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Influence of initial stresses on the critical velocity of the moving load acting in the interior of the hollow cylinder surrounded by an infinite elastic medium

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Abstract. The bi-material elastic system consisting of the pre-stressed hollow cylinder and pre-stresses surrounding infinite elastic medium is considered and it is assumed that the mentioned initial stresses in this system are caused with the compressing or stretching uniformly distributed normal forces acting at infinity in the direction which is parallel to the cylinder's axis. Moreover, it is assumed that on the internal surface of the cylinder the ring load which moves with constant velocity acts and within these frameworks it is required to determine the influence of the aforementioned initial stresses on the critical velocity of the moving load. The corresponding investigations are carried out within the framework of the so-called three-dimensional linearized theory of elastic waves in initially stresses bodies and the axisymmetric stress-strain state case is considered. The "moving coordinate system" method is used and the Fourier transform is employed for solution to the formulated mathematical problem and Fourier transformation of the sought values are determined analytically. However, the originals of those are determined numerically with the use of the Sommerfeld contour method. The critical velocity is determined from the criterion, according to which, the magnitudes of the absolute values of the stresses and displacements caused with the moving load approaches an infinity. Numerical results on the influence of the initial stresses on the critical velocity and interface normal and shear stresses are presented and discussed. In particular, it is established that the initial stretching (compressing) of the constituents of the system under consideration causes a decrease (an increase) in the values of the critical velocity.

Keywords: critical velocity; initial stresses; moving load; interface stresses; hollow cylinder; surrounding elastic medium

1. Introduction

A safe and reliable use of the modern high-speed underground trains and other types of underground moving wheels requires theoretical investigations of corresponding dynamical problems, one of which is the problem related to the dynamics of the ring moving load acting on the interior of the hollow cylinder surrounded with elastic or viscoelastic medium. This is because, underground structures into which such high-speed wheels move are modelled as infinite hollow cylinders surrounded by an elastic or viscoelastic medium. In order to improve the adequacy of the theoretical results to the real cases it is necessary to take into account the reference particularities of these systems in these investigations. One of these particularities is the initial stresses which appear in the constituents of the system "hollow cylinder + surrounding elastic medium". Namely, the investigation the influence of the initial stresses which appear under unidirectional compression (stretching) of the system "hollow cylinder + surrounding elastic medium" in the load moving direction on the critical velocity of this load and interface stresses which also appear as a result of this load, is the subject of the present paper.

For determination significance and contribution of the investigations made in the present work we attempt to make a brief review of the studies related to the dynamics of the moving load and we begin this review with the paper by Achenbach *et al.* (1967). Note that in this paper the dynamic response of the system consisting of the covering layer and half plane to a moving load was investigated under which the motion of the plate was described with the use of the Timoshenko theory, however, the motion of the half-plane was described by using the exact equations of the theory of linear elastodynamics and the plane-strain state was considered. Later investigations started in the paper by Achenbach *et al.* (1967), are developed in the papers by Dieterman and Metrikine (1997) and by Metrikine and Vrouwenvelder (2000) and many others listed therein. The review of the related investigations is also considered in the paper by Ouyang (2011).

It should be noted that up to now, a certain number of investigations have also been made for the dynamics of the moving load acting on initially stressed systems. As an example, for earlier investigation in this type it can be take the paper by Kerr (1983) in which the influence of the initial stresses on the values of the critical velocity of the moving load acting on an ice plate resting on water were

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taken into account. In this paper, the motion of the plate is described by employing the Kirchhoff plate theory.

In the paper by Metrikin and Dieterman (1999), under investigation of the lateral vibration of the beam on an elastic half-space due to a moving lateral time-harmonic load acting on the Euler-Bernoulli beam, the initial axial compression of this beam is also taken into consideration.

The influence of the initial stresses acting on the stratified half-plane on the critical velocity of the moving load which acts on the plate was studied in the papers by Babich *et al.* (1986, 1988, 2008a, b), in which the plane strain state was considered and the motion of the half-plane was written within the scope of the three-dimensional linearized theory of elastic waves in initially stressed bodies (see Guz 1999, 2004). At the same time, the motion of the covering layer, which does not have any initial stresses, as in the paper by Achenbach *et al.* (1967), was written by employing the Timoshenko plate theory.

Furthermore, in recent years it was made series investigations on the dynamics of the moving load acting on the layered systems and under these investigations not only the motion of the half-plane but also the motion of the covering layer was written within the scope of the exact equations of the three-dimensional linearized theory of elastic waves in initially stressed bodies. Now we review some of them and begin this review with the paper by Akbarov *et al.* (2007) in which the influence of the initial stresses in the covering layer and half-plane on the critical velocity of the moving load acting on the plate covering the half-plane, was studied. In the paper by Dincsoy *et al.* (2009) the same problem was studied for the system consisting of the covering layer, substrate and half-plane.
The response of the system consisting of the initially stressed orthotropic covering layer and initially stressed orthotropic half plane to the moving and oscillating moving load were investigated in the papers by Akbarov and Ilhan (2008, 2009), and by Ilhan (2012).

The paper by Akbarov and Salmanova (2009) deals with the dynamics of the oscillating moving load acting on the pre-strained bi-layered slab made of highly elastic material and resting on a rigid foundation was studied.

In the paper by Akbarov *et al.* (2015) the 3D problems on the dynamics of the moving and oscillating moving point-located load acting on the system consisting of a pre-stressed covering layer and half-space were investigated. As a result of this investigation, in particular, it was established that the minimal values of the critical velocities determined within the scope of the 3D formulation coincide with the critical velocity determined within the scope of the corresponding 2D formulation.

The detail consideration and analysis of the foregoing investigations was made in the monograph by Akbarov (2015). Note that up to now there are also some investigations on the moving and oscillating moving load acting on the hydro-elastic systems, which do not considered in the monograph by Akbarov (2015). These investigations were made in the papers by Akbarov and Ismailov (2015, 2016a, 2016b).

Now we consider a review of the investigations related to the dynamics of the moving load acting on a cylindrical surface which bounds the infinite region filled with homogeneous or piecewise homogeneous elastic materials. In the historical aspect, the first attempt in this field was made in the paper by Parnes (1969) in which a ring load moving with constant velocity in the axial direction along the interior of a circular bore in an infinite homogeneous elastic medium is investigated. In this paper the theoretical investigations are made in the 3D case, however corresponding numerical results on the displacement and stress distribution are presented for the axisymmetric case. The case where in the interior of the cylindrical cavity a torsional moving load acts is considered in another paper by Parnes (1980). Note that, in these papers the question related to the critical velocity is not considered. Rather, the question on the determination of the critical velocity of the moving load acting on infinite (as in the papers by Parnes (1969, 1980)) or semi-infinite mediums does not appear in the cases where these mediums are homogeneous. This is because, in the mentioned cases the critical velocity is known beforehand, so that in these cases the critical velocity coincides with the Rayleigh wave propagation velocity in the corresponding medium. Note that the question related to determination of the critical velocity relates only to moving load problems acting on the piece-wise inhomogeneous infinite (for instance, for the system consisting of a hollow cylinder surrounded with elastic medium) or semi-infinite (for instance, for the system consisting of a covering layer and half-space) bodies. What is more, the critical velocity in the aforementioned infinite and semi-infinite bodies appears only in the cases where the modulus of elasticity of the covering layer material is greater than that of the surrounding infinite medium or of the stratified semi-infinite medium. For instance, investigations carried out in the papers by Chonan (1981), Pozhuev (1980), Abdulkadirov (1981) and others relate namely to the piece-wise inhomogeneous infinite cylindrically layered systems.

Studies on the dynamic response of a cylindrical shell imperfectly bonded to a surrounding infinite elastic continuum under action of axisymmetric ring pressure which moves with constant velocity in the axial direction along the interior of the shell, is made by Chonan (1980). It is assumed that the shell and the continuum are joined together by a thin elastic bond and the axisymmetric problem is considered. The motion of the shell is described by thick shell theories and the motion of the surrounding elastic medium is described by the exact equations of linear elastodynamics. Numerical results on the critical speed of the moving load and on the radial displacement of the shell for the subcritical moving load are presented.

The paper by Pozhuev (1981) studies the moving load problem for the system consisting of a thin cylindrical shell and surrounding transversally isotropic infinite medium. A thin shell theory is employed for describing the motion of the cylinder, however the motion of the surrounding elastic medium is described with the exact equations of motion of elastodynamics for transversally isotropic bodies. Numerical results regarding displacements and a radial normal stress are presented, but in this paper, there are no numerical results related to the critical velocity of the moving load.

The study of low-frequency resonance axisymmetric longitudinal waves in a cylindrical layer surrounded by an elastic medium is made in the paper by Abdulkadirov (1981), in which under “resonance waves” the cases under which the relation $dc/dk=0$ takes place, is understood, where c is the wave propagation velocity and k is the wavenumber. The velocity of these “resonance waves” coincides with the critical velocity of the corresponding moving load. Some numerical examples of “resonance waves” are presented and discussed. Note that in this paper dispersion curves are obtained within the scope the exact equations of linear elastodynamics.

Besides of all these, in recent years numerical and analytical solution methods have been developed for studying the dynamical response of tunnel (modelled as a hollow elastic cylinder) + soil (modeled as surrounding elastic or viscoelastic medium) systems generated by the moving load acting on the interior of the tunnels (see, for instance, the papers by Forrest and Hunt (2006), Sheng *et al.* (2006), Hung *et al.* (2013), Hussein *et al.* (2014), Yuan *et al.* (2017) and others listed therein). However, in these papers the main attention is focused on studying the displacement distribution in soil caused by the moving load. This completes the review of the investigations the subjects of which relate to the subject of the present paper. It follows from this review that up to now systematic investigations of the critical velocity of a moving load acting on the interior of the hollow cylinder surrounded with elastic medium are absent. Moreover, this review shows that the corresponding investigations related to the “pre-stressed hollow cylinder + pre-stressed surrounding elastic medium” are absent completely. Taking this statement into consideration, in the present paper we attempt to investigate the critical velocity of the moving ring load acting on interior surface of the pre-stressed hollow cylinder surrounded with the pre-stressed elastic medium. It is assumed that the mentioned pre-stresses appear as a result of the action of the uniformly distributed normal forces acting at infinity in the direction of the cylinder axis along which the ring load moves. The investigations are made with employing, so-called three-dimensional linearized equations of elastic wave propagation in initially stressed bodies and the axisymmetric case is considered.

Note that the corresponding forced vibration problem is investigated in the paper by Akbarov and Mehdiyev (2017).

2. Formulation of the problem

Consider an infinite body consisting of a hollow circular cylinder with the thickness h and with the external radius R and of a surrounded infinite elastic medium. We associate the cylindrical $Or\theta z$ and Cartesian $Ox_1x_2x_3$ systems of coordinates (Fig. 1) with central axis of the cylinder and the position of points of this body we determine through Lagrange coordinates in these coordinate systems. The stress-strain state in this infinite body we present as a summation of two states which are named as an “initial” and a “perturbed” states. Assume that in the initial state this body is loaded at infinity with uniformly distributed normal forces with intensity q acting in the cylinder's axis direction, and the stress-strain state which appear as result of this

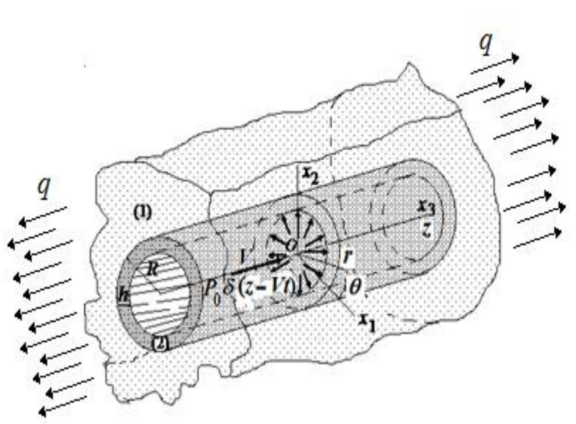


Fig. 1 The sketch of the pre-stressed system consisting of the hollow cylinder and surrounding elastic medium

action we take as initial stress-strain state. Below, the values related to the cylinder and to the surrounding elastic medium will be denoted by upper indices (2) and (1), respectively. Moreover, the values related to the initial state will be denoted by the additional upper index “0”. It is assumed that materials of the cylinder and surrounding medium are homogeneous, isotropic and linear elastic.

The values related to the initial state we determine within the framework of the classical linear theory of elastostatics. Note that, in general, in the initial state the stress state in the body under consideration is inhomogeneous one and this inhomogeneity is caused by the difference of the Poisson’s ratio of the materials of the cylinder (denote it by $\nu^{(2)}$) and surrounding medium (denote it by $\nu^{(1)}$). However, in the cases where $\nu^{(2)}=\nu^{(1)}$ in the initial state the stress state in the bi-material system shown in Fig. 1 is inhomogeneous and is determined as follows.

$$\sigma_{rr}^{(k)0}=\sigma_{\theta\theta}^{(k)0}=\sigma_{r\theta}^{(k)0}=\sigma_{z\theta}^{(k)0}=\sigma_{rz}^{(k)0}=0\,,$$
$$k=1,2\,,\quad \sigma_{zz}^{(1)0}=q\,,\quad \sigma_{zz}^{(2)0}=\frac{E^{(2)}}{E^{(1)}}\sigma_{zz}^{(1)0}\,. \tag{1}$$

Here and below the conventional notation is used.

Thus, in the initial state the stresses are determined through the expression given in (1). In the perturbed state, we assume that the body having the foregoing initial stresses is loaded by additional rotationally symmetric normal ring load with intensity P_0 which moves with constant velocity V in the direction of the cylinder's axis, i.e., in the direction of the Oz axis. We assume that $P_0 \ll q$ and, according to Eringen and Suhubi (1975), Guz (1999, 2004), Akbarov (2015), the strain-stress state caused by this additional loading we describe with the following three-dimensional linearized equations of elastic waves in initially stressed bodies in the axially symmetric case.

Equations of motion

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{\partial\sigma_{rz}^{(k)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(k)}-\sigma_{\theta\theta}^{(k)})+ \tag{2}$$

$$\sigma_{zz}^{(k)0}\frac{\partial^2 u_r^{(k)}}{\partial z^2}=\rho^{(k)}\frac{\partial^2 u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{rz}^{(k)}}{\partial r}+\frac{\partial\sigma_{zz}^{(k)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(k)}+$$

$$\sigma_{zz}^{(k)0}\frac{\partial^2 u_z^{(k)}}{\partial z^2}=\rho^{(k)}\frac{\partial^2 u_z^{(k)}}{\partial t^2}$$

Elasticity relations

$$\sigma_{nn}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{zz}^{(k)})+2\mu^{(k)}\varepsilon_{nn}^{(k)},$$
$$nn=rr;\theta\theta;zz\ ,\ \sigma_{rz}^{(k)}=2\mu^{(k)}\varepsilon_{rz}^{(k)} \tag{3}$$

Strain-displacement relations

$$\varepsilon_{rr}^{(k)}=\frac{\partial u_r^{(k)}}{\partial r},\ \varepsilon_{\theta\theta}^{(k)}=\frac{u_r^{(k)}}{r}\ ,\ \varepsilon_{zz}^{(k)}=\frac{\partial u_z^{(k)}}{\partial z},$$
$$\varepsilon_{rz}^{(k)}=\frac{1}{2}(\frac{\partial u_z^{(k)}}{\partial r}+\frac{\partial u_r^{(k)}}{\partial z}) \tag{4}$$

The Eqs. (2), (3) and (4) are the complete system of field equations of the three-dimensional linearized theory of elastic waves in initially stressed bodies in the case where the strains in the initial state so small that the corresponding strain-stress state can be determined within the scope of the classical linear theory of elastostatics.

Now we consider the formulation of the boundary and contact conditions for the values related to the aforementioned perturbed state, i.e., for the values which appear as a result of the action of the additional load which moves with constant velocity V. According to the foregoing description of the problem, the boundary conditions on the inner face surface of the cylinder can be formulated as follows

$$\sigma_{rr}^{(2)}\Big|_{r=R-h}=-P_0\delta(z-Vt),\ \sigma_{rz}^{(2)}\Big|_{r=R-h}=0. \tag{5}$$

Suppose that the contact conditions with respect to the forces and displacement are continuous

$$\sigma_{rr}^{(1)}\Big|_{r=R}=\sigma_{rr}^{(2)}\Big|_{r=R},\ \sigma_{rz}^{(1)}\Big|_{r=R}=\sigma_{rz}^{(2)}\Big|_{r=R},$$
$$u_r^{(1)}\Big|_{r=R}=u_r^{(2)}\Big|_{r=R}\ .\ u_z^{(1)}\Big|_{r=R}=u_z^{(2)}\Big|_{r=R}. \tag{6}$$

Besides all these, we assume that the moving load velocity is subsonic, i.e., the condition

$$V<\min\left\{c_2^{(1)};c_2^{(2)}\right\},\ c_2^{(k)}=\sqrt{\mu^{(k)}/\rho^{(k)}}\ ,\ k=1,2 \tag{7}$$

occurs and according to this condition, the following decay condition takes place.

$$\left|\sigma_{rr}^{(k)}\right|;\left|\sigma_{\theta\theta}^{(k)}\right|;\left|\sigma_{zz}^{(k)}\right|;\left|\sigma_{rz}^{(k)}\right|;\left|u_r^{(k)}\right|;\left|u_z^{(k)}\right|\to 0,$$
$$k=1,2\ \text{as}\ \sqrt{r^2+(z-Vt)^2}\to\infty \tag{8}$$

This completes formulation of the problem and consideration of the governing field equations.

3. Method of solution

We use the well-known, classical Lame (or Helmholtz) decomposition (see, for instance, Eringen and Suhubi (1975)) for solution to the system of Eqs. (2)-(4)

$$u_r^{(k)}=\frac{\partial\Phi^{(k)}}{\partial r}+\frac{\partial^2\Psi^{(k)}}{\partial r\partial z},$$
$$u_z^{(k)}=\frac{\partial\Phi^{(k)}}{\partial z}-\frac{\partial^2\Psi^{(k)}}{\partial r^2}-\frac{\partial\Psi^{(k)}}{r\partial r}. \tag{9}$$

Substituting the expressions in (9) into the Eqs. (2)-(4), doing corresponding mathematical manipulations we obtain the following equations for the functions $\Phi^{(k)}$ and $\Psi^{(k)}$.

$$\nabla^2\Phi^{(k)}+\sigma_{zz}^{(2)0}\frac{\partial^2\Phi^{(k)}}{\partial z^2}-\frac{1}{(c_1^{(k)})^2}\frac{\partial^2\Phi^{(k)}}{\partial t^2}=0,$$
$$\nabla^2\Psi^{(k)}+\sigma_{zz}^{(k)0}\frac{\partial^2\Psi^{(k)}}{\partial z^2}-\frac{1}{(c_2^{(k)})^2}\frac{\partial^2\Psi^{(k)}}{\partial t^2}=0, \tag{10}$$
$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},$$

where $c_1^{(k)}=\sqrt{(\lambda^{(k)}+2\mu^{(k)})/\rho^{(k)}}$ and $c_2^{(k)}=\sqrt{\mu^{(k)}/\rho^{(k)}}$. Note that in the case where $\sigma_{zz}^{(k)0}=0$ equations in (10) coincide with the corresponding classical equations given, for instance, in the monograph by Eringen and Suhubi (1975).

We introduce the moving coordinate system

$$r'=r,\ z'=z-Vt \tag{11}$$

which moves with the loading internal pressure and by rewriting all the foregoing equations with the coordinates r' and z', we obtain the following equations for the potentials $\Phi^{(k)}$ and $\Psi^{(k)}$

$$\nabla^2\Phi^{(k)}+\left(\frac{\sigma_{zz}^{(k)0}}{\lambda^{(k)}+2\mu^{(k)}}-\frac{V^2}{(c_1^{(k)})^2}\right)\frac{\partial^2\Phi^{(k)}}{\partial z^2}=0,$$
$$\nabla^2\Psi^{(k)}+\left(\frac{\sigma_{zz}^{(k)0}}{\mu^{(k)}}-\frac{V^2}{(c_2^{(k)})^2}\right)\frac{\partial^2\Psi^{(k)}}{\partial z^2}=0, \tag{12}$$

where the primes on the r and z have been omitted.

As a result of the coordinate transformation (11) the first condition in (5) transforms to the following one

$$\sigma_{rr}^{(2)}\Big|_{r=R-h}=-P_0\delta(z) \tag{13}$$

However, the other relations and conditions in (2)-(8) remain valid in the new coordinates determined by (11).

For simplicity of the consideration below we will use the dimensionless coordinates $\bar{r} = r/h$ and $\bar{z} = z/h$ instead of the coordinates r and z, respectively and the over-bar in $\bar{r}$ and $\bar{z}$ will be omitted.

Thus, we consider the solution to the considered boundary value problem which is reduced to the solution to the equations in (12). Using the Fourier transformation with respect to the coordinate z and taking the problem symmetry with respect to the point z=0 into consideration, the sought values can be presented as follows.

$$\left\{\Phi^{(k)};u_r^{(k)};\sigma_{nn}^{(k)};\varepsilon_{nn}^{(k)}\right\}(r,z)= \frac{1}{\pi}\int_0^\infty\left\{\Phi_F^{(k)};u_{rF}^{(k)};\sigma_{nnF}^{(k)};\varepsilon_{nnF}^{(k)}\right\}(r,s)\cos(sz)ds,$$

$$nn = rr;\theta\theta;zz, \tag{14}$$

$$\left\{\Psi^{(k)};u_z^{(k)};\sigma_{rz}^{(k)};\varepsilon_{rz}^{(k)};\right\}(r,z)= \frac{1}{\pi}\int_0^\infty\left\{\Psi_F^{(k)};u_{zF}^{(k)};\sigma_{rzF}^{(k)};\varepsilon_{rzF}^{(k)}\right\}(r,s)\sin(sz)ds$$

After substituting the expressions in (14) into the foregoing equations, relations and contact and boundary conditions, the corresponding ones for the Fourier transformations of the sought values are obtained. In this case the third and fourth relations in (4) and the condition (13) and the relations in (9) transform to the following ones

$$\varepsilon_{zzF}^{(k)} = su_{zF}^{(k)},\quad \varepsilon_{rzF}^{(k)} = \frac{1}{2}\left(\frac{du_{zF}^{(k)}}{dr} + su_{rF}^{(k)}\right),$$

$$\sigma_{rrF}^{(2)}\Big|_{r=R-h} = -P_0,$$

$$u_{rF}^{(k)} = \frac{d\Phi_F^{(k)}}{dr} + s\frac{d\Psi_F^{(k)}}{dr}, \tag{15}$$

$$u_{zF}^{(k)} = -s\Phi_F^{(k)} - \frac{\partial^2\Psi_F^{(k)}}{\partial r^2} - \frac{\partial\Psi_F^{(k)}}{r\partial r}.$$

Moreover, according to the above-noted transformation, we obtain the following equations for $\Phi_F^{(k)}$ and $\Psi_F^{(k)}$ from the equations in (12).

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - s^2\left(1 + \frac{\sigma_{zz}^{(k)0}}{\lambda^{(k)} + 2\mu^{(k)}} - \frac{V^2}{(c_1^{(k)})^2}\right)\right]\Phi_F^{(k)} = 0,$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - s^2\left(1 + \frac{\sigma_{zz}^{(k)0}}{\mu^{(k)}} - \frac{V^2}{(c_2^{(k)})^2}\right)\right]\Psi_F^{(k)} = 0. \tag{16}$$

However, the foregoing other equations and relations are also valid as are for the corresponding Fourier transformations.

Thus, consider the solution to the equations in (16) which, according to the condition (7), can be presented as follows

$$\Phi_F^{(2)} = A_1^{(2)}I_0(q_1^{(2)}r) + A_2^{(2)}K_0(q_1^{(2)}r),$$

$$\Phi_F^{(1)} = A_2^{(1)}K_0(q_1^{(1)}r),$$

$$\Psi_F^{(2)} = B_1^{(2)}I_0(q_2^{(2)}r) + B_2^{(2)}K_0(q_2^{(2)}r),$$

$$\Psi_F^{(1)} = B_2^{(1)}K_0(q_2^{(1)}r),$$

$$q_1^{(k)} = \sqrt{s^2\left(1 + \frac{\sigma_{zz}^{(k)0}}{\lambda^{(k)} + 2\mu^{(k)}} - \frac{V^2}{(c_1^{(k)})^2}\right)}, \tag{17}$$

$$q_2^{(k)} = \sqrt{s^2\left(1 + \frac{\sigma_{zz}^{(k)0}}{\mu^{(k)}} - \frac{V^2}{(c_2^{(k)})^2}\right)}.$$

where $I_0(x)$ and $K_0(x)$ are modified Bessel functions for the purely imaginary arguments of the first and second kind, respectively with zeroth order, $A_1^{(2)}$, $A_2^{(2)}$, $A_2^{(1)}$, $B_1^{(2)}$, $B_2^{(2)}$ and $B_2^{(1)}$ are unknown constants.

Thus, substituting the solutions (17) into the expressions in (15) and into the Fourier transformations of the expressions in (3) and (4) we obtain the following expressions for the Fourier transformation of the sought values.

$$u_{rF}^{(2)} = A_1^{(2)}q_1^{(2)}I_1(q_1^{(2)}r) - A_2^{(2)}q_1^{(2)}K_1(q_1^{(2)}r) + B_1^{(2)}sq_2^{(2)}I_1(q_2^{(2)}r) - B_2^{(2)}sq_2^{(2)}K_1(q_2^{(2)}r),$$

$$u_{rF}^{(1)} = -A_2^{(1)}q_1^{(1)}K_1(q_1^{(1)}r) - B_2^{(1)}sq_2^{(1)}K_1(q_2^{(1)}r),$$

$$u_{zF}^{(2)} = -A_1^{(2)}sI_0(q_1^{(2)}r) - A_2^{(2)}sK_0(q_1^{(2)}r) - B_1^{(2)}q_2^{(2)}I_0(q_1^{(2)}r) - B_2^{(2)}q_2^{(2)}K_0(q_1^{(2)}r),$$

$$u_{zF}^{(1)} = -A_2^{(1)}sK_0(q_1^{(1)}r) - B_2^{(1)}q_2^{(1)}K_0(q_1^{(1)}r),$$

$$\begin{aligned}\sigma_{rzF}^{(2)} = \mu^{(2)}\Big[& A_1^{(2)}\Big(0.5(q_1^{(2)})^2(I_0(q_1^{(2)}r) + I_2(q_1^{(2)}r)) - s^2I_0(q_1^{(2)}r)\Big) + \\ & A_2^{(2)}\Big(0.5(q_1^{(2)})^2(K_0(q_1^{(2)}r) + K_2(q_1^{(2)}r)) - s^2K_0(q_1^{(2)}r)\Big) + \\ & B_1^{(2)}\Big(0.5s(q_2^{(2)})^2(I_0(q_2^{(2)}r) + I_2(q_2^{(2)}r)) - sq_2^{(2)}I_0(q_2^{(2)}r)\Big) + \\ & B_2^{(2)}\Big(0.5s(q_2^{(2)})^2(K_0(q_2^{(2)}r) + K_2(q_2^{(2)}r)) - sq_2^{(2)}K_0(q_2^{(2)}r)\Big)\Big],\end{aligned} \tag{18}$$

$$
\begin{aligned}
\sigma_{rzF}^{(1)} = \mu^{(2)} \Bigg[& A_2^{(1)} \Big(0.5(q_1^{(2)})^2 (K_0(q_1^{(1)} r) + \\
& K_2(q_1^{(1)} r)) - s^2 K_0(q_1^{(1)} r) \Big) + \\
& B_2^{(1)} \Big(0.5 s (q_2^{(1)})^2 (K_0(q_2^{(1)} r) + K_2(q_2^{(1)} r)) \,, \\
& -s q_2^{(1)} K_0(q_2^{(1)} r) \Big) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{rrF}^{(2)} = 2\mu^{(2)} \Bigg[& A_1^{(2)} \Bigg[(1 + \frac{\lambda^{(2)}}{2\mu^{(2)}})(q_1^{(2)})^2 \times \\
& 0.5(I_0(q_1^{(2)} r) + I_2(q_1^{(2)} r)) + \\
& \frac{\lambda^{(2)}}{2\mu^{(2)}} (\frac{q_1^{(2)}}{r} I_1(q_1^{(2)} r) - s^2 I_0(q_1^{(2)} r)) \Bigg] + \\
& A_2^{(2)} \Bigg[(1 + \frac{\lambda^{(2)}}{2\mu^{(2)}})(q_1^{(2)})^2 \times \\
& 0.5(K_0(q_1^{(2)} r) + K_2(q_1^{(2)} r)) + \\
& \frac{\lambda^{(2)}}{2\mu^{(2)}} (-\frac{q_1^{(2)}}{r} K_1(q_1^{(2)} r) - s^2 K_0(q_1^{(2)} r)) \Bigg] + \\
& B_1^{(2)} \Bigg[(1 + \frac{\lambda^{(2)}}{2\mu^{(2)}}) s (q_2^{(2)})^2 \times \\
& 0.5(I_0(q_2^{(2)} r) + I_2(q_2^{(2)} r)) \\
& + \frac{\lambda^{(2)}}{2\mu^{(2)}} (\frac{s q_2^{(2)}}{r} I_1(q_2^{(2)} r) - s q_2^{(2)} I_0(q_2^{(2)} r)) \Bigg] + \\
& B_2^{(2)} \Bigg[(1 + \frac{\lambda^{(2)}}{2\mu^{(2)}}) s (q_2^{(2)})^2 \times \\
& 0.5(K_0(q_2^{(2)} r) + K_2(q_2^{(2)} r)) + \\
& \frac{\lambda^{(2)}}{2\mu^{(2)}} (-\frac{s q_2^{(2)}}{r} K_1(q_2^{(2)} r) - s q_2^{(2)} K_0(q_2^{(2)} r)) \Bigg] \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{rrF}^{(1)} = 2\mu^{(1)} \Bigg[& A_2^{(1)} \Bigg[(1 + \frac{\lambda^{(1)}}{2\mu^{(1)}})(q_1^{(1)})^2 \times \\
& 0.5(K_0(q_1^{(2)} r) + K_2(q_1^{(1)} r)) + \\
& \frac{\lambda^{(1)}}{2\mu^{(1)}} (-\frac{q_1^{(1)}}{r} K_1(q_1^{(1)} r) - s^2 K_0(q_1^{(1)} r)) \Bigg] + \\
& B_2^{(1)} \Bigg[(1 + \frac{\lambda^{(1)}}{2\mu^{(1)}}) s (q_2^{(1)})^2 \times \\
& 0.5(K_0(q_2^{(1)} r) + K_2(q_2^{(1)} r)) + \\
& \frac{\lambda^{(1)}}{2\mu^{(1)}} (-\frac{s q_2^{(1)}}{r} K_1(q_2^{(1)} r) - s q_2^{(1)} K_0(q_2^{(1)} r)) \Bigg] \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\theta\theta F}^{(2)} = 2\mu^{(2)} \Bigg[& A_1^{(2)} \Bigg[\frac{\lambda^{(2)}}{2\mu^{(2)}} ((q_1^{(2)})^2 \times \\
& 0.5(I_0(q_1^{(2)} r) + I_2(q_1^{(2)} r)) - \\
& s^2 I_0(q_1^{(2)} r)) + (1 + \frac{\lambda^{(2)}}{2\mu^{(2)}}) \frac{q_1^{(2)}}{r} I_1(q_1^{(2)} r) \Bigg] + \\
& A_2^{(2)} \Bigg[\frac{\lambda^{(2)}}{2\mu^{(2)}} ((q_1^{(2)})^2 0.5(K_0(q_1^{(2)} r) \\
& + K_2(q_1^{(2)} r)) - s^2 K_0(q_1^{(2)} r)) + \\
& (1 + \frac{\lambda^{(2)}}{2\mu^{(2)}})(-\frac{q_1^{(2)}}{r} K_1(q_1^{(2)} r)) \Bigg] + \\
& B_1^{(2)} \Bigg[\frac{\lambda^{(2)}}{2\mu^{(2)}} (s (q_2^{(2)})^2 0.5(I_0(q_2^{(2)} r) + \\
& I_2(q_2^{(2)} r)) - s q_2^{(2)} I_0(q_2^{(2)} r)) + \\
& (1 + \frac{\lambda^{(2)}}{\mu^{(2)}}) \frac{s q_2^{(2)}}{r} I_1(q_2^{(2)} r) \Bigg] + \\
& B_2^{(2)} \Bigg[\frac{\lambda^{(2)}}{2\mu^{(2)}} (s (q_2^{(2)})^2 0.5(K_0(q_2^{(2)} r) + \\
& K_2(q_2^{(2)} r)) - s q_2^{(2)} K_0(q_2^{(2)} r)) + \\
& (1 + \frac{\lambda^{(2)}}{\mu^{(2)}})(-\frac{s q_2^{(2)}}{r} K_1(q_2^{(2)} r) \Bigg] \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{\theta\theta F}^{(1)} = 2\mu^{(1)} \Bigg[& A_2^{(1)} \Bigg[\frac{\lambda^{(1)}}{2\mu^{(1)}} ((q_1^{(1)})^2 0.5 \times \\
& (K_0(q_1^{(1)} r) + K_2(q_1^{(1)} r)) - s^2 K_0(q_1^{(1)} r)) \\
& + (1 + \frac{\lambda^{(1)}}{2\mu^{(1)}})(-\frac{q_1^{(1)}}{r} K_1(q_1^{(1)} r)) \Bigg] + \\
& B_2^{(1)} \Bigg[\frac{\lambda^{(1)}}{2\mu^{(1)}} (s (q_2^{(1)})^2 0.5(K_0(q_2^{(1)} r) + \\
& K_2(q_2^{(1)} r)) - s q_2^{(1)} K_0(q_2^{(1)} r)) + \\
& (1 + \frac{\lambda^{(1)}}{\mu^{(1)}})(-\frac{s q_2^{(1)}}{r} K_1(q_2^{(1)} r) \Bigg] \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{zzF}^{(2)} = 2\mu^{(2)} \Bigg[& A_1^{(2)} \Bigg[\frac{\lambda^{(2)}}{2\mu^{(2)}} ((q_1^{(2)})^2 \times \\
& 0.5(I_0(q_1^{(2)} r) + I_2(q_1^{(2)} r)) + \\
& \frac{q_1^{(2)}}{r} I_1(q_1^{(2)} r)) - (1 + \frac{\lambda^{(2)}}{2\mu^{(2)}}) s^2 I_0(q_1^{(2)} r)) \Bigg] +
\end{aligned}
$$

$$A_2^{(2)}\left[\frac{\lambda^{(2)}}{2\mu^{(2)}}((q_1^{(2)})^2 0.5\times\right.$$

$$(K_0(q_1^{(2)}r)+K_2(q_1^{(2)}r))-\frac{q_1^{(2)}}{r}K_1(q_1^{(2)}r))-$$

$$\left.(1+\frac{\lambda^{(2)}}{2\mu^{(2)}})s^2K_0(q_1^{(2)}r)\right]+$$

$$B_1^{(2)}\left[\frac{\lambda^{(2)}}{2\mu^{(2)}}(s(q_2^{(2)})^2\times\right.$$

$$0.5(I_0(q_2^{(2)}r)+I_2(q_2^{(2)}r))+\frac{sq_2^{(2)}}{r}I_1(q_2^{(2)}r))$$

$$\left.-(1+\frac{\lambda^{(2)}}{2\mu^{(2)}})sq_2^{(2)}I_0(q_2^{(2)}r)\right]+$$

$$B_2^{(2)}\left[\frac{\lambda^{(2)}}{2\mu^{(2)}}(s(q_2^{(2)})^2 0.5(K_0(q_2^{(2)}r)+\right.$$

$$K_2(q_2^{(2)}r))-\frac{sq_2^{(2)}}{r}K_1(q_2^{(2)}r))-$$

$$\left.\left.(1+\frac{\lambda^{(2)}}{2\mu^{(2)}})sq_2^{(2)}K_0(q_2^{(2)}r)\right]\right],$$

$$\sigma_{zzF}^{(1)}=2\mu^{(1)}\left[A_2^{(1)}\left[\frac{\lambda^{(1)}}{2\mu^{(1)}}((q_1^{(1)})^2\times\right.\right.$$

$$0.5(K_0(q_1^{(2)}r)+K_2(q_1^{(2)}r))-$$

$$\left.\frac{q_1^{(1)}}{r}K_1(q_1^{(1)}r))-(1+\frac{\lambda^{(1)}}{2\mu^{(1)}})s^2K_0(q_1^{(1)}r)\right]+$$

$$B_2^{(1)}\left[\frac{\lambda^{(1)}}{2\mu^{(1)}}(s(q_2^{(1)})^2\times\right.$$

$$0.5(K_0(q_2^{(1)}r)+K_2(q_2^{(1)}r))\ -\frac{sq_2^{(1)}}{r}K_1(q_2^{(1)}r))$$

$$\left.\left.-(1+\frac{\lambda^{(1)}}{2\mu^{(1)}})sq_2^{(1)}K_0(q_2^{(1)}r))\right]\right]$$

Substituting the expressions in (18) into the Fourier transformations of the corresponding boundary and contact conditions in (5), (6) and (13) we obtain the following algebraic equations with respect to the unknown constants $A_1^{(2)}$, $A_2^{(2)}$, $A_2^{(1)}$, $B_1^{(2)}$, $B_2^{(2)}$ and $B_2^{(1)}$.

$$\sigma_{rrF}^{(2)}\Big|_{r=R-h}=-P_0\Rightarrow\alpha_{11}A_1^{(2)}+\alpha_{12}A_2^{(2)}+$$

$$\alpha_{13}B_1^{(2)}+\alpha_{14}B_2^{(2)}+\alpha_{15}B_1^{(1)}+\alpha_{16}B_2^{(1)}=-P_0,\qquad(19)$$

$$\sigma_{rzF}^{(2)}\Big|_{r=R-h}=0\Rightarrow\alpha_{21}A_1^{(2)}+\alpha_{22}A_2^{(2)}+$$

$$\alpha_{23}B_1^{(2)}+\alpha_{24}B_2^{(2)}+\alpha_{25}B_1^{(1)}+\alpha_{26}B_2^{(1)}=0,$$

$$\sigma_{rr}^{(1)}\Big|_{r=R}=\sigma_{rr}^{(2)}\Big|_{r=R}\Rightarrow\alpha_{31}A_1^{(2)}+\alpha_{32}A_2^{(2)}+$$

$$\alpha_{33}B_1^{(2)}+\alpha_{34}B_2^{(2)}+\alpha_{35}B_1^{(1)}+\alpha_{36}B_2^{(1)}=0,$$

$$\sigma_{rz}^{(1)}\Big|_{r=R}=\sigma_{rz}^{(2)}\Big|_{r=R}\Rightarrow\alpha_{41}A_1^{(2)}+\alpha_{42}A_2^{(2)}+$$

$$\alpha_{43}B_1^{(2)}+\alpha_{44}B_2^{(2)}+\alpha_{45}B_1^{(1)}+\alpha_{46}B_2^{(1)}=0,$$

$$u_r^{(1)}\Big|_{r=R}=u_r^{(2)}\Big|_{r=R}\Rightarrow\alpha_{51}A_1^{(2)}+\alpha_{52}A_2^{(2)}+$$

$$\alpha_{53}B_1^{(2)}+\alpha_{54}B_2^{(2)}+\alpha_{55}B_1^{(1)}+\alpha_{56}B_2^{(1)}=0,$$

$$u_z^{(1)}\Big|_{r=R}-u_z^{(2)}\Big|_{r=R}=\frac{FR}{\mu^{(1)}}\sigma_{rz}^{(1)}\Big|_{r=R}\Rightarrow$$

$$\alpha_{61}A_1^{(2)}+\alpha_{62}A_2^{(2)}+\alpha_{63}B_1^{(2)}+$$

$$\alpha_{64}B_2^{(2)}+\alpha_{65}B_1^{(1)}+\alpha_{66}B_2^{(1)}=0$$

Note that the coefficients α_{ij} in (19), where $i;j=1,2,3,...,6$ can be easily determined from the expressions in (18).

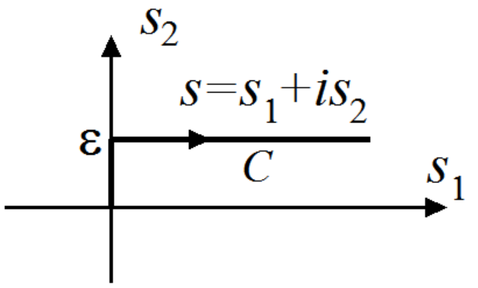
Thus, after solving the equations in (19) with respect to the unknowns $A_1^{(2)},A_2^{(2)},B_1^{(2)}$, $B_2^{(2)}$, $A_2^{(1)}$ and $B_2^{(1)}$ we determine completely the Fourier transformations of all the sought values and, substituting these values into the integrals in (14) and calculating these integrals, we determine the originals of the stresses and displacements which are caused by the action of the moving ring load acting on the interior of the hollow cylinder.

This completes the consideration of the solution method.

4. Numerical results and discussions

4.1 On the calculation algorithm

Numerical results on the critical velocity of the moving load and on the influence of the initial stresses in the cylinder and surrounding elastic medium on these critical velocities, as well as numerical results on the influence of the initial stresses on the interface stresses which appear as a result of the action of the moving load, are obtained through the numerical calculation of the integrals in (14). Note that the algorithm for this calculation is based on the Sommerfield contour method and is developed in the papers by Akbarov *et al.* (2015), Akbarov and Ismailov (2015, 2016a, 2016b) and other ones listed therein. Moreover, the mentioned algorithm is also detailed in the monograph by Akbarov (2015). Therefore, here we do not consider detailed description of this algorithm and note that the used Sommerfield contour is selected such as in Fig. 2, according to which the calculation of the integrals in (14) are reduced to the calculation of the following ones.



c

Fig. 2 The sketch of the Sommerfeld contour

$$\left\{\Phi^{(k)}; u_r^{(k)}; \sigma_{nn}^{(k)}; \varepsilon_{nn}^{(k)}\right\}(r,z) \approx$$
$$\frac{1}{\pi}\mathrm{Re}\int_0^{+\infty}\left\{\Phi_F^{(k)}; u_{rF}^{(k)}; \sigma_{nnF}^{(k)}; \varepsilon_{nnF}^{(k)}\right\}\times$$
$$(r,z)\cos((s_1+i\varepsilon)z)ds_1,$$
$$\left\{\Psi^{(k)}; u_z^{(k)}; \sigma_{rz}^{(k)}; \varepsilon_{rz}^{(k)};\right\}(r,z) \approx \tag{20}$$
$$\frac{1}{\pi}\mathrm{Re}\int_0^{+\infty}\left\{\Psi_F^{(k)}; u_{zF}^{(k)}; \sigma_{rzF}^{(k)}; \varepsilon_{rzF}^{(k)}\right\}\times$$
$$(r,z)\sin((s_1+i\varepsilon)z)ds_1.$$

Note that during calculation of the integrals in (20) the improper integral $\int_0^{+\infty}(\bullet)ds_1$ is replaced with the corresponding definite integral $\int_0^{S_1^*}(\bullet)ds_1$ and the values of S_1^* are determined from the corresponding convergence requirement. Moreover, during the calculation of the integral $\int_0^{S_1^*}(\bullet)ds_1$, the interval $[0, S_1^*]$ is divided into a certain number (denote this number through N) of shorter intervals and within each of these intervals the integrals are calculated by the use of the Gauss algorithm with ten integration points. The values of the integrated functions at these integration points are calculated through the solution of the Eq. (19). All these procedures are performed automatically in the PC by use of the corresponding programs constructed by the authors of the present paper in MATLAB. Numerical results presented in the present paper are obtained in the case where N=200, $S_1^* = 9$ and ε=0.001 under which these results have sufficient high accuracy in the convergence sense and in the trustiness sense.

4.2 Numerical results related to the influence of the initial stress on the critical velocity

We introduce the notation

$$\sigma_{rr}(z) = \sigma_{rr}^{(1)}(R,z) = \sigma_{rr}^{(2)}(R,z) \tag{21}$$

First, we note that the critical velocity is determined from the following criterion: the critical velocity is the velocity with approaching to which the absolute values of the interface stress $\sigma_{rr}(z)$ (21) (or any quantities characterizing the displacement and stress-strain state of the system under consideration in the perturbed state) increase indefinitely. This criterion is general one and can be applied for the cases where the materials of the hollow cylinder and surrounding elastic medium are viscoelastic ones.

Numerical results which will be discussed below are obtained in the following three cases

$$\text{Case 1.}\quad E^{(1)}/E^{(2)} = 0.35,\ \rho^{(1)}/\rho^{(2)} = 0.1,\quad \nu^{(1)} = \nu^{(2)} = 0.25. \tag{22}$$

$$\text{Case 2.}\quad E^{(1)}/E^{(2)} = 0.05,\ \rho^{(1)}/\rho^{(2)} = 0.01,\quad \nu^{(1)} = \nu^{(2)} = 0.25. \tag{23}$$

$$\text{Case 3.}\quad E^{(1)}/E^{(2)} = 0.5,\ \rho^{(1)}/\rho^{(2)} = 0.5,\quad \nu^{(1)} = \nu^{(2)} = 0.3. \tag{24}$$

As follows from the relations (22)-(24) that the Poisson's ratio of all the selected pairs of materials are equal to each other and therefore for these pairs the homogeneity of the initial stresses, i.e. the relations in (1) are satisfied exactly. Note that Case 2 was also considered in the paper by Abdulkadirov (1981) under h/R=0.5 and Case 3 was considered in the paper by Babich *et al.* (1986) under $h/R\to 0$. Consequently, the critical velocity obtained for Case 2 under h/R=0.5 must coincide with the corresponding one obtained in the paper by Abdulkadirov (1981) and the critical velocities obtained for Case 3 must approach to the critical velocity obtained by Babich *et al.* (1986) with decreasing the ratio h/R. Note that these predictions relate only to the case where the initial stresses in the cylinder and surrounding elastic medium are absent. Moreover, note that for selected pairs of materials the relations

$$\frac{c_2^{(1)}}{c_2^{(2)}} = 3.5 \text{ in Case 1},\quad \frac{c_2^{(1)}}{c_2^{(2)}} = 5 \text{ in Case 2},\quad \text{and } \frac{c_2^{(1)}}{c_2^{(2)}} = 1.0 \text{ in Case 3} \tag{25}$$

takes place and according to these relation, it can be concluded that if $V/c_2^{(2)} < 1$, then the moving velocity of the ring load is subsonic.

Thus, we consider the results related to the influence of the initial stress on the values of the dimensionless critical velocity

$$c_{cr} = V_{cr}/c_2^{(2)} \tag{26}$$

and for estimation this influence we introduce the following notation.

Table 1 The influence of the dimensionless initial stress $\eta = \sigma_{zz}^{(2)0} / \mu^{(2)}$ on the values of the dimensionless critical velocity $c_{cr} = V_{cr} / c_2^{(2)}$ in Case 1

η	h/R			
	0.5	0.2	0.1	0.05
0.000	0.935	0.864	0.843	0.836
–0.001	0.935	0.863	0.843	0.835
+0.001	0.936	0.864	0.844	0.837
–0.005	0.932	0.861	0.840	0.832
+0.005	0.938	0.867	0.847	0.839
–0.01	0.930	0.856	0.837	0.829
+0.01	0.942	0.870	0.850	0.842
–0.02	0.926	0.851	0.830	0.822
+0.02	0.946	0.876	0.856	0.849
–0.03	0.919	0.845	0.824	0.816
+0.03	0.951	0.881	0.862	0.855
–0.04	0.915	0.838	0.817	0.809
+0.04	0.958	0.888	0.869	0.861
–0.05	0.908	0.832	0.810	0.802
+0.05	0.964	0.894	0.875	0.868

Table 2 The influence of the dimensionless initial stress $\eta = \sigma_{zz}^{(2)0} / \mu^{(2)}$ on the values of the dimensionless critical velocity $c_{cr} = V_{cr} / c_2^{(2)}$ in Case 2

η	h/R			
	0.5	0.2	0.1	0.05
0.000	0.826 prest. 0.826 by Abdulkadirov (1981)	0.617	0.529	0.488
–0.001	0.825	0.616	0.528	0.487
+0.001	0.826	0.618	0.530	0.489
–0.005	0.823	0.613	0.524	0.483
+0.005	0.829	0.621	0.533	0.493
–0.01	0.820	0.609	0.519	0.478
+0.01	0.832	0.625	0.533	0.498
–0.02	0.813	0.601	0.510	0.467
+0.02	0.838	0.633	0.547	0.508
–0.03	0.807	0.592	0.500	0.457
+0.03	0.844	0.641	0.556	0.516
–0.04	0.801	0.584	0.490	0.446
+0.04	0.849	0.649	0.565	0.527
–0.05	0.795	0.575	0.479	0.434
+0.05	0.855	0.656	0.574	0.537

$$\eta = \sigma_{zz}^{(2)0} / \mu^{(2)} \tag{27}$$

These results are given in Tables 1 (for Case 1), 2 (for Case 2) and 3 (for Case 3) which are obtained for various values of the parameter η and the ratio h/R. In Tables 2 and 3 in particular cases the corresponding results which mentioned above and obtained in the papers by Abdulkadirov (1981) and Babich *et al.* (1986) are also given. The comparison of the present results with corresponding ones obtained in the papers by Abdulkadirov (1981) and Babich *et al.* (1986) shows the trustiness of the foregoing prediction and in this way, it is proven the validity of the used PC programs and algorithm which are used under obtaining the discussed numerical results.

Table 3 The influence of the dimensionless initial stress $\eta = \sigma_{zz}^{(2)0} / \mu^{(2)}$ on the values of the dimensionless critical velocity $c_{cr} = V_{cr} / c_2^{(2)}$ in Case 3

η	h/R			
	0.5	0.2	0.1	0.05
0.000	0.939	0.874	0.854	0.847 prest. 0.832 (by Babich *et al.* (1986))
–0.001	0.939	0.873	0.854	0.846
+0.001	0.940	0.874	0.855	0.847
–0.005	0.937	0.871	0.851	0.844
+0.005	0.942	0.877	0.857	0.849
–0.01	0.934	0.868	0.848	0.841
+0.01	0.944	0.879	0.860	0.852
–0.02	0.928	0.862	0.842	0.835
+0.02	0.950	0.885	0.866	0.858
–0.03	0.923	0.856	0.835	0.829
+0.03	0.955	0.891	0.872	0.864
–0.04	0.918	0.851	0.831	0.823
+0.04	0.963	0.896	0.877	0.870
–0.05	0.912	0.845	0.824	0.815
+0.05	0.968	0.902	0.883	0.876

Thus, we turn to the discussion of the results given in Tables 1, 2 and 3, according to which, it can be concluded that an initial stretching of the constituents of the system under consideration causes an increase, however an initial compression of these constituents causes a decrease in the values of the critical velocities c_{cr} (26).

This conclusion agrees in the qualitative sense with the corresponding ones obtained in the papers by Babich *et al.* (1986), Akbarov *et al.* (2015) and other ones detailed in the monograph by Akbarov (2015).

Also, these results show that the magnitude of the influence of the initial stresses on the values of the critical velocities increase with decreasing the ratio h/R. Moreover, the mentioned influence in Case 2 is more considerable than that obtained in Case 1 and Case 3. So that, in Case 1 and in Case 3 the influence of the initial stress on the critical velocity is not more than 5-6%, however this influence in Case 2 may be greater than 17%.

This completes the consideration of the numerical results related to the influence of the initial stresses on the critical velocity.

4.3 Numerical results related to the influence of the initial stresses on the interface stress distribution

Consider numerical results which illustrate the influence

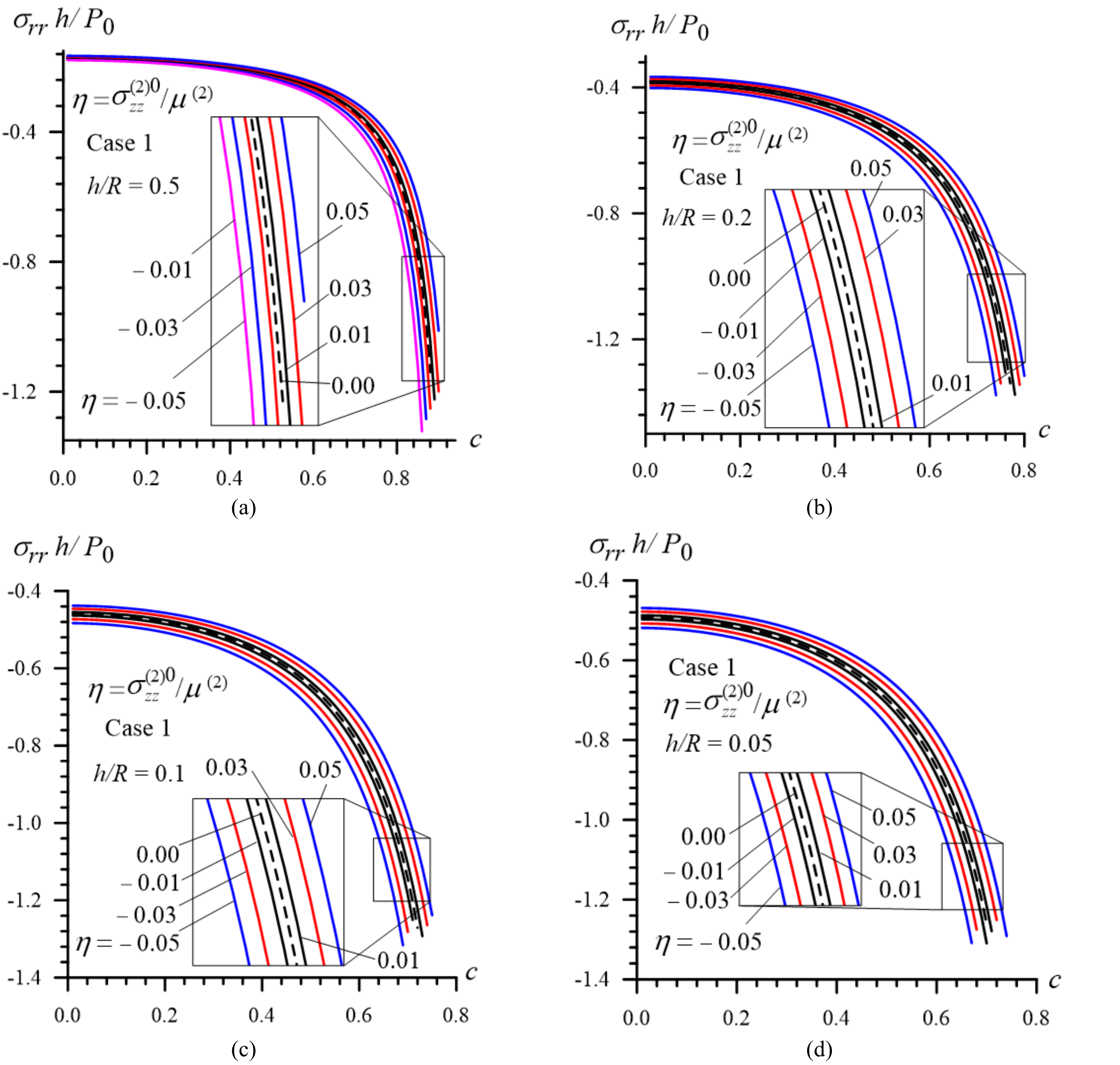


Fig. 3 Response of the interface normal stress at the point z/h=0 to the moving load velocity in Case 1 (22) under h/R=0.5 (a), 0.2 (b), 0.1 (c) and 0.05 (d)

of the parameter η (27) (i.e., the influence of the initial stresses) on the response of the interface normal stress $\sigma_{rr}(z)$ to the dimensionless velocity $c = V / c_2^{(2)}$ of the moving load. For this purpose, we consider the graphs of the dependence between $\sigma_{zz}(=\sigma_{zz}(0))$ and c constructed for various values of the parameter η. These graphs are given in Figs. 3, 4 and 5 which relate to Case 1 (22), Case 2 (23) and Case 3 (24), respectively. Note that in these figures the graphs grouped by letters a, b, c and d correspond the cases where h/R=0.5, 0.2, 0.1 and 0.05, respectively. Moreover, note that under construction these graphs it is assumed that $c<c_{cr}$, i.e., the velocity of the moving load is less than the corresponding critical velocity.

Thus, it follows from Figs. 3, 4 and 5 that the absolute values of the interface normal stress increase (decrease) as a result of the initial compression (of the initial stretching) and the magnitude of this increase (decrease) becomes more considerable as the moving velocity approaches the critical one. Moreover, these results show that the magnitude of the influence of the initial stresses on the values of the interface normal stress becomes more significantly with decreasing of the ratio h/R. For instance, according to Fig. 4(d), in the case where h/R=0.05 in Case 2 as result of the initial compression the absolute values of the stress under consideration may be greater two times than that obtained in the case where the mentioned initial compression is absent. The comparison the graphs grouped by the letters a, b, c and d shows that, as a result of the decrease in the values of the ratio h/R the mentioned influence becomes significantly not only for the cases where the moving velocity approaches to the corresponding critical velocity,

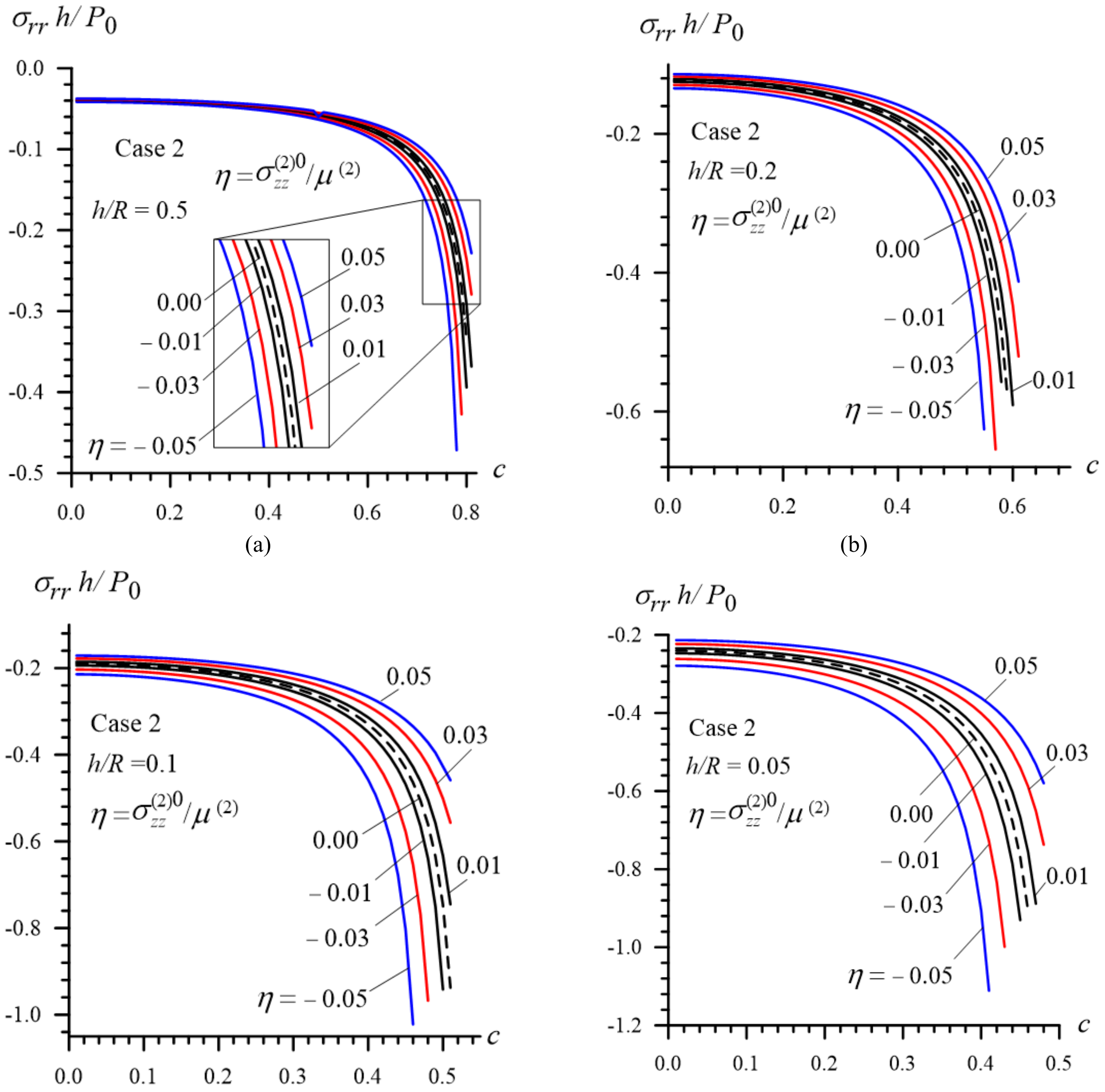


Fig. 4 Response of the interface normal stress at the point z/h=0 to the moving load velocity in Case 2 (23) under h/R=0.5 (a), 0.2 (b), 0.1 (c) and 0.05 (d)

but also for the cases where moving velocity is near to the 0.01. At the same time, the comparison of the results illustrated with each other shows that in Case 2 the influence of the initial stresses on the values of the interface normal stress is more significantly than that in Case 1 and Case 3. Consequently, the magnitude of the influence of the initial stresses on the values of the normal stress depends significantly also on the mechanical properties of the constituents of the system under consideration.

Now we consider numerical results related to the distribution of the interface normal and shear stresses with respect to the z/h. Epures of these distributions are given in Figs. 6 and 7 for the interface normal stress σ_{rr} (21) and for the interface shear stress $\sigma_{rz}(z) = \sigma_{rz}^{(1)}(z,R) = \sigma_{rz}^{(2)}(z,R)$, respectively. In these figures the graphs grouped by letters a and b relate to the cases where c=0.5 and 0.7, respectively and Case 1 under h/R=0.2 is considered.

Epures of the distribution of the interface normal stress and interface shear stress obtained in the case where h/R=0.05 are given in Fig. 8(a) and Fig. 8(b), respectively. Note that under construction the graphs given in Fig. 8 it is also Case 1 is considered and it is assumed that c=0.7.

Thus, it follows from Fig. 6 that in a certain distance from the point at which the moving load acts at behind and ahead of this point the interface normal stress becomes stretched one. Note that this moment can play important role in the adhesion strength of the system “hollow cylinder

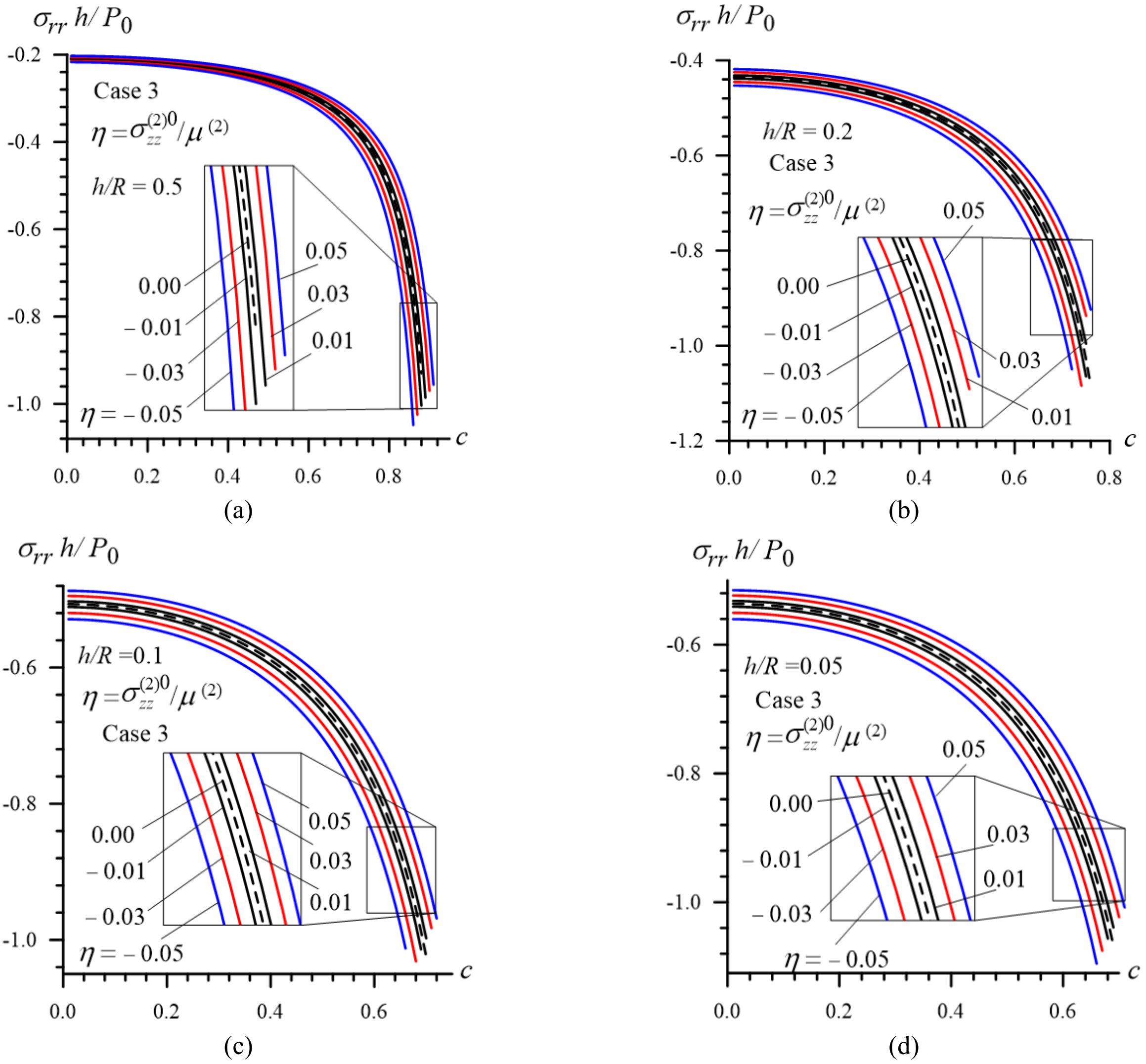


Fig. 5 Response of the interface normal stress at the point z/h=0 to the moving load velocity in Case 3 (24) under h/R=0.5 (a), 0.2 (b), 0.1 (c) and 0.05 (d)

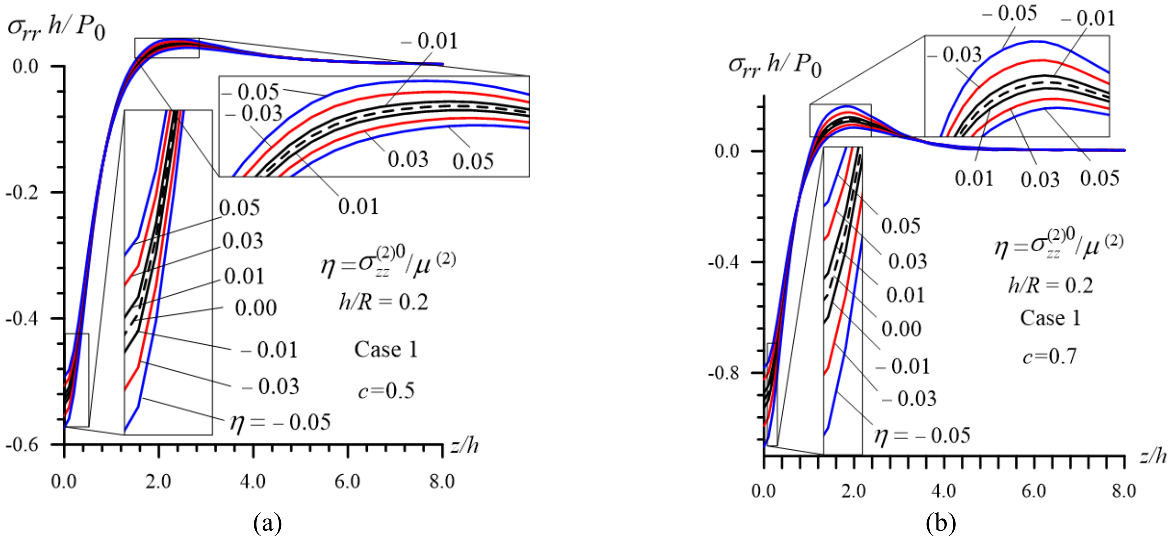
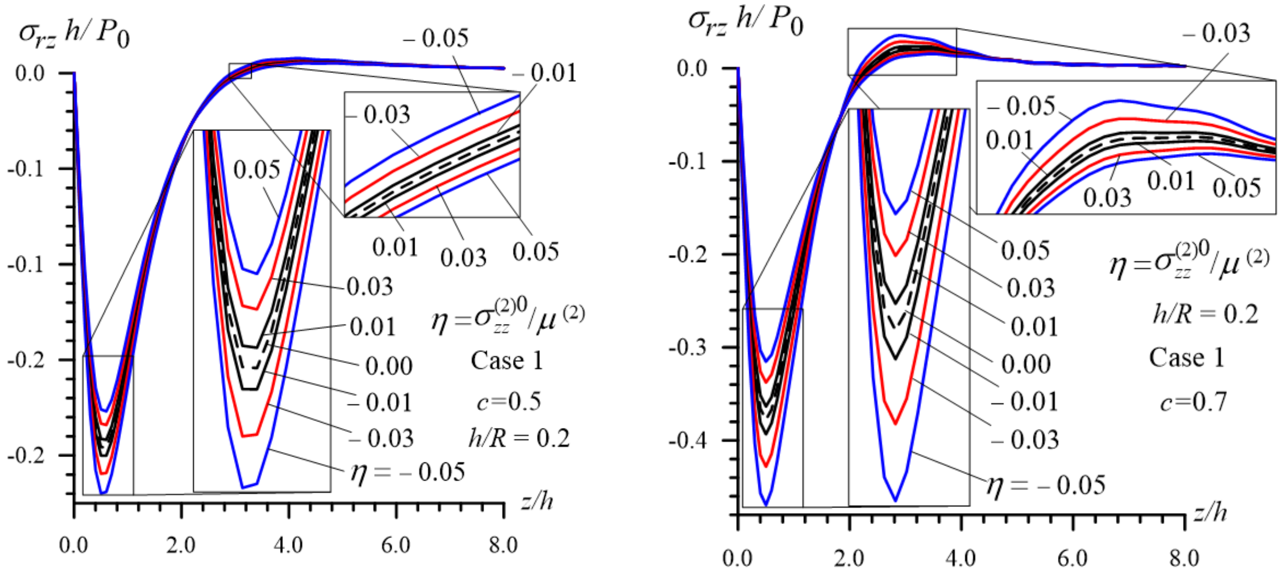


Fig. 6 Distribution of the interface normal stress with respect to z/h under c=0.5 (a) and 0.7 (b) in Case 1

+surrounding elastic medium" and the results given in Fig. 6 shows that the values of this stretching normal stress



(a) (b)

Fig. 7 Distribution of the interface shear stress with respect to z/h under c=0.5 (a) and 0.7 (b) in Case 1 for h/R=0.2

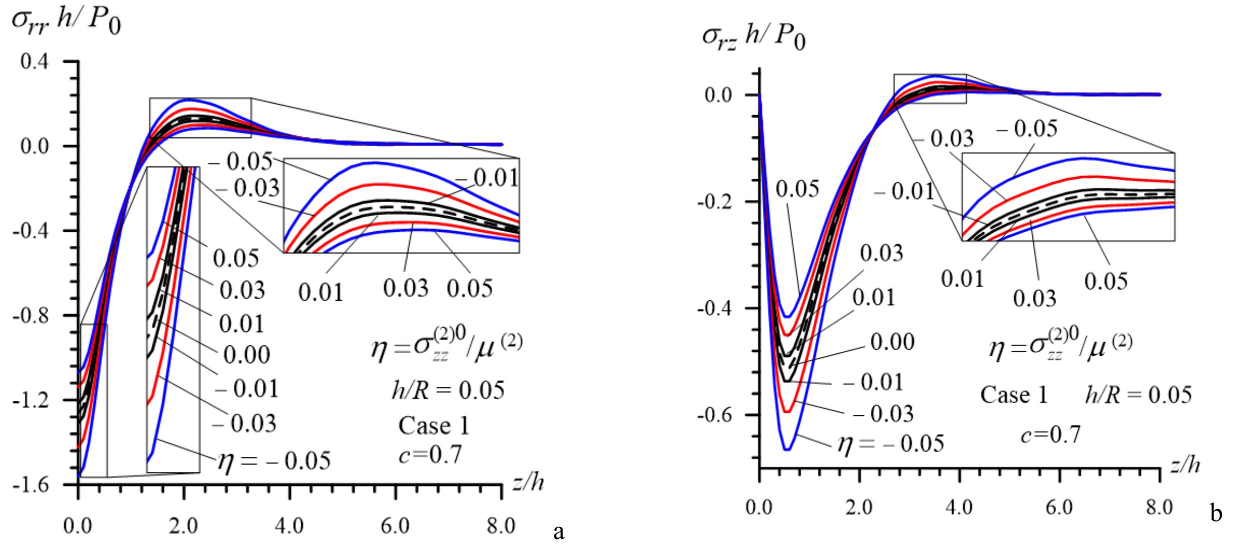


Fig. 8 Distribution of the interface normal (a) and shear (b) stresses with respect to z/h under h/R=0.05 and c=0.7 in Case 1

increase (decrease) with initial compression (with initial stretching) of this system. Moreover, the comparison of the results given in Fig. 6(a) with corresponding ones given in Figs. 6(b) and 8(a) shows that an increase in the values of the velocity of the moving load and a decrease in the values of the ratio h/R cause an increase in the values of the aforementioned stretching interface normal stress, as well cause an increase of the magnitude of the influence of the initial stresses on the values of this stress Analysis of the graphs given in Fig. 7 and in Fig. 8(b) shows that the influence of the initial stresses on the absolute values of the interface shear stress is similar in the quantitative sense with that related to the interface normal stress. These graphs also show that the shear stress has its absolute maximum value in a certain distance from the point at which the moving load acts. The comparison of the graphs given in Fig. 7(a) with corresponding ones given in Fig. 7(b) and 8(b) shows that the influence of the initial stresses on the values of the interface shear stress increases with increasing the load moving velocity and with decreasing the ratio h/R.

Finally, we note the following statement. For this purpose, we recall that the coordinate z with respect to which the distribution of the interface stresses is illustrated in Figs. 6, 7 and 8, is the coordinate in the moving coordinate system determined by expressions in (11), i.e., the z/h in these figures is the z'/h. Consequently, for more correct explanation of the results given in Figs. 6, 7 and 8 we must take into consideration $(z-Vt)/h$ (where z is a coordinate in the reference coordinate system) instead of z/h. According to this consideration, if we fix the time $t(=t^*)$ then these figures illustrate the distribution of the stresses with respect to spatial coordinate ($z'=z-Vt^*$) which shows the distance in the cylinder's axis direction from the point at which the moving load acts. If we fix the spatial coordinate

$z=z^*$ in the reference coordinate system, then, according to $z'=z^*-Vt$, the Figs. 6, 7 and 8 illustrate the change of the stresses with respect to time at the mentioned fixed point. Consequently, the results given in Figs. 6, 7 and 8 illustrate not only the distribution of the interface stresses with respect to the spatial coordinate, but also these graphs illustrate the change of these stresses with respect to time.

This completes the considerations and analysis of the numerical results.

5. Conclusions

Thus, in the present paper the problems related to the dynamics of the moving normal ring load acting on the interior of the pre-stressed hollow cylinder and the pre-stressed surrounding elastic medium is investigated with employing the three-dimensional linearized theory of elastic waves in initially stressed bodies. It is assumed that the initial stresses in the constituents of the system under consideration appear as a result of the action of the uniformly distributed normal forces applied at infinity in the cylinder's axis direction which coincides with load moving direction and the case where the initial stresses are homogeneous, is considered. The equations for the potentials which enter into the classical Lame decomposition for displacements are obtained for the considered case which coincide with corresponding ones used the classical elastodynamics under absent of the initial stresses. The Fourier transform is employed with respect to the spatial axial coordinate and Fourier transformation of sought values are determined analytically, however originals of those are found numerically for which corresponding algorithm and PC programs are developed and composed by authors. Numerical results on the influence of the initial stresses on the values of the critical velocity and on the distribution of the interface stresses are presented and discussed. According to these results, it can be made the following concrete conclusions:

- The initial stretching (compression) of the hollow cylinder and surrounding elastic medium causes an increase (a decrease) in the values of the critical velocity;
- The magnitude of the aforementioned influence increase with decreasing of the ratio h/R where h is a thickness and R is an external radii of the hollow cylinder. At the same time, the values of the critical velocity decrease with the ratio h/R;
- The influence of the initial stresses on the values of the critical velocity depends also significantly on the mechanical properties of the materials of the cylinder and surrounding elastic medium;
- Absolute values of the interface normal and shear stresses increase (decrease) with initial compression (with initial stretching) of the constituents of the system under consideration;
- The magnitude of the influence of the initial stresses on the absolute values of the interface stresses increase with decreasing of the ratio h/R. At the same time, this magnitude increases significantly in the cases where the load moving velocity approaches the corresponding critical velocity;
- An initial compression (stretching) of the system under consideration causes an increase (a decrease) of the stretching interface normal stress which appear in a certain distance from the point at which the moving load acts. The adhesion strength of the considered system can depend significantly on the values of this normal stress;
- The results related to the distribution of the interface stresses with respect to the axial coordinate in the moving coordinate system can be also taken as the change of these stresses with respect to time at a certain point in the reference system of coordinates.

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