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Structural reliability assessment using an enhanced adaptive Kriging method

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Abstract. Reliability assessment of complex structures using simulation methods is time-consuming. Thus, surrogate models are usually employed to reduce computational cost. AK-MCS is a surrogate-based Active learning method combining Kriging and Monte-Carlo Simulation for structural reliability analysis. This paper proposes three modifications of the AK-MCS method to reduce the number of calls to the performance function. The first modification is related to the definition of an initial Design of Experiments (DoE). In the original AK-MCS method, an initial DoE is created by a random selection of samples among the Monte Carlo population. Therefore, samples in the failure region have fewer chances to be selected, because a small number of samples are usually located in the failure region compared to the safe region. The proposed method in this paper is based on a uniform selection of samples in the predefined domain, so more samples may be selected from the failure region. Another important parameter in the AK-MCS method is the size of the initial DoE. The algorithm may not predict the exact limit state surface with an insufficient number of initial samples. Thus, the second modification of the AK-MCS method is proposed to overcome this problem. The third modification is relevant to the type of regression trend in the AK-MCS method. The original AK-MCS method uses an ordinary Kriging model, so the regression part of Kriging model is an unknown constant value. In this paper, the effect of regression trend in the AK-MCS method is investigated for a benchmark problem, and it is shown that the appropriate choice of regression type could reduce the number of calls to the performance function. A stepwise approach is also presented to select a suitable trend of the Kriging model. The numerical results show the effectiveness of the proposed modifications.

Keywords: structural reliability; surrogate model; Kriging; Monte-Carlo simulation; design of experiments

1. Introduction

The main purpose of reliability analysis is to compute failure probabilities for incorporating uncertainties in real engineering problems. The probability of failure is defined as (Melchers 1999)

$$\mathrm{P_f} = \mathrm{P}(\mathrm{G}(\boldsymbol{X}) \le 0) = \int_{\mathrm{G(X)} \le 0} \mathrm{f}_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{1}$$

where $\boldsymbol{X}$ is a vector of random variables; $\mathrm{f}_{\boldsymbol{X}}$ denotes the joint probability density function, and $\mathrm{G}(\boldsymbol{X})$ is the performance function defined such that $\mathrm{G}(\boldsymbol{X}) \le 0$ denotes the failure domain and $\mathrm{G}(\boldsymbol{X}) = 0$ is called limit state surface. Direct computation of the above integral is difficult for real problems, so various methods have been proposed to approximate the integral. Among these methods, FORM and SORM use an approximation of the limit state surface around the "most probable failure point (MPFP)" based on a Taylor series expansion (Madsen *et al.* 1986, Zhao and Ono 2001). These analytical methods require calculation of the gradient of the limit state function, so they are not appropriate for highly nonlinear and non-differentiable functions.

Monte Carlo Simulation (MCS) is another method for approximating probabilities of failure (Melchers 1999). This method is based on the random sampling according to probability distribution functions. After the evaluation of the performance function for each sample, the probability of failure can be predicted by the ratio of failed samples to the total number of samples as

$$\hat{P}_f = \frac{n_f}{N_{MC}} = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} I_{G \le 0}(\boldsymbol{x}_{MC,i}) \tag{2}$$

where n_f is the number of failure samples, N_{MC} is the total number of random samples, $\boldsymbol{x}_{MC,i}$ $(i = 1,2,\dots,N_{MC})$ is a sequence of random samples drawn from the distribution of $\mathrm{f}_{\boldsymbol{X}}$, and $\mathrm{I}_{\mathrm{G} \le 0}$ is equal to 1 or 0 for samples located inside and outside the failure domain, respectively. The accuracy of estimation can be measured by the coefficient of variation as (Lemaire 2009)

$$C.O.V_{\hat{P}_f} = \sqrt{\frac{1 - \hat{P}_f}{N_{MC}.\hat{P}_f}} \tag{3}$$

The required number of samples to obtain a target coefficient of variation in Eq. (3) rapidly increases with decreasing failure probability. Thus, the main drawback of the Crude Monte Carlo simulation is significant numerical efforts to estimate failure probabilities with a good accuracy. To deal with such a drawback, several variance reduction techniques have been proposed in the literature such as importance sampling (Engelund and Rackwitz 1993), directional simulation (Melchers 1994) or subset

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simulation (Au and Beck 2001). Despite these improvements, Monte Carlo simulations are time-consuming. Therefore, the methods above may not be practical for an expensive-to-evaluate performance function. For example, if a finite element analysis (FEA) is required to evaluate the performance function, these methods may not be appropriate. In such a case, using an approximate or surrogate model instead of the main performance function is desirable. Several methods for approximation of the performance function have been presented such as RSM (Rajashekhar and Ellingwood 1993, Dizangian and Ghasemi 2016, Fang and Tee 2017), ANN (Cardoso *et al.* 2008, Bucher and Most 2008), SVM (Hurtado and Alvarez 2003, Basudhar and Missoum 2008) and Kriging (Kaymaz 2005, Bichon *et al.* 2008, Echard *et al.* 2011, Schöbi *et al.* 2016).

Among these methods, the response surface methodology (RSM) is widely used in structural reliability analysis. In this method, the first or second order polynomials are used to approximate the real performance function. Artificial Neural Network (ANN) is another method for function approximation, and several researchers have used this method for structural reliability assessment. Cardoso *et al.* (2008) have shown that ANN is a versatile method that can approximate non-linear functions. Comparing ANN with other surrogate methods shows that the accuracy of the various approaches depends on specific problems (Bucher and Most 2008). Another surrogate model is Support Vector Machine (SVM), which is widely used for classification and regression applications (Vapnik 1998). The main feature of SVM is its ability to separate data samples with the largest margin (Hurtado and Alvarez 2003). Kriging model or Gaussian process regression is another popular surrogate method due to its flexibility for highly nonlinear problems. This model was originally developed in geostatistics by Daniel Krige (Krige 1951) and then became very popular for fitting surrogates to expensive computer simulations (Jones *et al.* 1998).

Surrogate models are built from a limited number of evaluations of the original computational model, which is called the design of experiments (DoE). The appropriate selection of the DoE is so important to build an accurate surrogate (Sacks *et al.* 1989). The region of failure is usually relatively small compared to the domain of the whole variable, and a small number of samples may be located in the region of interest. Therefore, iterative training processes or adaptive methods have been proposed to improve the training process of surrogate models with a lower number of samples (Shao and Murotsu 1997, Hurtado and Alvarez 2003, Schueremans and Van Gemert 2005, Basudhar and Missoum 2008, Bourinet *et al.* 2011). Shao and Murotsu have presented an active learning algorithm, where the ANN starts training with a limited number of samples, and more samples are added only in the most important region. Schueremans and Van Gemert have used an adaptive ANN method that refined during the reliability analysis. An adaptive SVM method has also been proposed for the structural reliability assessment (Hurtado and Alvarez 2003), in which the margin is formed with initial random samples, whereas next training samples are selected near the margin.

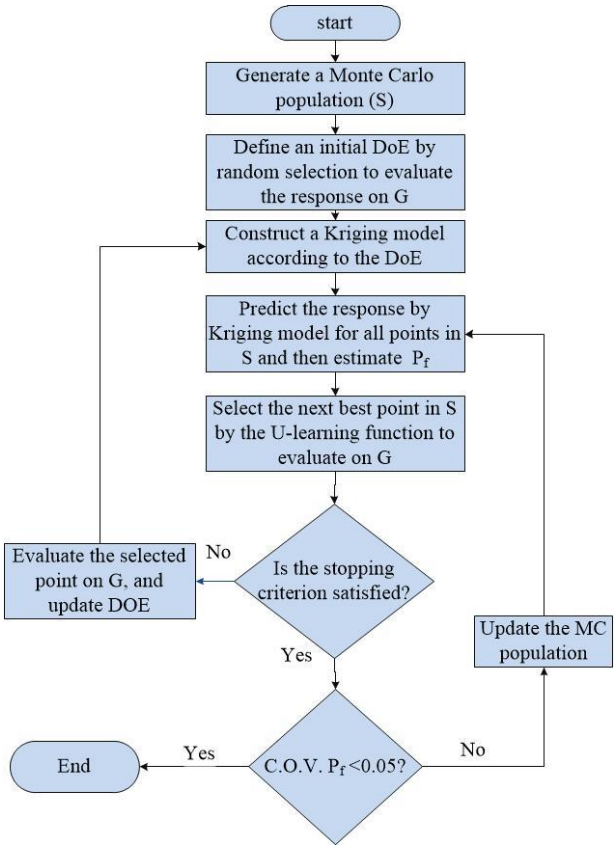
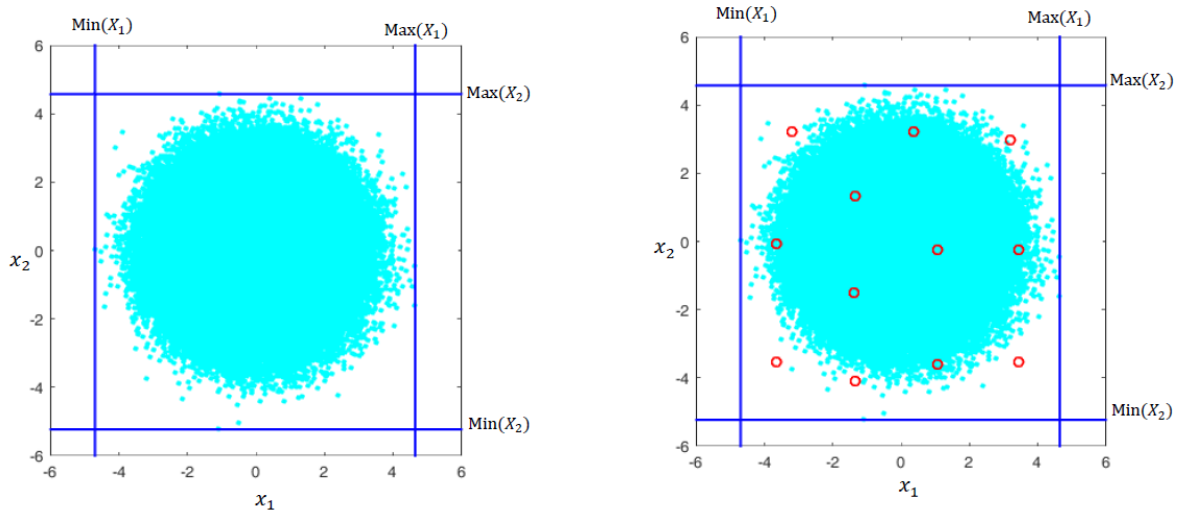


Fig. 1 Flowchart of the AK-MCS method

Recently, some researchers have presented adaptive DoE methods with Kriging surrogate model (Bichon *et al.* 2008, Echard *et al.* 2011, Lv *et al.* 2015, Schöbi *et al.* 2016, Sun *et al.* 2017, Gaspar *et al.* 2017, Lelièvre *et al.* 2018). The unique property of Kriging is its ability to determine the variance of predictions. This property allows developing adaptive algorithms based on the variance of Kriging model. Bichon *et al.* (2008) have proposed an Efficient Global Reliability Analysis (EGRA), which approximates limit state functions with good accuracy. This method begins with a Kriging model built with a small number of samples and then chooses where to generate subsequent samples with an expected feasibility function (EFF). Echard *et al.* (2011) have presented an Active learning method that combines Kriging and Monte Carlo Simulation (AK-MCS) method to evaluate the reliability of structures. Active learning means that the Kriging model is updated by adding a new point to the DoE. Schöbi *et al.* (2016) have used an adaptive Polynomial-Chaos Kriging method to assess the probability of failure and showed the efficiency of the proposed PC-Kriging method for rare events.

The AK-MCS is a powerful method for structural reliability assessment, in which the number of calls to performance function is usually less in comparison with other traditional methods. This paper presents three modifications of the AK-MCS method to reduce the number of function calls. These modifications include: a method for defining an initial DoE, a new strategy for adding samples in the case of an insufficient size of initial DoE, and finally a stepwise regression trend for the AK-MCS method.



(a) Determine upper and lower values of variables in the MC population

(b) Uniform sampling in the specified domain

Fig. 2 Two steps of uniform sampling of DoE (empty red circles mark the initial DoE)

The remainder of this paper is organized as follows: Section 2 gives a brief overview of the Kriging model and AK-MCS method for structural reliability analysis. Section 3 presents the proposed modifications to improve the AK-MCS method. Four examples are employed in Section 4 to illustrate the efficiency of proposed modifications. Section 5 is the conclusion.

2. Structural reliability assessment using Kriging meta-model

2.1 Kriging model

Consider a design of experiments $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$ with $\boldsymbol{x}_i \in \mathbb{R}^n$, and $\mathbf{Y}$ with $Y_i = G(\boldsymbol{x}_i) \in \mathbb{R}$ to build a Kriging surrogate model. Kriging estimates the value of a function as a combination of known functions $\mathbf{f}(\boldsymbol{x})$ and departures as (Jones *et al.* 1998)

$$\mathrm{G}(\boldsymbol{x}) = \mathbf{f}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\beta} + \mathrm{Z}(\boldsymbol{x}) \tag{4}$$

where $\mathbf{f}(\boldsymbol{x}) = [\mathrm{f}_1(\boldsymbol{x}), \ldots, \mathrm{f}_p(\boldsymbol{x})]^{\mathrm{T}}$ represents the basic functions, $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_p]^{\mathrm{T}}$ is the vector of regression coefficients, and $\mathrm{Z}(\boldsymbol{x})$ denotes a Gaussian process with zero mean and covariance model defined by

$$\mathrm{cov}(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{j}) = \sigma^2 \mathrm{R}(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{j}) \quad \mathrm{i,j} = 1, \ldots, \mathrm{k} \tag{5}$$

where σ^2 is the process variance and $\mathrm{R}(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{j})$ denotes the correlation function between the points $\boldsymbol{x}_\mathrm{i}$ and $\boldsymbol{x}_\mathrm{j}$.

The most widely used correlation model is Gaussian correlation function, which can be expressed as

$$\mathrm{R}(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{j}) = \prod_{\mathrm{l}=1}^{\mathrm{n}} \exp(-\theta_\mathrm{l}(x_{\mathrm{i,l}} - x_{\mathrm{j,l}})^2) \tag{6}$$

where $x_{\mathrm{i,l}}$ and $x_{\mathrm{j,l}}$ are the ^{l}th coordinates of the points $\boldsymbol{x}_\mathrm{i}$ and $\boldsymbol{x}_\mathrm{j}$, and θ_l is the correlation parameter, which is determined using the Maximum Likelihood method as

$$\mathrm{L}(\theta) = (2\pi\sigma^2)^{-\frac{\mathrm{k}}{2}}|\mathbf{R}|^{-\frac{1}{2}}\exp(-\frac{\mathrm{k}}{2}) \tag{7}$$

where $\mathrm{R_{ij}} = \mathrm{R}(\boldsymbol{x}_\mathrm{i}, \boldsymbol{x}_\mathrm{j})$, $\mathrm{i,j} = 1, \ldots, \mathrm{k}$ is the matrix of correlation between each pair of points in the design of experiments.

Regression coefficients and the variance of the process are determined using the least squares method as (Sacks *et al.* 1989)

$$\boldsymbol{\beta} = (\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{F})^{-1}\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{Y} \tag{8}$$

$$\sigma^2 = \frac{1}{k}(\boldsymbol{Y} - \boldsymbol{F\beta})^T\boldsymbol{R}^{-1}(\boldsymbol{Y} - \boldsymbol{F\beta}) \tag{9}$$

where $\mathrm{F_{ij}} = \mathrm{f_j(x_i)}$, $\mathrm{i} = 1, \ldots, \mathrm{k}$, $\mathrm{j} = 1, \ldots, p$ is the information matrix.

The mean of prediction $\mu_{\hat{G}}(\boldsymbol{x}_0)$ and its variance $\sigma^2_{\hat{G}}(\boldsymbol{x}_0)$ for an arbitrary input sample point $\boldsymbol{x}_0$ is expressed as

$$\mu_{\hat{G}}(\boldsymbol{x}_0) = \mathbf{f}(\mathbf{x}_0)^{\mathrm{T}}\boldsymbol{\beta} + \mathbf{r}_0^{\mathrm{T}}(\mathbf{x}_0)\mathbf{R}^{-1}(\mathbf{Y} - \mathbf{F}\boldsymbol{\beta}) \tag{10}$$

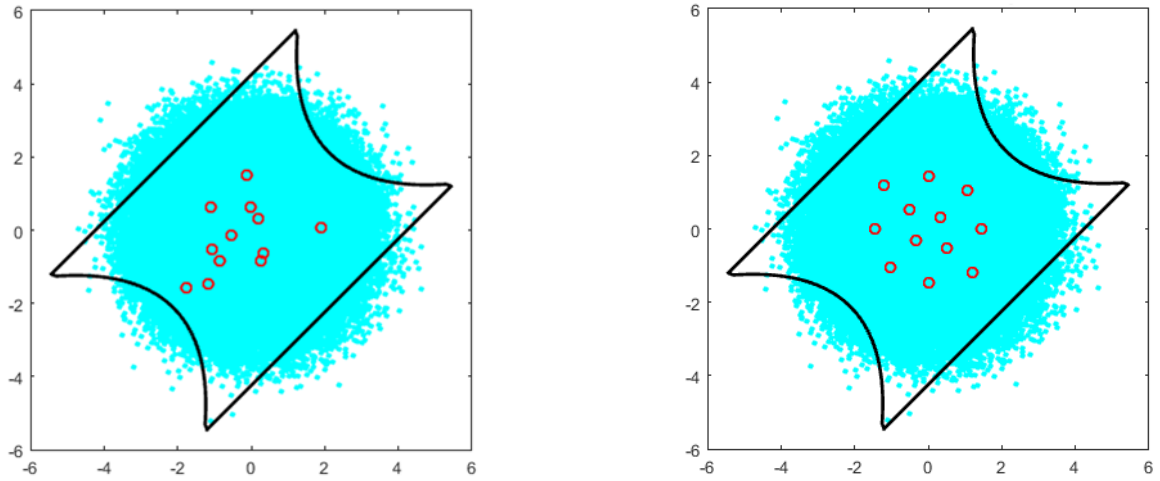
$$\sigma^2_{\hat{G}}(\boldsymbol{x}_0) = \sigma^2[1 - \mathbf{r}_0^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{r}_0 + \mathbf{u}^{\mathrm{T}}(\mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{F})^{-1}\mathbf{u}] \tag{11}$$

where $\mathbf{u} = \mathbf{F}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{r}_0 - \mathbf{f}(\boldsymbol{x}_0)$ and the vector of $\mathbf{r}_0 = [\mathrm{R}(\boldsymbol{x}_0, \boldsymbol{x}_1), \ldots, \mathrm{R}(\boldsymbol{x}_0, \boldsymbol{x}_\mathrm{k})]^{\mathrm{T}}$ represents the correlation between the new sample point $\boldsymbol{x}_0$ and the experimental design points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$.

The implementation of the Kriging meta-modeling technique can be found in Matlab toolbox DACE (Lophaven *et al.* 2002), which is used in this research to construct Kriging models.

2.2 AK-MCS method

The AK-MCS method includes Monte Carlo simulation and adaptive Kriging meta-model to evaluate the reliability of structures (Echard *et al.* 2011). It consists of 10 stages:



(a) Random selection method (b) K-means clustering method

Fig. 3 Defining of initial DoE by selection from the MC population for the four-branch benchmark problem

1) Generate a Monte Carlo population (S) in the design space.
2) Define an initial design of experiments (DoE) by random selection of N1 points among the population (S), and evaluate these points on the performance function (G).
3) Construct a Kriging model with the toolbox of DACE according to the DoE and corresponding function values using ordinary Kriging.
4) Predict the MC population (S) by the Kriging model and then estimate P_f.
5) Identify the best next training sample in the S by a learning function.
6) Check the stopping criterion.
7) If the stopping criterion in step 6 is not satisfied, call the performance function (G) to evaluate the best sample. Then update the previous DoE with the best sample, and repeat stages 3 to 7 until the stopping criterion is satisfied.
8) Compute the coefficient of variation $(C.O.V)$ of P_f by Eq. (3).
9) If $C.O.V_{P_f} > 0.05$, update the MC population by adding more samples to the current population and go back to stage 4.
10) End AK-MCS.

Fig. 1 summarizes these steps as a flowchart. The learning function in stage 5 is defined using the prediction value $\mu_{\hat{G}}(\boldsymbol{x})$ and the variance of prediction $\sigma^2_{\hat{G}}(\boldsymbol{x})$. To select a sample with high variance of prediction and near the limit state surface ($\mu_{\hat{G}}(\boldsymbol{x}) \approx 0$), the learning function $U(\boldsymbol{x})$ can be defined as (Echard *et al.* 2011)

$$U(\boldsymbol{x}) = |\mu_{\hat{G}}(\boldsymbol{x})|/\sigma_{\hat{G}}(\boldsymbol{x}) \tag{12}$$

The U-learning function is computed for the MC population. Then a sample with the minimum value of U-function is considered as the best sample to enrich the DoE. The prediction of the Kriging model for each point $\boldsymbol{x}_0$ has a normal distribution with the mean value of $\mu_{\hat{G}}(\boldsymbol{x}_0)$ and variance of $\sigma^2_{\hat{G}}(\boldsymbol{x}_0)$. So $U(\boldsymbol{x})$ is equivalent to the reliability index corresponding to the probability of misclassification. Based on this property, Echard *et al.* (2011) have defined the stopping criterion (stage 6) as

$$\min[U(\boldsymbol{x})] \geq 2 \qquad \forall \boldsymbol{x} \epsilon S \tag{13}$$

So, the probability of misclassification must be smaller than $\Phi(-2) \approx 2\%$ for all samples to stop iterations.

Schöbi *et al.* (2016) have shown this stopping criterion is conservative and proposed another criterion. Three failure boundaries were used to define the stopping criterion:

1) The limit state surface using mean values of predictions ($\mu_{\hat{G}}(x) = 0$)
2) The lower boundary of the limit state surface using $\mu_{\hat{G}}(\boldsymbol{x}) - k\sigma_{\hat{G}}(\boldsymbol{x}) = 0$
3) The upper boundary of the limit state surface using $\mu_{\hat{G}}(\boldsymbol{x}) + k\sigma_{\hat{G}}(\boldsymbol{x}) = 0$

where k defines the confidence level, typically equal to $1.96 = \Phi^{-1}(97.5\%)$. So, the stopping criterion can be defined as (Schöbi *et al.* 2016)

$$\frac{\hat{P}_f^+ - \hat{P}_f^-}{\hat{P}_f^0} \leq \epsilon_{\hat{P}_f} \tag{14}$$

where $\hat{P}_f^+$ and $\hat{P}_f^-$ are the upper and lower bounds of failure probabilities, respectively; $\hat{P}_f^0$ is failure probability using the mean values of predictions $\mu_{\hat{G}}(\boldsymbol{x})$, and $\epsilon_{\hat{P}_f} = 5\%$ is the acceptable threshold error. The bounds of failure probabilities were defined as

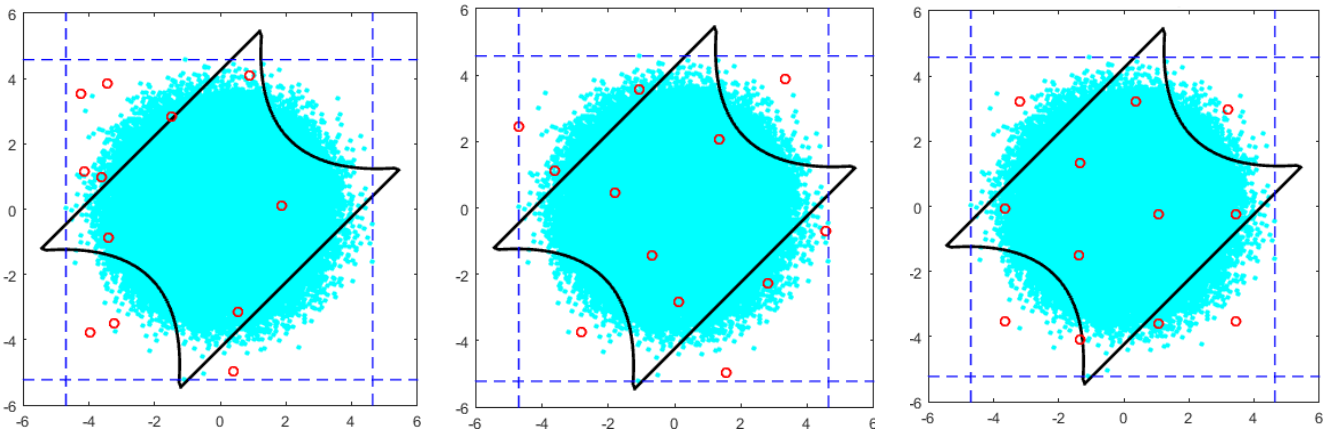
$$\hat{P}_f^{\pm} = P[\mu_{\hat{G}}(\boldsymbol{x}) \mp k\sigma_{\hat{G}}(\boldsymbol{x}) \leq 0] \tag{15}$$

and failure probability as

$$\hat{P}_f^0 = P[\mu_{\hat{G}}(\boldsymbol{x}) \leq 0] \tag{16}$$

They have shown that the stopping criterion in Eq. (14) leads to accurate results despite the smaller number of function calls when compared to the stopping criterion in Eq. (13).

In the AK-MCS method, two types of error in prediction



(a) URS method (b) LHS method (c) CVT method

Fig. 4 Defining of the initial DoE by uniform sampling in the defined domain for the four-branch benchmark problem

of failure probability may occur:

1) Error due to an insufficient number of MC samples: To control this error, the coefficient of variation of the prediction $(C.O.V_{P_f})$ should be less than 0.05.

2) The error of misclassification: To control this error, the stopping criterion of Eq. (13) or Eq. (14) are used. As mentioned before, according to Eq. (13), the probability of misclassification must be smaller than $\Phi(-2) \approx 2\%$ for all samples. Also, $\epsilon_{\hat{P}_f} = 5\%$ is used in Eq. (14) as a stopping criterion.

Therefore, it is not expected that the AK-MCS method will give an exact probability of failure.

3. Evaluation of effective parameters in the AK-MCS method

This section evaluates three effective parameters in the AK-MCS method. These parameters include the following: 1) method of defining an initial DoE, 2) size of the initial DoE, and 3) regression trend of the Kriging model.

3.1 Method of defining the initial DoE

The method of defining the initial DoE is an important factor in the AK-MCS method. Therefore, the conventional and proposed methods are introduced and compared by attempting a sample problem.

3.1.1 The conventional method

In the original AK-MCS method, the initial DoE is defined by a random selection of points among the MC population (Echard *et al.* 2011). The probability of choosing DoE points from the failure region is small by the random selection method because small portions of samples may be located in the failure region.

Schöbi *et al.* (2016) have used another method to generate an initial DoE. Their method is based on the Latin Hypercube Sampling (LHS) algorithm in the unit hypercube and transforming into the variable space by an inverse cumulative distribution function. Because of using an inverse cumulative distribution function, most of the defined samples may be located near the mean value and are likely to be in the safe region.

Another strategy to have a well-behaved initial DoE for selection of samples from the MC population is K-means clustering method (Hartigan and Wong 1979). In this case, centers of clusters are considered as an initial DoE. Centers of clusters will be near to the mean of random variables because of the high density of samples near the mean value. Therefore, samples in the failure region may not be selected as centers of clusters.

In all the methods above, the AK-MCS algorithm may start with the samples located in the safe region. So, the predicted probability of failure may not be accurate enough in the early iterations.

3.1.2 Proposed method for defining the initial DoE

The proposed method for defining an initial DoE consists of two stages:

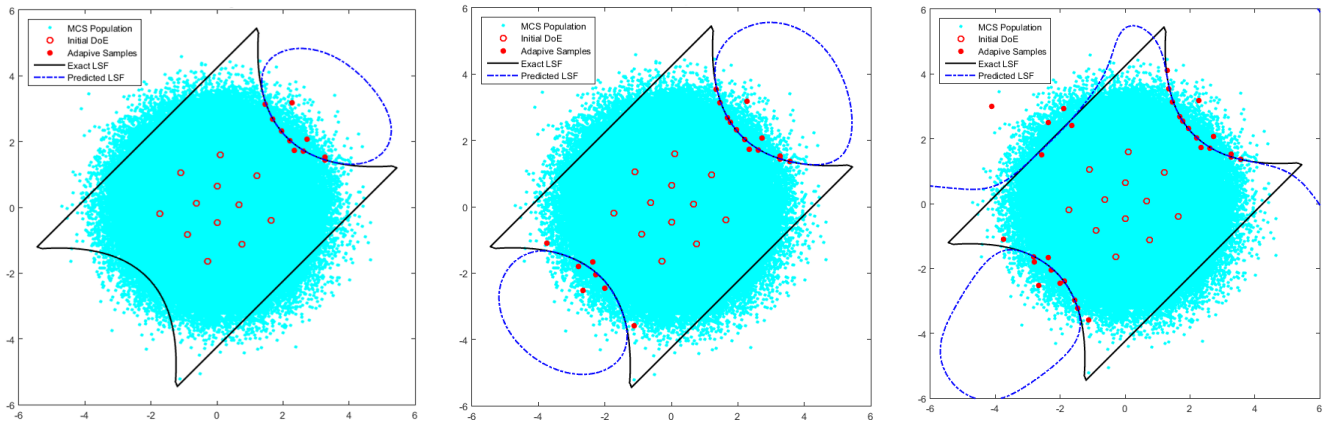
1) Determine the upper and lower values (bounds) of each variable in the MC population

2) Uniform sampling in the specified domain

Fig. 2 shows these stages for random variables with standard normal distribution in the two-dimensional case. $\mathrm{Min}(\boldsymbol{x}_1)$ and $\mathrm{Max}(\boldsymbol{x}_1)$ are minimum and maximum values of $\boldsymbol{x}_1$ random variable in the MC population, respectively. Bounds of other random variables are determined similarly.

For uniform sampling in the specified domain, different sampling strategies could be used. Uniform Random Sampling (URS) is one of the popular methods for generating uniform samples in the domain (Melchers 1999, Rashki *et al.* 2012). Other space-filling design methods include Latin Hypercube Sampling (LHS) (Butler 2001), Centroidal Voronoi Tessellations (CVT) (Romero *et al.* 2006), etc.

In this study, three uniform sampling methods including URS, LHS, and CVT are considered for uniform sampling in the specified domain. In the URS method, the uniform distribution is used to generate random samples within the predefined intervals. The basic idea of the LHS method is that no two samples can have the same value or level for a specific variable. The CVT method also provides a uniform



(a) Iteration=10 (b) Iteration=20 (c) Iteration=30

Fig. 6 Iterations of adaptive experimental design with K-means clustering method to select initial DoE for the four-branch benchmark problem (Blue lines denote the predicted limit states, and black lines are the real limit state. The empty circles mark the initial DoE, whereas the filled circles mark the adaptive samples)

sampling of the space. In the CVT method, samples are placed such that they lie at the centroids of the respective Voronoi cells (Basudhar 2011).

3.1.3 Comparing two strategies via a sample problem

The method of defining an initial DoE is investigated for a sample problem with two strategies. Consider a benchmark problem with four distinct limit states (four-branch function) as follows (Waarts 2000)

$$G(x_1, x_2) = \min \begin{Bmatrix} 3 + \dfrac{(x_1 - x_2)^2}{10} - \dfrac{(x_1 + x_2)}{\sqrt{2}} \\ 3 + \dfrac{(x_1 - x_2)^2}{10} + \dfrac{(x_1 + x_2)}{\sqrt{2}} \\ (x_1 - x_2) + 6/\sqrt{2} \\ (x_2 - x_1) + 6/\sqrt{2} \end{Bmatrix} \quad (17)$$

where x_1 and x_2 are standard normal distributed random variables.

The size of the initial design of experiments and MC population are considered $N_1 = 12$ and $n_{MC} = 10^6$, respectively. These values have been used by Schöbi *et al.* (2016) to compare the original Kriging method with the PC-Kriging method.

The initial DoE by the conventional selection method is shown in Fig. 3 for two cases of random selection method and the K-means clustering method. It can be seen that the K-means clustering method provides a better distribution for the initial DoE. In both cases, the initial samples are located in the safe region.

Fig. 4 shows the initial designs of experiments by the proposed uniform sampling method for three cases. As can be seen, the distribution of initial samples is almost uniform, and some of them are located in the failure region. Also, the CVT method provides a better uniform sampling of the space.

Fig. 5 compares the convergence histories of the AK-MCS algorithm with different strategies for defining the initial DoE. It is seen that the uniform sampling methods converge faster than the selection methods. Therefore, the

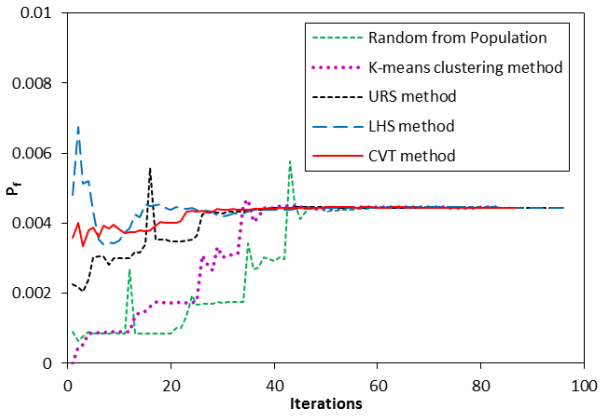
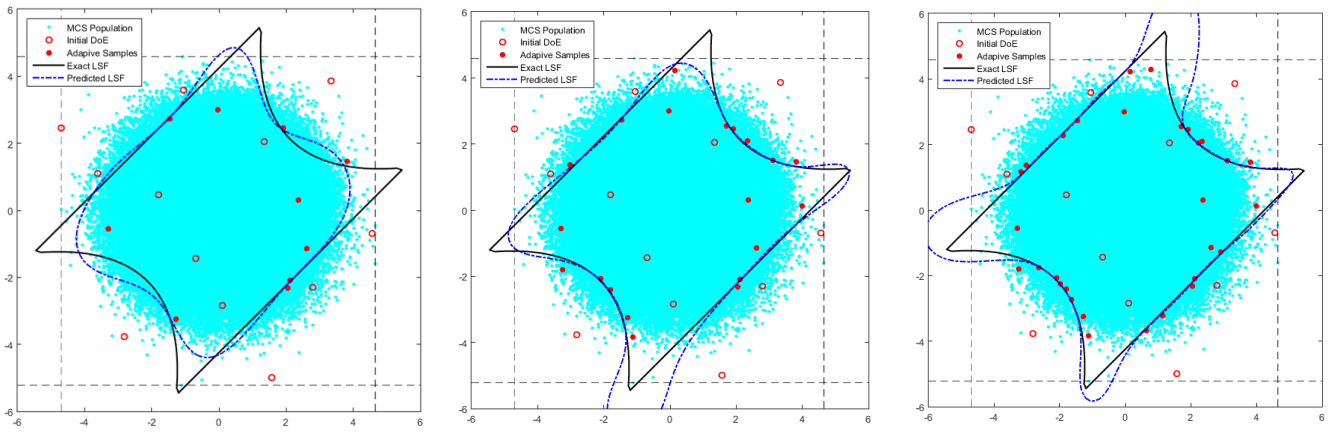


Fig. 5 Convergence histories with different methods of defining the initial DoE for the benchmark problem

approximate probability of failure is obtained with less iteration by uniform sampling methods, whereas selection methods need more iteration to reach the same results. Among the uniform sampling methods, the LHS and CVT methods converge faster than the URS. Also, the K-means clustering selection method has a better performance than the random selection method.

To compare two strategies, predicted limit states at different iterations are shown in Figs. 6-7 for selection method (K-means clustering) and uniform sampling method (LHS), respectively. The empty red circles show the initial DoE, whereas the red filled circles show the additional adaptive samples. As can be seen in Fig. 6, the failure boundaries are explored one by one in the case of selection method, and only three failure boundaries are discovered at iteration of 30. Schöbi *et al.* (2016) have reported this issue previously for the ordinary Kriging and presented an adaptive PC-Kriging method (APCK- MCS) to solve it. In the current study, improvement is achieved by uniform sampling methods instead of selection methods. As shown in Fig. 7, all four-failure boundaries are discovered after ten iterations by LHS sampling method, so the algorithm gives better results in the early iterations.



(a) Iteration=10 (b) Iteration=20 (c) Iteration=30

Fig. 7 Iterations of adaptive experimental design with LHS sampling method for defining initial DoE for the four-branch benchmark problem

Table 1 Four-branch function: Results using different stopping criteria and defining methods of initial DoE

Method	Stopping criteria	Defining method of initial DoE	N_{call}	P_f	$\epsilon_{P_f}(\%)$
Monte Carlo (Echard *et al.* 2011)	-	-	10^6	0.004416	-
AK-MCS (Echard *et al.* 2011)	Eq. (13)	Selection method (Random)	126	0.004416	0.0000
Monte Carlo	-	-	10^6	0.004448	-
AK-MCS	Eq. (13)	Selection method (Random)	12+83=95	0.004447	0.0225
		Selection method (K-means)	12+83=95	0.004446	0.0450
		Proposed uniform sampling (CVT)	12+93=105	0.004448	0.0000
		Proposed uniform sampling (LHS)	12+96=108	0.004450	0.0450
		Proposed uniform sampling (CVT)	12+86=98	0.004414	0.7644
AK-MCS	Eq. (14)	Selection method (Random)	12+60=72	0.004450	0.0449
		Selection method (K-means)	12+52=64	0.004445	0.0674
		Proposed uniform sampling (RUS)	12+45=**57**	0.004446	0.0450
		Proposed uniform sampling (LHS)	12+46=**58**	0.004447	0.0225
		Proposed uniform sampling (CVT)	12+48=**60**	0.004446	0.0450

Table 1 summarizes the results of the considered benchmark problem. The results are presented for both stopping criteria. The relative error (RE) is calculated based on the probability of failure for the same MC population evaluated by the original performance function as

$$\epsilon_{P_f}(\%) = \frac{\left|\widehat{P}_f - P_{f,MC}\right| * 100}{P_{f,MC}} \quad (18)$$

where $\widehat{P}_f$ and $P_{f,MC}$ are predicted and exact probabilities of the failure, respectively. In the case of using Eq. (13) as a stopping criterion, the number of calls to the performance function (N_{call}) is not reduced by uniform sampling methods for defining the initial DoE (despite the fast convergence) due to conservatively of this criterion (Schöbi *et al.* 2016). On the other hand, the stopping criterion of Eq. (14) and uniform sampling methods tend to better results. Comparing the number of calls to the performance function (N_{call}) with reference value shows the efficiency of the proposed uniform sampling method for defining the initial DoE. Furthermore,

Table 2 Four-branch function: Results using different size of initial DoE by the AK-MCS method

Number of initial samples	Number of adaptive samples	Total number of samples (N_{call})	P_f	$\epsilon_{P_f}(\%)$
2	7	9	0.000834	**81.2500**
4	22	26	0.001734	**61.0162**
6	29	35	0.001737	**60.9487**
8	53	61	0.004353	2.1358
10	63	73	0.004414	0.7644
12	60	72	0.004450	0.0450

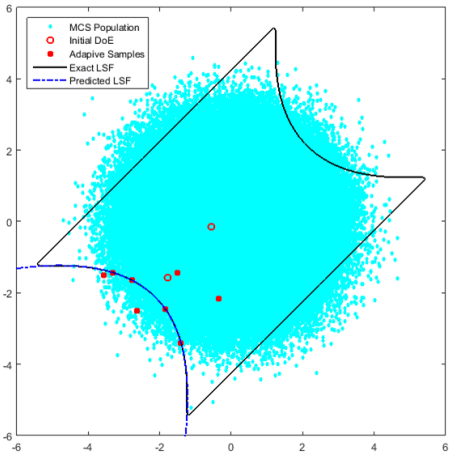


Fig. 8 Final iteration of the AK-MCS method with initial DoE size of two

the difference of the N_{call} between three uniform sampling methods is not significant. According to the results, the relative error for all cases is less than the predefined value of 5%, so all results are acceptable.

3.2 The sensitivity of the AK-MCS method on the size of initial DoE

3.2.1 Original AK-MCS method

In the original AK-MCS method, the algorithm starts with a very small size of the initial DoE. For instance, Echard *et*

Table 3 Four-branch function: Results using different size of initial DoE by the modified AK-MCS method

Number of initial samples	Number of adaptive samples	Total number of samples(N_{call})	P_f	ϵ_{P_f}(%)
2	50	52	0.004396	1.1691
4	56	60	0.004455	0.1574
6	57	63	0.004384	1.4388
8	57	65	0.004428	0.4496
10	65	75	0.004408	0.8993
12	61	73	0.004447	0.0225

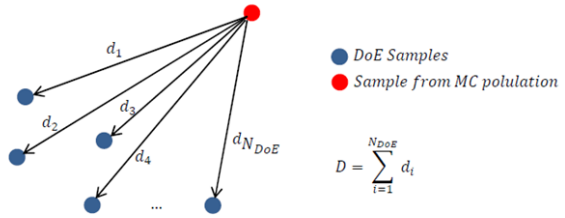


Fig. 9 Calculation of distance of a sample in the MC population to the samples of DoE

al. (2011) have suggested a dozen of points for the initial DoE. If the sample distribution of initial DoE doesn't properly cover the space, or the size of initial DoE is not large enough, the AK-MCS algorithm may not discover all of the failure boundaries in the limit state surface. To illustrate the effect of the size of an initial DoE, consider the four-branch function which is defined in the previous section. The total number of required samples (N_{call}) for different size of the initial design are shown in Table 2. The initial DoE is created with random selection method and Eq. (14) is considered as a stopping criterion. The results show that the relative error is so high for the lower size of the initial DoE.

Fig. 8 illustrates the final predicted limit state with an initial DoE size of two. As can be seen, only one branch of the limit state is found with this sample size of the initial DoE. Therefore, the predicted limit state is not accurate enough to compute the probability of failure.

3.2.2 The proposed method for scattered DoE

To overcome the aforementioned drawback of the AK-MCS, a modified approach is proposed by adding the endmost sample of the MC population from the current DoE samples. The proposed method includes three steps after the convergence of the original AK-MCS algorithm:

1) Compute the distance of each sample in the MC population to the samples of DoE (see Fig. 9)
2) Calculate the summation of distances for each sample of the MC population
3) Find a sample which corresponds to the maximum of summation of distances

So, a candidate sample is selected to add to the DoE as follows

$$\boldsymbol{x}_{\text{candidate}} = argmax_x D(\boldsymbol{x}) = argmax_{\boldsymbol{x}} \sum_{i=1}^{N_{DoE}} d_i(\boldsymbol{x}) \quad (19)$$

$$= argmax_{\boldsymbol{x}} \sum_{i=1}^{N_{DoE}} \sum_{l=1}^{n} \left(x_l - \mathrm{x_{i,l}}\right)^2 \qquad \forall \boldsymbol{x} \epsilon S$$

where $\boldsymbol{x}$ is a sample from MC population, $\boldsymbol{x}_i$ is a sample in the DoE, x_l and $x_{i,l}$ are the ^{l}th coordinates of points $\boldsymbol{x}$ and $\boldsymbol{x}_i$, respectively, N_{DoE} is the number of samples in the DoE, and n is the dimension of the problem.

Briefly, the endmost sample in the MC population from the current samples (DoE) is added to the DoE after convergence of the original AK-MCS method. Then, the Kriging model is updated according to the new DoE. If adding a new sample, has a considerable effect on the estimated probability of failure (P_f), other samples are needed to increase the accuracy of the surrogate model. So, the iterative process continues until the convergence of the algorithm is achieved (see Fig. 14).

Fig. 10 shows the added sample and the updated model for the four-branch problem. As can be seen, by adding one sample to DoE, the algorithm finds another mode of failure. So, the original AK-MCS method could find other adaptive samples, and the algorithm converges to the exact limit state in the next iterations.

The results in Table 3 show the reduction of relative error from 81.25% to less than 1.5% by the modified AK-MCS method for initial DoE size of two. The number of function evaluations and relative error for predicted P_f are compared in Fig. 11 for the original and modified AK-MCS method. As can be seen, the relative error reduces dramatically in the fewer size of initial DoE by the modified AK-MCS method. Therefore, the sensitivity of the AK-MCS method to the size of initial DoE could be reduced by the modified approach.

3.3 The trend of Kriging model

In the original AK-MCS method, the regression trend of Kriging model has been assumed to be an unknown constant value (Echard *et al.* 2011). In this section, different regression models with polynomial orders of zero, one and two are considered in the AK-MCS method. As defined in Eq. (4), the regression part of Kriging model is in the form $\mathbf{f}(\boldsymbol{x})^{\mathrm{T}}\beta$, where $\mathbf{f}(\boldsymbol{x}) = [\mathrm{f}_1(\boldsymbol{x}), \ldots, \mathrm{f}_p(\boldsymbol{x})]^{\mathrm{T}}$ represents the basic functions and $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_p]^{\mathrm{T}}$ is the vector of regression coefficients. Therefore, the polynomials of orders 0, 1 and 2 can be defined as (Lophaven *et al.* 2002)

Constant Regression ($\mathrm{p} = 1$)

$$\mathbf{f}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\beta}=\beta_0 \quad (20)$$

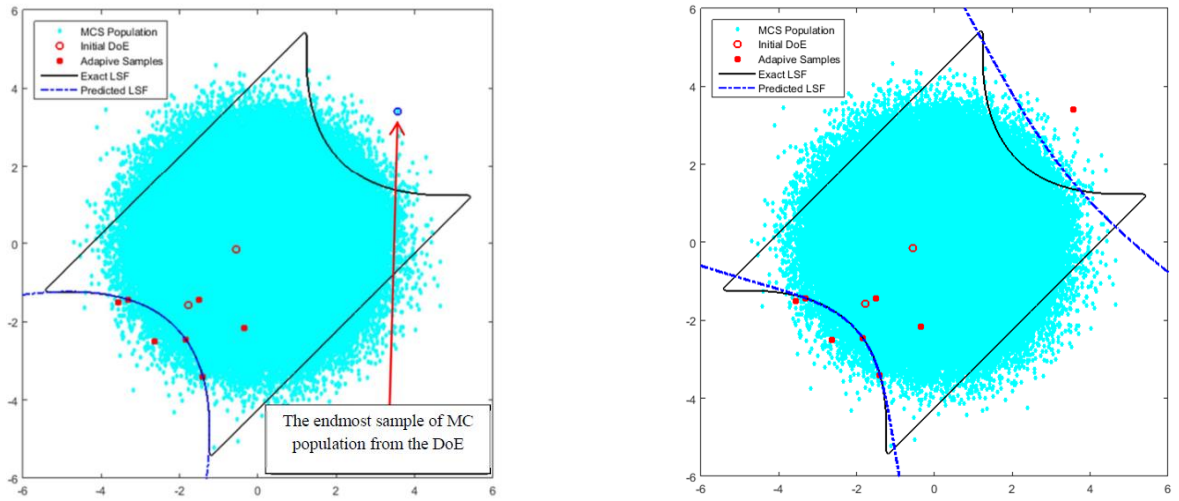
Linear Regression ($\mathrm{p} = \mathrm{n} + 1$)

$$\mathbf{f}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\beta}=\beta_0+\sum_{\mathrm{i}=1}^{\mathrm{n}} \beta_{\mathrm{i}}\, x_{\mathrm{i}} \quad (21)$$

Quadratic Regression ($\mathrm{p} = \frac{(n+1)(n+2)}{2}$)

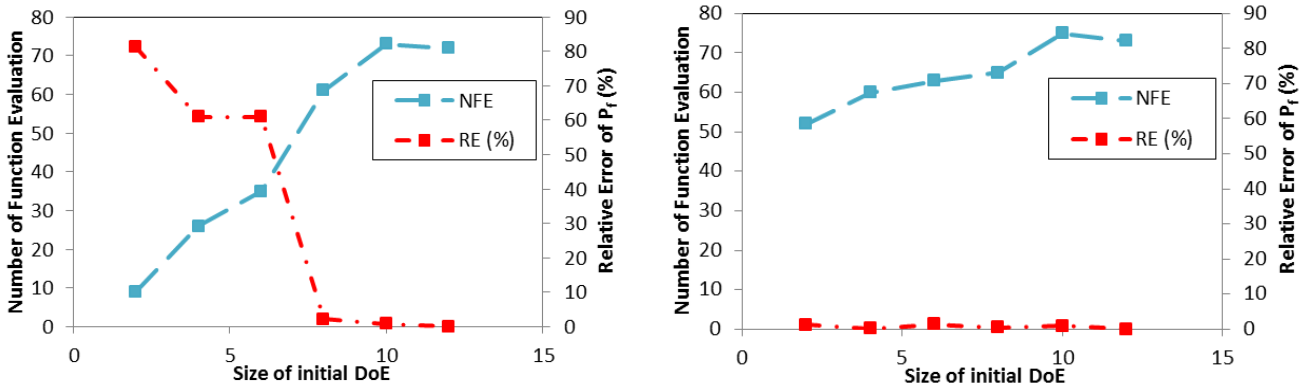
$$\mathbf{f}(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{\beta}=\beta_0+\sum_{\mathrm{i}=1}^{n} \beta_{\mathrm{i}}\, x_{\mathrm{i}}+\sum_{\mathrm{i}=1}^{n}\sum_{\mathrm{j}=1}^{n} \beta_{\mathrm{ij}} x_{\mathrm{i}} x_{\mathrm{j}} \quad (22)$$

where n is the dimension of the problem (number of random variables), and p is the number of terms in the



(a) Finding the endmost sample in MC to samples of DoE (Iteration=7) (b) Predicted LSF after adding the farthest sample to DoE (Iteration=8)

Fig. 10 Illustration of adding one sample in the proposed method after the convergence of the algorithm



(a) Original AK-MCS method (b) Modified AK-MCS method

Fig. 11 Number of function evaluation and relative error of P_f using different size of initial DoE for the four-branch benchmark problem

regression model.

Unknown coefficients (β_i) can be calculated according to Eq. (8). The minimum size of DoE could not be less than p, because the number of unknown coefficients (β_i) is equal to p. For example, if the number of random variables is equal

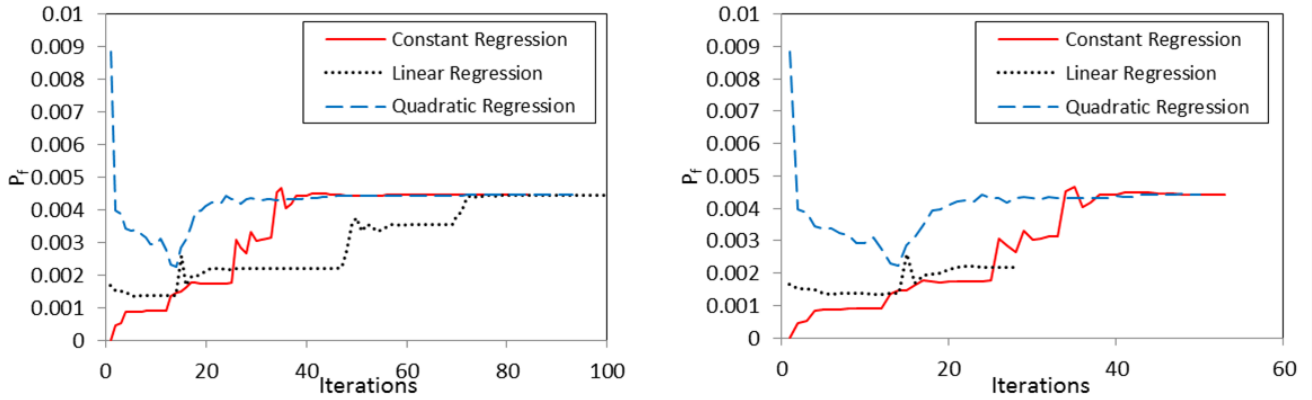
to 10, the number of samples in the DoE should not be less than (10+1)(10+2)/2=66 for the quadratic regression. Therefore, for high dimensional problems, the quadratic regression may not be a good choice as a trend of the Kriging model because it requires a large size of initial DoE.

To investigate the effect of regression part of Kriging model in the AK-MCS method, previously defined problem is considered. The initial DoE is defined with the selection method (K-means clustering), and three types of regression models are considered for comparison. Convergence histories in Fig. 12 show that the AK-MCS method with the quadratic regression trend converges faster for this problem. In the case of linear regression trend with the stopping criterion of Eq. (14), the algorithm has stopped before an accurate failure probability could be predicted.

Table 4 summarizes the results of the AK-MCS method with different regression trends using two stopping criteria. According to the results, P_f is predicted with a good accuracy for different regression trends in the case of using Eq. (13) as a stopping criterion, whereas the relative error is high with the linear regression trend and the stopping criterion of Eq. (14). Therefore, using Eq. (14) as a stopping criterion may be not efficient for some cases.

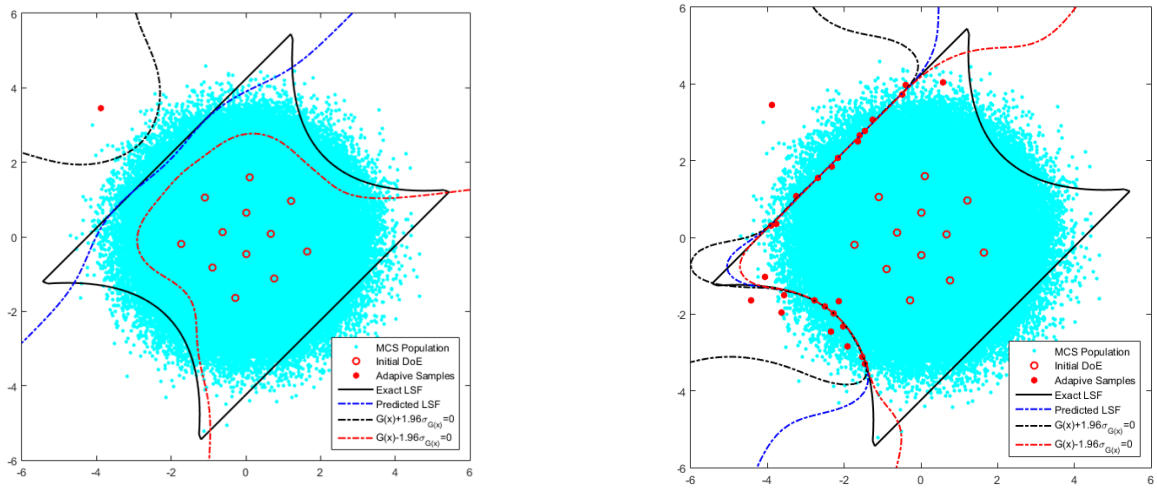
Fig. 13 shows the predicted limit state for this case (linear regression trend with Eq. (14) as a stopping criterion for the sample problem) in the first and final iterations. The upper and lower boundaries of estimations are also illustrated. These boundaries are used to define the stopping criterion according to Eq. (15). As can be seen, the predicted limit state in the first iteration is similar to the linear regression. Although it improves in the next iterations, not all branches of the limit state are found in the final iteration.

As it can be seen, the samples of MC population between the upper and lower boundaries of limit state surface in the final iteration have not a considerable effect on finding other



(a) Stopping criterion of Eq. (13) (b) Stopping criterion of Eq. (14)

Fig. 12 Convergence histories of the AK-MCS method with different regression trends and two stopping criteria for the four-branch problem



(a) Iteration=1 (b) Iteration=27 (final iteration)

Fig. 13 Prediction of the AK-MCS method with linear regression trend and stopping criterion of Eq. (14)

branches of limit state. To overcome this drawback, the defined method in the previous section can be used for adding more samples (section 3.2.2).

Although the type of regression trend could be effective in the number of calls to the performance function and the accuracy of the estimated P_f, there are some limitations in using a higher order regression trend. As mentioned above, the minimum size of initial DoE is increased with linear and quadratic regression trends in comparison with the ordinary Kriging (constant regression). So, a stepwise method is proposed to achieve better results for higher dimensional problems as

$$\mathbf{f}(x)^{\mathrm{T}}\boldsymbol{\beta} = \begin{cases} \beta_0, & if\ \mathrm{N_{DoE}} < n+1 \\ \beta_0 + \sum_{i=1}^{n} \beta_i x_i, & if\ n+1 \le \mathrm{N_{DoE}} < (n+1)(n+2)/2 \\ \beta_0 + \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} x_i x_j, & \text{if } \mathrm{N_{DoE}} \ge (n+1)(n+2)/2 \end{cases} \quad (23)$$

Table 4 Four-branch function: Results using different regression trends and stopping criteria

Regression Trend	Stopping criteria	Total number of samples (N_{call})	P_f	ϵ_{P_f}(%)
Constant	Eq. (13)	12+83=95	4.449×10^{-3}	0.0225
Linear	Eq. (13)	12+103=115	4.447×10^{-3}	0.0225
Quadratic	Eq. (13)	12+92=104	4.448×10^{-3}	0.0000
Constant	Eq. (14)	12+52=64	4.414×10^{-3}	0.7644
Linear	Eq. (14)	12+27=49	2.180×10^{-3}	50.9892
Quadratic	Eq. (14)	12+49=61	4.427×10^{-3}	0.4721

where N_{DoE} is the size of DoE and n is the number of random variables.

In the proposed method, the size of DoE is calculated in each iteration, and then the regression trend is selected based on the size of DoE. Therefore, the type of regression trend may change with iterations. Also, if the algorithm converges before the $\mathrm{N_{DoE}}$ reaches $(n+1)(n+2)/2$, there is no need to use the quadratic regression trend.

Table 5 Example 1-random variables of the nonlinear oscillator

Variable	Distribution	Mean	Standard deviation
m	Normal	1	0.05
c_1	Normal	1	0.1
c_2	Normal	0.1	0.01
r	Normal	0.5	0.05
F_1	Normal	1	0.2
t_1	Normal	1	0.2

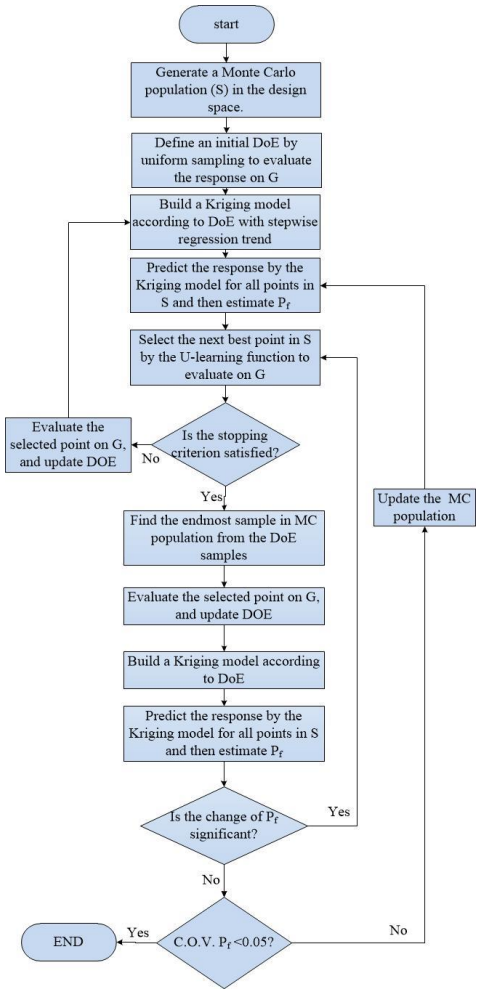


Fig. 14 Flowchart of the enhanced AK-MCS method

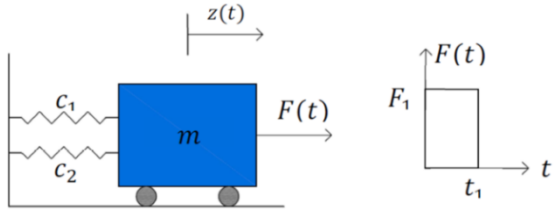


Fig. 15 Example 1-nonlinear oscillator

However, if N_{DoE} become greater than $(n+1)(n+2)/2$, all samples of DoE (includes initial and adaptive samples) will be used as DoE by the Kriging model with the quadratic regression trend. Therefore, the procedure can converge faster than AK-MCS method with the quadratic regression trend and the regular initial DoE (without adaptive samples).

All mentioned modifications of the AK-MCS method are summarized as a flowchart in Fig. 14. In summary, three modifications are presented in this paper: The initial DoE is defined with uniform sampling method as described in section 3.1. The endmost sample is added to DoE after the convergence of the algorithm according to the method described in section 3.2, and finally, the stepwise regression trend used according to Eq. (23) in section 3.3. Briefly, the proposed modification could be effective in three cases as follows:

1) Defining the initial DoE with the uniform sampling method helps to cover the entire domain of variables. So using such an initial DoE could provide a better approximation in the earlier iterations.

2) Adding the endmost sample of the MC population to the DoE could help to discover other failure boundaries. This strategy could be effective for problems with multiple failure boundaries.

3) In some problems, using a linear or quadratic trend could reduce the number of calls to the performance function. Using a stepwise regression trend could be sound in such cases.

The efficiency of the proposed enhanced AK-MCS method is shown for some benchmark problems in Section 4.

4. Applications

In this section, two analytical example and two structural examples are employed to validate the efficiency of the proposed modifications. In these examples, Eq. (14) is used as a stopping criterion due to its efficiency (Schöbi *et al.* 2016), and the size of initial DoE is considered equal to12 according to suggested value in (Echard *et al.* 2011).

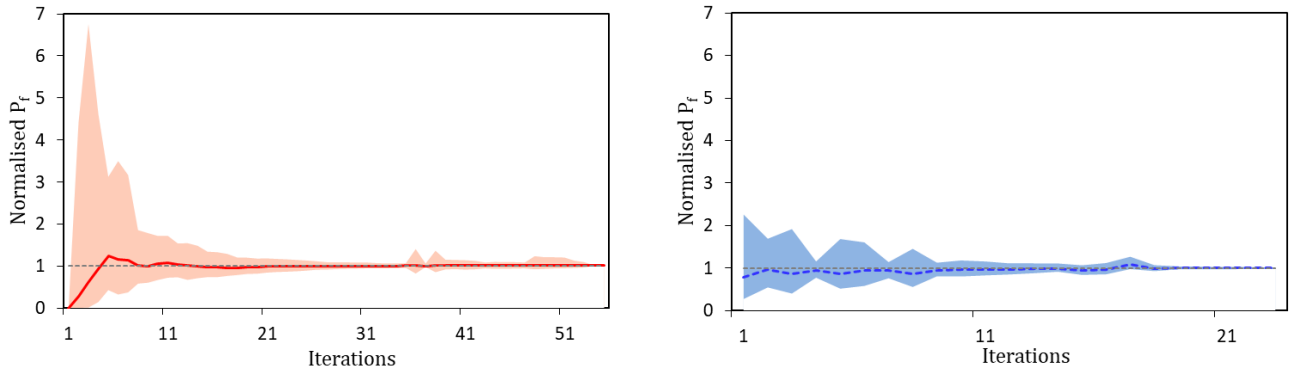
4.1 Example 1: Dynamic response of a nonlinear oscillator

The first example is a problem with a moderate number of random variables (Echard *et al.* 2011). It deals with a nonlinear un-damped single degree of freedom system (see Fig. 15). This problem is also studied by Pan and Dias (2017) using Adaptive Support Vector Machine and the Monte Carlo Simulation (ASVM-MC) method. The performance function is defined as

$$G(c_1, c_2, m, r, t_1, F_1)=3r-|z_{max}|=3r-\left|\frac{2F_1}{m\omega_0^2}\sin\left(\frac{\omega_0^2 t_1}{2}\right)\right| \quad (24)$$

where $\omega_0 = \sqrt{(c_1 + c_2)/m}$. The six random variables listed in Table 5.

The obtained results including the failure probability (P_f), the number of calls to the performance function (N_{call}), and the relative errors of failure probability ϵ_{P_f} are summarized in Table 6. It is seen that the enhanced AK-MCS method requires fewer calls to the performance function than the presented values in the references (Echard *et al.* 2011, Pan and Dias 2017). The enhanced AK-MCS needs only 35 calls



(a) AK-MCS method (b) Enhanced AK-MCS method

Fig. 16 Example 1-Convergence histories of the AK-MCS and enhanced algorithms for the nonlinear oscillator

Table 6 Results for Example 1: Nonlinear oscillator problem

Method	N_{call}	P_f	ϵ_{P_f}(%)
AK-MCS (Echard *et al.* 2011)	58	0.02834	-
ASVM-MCS (Pan and Dias 2017)	56	0.02790	-
MCS	2×10^4	0.02785	-
AK-MCS	12+55=67	0.02805	0.7181
Enhanced AK-MCS	12+23=35	0.02800	0.5386

to the performance function, whereas the AK-MCS and ASVM-MC require more calls (58 and 56 calls, respectively). Also, the predicted value is very close to the exact value using the MC simulation.

Fig. 16 shows the convergence histories of the AK-MCS and enhanced methods. It is seen that the enhanced AK-MCS method converges faster than the original AK-MCS method. The upper and lower bounds of failure ($\widehat{P}_f^+$ and $\widehat{P}_f^-$ according to Eq. (15)) have also been indicated in the Fig. 16. As can be seen, the difference between the upper and lower bounds (shading area) of failure probabilities is significant in the AK-MCS method, so more iterations are required to satisfy the stopping criterion of Eq. (14).

4.2 Example 2: High-dimensional example

The second example aims to apply the enhanced AK-MCS method for a high-dimensional problem (Echard *et al.* 2011). It consists of independent lognormal random variables with mean value $\mu = 1$ and standard deviation σ=0.2. This example is also studied by some researchers (Bourinet *et al.* 2011, Pan and Dias 2017). The performance function reads as follows

$$G(X_1, \ldots, X_n)=(n+3\sigma\sqrt{n})-\sum_{i=1}^{n} X_i \quad (25)$$

The dimension of the problem is n=40 in this study. Obtained results are provided in Table 7. According to the results, the proposed enhanced method needs only 43 calls to the performance function, which is better than the reported values of 112, 341 and 3729 (Echard *et al*. 2011, Bourinet *et al*. 2011, Pan and Dias 2017).

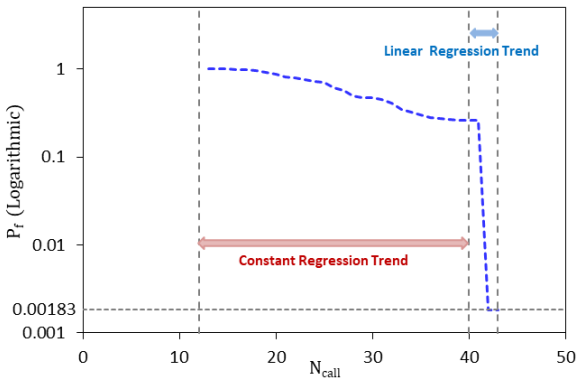


Fig. 17 Example 2-Convergence of P_f by enhanced AK-MCS method for the high dimensional problem

Table 7 Results for Example 2: High dimensional problem

Method	N_{call}	P_f	ϵ_{P_f}(%)
MCS (Echard *et al.* 2011)	3×10^5	0.001813	-
AK-MCS (Echard *et al.* 2011)	112	0.001813	0.0000
MCS (Pan and Dias 2017)	10^6	0.001820	-
ASVM-MCS (Pan and Dias 2017)	341	0.001780	2.2000
SS-SVM (Bourinet *et al.* 2011)	3729	0.001950	-
MCS	3×10^5	0.001830	-
Enhanced AK-MCS	12+31=43	0.001830	0.0000

Convergence history in Fig. 17 shows that the changing of the trend from constant regression to linear regression causes fast convergence. According to the proposed stepwise regression model, this change occurs when the size of DoE (which is equal to N_{call}) reaches the value of $n + 1 = 41$. Therefore, using a linear regression could reduce the number of calls to the performance function in this example.

4.3 Example 3: Two-dimensional 23 bars truss

This example includes a two-dimensional 23-bars truss structure, which has been studied by Schöbi *et al.* (2016). As shown in Fig. 18, it consists of 11 horizontal bars and 12 diagonal bars. The geometry is deterministic, whereas the material properties and the loadings are modeled

Table 8 Example 3-Random variables of truss structure

Variable	Distribution	Mean	Standard deviation
E_1, E_2(Pa)	Lognormal	2.1×10^{11}	2.1×10^{10}
A_1(m^2)	Lognormal	2.0×10^{-3}	2.0×10^{-4}
A_2(m^2)	Lognormal	1.0×10^{-3}	1.0×10^{-4}
$P_1, \ldots, P_6$(N)	Gumbel	5.0×10^{4}	7.5×10^{3}

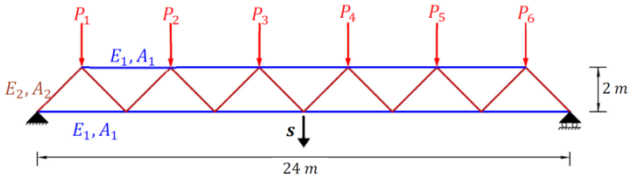


Fig. 18 Example 3-Two-dimensional 23 bar truss

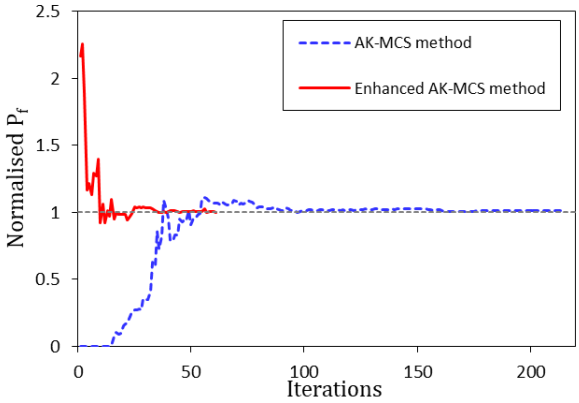


Fig. 19 Example 3-Convergence histories of the AK-MCS and enhanced algorithms for 23 bars truss

stochastically. Ten random input variables form the input vector as

$$\boldsymbol{x} = [\mathrm{E}_1, \mathrm{E}_2, \mathrm{A}_1, \mathrm{A}_2, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4, \mathrm{P}_5, \mathrm{P}_6] \quad (26)$$

where $\mathrm{A}_1, \mathrm{E}_1$ denote horizontal bars cross-section and Young's modulus respectively, while $\mathrm{A}_2, \mathrm{E}_2$ denote diagonal bars', and $\mathrm{P}_1, \ldots, \mathrm{P}_6$ are vertical loads acting on the nodes of the upper part of the structure. These ten random variables are independent, and their distribution parameters are given in Table 8.

A finite-element model is used to calculate the mid-span deflection, denoted by $\mathrm{s}(\boldsymbol{x})$, as a function of random variables. The deflection is defined as positive in the direction indicated in Fig. 18. The threshold of $\mathrm{s}(\boldsymbol{x})$ is considered equal to 0.10 m. Hence the performance function can be defined as

$$\mathrm{G}(\boldsymbol{x}) = 0.10 - |\mathrm{s}(\boldsymbol{x})| \quad (27)$$

The results of AK-MCS and enhanced methods are summarized in Table 9. The results show the effectiveness of the enhanced method. The number of calls to the performance function for the original AK-MCS and the enhanced AK-MCS methods are 225 and 72, respectively. Schöbi *et al.* (2016) reported the number of calls equal to 170 for the APCK-MCS method, so the number of calls to performance function using the enhanced method is considerably lower than the reported value. The relative error of the failure probability for the enhanced method is equal to 0.9368%, which is acceptable because of using the value of $\epsilon_{\hat{\mathrm{P}}_f}$=5% as an accuracy criterion in Eq. (14). It should be noted that the improvement in term of calls to the performance function is considerable, whereas the error of the prediction is negligible.

Table 9 Results for Example 3: 23 bars truss problem

Method	N_{call}	P_f	ϵ_{P_f}(%)
APCK-MCS (Schöbi *et al.* 2016)	170	0.0432	0.6693
MCS	10^4	0.0427	-
AK-MCS	12+213=225	0.0428	0.2342
Enhanced AK-MCS	12+60=72	0.0423	0.9368

Table 10 Example 3-Random variables of the frame structure

Variable	Distribution	Mean	Standard deviation
A_1(m^2)	Lognormal	0.25	0.025
A_2(m^2)	Lognormal	0.16	0.016
A_3(m^2)	Lognormal	0.36	0.036
A_4(m^2)	Lognormal	0.20	0.020
A_5(m^2)	Lognormal	0.15	0.015
P(kN)	Type I Largest	30.0	7.5

Table 11 Results for Example 4: Twelve-story steel frame

Method	N_{call}	P_f	ϵ_{P_f}(%)
MCS (Cheng and Xiao 2005)	2×10^3	0.0751	-
MCS	10^4	0.0737	-
AK-MCS	12+90=102	0.0738	0.1357
Enhanced AK-MCS	12+19=31	0.0739	0.2713

Fig. 19 shows that the enhanced method converges faster than the original AK-MCS method. As can be seen, the number of required iterations decreases significantly from 213 in the original AK-MCS method to 60 in the enhanced method. This improvement is due to the use of the uniform sampling method and the stepwise regression trend.

4.4 Example 4: A three-bay twelve-story steel frame

The last example consists of a steel portal frame structure with twelve stories and three bays as shown in Fig. 20 (Cheng and Xiao 2005). Cross-sectional areas (A_i) and horizontal load (P) are independent random variables, and their distribution information is listed in Table 10. The sectional moments of inertia are expressed as $\mathrm{I}_\mathrm{i} = \alpha_\mathrm{i}\mathrm{A}_\mathrm{i}^2$ (α_1=α_2=α_3=0.08333, α_4=0.26670, α_5=0.200).

Fig. 20 shows the element types. The Young's modulus (E) is treated as deterministic, E=2.0×10^{7}kN/m^2. The performance function reads as follows

$$\mathrm{G}(A_1,A_2,A_3,A_4,A_5,P)=0.096-u_A(A_1,A_2,A_3,A_4,A_5,P) \quad (28)$$

where u_A is the horizontal displacement at node A.

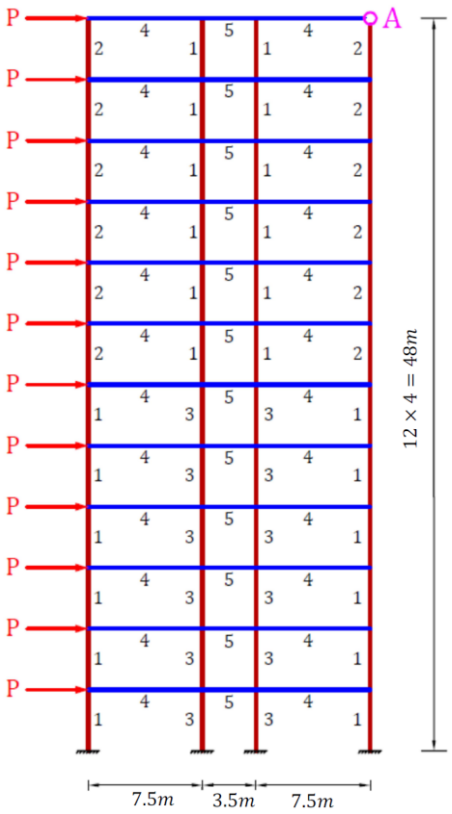


Fig. 20 Example 4-Three-bay twelve-story steel frame

The reliability results in Table 11 show the efficiency of the proposed method. It is seen that the number of calls to the performance function (N_{call}) is 102 in the original AK-MCS method, whereas it decreases to 31 in the enhanced method (including 12 samples of initial DoE and 19 adaptive samples). Comparison of convergence histories in Fig. 21 shows that the enhanced AK-MCS method converges faster than the original method, and the number of calls to the performance function (N_{call}) is also reduced significantly. The relative error of the failure probability is not significant by the enhanced method. Also, the probability of failure is estimated with a good accuracy with only 31 calls to the performance function. Therefore, the results show the effectiveness of the enhanced AK-MCS method for reliability assessment of the frame structure.

5. Conclusions

This paper proposes an improved AK-MCS method to reduce the number of calls to the original performance function. The proposed method includes three modifications:

1) method of defining an initial DoE, 2) adding more samples in the case of incorrect prediction, and 3) a stepwise regression trend.

The method of defining the initial DoE is important in the AK-MCS method, so the uniform sampling method is proposed instead of the selection method in the original AK-MCS method. The proposed strategy could reduce the number of function calls. The size of initial DoE is another important factor in the application of AK-MCS method. The

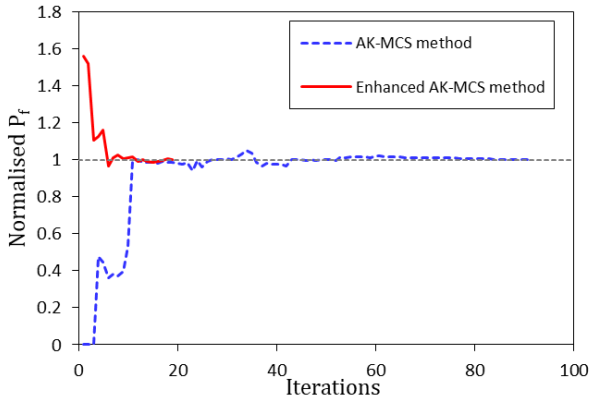


Fig. 21 Example 4-Convergence histories of the AK-MCS and enhanced methods for the twelve-story steel frame

algorithm may not converge to exact limit state with an inappropriate size of the initial DoE. Therefore, a new strategy is proposed for this case by adding the endmost sample of MC population to the DoE, and its efficiency is shown for four-branch function.

The original AK-MCS method uses the ordinary Kriging model in the algorithm, so the regression trend of Kriging model is an unknown constant regression. In this paper, different regression trends (constant regression, linear regression, and quadratic regression) are evaluated for the benchmark problem, and it is shown that the appropriate choice of trend can be effective in reducing the number of function calls. In addition, the required size of the initial DoE increases rapidly with the growth of dimension in the case of using the quadratic regression trend. Therefore, a stepwise regression method is proposed to overcome this drawback.

The efficiency of the proposed method is shown for some benchmark examples. In the benchmark problems, the number of calls to performance function reduced significantly. The reduction of function calls is near to 70% for the studied truss and frame structures. Therefore, the proposed enhanced method can be effective in prediction of the failure probability with less number of calls to the performance function.

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