial x} = I_0 \frac{d^2 u_0}{dt^2} - I_1 \frac{d^3 w_b}{dx dt^2} - J_1 \frac{d^3 w_s}{dx dt^2} \tag{16a}$$

$$B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + Y_{13}\frac{\partial^2 \varphi_z}{\partial x^2} + K_s\left(\frac{\partial^2 (w_b + w_s)}{\partial x^2}\right)$$
$$-K_w (w_b + w_s) = I_0\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + J_0 \frac{\partial^2 \varphi_z}{\partial t^2} + I_1 \frac{\partial^3 u_0}{\partial x d t^2} \tag{16b}$$
$$-I_2 \frac{\partial^4 w_b}{\partial x^2 dt^2} - J_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2}$$

$$B_{11}^s \frac{\partial^3 u_0}{\partial x^3} - D_{11}^s \frac{\partial^4 w_b}{\partial x^4} - H_{11}\frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + \left(Y_{13}^s + A_{55}^s\right)\frac{\partial^2 \varphi_z}{\partial x^2}$$
$$+K_s\left(\frac{\partial^2 (w_b + w_s)}{\partial x^2}\right) - K_w (w_b + w_s) = I_0\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + J_0 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{16c}$$
$$+J_1 \frac{\partial^3 u_0}{\partial t^2} - J_2 \frac{\partial^4 w_b}{\partial x^2 \partial t^2} - K_2 \frac{\partial^4 w_s}{\partial x^2 \partial t^2}$$

$$-X_{13}\frac{\partial u_0}{\partial x} + Y_{13}\frac{\partial^2 w_b}{\partial x^2} + \left(Y_{13}^s + A_{55}^s\right)\frac{\partial^2 w_s}{\partial x^2} + A_{55}^s \frac{\partial^2 \varphi_z}{\partial x^2} - Z_{33}\varphi_z$$
$$= J_0\left(\frac{\partial^2 w_b}{\partial t^2} + \frac{\partial^2 w_s}{\partial t^2}\right) + K_0 \frac{\partial^2 \varphi_z}{\partial t^2} \tag{16d}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$A_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} dz,\ B_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z dz,$$
$$B_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} f dz,\quad X_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} g' dz, \tag{17a}$$

$$D_{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z^2 dz,\ D_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} z.f dz,$$
$$Y_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13} z.g' dz,\ H_{11}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11} f^2 dz, \tag{17b}$$

$$Y_{13}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}.f.g'.dz,\ Z_{33} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{13}\left[g'\right]^2 dz,$$
$$A_{55}^s = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{55} g^2 dz, \tag{17c}$$

4. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The boundary conditions for simply supported and clamped edge condition are

$$w_b = w_s = \varphi_z = 0 \ \text{ at } \ x = 0, L \tag{18}$$

The following representation for the displacement quantities, that satisfy the above boundary conditions, is appropriate in the case of our problem

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \\ \varphi_z \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ w_{bm} \sin(\lambda x) e^{i\omega t} \\ w_{sm} \sin(\lambda x) e^{i\omega t} \\ w_{zm} \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \tag{19}$$

where U_m , W_{bm} , W_{sm} and W_{zm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, and $\lambda = m\pi / L$.

Substituting the expressions of u_0 , w_b , w_s , φ_z from Eq. (19) into the equations of motion of Eq. (16), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{12} & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}\right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \\ W_{zm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{20}$$

where

$$\begin{aligned} &a_{11} = A_{11}\lambda^2, a_{12} = -B_{11}\lambda^3, a_{13} = -B_{11}^s \lambda^3, \\ &a_{14} = -X_{13}\lambda, a_{22} = D_{11}\lambda^4 + K_w + K_s \lambda^2, \\ &a_{23} = D_{11}^s \lambda^4 + K_w + K_s \lambda^2, a_{24} = Y_{13}\lambda^2, \\ &a_{33} = H_{11}^s \lambda^4 + A_{55}^s \lambda^2 + K_w + K_s \lambda^2, \\ &a_{34} = Y_{13}^s \lambda^2 + A_{55}^s \lambda^2, a_{44} = A_{55}^s \lambda^2 + Z_{33} \end{aligned} \tag{21}$$

and

$$\begin{aligned} &m_{11} = I_0, m_{12} = -I_1\lambda, m_{13} = J_1\lambda, m_{14} = 0, m_{22} = I_0 + I_2\lambda^2, m_{23} = I_0 + J_2\lambda^2, \\ &m_{24} = J_0, m_{33} = I_0 + K_2\lambda^2, m_{34} = J_0, m_{44} = K_0 \end{aligned} \tag{22}$$

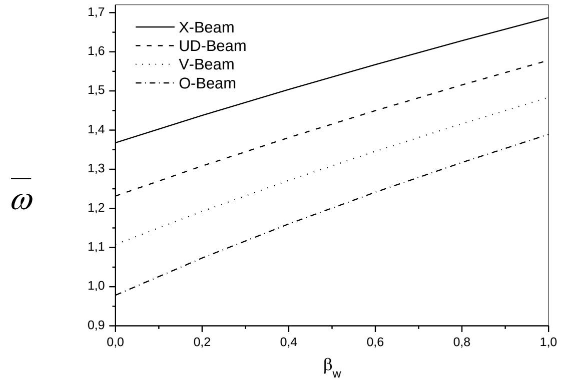

Fig. 2 Effect of Winkler modulus parameter on the fundamental frequencies of CNTRC beams ($L/h=10$; $\beta_s=0; V^*_{cnt}=0.12$)

5. Results and discussion

In this section, numerical results of vibration behavior of CNTRC beams are presented and discussed. The effective material characteristics of CNTRC beams at ambient temperature employed throughout this work are given as follows. Poly methyl methacrylate (PMMA) is utilized as the matrix and its material properties are: $\nu^p=0.3$; $\rho^p=1190kg/m^3$ and $E^p=2.5GPa$. For reinforcement material, the armchair (10, 10) SWCNTs is chosen with the following properties (Tagrara *et al.* 2015, Yas and Samadi 2012): $\nu^{cnt}=0.19$; $\rho^{cnt}=1400kg/m^3$; $E_{11}^{cnt}=600GPa$; $E_{22}^{cnt}=10GPa$ and $G_{12}^{cnt}=17.2GPa$.

For convenience, the following nondimensionalization is employed

$$\bar{\omega}=\omega L\sqrt{\frac{I_{00}}{A_{110}}} \tag{23}$$

where A_{110} and I_{00} are A_{11} and I_0 of beam made of pure matrix material, respectively.

5.1 Results for vibration analysis of CRTRC beams

In order to prove the validity of the present formulation in the case of vibration analysis with $\varepsilon_z\neq 0$, the computed frequencies of CNTRC beams are numerically compared with those of Wattanasakulpong and Ungbhakorn (2013), Yas and Samadi (2012) and Tagrara *et al.* (2015) in Table 1. It can be observed that our results with $(\varepsilon_z\neq 0)$ are in an excellent agreement to those predicted using the higher order shear deformation theory of Wattanasakulpong and Ungbhakorn (2013) and Tagrara *et al.* (2015) with

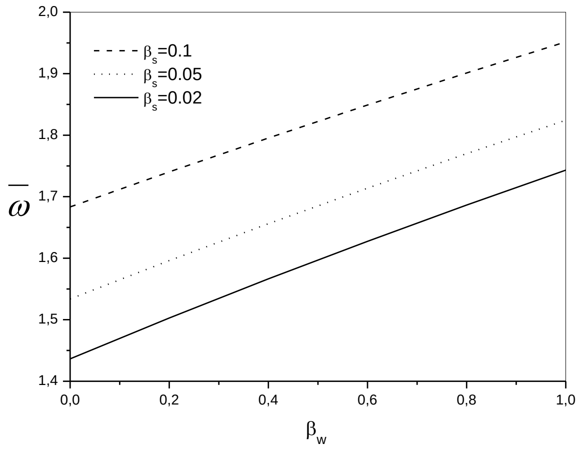

Fig. 3 Dimensionless fundamental frequencies of X-Beam on elastic foundation with various spring constant factors ($L/h=10; V^*_{cnt}=0.12$)

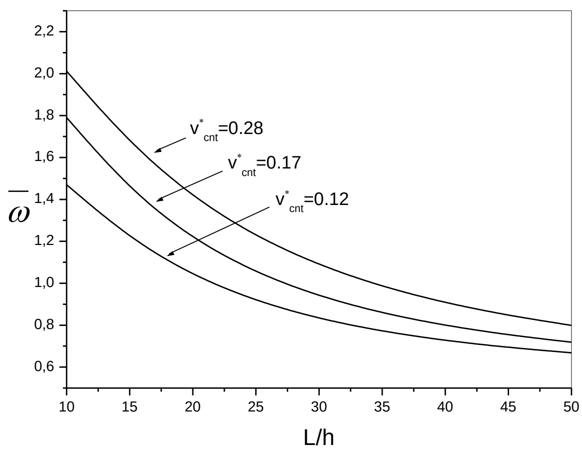

Fig. 4 Dimensionless fundamental frequencies of X Beam on elastic foundation with various thickness ratios ($\beta_w=0.1; \beta_s=0.02$)

Table 1 Comparison of fundamental frequencies for CNTRC beam with and without elastic foundation ($L/h=15$, $V^*_{cnt}=0.12$).

Source	$\beta_w=0$, $\beta_s=0$				$\beta_w=0.1$, $\beta_s=0.02$			
	UD	O	X	V	UD	O	X	V
FSDBT (Wattanasakulpong and Ungbhakorn 2013)	0.9976	0.7628	1.1485	0.8592	1.1339	0.9339	1.2688	1.0142
TSDBT (Wattanasakulpong and Ungbhakorn 2013)	0.9749	0.7446	1.1163	0.8443	1.1140	0.9192	1.2397	1.0016
Yas and Samadi (2012)	0.9753	0.7527	1.1150	0.9453	1.1144	0.9258	1.2386	1.0883
Tagrara *et al.* (2015)	0.9749	0.7446	1.1163	0.8442	1.1140	0.9192	1.2397	1.0015
Present ($\varepsilon_z\neq 0$)	0.9458	0.7189	1.0882	0.8258	1.0885	0.8982	1.2144	0.9859

$(\varepsilon_z = 0)$. However, the small difference found between the results is due to that the theories presented by Wattanasakulpong and Ungbhakorn (2013) and Tagrara *et al.* (2015) ignore the thickness stretching effect.

Fig. 2 illustrate the variation of the fundamental frequency parameter ($\bar{\omega}$) with stretching effect of different types of CNTRC beams with Winkler modulus parameter. It can be deduced from Fig. 2 that frequency of the X-Beam are higher than those of beams with other CNTs distributions. The effects of Pasternak shear modulus and CNT volume fractions on frequency parameter of the X-Beam are showns in Figs. 3 and 4, respectively. It can be deduced from Fig. 3 that the frequencies increase almost linearly as the increase of the spring constant factors. It is seen that frequencies increase linearly as the spring constant factors increase. In addition, the increase of CNT volume fractions conducts to an increase of frequencies. It is seen that the increase of thickness ratios leads to a decrease of frequencies, especially in the range of L/h = 10 to 30.

6. Conclusions

In this paper, based on the refined beam theory, the free vibration with stretching effect of nanocomposite beams reinforced by single-walled carbon nanotubes resting on an elastic foundation have been studied. The equations of motion have been determined through the Hamilton's principle. The material properties have been estimated though the rule of mixture. The numerical results reveal that the distribution of CNT, foundation stiffness, stretching effect and volume fraction of CNT have significant effects on the natural frequencies of the CNTRC beams. The obtained results show the beams with FG-X distribution have higher fundamental frequency in comparison with other distributions. Increase in the spring constant factors of the elastic foundation results in increase of the frequency of the beam. The frequencies increase almost linearly as the increase of the spring constant factors. It is seen that frequencies increase linearly as the spring constant factors increase. In addition, the increase of CNT volume fractions conducts to an increase of frequencies. For the effect of thickness ratio, it is revealed that the frequencies of CNTRC beams decrease as the increase of the thickness ratios.

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Nomenclature

E_{11}^{cnt}, E_{22}^{cnt}, E^{p}	elastic moduli for CNT and matrix
E_{11}, E_{22}	elastic moduli for a nanocomposite
G_{12}^{cnt}, G^{p}	shear modulus for carbon nanotube and matrix
V_{cnt} , V_{p}	volume fractions of carbon nanotube and matrix
W_{cnt}	mass fraction of carbon nanotube
η_i	carbon nanotube efficiency parameters
u , w	axial displacement and transverse deflection
u_0 , w_b w_s	axial displacement, the bending and shear components of transverse displacement along the mid-plane of the beam
L , b h	length, width and height of the beam
σ_x, τ_{xz}	normal stress and shear stress

$A_{11}, B_{11}, D_{11}, A_{55}$	beam stiffness components.
$I_0, I_1, I_2, J_0, J_1, J_2, K_0, K_2$	inertia term
$K_w, K_s, \beta_w, \beta_s$	Winkler and shearing layer spring constants and the corresponding spring constant factors
$\bar{\omega}$	dimensionless natural frequency